## GENERALIZING THE HIGHEST WEIGHT PROCEDURE FROM $\mathfrak{su}(2)$ TO $\mathfrak{su}(3)$

We found the representations of $\mathfrak{su}(2)$ by analysing the decompositions of their vector spaces into eigenspaces with respect to the diagonalizable generator $H$. The correct generalization of this procedure can be fully understood by considering the next simple example, $\mathfrak{su}(3)$. Any other semi-simple Lie algebra $\mathfrak{g}$ can be analysed in exactly the same manner. In order to emphasize the general validity of the concepts, I will often simply write $\mathfrak{g}$ instead of $\mathfrak{su}(3)$, although explicit computations will always be performed for the example Lie algebra $\mathfrak{su}(3)$. The good news is: If you have understood how this general method works, you don't have to learn any other additional concepts in order to understand Lie algebras.

**Setup.** The Lie group $SU(3)$ is the group of unitary $3\times 3$ matrices with determinant one. It is a nice exercise, to consider instead the complex Lie group $SL(3,\mathbb{C})$, which is the group of $3\times 3$ matrices with determinant one. For the latter group, the complex dimension is therefore $\dim_{\mathbb{C}} SL(3,\mathbb{C}) = 8$. The corresponding Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ is given by the traceless $3\times 3$ matrices. We decompose this Lie algebra as follows: $\mathfrak{sl}(3,\mathbb{C}) = \mathfrak{h}\oplus\mathfrak{n}_+\oplus\mathfrak{n}_-$, where $\mathfrak{h}$ is the *Cartan subalgebra*, $\mathfrak{h}\subset\mathfrak{g}$, which is the maximal Abelian subalgebra. Remember that mutually commuting diagonalizable matrices can be diagonalized simultaneously. The Cartan subalgebra takes the place of the single element $H$ in the $\mathfrak{su}(2)$ algebra. In our example, we have

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} : a_1+a_2+a_3 = 0 \right\},$$

i.e. $\dim_{\mathbb{C}}\mathfrak{h} = 2$. The dimension of the Cartan subalgebra is also called the *rank* of the Lie algbera $\mathfrak{g}$. The other two subalgebras are spanned by the generators $E_{i,j}$. These are matrices which have precisely one non-zero entry, namely a one at the place where the $i$-th row and the $j$-th column intersect, i.e. $(E_{i,j})_{kl} = \delta_{ik}\delta_{jl}$. Thus, we have $\mathfrak{n}_+ = \mathrm{span}\{E_{1,2},E_{1,3},E_{2,3}\}$, and $\mathfrak{n}_- = \mathrm{span}\{E_{2,1},E_{3,1},E_{3,2}\}$. Note that all this remains true for $\mathfrak{su}(3)$, but we leave it as an exercise to you to reduce the complex vector space to a real one.

**Definitions.** Let $V$ be an arbitrary representation of $\mathfrak{g}$. An *eigenvector* $v\in V$ of $\mathfrak{h}\subset\mathfrak{g}$ is a vector $v$, which is an eigenvector for each $H\in\mathfrak{h}$,

$$H(v) = \alpha(H)\,v\,, \qquad (*)$$

where $\alpha(H)$ is a scalar which depends linearly on $H$, i.e. $\alpha\in\mathfrak{h}^*$.
An *eigenvalue* of the action of $\mathfrak{h}\subset\mathfrak{g}$ is an element $\alpha\in\mathfrak{h}^*$, such that there exists a $v\in V$, $v\neq 0$, for which $(*)$ holds. An *eigenspace* associated to $\alpha\in\mathfrak{h}^*$ is the subspace $V_\alpha$ of all $v\in V$, for which $(*)$ holds.
The crucial generalization from the case $\mathfrak{su}(2)$, or $\mathfrak{sl}(2,\mathbb{C})$, to a generic semi-simple Lie algebra $\mathfrak{g}$ is the ansatz, that any finite dimensional representation $V$ of $\mathfrak{g}$ has a decomposition $V = \bigoplus_\alpha V_\alpha$, where $V_\alpha$ is eigenspace of $\mathfrak{h}\subset\mathfrak{g}$, and $\alpha$ runs through a finite subset of $\mathfrak{h}^*$.

