Lecture Theoretical Physics III – fall term 2002/2003 – Michael Flohr

Revision Exam

March 21st, 2003

\# 1 *Addition of angular momenta* (18 points)

Do the addition of angular momenta for $j_1 = 1$, $j_2 = \frac{1}{2}$ explicitly. Start with the states $|1, m_1\rangle \otimes |\frac{1}{2}, m_2\rangle$ and use the action of the raising and lowering operators.

\# 2 *Wigner-Eckart theorem* (4+6+5+3=18 points)

(1) What is a spherical tensor operator of rank r? State an example for $r = 1$.

(2) Give the statements of the Wigner-Eckart theorem and illustrate them briefly.

(3) Which operator is responsible for magnetic dipole transitions? Show which values for $\Delta\ell$ are possible due to the Wigner-Eckart theorem.

(4) Deduce a further restriction on $\Delta\ell$ from the invariance under parity.

\# 3 *Landé factor* (5+2+10+2=19 points)

Consider an atom in a magnetic field $\vec{B} = B\vec{e}_3$ (constant in time, homogeneous in space). Let the magnetic field be so weak that the perturbation Hamiltonian is given by $H' = \omega(L_3 + 2S_3)$, where $\omega = \frac{eB}{2mc}$ is called Larmor frequency.

(1) Let the Hamiltonian H of the atom contain the spin–orbit coupling. Why are then H, $\vec{L}^{\,2}$, $\vec{S}^{\,2}$, $\vec{J}^{\,2}$ and J_3 a complete set of commuting observables?

(2) Why does an energy level have the degeneracy $2j + 1$ at least?

(3) With the projection theorem, show that $H' = g_L\omega J_3$ holds on the subspace of states with fixed E, j, ℓ and s. Compute the Landé factor g_L.

(4) How do the energy eigenvalues of the full Hamiltonian $H + H'$ look like?

\# 4 *Scattering theory* (3+10+13+2=28 points)

(1) How does the differential cross section for elastic scattering on a potential $V(\vec{r})$ look like in Born approximation?

(2) Perform the angular integration in (1) for a radially symmetric potential and compute $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ for a spherically symmetric potential well (depth V_0, radius r_0).

(3) A particle with mass m is scattered on an idealised (i.e. infinitely thin) spherical shell $V(r) = V_0\delta(R - r)$. We consider S wave scattering only. Compute the scattering phase $\delta_{\ell=0}(k)$ by solving the Schrödinger equation exactly.

(4) State the scattering amplitude and the differential cross section for (3).

\# 5 *Many particle theory* (8+6=14 points)

(1) Which algebraic relations do creation and annihilation operator for fermions satisfy? State the occupation number operator. Show which eigenvalues are possible.

(2) Consider a many particle system with dispersion relation $\varepsilon(\vec{p})$ and a two-particle interaction V, i.e. V depends only on the relative distance of the particles. State the Hamiltonian, written in terms of creation and annihilation operators.

\# 6 *Questions on relativistic quantum mechanics* (5+7+8+7+4+6+13+3=53 points)

(1) Which algebra do the γ matrices satisfy? Show $\mathrm{Tr}\,\gamma^\mu = 0$ in general. Which γ^μ are hermitian or antihermitian in standard representation, respectively?

(2) Give the covering map for the Lorentz group. State the two inequivalent representations of the Lorentz group for spin $\frac{1}{2}$ on two-component spinors.

(3) Deduce from the Lorentz covariance of the Dirac equation: $\psi'(x') = S\psi(x)$ and the properties of the matrix S. Give S explicitly in Weyl representation.

(4) How do the two Weyl equations look like? Show, which particles are described by them in very good approximation.
(5) State the Pauli equation. What does that equation predict for the energy of a particle with spin $\frac{1}{2}$ in a magnetic field?
(6) State the parity operator for the Dirac equation. Give reasons for the answer.
(7) Give the Dirac pseudo scalar and the Dirac vector current. Show how they transform under proper orthochrone Lorentz transformations and under parity.
(8) Sketch the energy levels for the solution of the Dirac equation for the hydrogen atom. Which degeneracy is not broken until the Lamb shift is taken into account?

