TOPOLOGICAL SPACES

One starts with *sets* S of points. If a point s_i is an element of S, one writes $s_i \in S$. Let S_1 and S_2 be two sets of points. One says that S_1 is a *subset* of S_2, if every point of S_1 is also contained in S_2, and denotes this as $S_1 \subset S_2$. If there exist points in S_2, which are not contained in S_1, then S_1 is called a *proper subset* of S_2. The set which does contain no points is called *empty set*, denoted by the symbol $\emptyset$. The *union* $S_1 \cup S_2$ of two sets S_1 and S_2 contains all points which are either in S_1 or in S_2. The *intersection* $S_1 \cap S_2$ of two sets S_1 and S_2 contains all poitns which are contained in both set, S_1 and S_2.

Topological space. A *topological space* T is a set of points which is equipped with a *topology* $\mathcal{T}$. A topology $\mathcal{T}$ is a selection (i.e. a set) of subsets $S_1, S_2, \ldots$ of T, i.e. $S_i \subset T$, $S_i \in \mathcal{T}$, which satisfies the following three axioms:

- **[T1]** The empty set $\emptyset$ and the whole space T belong to $\mathcal{T}$, i.e. $\emptyset \in \mathcal{T}$ and $T \in \mathcal{T}$.
- **[T2]** Finite intersections of elements of $\mathcal{T}$ are again contained in $\mathcal{T}$, i.e. $\bigcap_{i\in I} S_i \in \mathcal{T}$ for $|I| < \infty$.
- **[T3]** Arbitrary unions of elements of $\mathcal{T}$ are again contained in $\mathcal{T}$, i.e. $\bigcup_{i\in I} S_i \in \mathcal{T}$.

The elements S_i of the topology $\mathcal{T}$ are called *open sets*. This concept is too general for the definition of manifolds. One needs one further axion, which expresses the concept of *separability*. This means that one can always find for two differen points p, q in T two open subsets S_p and S_q, which contain p and q respectively, but which do not overlap. More precisley, this reads:

- **[T4]** If $p \in T$ and $q \in T$ with $p \neq q$ are two different point, then there exist $S_p \in \mathcal{T}$ and $S_q \in \mathcal{T}$ with the properties $p \in S_p$, $q \in S_q$, and $S_p \cap S_q = \emptyset$.

A topological space, which additionally satisfies **[T4]**, is called *Hausdorff space*. An open set S_p, which contains the point p, is also called *neighborhood* of p. To emphasize that T is a topological space, one typically gives the tuple $(T, \mathcal{T})$ instaed.

MANIFOLDS

The topological concept of continuity is very easy to define: A map $\phi : (T, \mathcal{T}) \to (U, \mathcal{U})$ of a topological space T to another topological space U is called *continuous*, if the pre-image of every open set in U is an open set in T. Now we have everything in place in order to define manifolds.

Differentiable manifolds. A *differentiable manifold* $\mathcal{M}$ is a Hausdorff space $(T, \mathcal{T})$ together with a set Φ of maps $\phi_p \in \Phi$, $\phi_p : T \to \mathbb{R}^n$, $p \in T$, with the following four properties:

- **[M1]** ϕ_p is a one-to-one map of an open set T_p with $p \in T_p$ onto an open set in $\mathbb{R}^n$.
- **[M2]** The union $\bigcup_p T_p = T$.
- **[M3]** If $T_p \cap T_q$ is not empty, then $\phi_p(T_p \cap T_q)$ is an open set in $\mathbb{R}^n$, and $\phi_q(T_p \cap T_q)$ is an open set in $\mathbb{R}^n$ different from $\phi_p(T_p \cap T_q)$. The map $\phi_p \circ \phi_q^{-1}$ must be continuous and differentiable.
- **[M4]** The maps $\phi_p \circ \phi_q^{-1}$ and $\phi_q \circ \phi_p^{-1}$ from axiom **[M3]** are maps in Φ. This property is also called *maximality*.

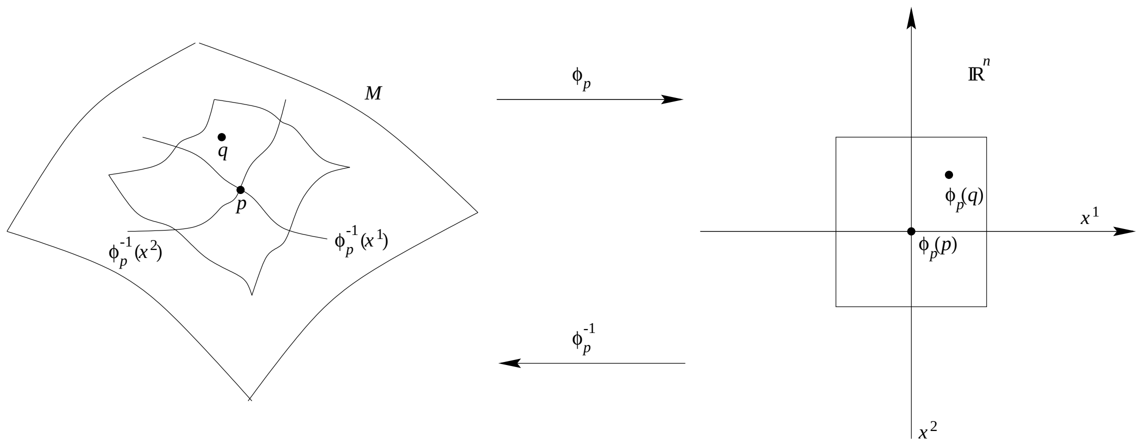
The property **[M1]** admits to construct a coordinate system for the neighborhood of any point p. The point p is mapped to the origin of $\mathbb{R}^n$ with the help of ϕ_p. Each point q near p is thus mapped to a point $\phi_p(q)$ near $\mathbf{0} = \phi_p(p)$. In this way, one can associate the coordinates $\phi_p^i(q)$, $i = 1, \ldots, n$ of $\phi_p(q) \in \mathbb{R}^n$ with the original point $q \in T$. This association yields a (local) coordinate system for the whole space T.