**Adjoint Representation.** To find the analoga of $E_\pm\in\mathfrak{su}(2)$, observe that the Lie bracket $[H,E_\pm] = \pm 2E_\pm$ defines the elements $E_\pm$ as eigenvectors of the adjoint action of $H$ on $\mathfrak{su}(2)$. Note our change in normalization. The normalization choosen here is more appropriate, if you wish to switch between the real and the complex case, i.e. between considering $\mathfrak{su}(2)$ or $\mathfrak{sl}(2,\mathbb{C})$. Let us therefore consider the adjoint representation of $\mathfrak{sl}(3,\mathbb{C})$, which has a decomposition

$$\mathfrak{g} = \mathfrak{h}\oplus\left(\bigoplus_\alpha \mathfrak{g}_\alpha\right),$$

where $\alpha$ runs through a finite subset of $\mathfrak{h}^*$, and $\mathfrak{h}$ acts on each space $\mathfrak{g}_\alpha$ by scalar multiplication, i.e.

$$\forall H\in\mathfrak{h}\,,\ \forall Y\in\mathfrak{g}_\alpha : [H,Y] \equiv \mathrm{ad}(H)(Y) = \alpha(H)\,Y\,.$$

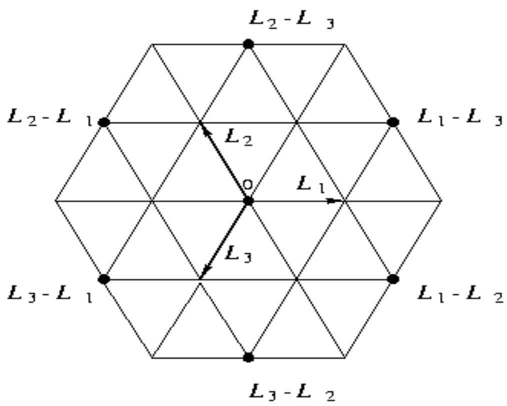
Let $M$, $(M)_{kl} = m_{kl}$, be an arbitrary matrix. Its commutator with a diagonal matrix $D$, $(D)_{kl} = a_k\delta_{kl}$ is $([D,M])_{kl} = (a_k - a_l)m_{kl}$. This shall now be a scalar multiple of $M$ for all $D$. The only possibility for this is the choice $M = E_{i,j}$. Thus, the matrices $E_{i,j}$ precisley generate the eigenspaces of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. With the above explicit definition of $\mathfrak{h}$ for $\mathfrak{sl}(3,\mathbb{C})$ the dual space is defined as

$$\mathfrak{h}^* = \mathrm{span}_{\mathbb{C}}\{L_1,L_2,L_3\}/\{L_1+L_2+L_3 = 0\}\,,$$

where the linear functionals $L_i$ yield the dual basis to the standard basis of the diagonal $3 \times 3$ matrices, i.e.

$$L_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i \, .$$

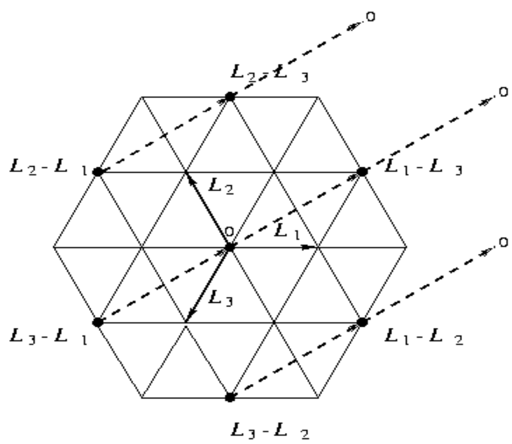
We conclude that the linear functionals $\alpha \in \mathfrak{h}^*$, which occur in the decomposition of $\mathfrak{sl}(3,\mathbb{C}) = \mathfrak{h} \oplus (\bigoplus_\alpha \mathfrak{g}_\alpha)$, are the six functionals $L_i - L_j$, $1 \leq i \neq j \leq 3$. The spaces $\mathfrak{g}_{L_i - L_j}$ are spanned by one element each, namely by $E_{i,j}$.




