Exercises XII

January 20th

E12.1 *Mott scattering*

Consider an electron, confined to a volume V, in an external field $A_\mu(x)$. As a basis for the free solutions of the Dirac equation with momentum p and spin s we choose

$$\psi(x) = \sqrt{\frac{mc^2}{EV}}\, u(p,s)\, \mathrm{e}^{-\frac{i}{\hbar}\langle p,x\rangle} \text{ (electron)}, \quad \psi(x) = \sqrt{\frac{mc^2}{EV}}\, v(p,s)\, \mathrm{e}^{\frac{i}{\hbar}\langle p,x\rangle} \text{ (positron)}$$

with ($v(p,s)$ is $u(p,s)$, the first two components interchanged with the last two)

$$u(p,\tfrac{1}{2}) = \sqrt{\tfrac{E+mc^2}{2mc^2}} \begin{pmatrix} 1 \\ 0 \\ \frac{c\,p^z}{E+mc^2} \\ \frac{c\,p^+}{E+mc^2} \end{pmatrix}, \quad u(p,-\tfrac{1}{2}) = \sqrt{\tfrac{E+mc^2}{2mc^2}} \begin{pmatrix} 0 \\ 1 \\ \frac{c\,p^-}{E+mc^2} \\ \frac{-c\,p^z}{E+mc^2} \end{pmatrix}, \quad p^\pm = p^x \pm \mathrm{i}\, p^y.$$

We consider the external field $A_\mu(x)$ only in lowest order perturbation theory (Born approximation). Then the transition matrix element between ψ_i and ψ_f is

$$S_{fi} = -\frac{\mathrm{i}e}{\hbar c}\int \mathrm{d}^4x\, \overline{\psi}_f(x)\gamma^\mu A_\mu(x)\psi_i(x).$$

(1) Show that the matrix element is in principle determined by the Fourier transform of the four potential, taken at the four momentum transfer:

$$S_{fi} = -\frac{\mathrm{i}e}{V}\frac{mc}{\hbar}\frac{1}{\sqrt{E_iE_f}}(\overline{u}_f\gamma^\mu u_i)(2\pi)^2(\mathcal{F}A_\mu)\left(\frac{p_i-p_f}{\hbar}\right).$$

(2) In particular, consider a Coulomb potential $A_0(x) = -\frac{Ze}{r}$, $\vec{A} = 0$. Using the momentum transfer $\vec{q} = \vec{p}_i - \vec{p}_f$, show that

$$S_{fi} = \mathrm{i}\,4\pi\frac{Ze^2}{V}\frac{mc^2}{\sqrt{E_iE_f}}(\overline{u}_f\gamma^0u_i)\frac{\hbar^2}{|\vec{q}|^2}2\pi\delta(E_f-E_i).$$

(3) Why does $N = \frac{V\,\mathrm{d}^3p_f}{(2\pi\hbar)^3}$ give the number of states in the momentum interval d^3p_f? Thus, the transition probability per particle is given by $N|S_{fi}|^2$.

(4) Compute the number R of transitions to the momentum interval d^3p_f per time:

$$R = \frac{4Z^2\alpha^2(\hbar c)^2}{V}\frac{m^2c^4}{E_iE_f}|\overline{u}_f\gamma^0u_i|^2\frac{1}{|\vec{q}|^4}\mathrm{d}^3p_f\delta(E_f-E_i).$$

(5) Why is $v_i = \frac{|\vec{p}_i|c^2}{E}$ the velocity of incoming particles and $j = \frac{v_i}{V}$ their current density? Compute the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{4Z^2\alpha^2m^2(\hbar c)^2}{|\vec{q}|^4}|\overline{u}_f\gamma^0u_i|^2.$$

(6) In most experiments one will not know the spin polarizations. Thus we have to sum the outgoing states and to average the incoming ones:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{4Z^2\alpha^2m^2(\hbar c)^2}{|\vec{q}|^4}\frac{1}{2}\sum_{s_f,s_i}|\overline{u}_f\gamma^0u_i|^2.$$

Show that this averaging can be written as a trace:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{4Z^2\alpha^2 m^2(\hbar c)^2}{2|\vec{q}\,|^4}\mathrm{Tr}\left(\gamma^0 \frac{\gamma_\mu p_i^\mu + mc}{2mc}\gamma^0 \frac{\gamma_\mu p_f^\mu + mc}{2mc}\right).$$

