Statistical Physics Exam

1 Series of questions

1.1 Fermions:

a) Write down the 2-particle wave function for two Fermions with single particle states $\phi_i(x_j), i, j = 1, 2$. (2 P.)

b) What is a Slater determinant? (2 P.)

c) Compute the Fermi-energy ϵ_F of a system of N Fermions in its ground state as a function of N (at zero temperature). (4 P.)

1.2 Second quantization:

a) Express the particle density and the total particle number in terms of the field operators. (2 P.)

b) Write down a generic Hamiltonian of a system of N particles within a potential $U(x)$ and a 2-particle interaction $V(x - x')$ in second quantized form in the momentum representation. (4 P.)

1.3 Fermions II:

a) Express the pair-distribution function $g_{\sigma\sigma'}(x - x')$ in terms of particle-density operators.
Hint: Use the definition $\langle\phi_0|\Psi^\dagger_\sigma(x)\Psi^\dagger_{\sigma'}(x')\Psi_{\sigma'}(x')\Psi_\sigma(x)|\phi_0\rangle = \left(\frac{n}{2}\right)^2 g_{\sigma\sigma'}(x - x')$. (2 P.)

b) Compute the pair-distribution function for free fermions, discuss separately the cases $\sigma = \sigma'$ and $\sigma \neq \sigma'$.
Hint: Work in momentum representation. (6 P.)

1.4 Density matrix and correlators:

a) Write down the density matrix ρ for a canonical ensemble. What is the partition function Z? (2 P.)

b) Express the expectation value $\langle O\rangle$ of an observable O with the help of the density matrix. (2 P.)

c) Prove that, if the Hamiltonian H is not explicitly time-depedent, the correlation functions satisfy $\langle A(t)B(t')\rangle = \langle A(t - t')B(0)\rangle$.
Hint: Go to the Heisenberg picture. (4 P.)

2 Exercices

2.1 Heisenberg model:

The Heisenberg model of a ferromagnet is defined by the Hamiltonian

$$H = -\frac{1}{2}\sum_{l,l'} J(|l-l'|)\,\vec{S}_l \cdot \vec{S}_{l'},$$

where l and l' are nearest neighbor sites on a cubic lattice. In the large spin approximation ($S \gg 1$), the *Holstein-Primakoff transformation*

$$\begin{aligned} S_i^+ &= \sqrt{2S}\phi(\hat{n}_i)a_i, \\ S_i^- &= \sqrt{2S}a_i^\dagger\phi(\hat{n}_i), \\ S_i^z &= S - \hat{n}_i\,, \end{aligned}$$

with the number S denoting the total spin, $\phi(\hat{n}_i) = \sqrt{1-\hat{n}_i/2S}$ and $\hat{n}_i = a_i^\dagger a_i$, can be used to express the Hamiltonian in terms of Bose operators a_i.

a) Show that the commutation relations for the spin are satisfied. (4 P.)

b) Write down the Hamiltonian to second order (harmonic approximation) in terms of the Bose operators a_i by regarding the square-roots in the above transformation as a short hand for the series expansion. (6 P.)

c) Diagonalize H by means of a Fourier transform and determine the dispersion relation of the spin waves (magnons). (6 P.)

2.2 Bogoliubov theory of the Bose liquid:

In Bogoliubov's theory, the Hamiltonian of the Bose liquid reads:

$$H_2 = \sum_k \left(\frac{\hbar^2 k^2}{2m} - \mu\right) a_k^\dagger a_k + \frac{n_0 g}{2}(a_k^\dagger a_{-k}^\dagger + 4a_k^\dagger a_k + a_{-k}a_k)$$

where $\mu = n_0 g$ being the chemical potential.

We introduce the Bogoliubov transformation:

$$\begin{aligned} a_k &= u_k\alpha_k + v_k\alpha_{-k}^\dagger \\ a_k^\dagger &= u_k\alpha_k^\dagger + v_k\alpha_{-k}. \end{aligned}$$

where u_k and v_k are real and obey $u_k^2 - v_k^2 = 1$.

The idea is to rewrite the Hamiltonian in terms of these new operators and then to vary the parameters u_k and v_k to make it diagonal.

a) Writing the Bogoliubov transformation in the matrix form,

$$\begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix}$$

show that the pair of equations can be inverted to yield

$$\begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ -v_k & u_k \end{pmatrix} \begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix}$$ (4 P.)

b) Rewrite the Hamiltonian H_2 in the matrix form

$$H_2 = \sum_k \begin{pmatrix} a_k^\dagger & a_{-k} \end{pmatrix} \begin{pmatrix} \epsilon_k + n_0 g & n_0 g/2 \\ n_0 g/2 & 0 \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix}$$

where $\epsilon_k = \hbar^2 k^2/2m$. (4 P.)

c) Use the inverse of the Bogoliubov transformation to express H_2 in terms of the $b's$ operators, namely you should find:

$$H_2 = \sum_k \begin{pmatrix} b_k^\dagger & b_{-k} \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix}$$

where the coefficients M_{ij} have to be computed explicitly. (8 P.)

d) Show that the condition for the transformed matrix to be diagonal is that

$$\frac{2u_k v_k}{u_k^2 + v_k^2} = \frac{n_0 g}{\epsilon_k + n_0 g}$$

must be satisfied. (4 P.)

e) Show that the trace of the M matrix is

$$E = (\epsilon_k + n_0 g)(u_k^2 + v_k^2) - 2 n_0 g u_k v_k \,.$$

Using the representation $u_k = \cosh(\theta)$ and $v_k = \sinh(\theta)$, show from **d)** that

$$\tanh(2\theta) = \frac{n_0 g}{\epsilon_k + n_0 g}$$

and hence prove that

$$E = \sqrt{\epsilon_k(\epsilon_k + 2n_0 g)}\,,$$

consistent with the Bogoliubov quasiparticle dispersion given in the lecture. (4 P.)

Total: (70 P.)