This drawing contains more or less all the information on the structure of the Lie algbra $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$. For example, let $X \in \mathfrak{g}_\alpha$. To find out to where $\mathrm{ad}(X)$ would map $Y \in \mathfrak{g}_\beta$, we only have to repeat our *fundamental computation* for an arbitrary $H \in \mathfrak{h}$,

$$\begin{aligned} [H,[X,Y]] &= [X,[H,Y]] + [[H,X],Y] \\ &= [X, \beta(H)\, Y] + [\alpha(H)\, X, Y] \\ &= (\alpha(H) + \beta(H))[X,Y] \, . \end{aligned}$$

Thus, $[X,Y] = \mathrm{ad}(X)(Y)$ is again an eigenvector of $\mathfrak{h}$, and it has eigenvalue $\alpha + \beta$, i.e. $\mathrm{ad}(\mathfrak{g}_\alpha) : \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}$. Since $\mathrm{ad}(\mathfrak{g}_\alpha)$ maps eigenspaces to eigenspaces, the decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_\alpha \mathfrak{g}_\alpha)$ is preserved. In our picture, all the $\mathfrak{g}_\alpha$ act by translation. Hence, the action of $\mathfrak{g}_{L_1 - L_3}$ is given by $\mathrm{ad}(\mathfrak{g}_{L_1-L_3})(\mathfrak{g}_{L_3-L_1}) \subset \mathfrak{h}$, $\mathrm{ad}(\mathfrak{g}_{L_1-L_3})(\mathfrak{h}) \subset \mathfrak{g}_{L_1-L_3}$, $\mathrm{ad}(\mathfrak{g}_{L_1-L_3})(\mathfrak{g}_{L_1-L_3}) = 0$ etc., i.e.




**Arbitrary representation.** This holds true in an analogous way for any representation $V$ of $\mathfrak{sl}(3,\mathbb{C})$. The representation has a decomposition $V = \bigoplus_\alpha V_\alpha$, and the $\mathfrak{g}_\alpha$ map $V_\beta$ to $V_{\alpha+\beta}$, since

$$\begin{aligned} H(X(v)) &= X(H(v)) + [H,X](v) \\ &= X(\beta(H)\, v + (\alpha(H)\, X)(v) \\ &= (\alpha(H) + \beta(H))X(v) \, . \end{aligned}$$

We can picture the $V_\alpha$ by points in a (planar) diagram, which are maped into each other by the $\mathfrak{g}_\alpha$ through translation. Therefore, the eigenvalues $\alpha$, which occur in the decomposition of an irrep $V$, differ from each other by *integer* linear combinations of the vectors $L_i - L_j \in \mathfrak{h}^*$. This motivates some further very important definitions.

**More definitions.** The *roots* are the set $R = \{L_i - L_j : i \neq j\}$. Their integer linear combinations span the *root lattice* $\Lambda_R = \bigoplus_{L \in R} \mathbb{Z} L$. Note that with this convention zero is not a root.

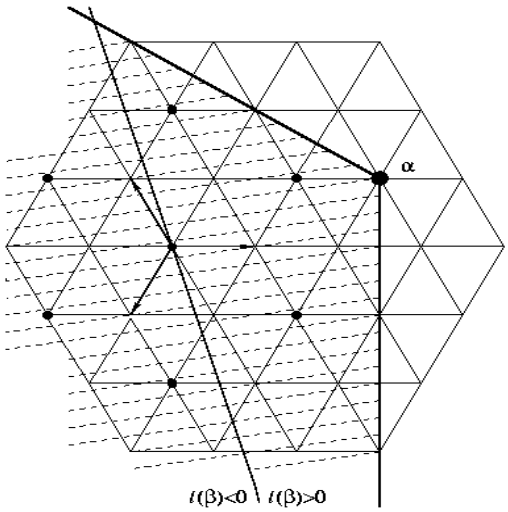
The eigenvalues $\alpha$, which occur in the decomposition of a representation $V = \bigoplus_\alpha V_\alpha$, are called the *weights* of the representation. The differences of weights of a representation are always $\alpha - \alpha' \in \Lambda_R$. Further, the eigenvectors $v \in V_\alpha$ are called *weight vectors* to the weights $\alpha$, and the eigenspaces $V_\alpha$ are called *weight spaces*. Thus, the roots are the weights of the adjoint representation. Consequently, the $\mathfrak{g}_\alpha$ are also called *root spaces*.