(7) Why does the trace of a product of an odd number of γ^μ vanish? Show the identity $\mathrm{Tr}(a_\mu \gamma^\mu b_\nu \gamma^\nu) = 4\langle a, b\rangle$ and generalize it. Deduce

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z^2\alpha^2\hbar^2}{2|\vec{q}\,|^4 c^2}(8E_i E_f - 4\langle p_i, p_f\rangle c^2 + 4m^2c^4).$$

(8) Show that $\langle p_i, p_f\rangle = \frac{E^2}{c^2} - p^2\cos\vartheta$ and $|\vec{q}\,|^2 = 4p^2\sin^2\frac{\vartheta}{2}$, where $E_i = E_f = E$, $|\vec{p}_i| = |\vec{p}_f| = p$ and ϑ is the scattering angle. Deduce Mott's cross section ($\beta = \frac{v}{c}$):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z^2\alpha^2\hbar^2}{4p^2\beta^2\sin^4\frac{\vartheta}{2}}\left(1 - \beta^2\sin^2\frac{\vartheta}{2}\right) = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Rutherford}}\left(1 - \beta^2\sin^2\frac{\vartheta}{2}\right).$$

Homework XII

Return: January 27th

H12.1 *Pair creation*

We would like to consider a process similar to E12.1, namely the creation of an electron–positron pair by an external field.

(1) Show that the matrix element is in principle determined by the Fourier transform of the four-potential, taken at the total four-momentum $P = p + p'$:

$$S_{fi} = -\frac{\mathrm{i}e}{V}\frac{mc}{\hbar}\frac{1}{\sqrt{EE'}}\overline{v}(p', s')\gamma^\mu u(p, s)(2\pi)^2(\mathcal{F}A_\mu)\left(\frac{P}{\hbar}\right).$$

The electron has four momentum p, spin s and spinor u, the positron p', s', v.

(2) Perform the sum over the particle spins in $|S_{fi}|^2$ and show using $K = \frac{P}{\hbar}$:

$$|S_{fi}|^2 = \frac{(2\pi)^4 e^2 m^2 c^2}{\hbar^2 V^2 EE'}(\mathcal{F}A_\mu)(K)(\mathcal{F}A_\nu)(-K)\,\mathrm{Tr}\left(\gamma^\mu \frac{\gamma^\kappa p_\kappa + mc}{2mc}\gamma^\nu \frac{\gamma^\kappa p'_\kappa - mc}{2mc}\right).$$

(3) Compute the trace using the rules from E12.1.7 and show

$$|S_{fi}|^2 = \frac{(2\pi)^4 e^2}{\hbar^2 V^2 EE'}(\mathcal{F}A_\mu)(K)(\mathcal{F}A_\nu)(-K)(-m^2c^2 g^{\mu\nu} + p^\mu p'^\nu + p^\nu p'^\mu - g^{\mu\nu}\langle p, p'\rangle).$$

(4) Why is $N = \frac{V^2\mathrm{d}^3p\,\mathrm{d}^3p'}{(2\pi\hbar)^6}$ the number of states in the momentum interval $\mathrm{d}^3p\,\mathrm{d}^3p'$?

(5) Compute the Fourier transform for $A_0 = 0$, $\vec{A} = (a\cos\omega t)\vec{e}_z$, $a \in \mathbb{R}$.

(6) Calculate the number of produced pairs as $N|S_{fi}|^2$:

$$\frac{e^2}{24\pi\hbar^2c^3}|a|^2\omega^2\sqrt{1 - \frac{4m^2c^4}{\hbar^2\omega^2}}\left(1 + \frac{2m^2c^4}{\hbar^2\omega^2}\right)(2\pi)^4\delta\left(\frac{\vec{p} + \vec{p}\,'}{\hbar}\right)\delta\left(\frac{E + E'}{\hbar} - \omega\right).$$

(7) Argue why the number R of pairs per volume and time unit is

$$R = \frac{e^2}{24\pi\hbar^2c^3}|a|^2\omega^2\sqrt{1 - \frac{4m^2c^4}{\hbar^2\omega^2}}\left(1 + \frac{2m^2c^4}{\hbar^2\omega^2}\right).$$

(8) How does the angular distribution of the particle momenta look like?

(9) In which cases can pair production take place at all?

(30 points)