Name: ____________________, Student Number: ____________________

*** Important: Fill in and submit with your exam!!! ***

■ 1 *Gauge transformations.* An electromagnetic field is described by the vector potential $\boldsymbol{A}$ and the electrical potential Φ. A particle of mass m and charge q moves in this field. Then, in this case, the Hamiltonian reads

$$H = \frac{1}{2m}(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A})^2 + q\Phi\,.$$

(1) How would H read, if the particle had spin $\frac{1}{2}$? 2:10

(2) A homogenous, time-independent magnetic field $\boldsymbol{B} = (0, 0, B)^t$ allows the choice

$$\boldsymbol{A} = \frac{1}{2}\boldsymbol{B} \times \boldsymbol{x}\,.$$

However, one might also choose $\boldsymbol{A}' = (-Bx_2, 0, 0)^t$. Why? 2:10

(3) Show, how the wave functions of the Schrödinger equation with the choice $\boldsymbol{A}'$ have to be transformed such that they become equivalent to the wave functions for the choice $\boldsymbol{A}$. 6:10

■ 2 *Spin orbit coupling.* In an atom, the coupling between the spin $\boldsymbol{S}$ of an electron and its angular momentum $\boldsymbol{L}$ is given by the operator $H_{LS} = \xi \boldsymbol{L}\cdot\boldsymbol{S}$.

(1) Which values can the total angular momentum $\boldsymbol{J}$ assume? 1:10

(2) Which eigen values has H_{LS} ? 3:10

(3) Assume now $\ell = 1$, and find the normalized eigen states of H_{LS} for this case. 6:10
[Hint: $J_-|j, m_j; L, S\rangle = \sqrt{j(j+1) - m_j(m_j-1)}|j, m_j - 1; L, S\rangle$. Note, using $J_- = L_- + S_-$, you can write down the states in the orthonormal basis $|\ell, m\rangle|S, m_S\rangle$.]

■ 3 Let $a^\dagger$ and a form a pair of creation and annihilation operators with the standard commutator $[a, a^\dagger] = 1$.

(1) Show that the operators

$$L_z = \hbar(\ell - a^\dagger a)\,, \quad L_+ = \hbar\sqrt{2\ell - a^\dagger a}\,a\,, \quad L_- = a^\dagger\hbar\sqrt{2\ell - a^\dagger a}$$

satisfy the angular momentum algebra, i.e. $[L_z, L_\pm] = \pm\hbar L_\pm$ and $[L_+, L_-] = 2\hbar L_z$. 7:10

(2) Check that the relation $\boldsymbol{L}^2 = \hbar^2\ell(\ell+1)$ is fulfilled. 3:10

■ 4 Let the Hamiltonian of an anharmonic oscillator be given as

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \alpha\frac{m^2\omega^2}{\hbar}x^4\,,$$

where $\alpha > 0$.

(1) Find the corrections to the energy in first order perturbation theory, i.e. the corrections to the eigen values $E_n = \hbar\omega(n + \frac{1}{2})$? 10
[Hint: $a = \sqrt{\frac{\omega m}{2\hbar}}\,x + \frac{\mathrm{i}}{\sqrt{2\omega m\hbar}}\,p\,, \quad a^\dagger = \sqrt{\frac{\omega m}{2\hbar}}\,x - \frac{\mathrm{i}}{\sqrt{2\omega m\hbar}}\,p\,.$]

■ 5 *Commutators and Heisenberg picture.* A much simplified model for electrons on a chain is the Hubbard Hamiltonian

$$H = -t\sum_{i,\alpha=\uparrow,\downarrow}\left(c^\dagger_{i+1,\alpha}c_{i,\alpha} + c^\dagger_{i,\alpha}c_{i+1,\alpha}\right) + U\sum_i c^\dagger_{i,\uparrow}c_{i,\uparrow}c^\dagger_{i,\downarrow}c_{i,\downarrow}\,.$$

Here, the fermionic creation and annihilation operators obey the standard anti-commutators $\{c_{i,\alpha}, c^\dagger_{j,\beta}\} = \delta_{i,j}\delta_{\alpha,\beta}$.

(1) Calculate the commutator of $c^\dagger_{j,\beta}$ with H. [6:10]

(2) Now, write down the commutator of $c_{j,\beta}$ with H, without long calculations. [1:10]

(3) Give the equations of motion for the creation and annihilation operators in the Heisenberg picture. [3:10]

■ 6 *Second quantization.* Give, in second quantization for non-relativistic particles, the operators for the following observables:

(1) the particle density $\rho(x)$ and its Fourier transform $\hat{\rho}(q) = \sum_x \mathrm{e}^{-\mathrm{i}qx}\rho(x)$, [4:10]

(2) the particle current density $\boldsymbol{j}(x)$ and its Fourier transform $\hat{\boldsymbol{j}}(q) = \sum_x \mathrm{e}^{-\mathrm{i}qx}\boldsymbol{j}(x)$, [4:10]

(3) and the kinetic energy E. [2:10]

■ 7 A one-dimensional harmonic oscillator of charge q is assumed to be in its ground state at the time $t_a = -\infty$. At this time, a homogenous time-dependent electric field

$$V_t = -qF\mathrm{e}^{-\alpha t^2}x$$

is switched on.

(1) Compute for $t \to +\infty$ to first order perturbation theory the transition probabilities $P_{|0\rangle\to|n\rangle}$ for $n > 0$. [10]

■ 8 *Relativistic quantum mechanics.* A free particle with fixed momentum in z-direction, i.e. $\boldsymbol{p} = (0, 0, p_z)^t$, satisfies the Dirac equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r}, t) = H_{\mathrm{Dirac}}\psi(\boldsymbol{r}, t)\,, \quad H_{\mathrm{Dirac}} = c\boldsymbol{\alpha}\cdot\boldsymbol{p} + mc^2\beta\,,$$

with $\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ and $\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}$.

(1) Make the ansatz $\psi(\boldsymbol{r}, t) = \psi(\boldsymbol{r})\exp(-\mathrm{i}Et/\hbar)$, $\psi(\boldsymbol{r}) = \exp(\mathrm{i}p_z z/\hbar)u$, in order to obtain a four-dimensional homogenous system of equations for fixed momentum in z-direction. [4:10]

(2) Which values of E allow the existence of non-trivial solutions? [3:10]

(3) How do then the 4-spinors u look like? [3:10]
[Hint: It is not necessary to normalize the solutions.]

□ TOTAL: $\Sigma =$ [80]

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GOOD LUCK!

1	2	3	4	5	6	7	8	Σ
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