7 *Gauge invariance of the Dirac equation* (8 points)

Consider a charged particle with spin $\frac{1}{2}$ in an electromagnetic potential $A^\mu(x)$. Show how the Dirac spinor of the particle has to transform under gauge transformations $A^\mu(x) \to A^\mu(x) + \partial^\mu \lambda(x)$ such that the Dirac equation remains valid.

8 *Particle in a light pulse* (2+6+4+3+4+11+5+5+2=42 points)

Consider a particle with spin $\frac{1}{2}$ and mass m in an electromagnetic four-potential $A_0 = A_1 = A_3 = 0$, $A_2 = \phi(x^1 - x^0)$. We define $u := x^1 - x^0$.

(1) Why can one take $\psi(x) = \psi_0(u)\exp(\mathrm{i}(k_+(x^1 + x^0) + k_2 x^2 + k_3 x^3))$ as an ansatz?
(2) Using the ansatz (1), convert the Dirac equation to the form ($\alpha^k = \gamma^0\gamma^k$, $\beta = \gamma^0$):

$$-\mathrm{i}(1-\alpha^1)\frac{\partial \psi_0}{\partial u} - k_+(1+\alpha^1)\psi_0 = \left(\alpha^2\left(k_2 - \frac{q}{\hbar c}\phi(u)\right) + \alpha^3 k_3 + \beta\frac{mc}{\hbar}\right)\psi_0.$$

(3) Why is it possible to choose the following form for the matrices α^i and β?

$$\alpha^1 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}.$$

(4) Using (3) and the decomposition $\psi_0 = \binom{\xi}{\eta}$ into two-component spinors, show:

$$-2\mathrm{i}\frac{\partial \eta}{\partial u} = \lambda\xi \quad \text{und} \quad -2k_+\xi = \lambda\eta, \quad \text{where} \quad \lambda = \sigma^1\left(k_2 - \frac{q}{\hbar c}\phi(u)\right) + \sigma^2 k_3 + \sigma^3\frac{mc}{\hbar}.$$

(5) Show that

$$\eta(u) = \exp\left(-\frac{\mathrm{i}}{4k_+}\left(\left(k_3^2 + \left(\frac{mc}{\hbar}\right)^2\right)u + \int_0^u \mathrm{d}u'\left(k_2 - \frac{q}{\hbar c}\phi(u')\right)^2\right)\right)\chi$$

is a solution of (4), where χ is a two-component spinor depending on k_+, k_2, k_3.
(6) For $\phi = 0$, one naturally gets a free spinor $\psi_{\vec{k}}$ with wave vector $\vec{k}$. Show that

$$k_1 = k_+ - \frac{1}{4k_+}\left(k_2^2 + k_3^2 + \left(\frac{mc}{\hbar}\right)^2\right) \quad \text{and} \quad k_0 = k_+ + \frac{1}{4k_+}\left(k_2^2 + k_3^2 + \left(\frac{mc}{\hbar}\right)^2\right)$$

and check that the four-vector k is on the mass shell. How does the sign of k_0 depend on k_+, k_2 and k_3? What happens for $k_+ = 0$?
(7) Let $\phi(u) = 0$ for large negative u and $\phi(u) = a = \text{const.}$ for large positive u. Why does such a ϕ correspond to a light pulse? Apparently, $\psi(u) = \psi_{\vec{k}}$ for large negative u. Show for large positive u: $\psi(u) = \mathrm{e}^{\mathrm{i}\frac{q}{\hbar c}ax^2}\psi_{\vec{k}'}$, with $\vec{k}' = \vec{k} - \frac{q}{\hbar c}a\vec{e}_2$.
(8) Interpret the phase factor and the change in momentum in (7).
(9) Can such a pulse produce particles? Give reasons for the answer.

Good Luck!