**Highest weights.** We learned in the lecture that there exist so-called *highest weights* among the weights $\alpha$, which have the property that there exists a corresponding *highest weight vector* $v \in V$, which is annihilated by one half of all the roots. Therefore we have

LEMMA: $\exists v \in V$ : (i) $\exists \alpha \in \mathfrak{h}^* : v \in V_\alpha$; (ii) $\forall 0 < i < j : E_{i,j}(v) = 0$. $\square$

As explained in the lecture, the ordering in (ii) is arbitrary, other ordering prescriptions define other highest weight vectors. More precisely, on has a linear function $\ell : \mathfrak{h}^* \to \mathbb{C}$, which can be choosen real if restriced to the integer linear combinations of the $L_i$. The roots $\alpha$ for which $\ell(\alpha) > 0$, are called *positive* roots. In our example, $\ell(a_1L_1 + a_2L_2 + a_3L_3) = aa_1 + ba_2 + ca_3$ with $a > b > c$ arbitrary real numbers, such that $\{\gamma : \ell(\gamma = 0)\}$ is irrational to $\Lambda_R$, is a suitable choice. Furthermore, we define $H_{i,j} = [E_{i,j}, E_{j,i}] = E_{i,i} - E_{j,j}$. There are special roots among all the positive roots, which have the property that they cannot be written as a sum of two other positive roots. Such roots are caleld *simple* roots. In our example, $L_2 - L_3$ and $L_1 - L_2$ are simple roots, while $L_1 - L_3 = (L_1 - L_2) + (L_2 - L_3)$ is not simple. Moreover, the following important fact holds: An irrep $V$ is completely generated from a highest weight vector $v \in V$ through the images of $v$ under successively acting with the $E_{j,i}$, $0 < i < j$. This has several immediate consequences:

(1) All $\beta \in \mathfrak{h}^*$, which occur in the decomposition of $V$, lie in a cone whose point is $\alpha$. For $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$, this is a 1/3-cone:




(2) $\dim V_\alpha = 1$, i.e. the highest weight vector $v$ is unique up to normalization.

(3) $\dim V_{\alpha+n(L_2-L_1)} = \dim V_{\alpha+n(L_3-L_2)} = 1$, since these spaces can only be generated by the action of $(E_{2,1})^n(v)$ or $(E_{3,2})^n(v)$, respectively. More generally, this holds true for all spaces, whose weights lie on the boundary of the cone.

Conversely, it is also true for an arbitrary representation $V$ and a highest weight vector $v \in V$ that the subrepresentation $W \subset V$ generated from images of $v$ under successive action of the $E_{j,i}$, $0 < i < j$, is irreducible. This can easily be proven by complete induction. Taking all together we have

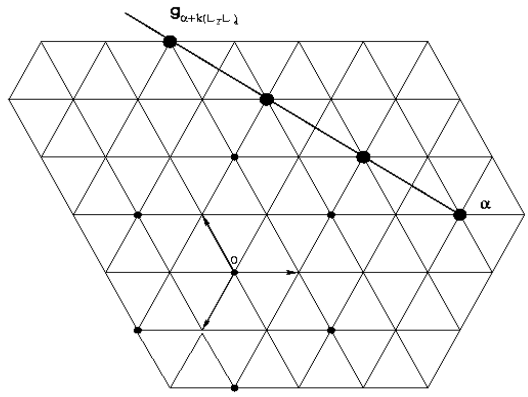
PROPOSITION: Any irrep $W$ of $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ possesses an up to normalization unique highest weight vector. The set of highest weight vectors of an arbitrary representation $V$ forms a union of linear subspaces $\Psi_W$ corresponding to the irreps $W$ in $V$, where $\dim \Psi_W$ is the multiplicity of $W$ in $V$. $\square$