Axiom **[M2]** ensures that such a local coordinate system can indeed be constructed for each point in T.

Axiom **[M3]** makes use of the map $\phi_p \circ \phi_q^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, which can be studied with standard methods from differential calculus. If here, and in the following, a map ϕ is called differentiable, we always mean that $\phi \in \mathcal{C}^\infty$, i.e. that ϕ is smooth.

Manifolds are a very important concept. They are just sufficiently benign to look everywhere locally like the Euklidean space $\mathbb{R}^n$. Therefore, we can use all mehtods and concepts, which we know for the study of Euklidean spaces $\mathbb{R}^n$, since we can transfer them via the axioms **[M1]** to **[M3]** to manifolds. In particular, the *dimension* of a manifold $\mathcal{M}$ is just the dimension of the space $\mathbb{R}^n$, namely n.

Complex manifolds. One can define other sorts of manifolds in the same manner. The property of smoothness is then replaced by other properties in the same consistent way. For example, we obtain a complex manifold, if we replace in the above axioms everywhere $\mathbb{R}^n$ by $\mathbb{C}^n$, and at the end of **[M3]** the word differentiable by *complex analytic*.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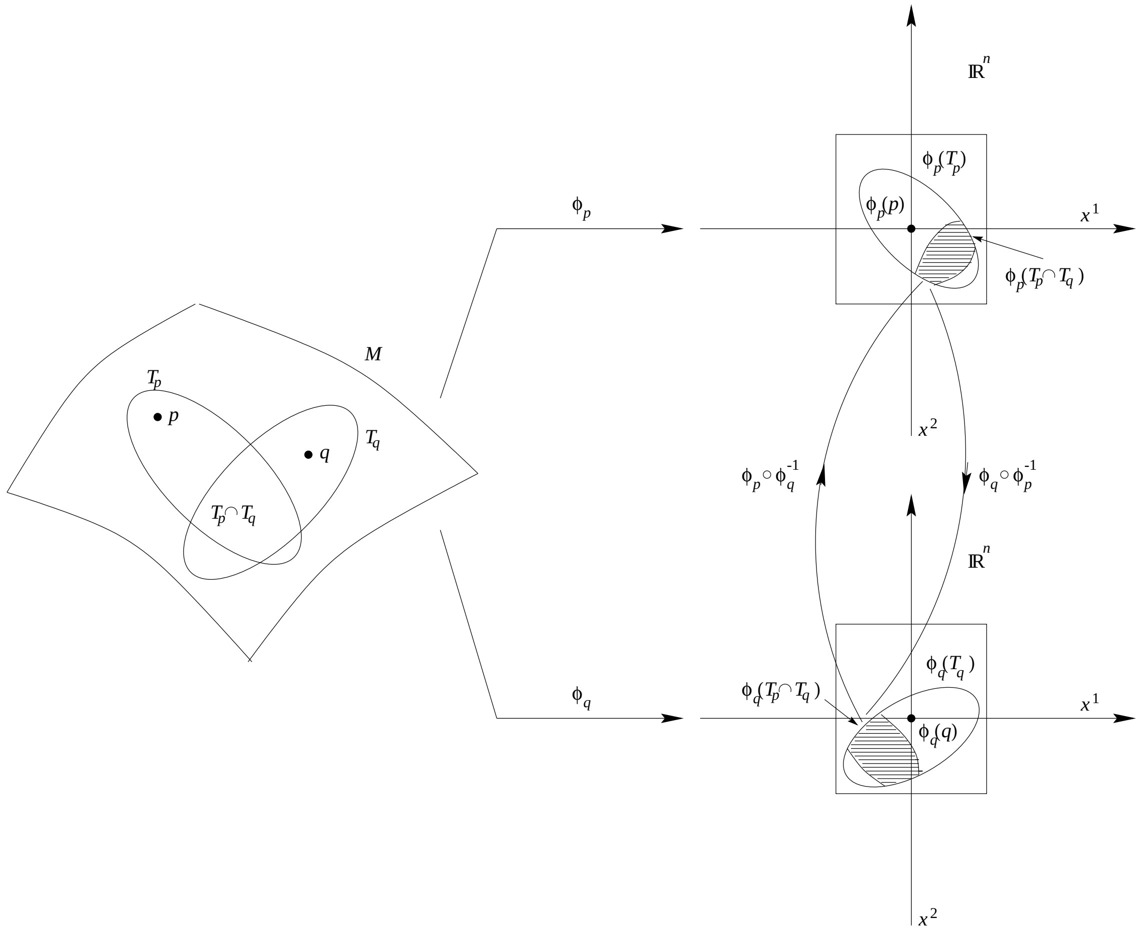
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