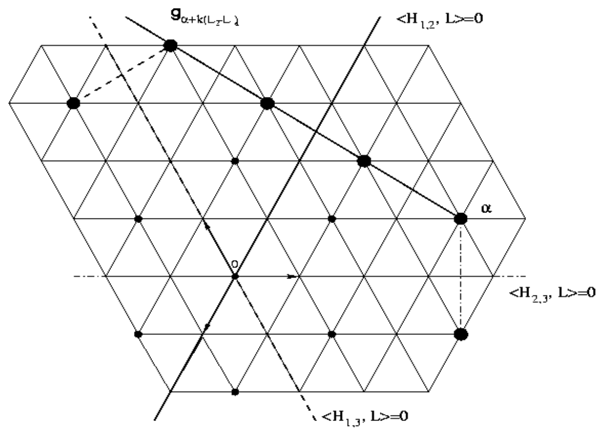
**Convex hull.** One sees that $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus H_\alpha \oplus \mathfrak{g}_{-\alpha} \cong \mathfrak{sl}(2, \mathbb{C})$. Note that in the lecture, we did write $\mathfrak{g}_\alpha$ for the $\mathfrak{s}_\alpha$, since we did not deal directly with the strictly upper or lower triangular Lie algebras which we use in the approach of this handout. In our example, we explicitly have that $\mathfrak{s}_{L_i - L_j} = \mathrm{span}\{E_{i,j}, H_{i,j}, E_{j,i}\}$ is a subalgebra of $\mathfrak{sl}(3, \mathbb{C})$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ for $i < j$. Analysing representations with respect to these subalgebras, one can exploit the condition $\dim V < \infty$ in the same way as in our dealing with the representations of $\mathfrak{su}(2)$. Let us, for instance, consider $\mathfrak{s}_{L_1 - L_2}$. The space $W = \bigoplus_k \mathfrak{g}_{\alpha + k(L_2 - L_1)}$ is preserved under $\mathfrak{s}_{L_1 - L_2}$, thus it is a representation of $\mathfrak{s}_{L_1 - L_2} \cong \mathfrak{sl}(2, \mathbb{C})$. Therefore, the eigenvalues of $H_{1,2}$ are integers on $W$ and symmetric with respect to zero.[1] Hence, the string of points in our diagram, which starts at $\alpha$ and leads in the direction of $L_2 - L_1$, must be symmetric

[1] Note once more that our normalization for the generators of $\mathfrak{sl}(2, \mathbb{C})$ differs by a factor of two from the one choosen for $\mathfrak{su}(2)$ in the lectures, which is the reason why all eigenvalues of $H$ turn out to be integers.

with respect to a line $\langle H_{1,2}, L\rangle$ in the plane $\mathfrak{h}^*$. It is no accident that $L \perp (L_1 - L_2)$ also in our diagram. The string of points $\mathfrak{g}_{\alpha+k(L_2-L_1)}$ has finite length and is invariant and a reflection with respect to the line $\langle H_{1,2}, L\rangle$.

In general one can consider $\mathfrak{s}_{L_i-L_j} = \mathrm{span}\{E_{i,j}, E_{j,i}, H_{i,j} = [E_{i,j}, E_{j,i}]\} \cong \mathfrak{sl}(2,\mathbb{C})$, which are all subalgebras of $\mathfrak{sl}(3,\mathbb{C})$. Thus, also the string of points $\mathfrak{g}_{\alpha+k(L_3-L_2)}$ must be invariant under reflection at the line $\langle H_{2,3}, L\rangle = 0$. Let $m$ be the smallest number, for which $(E_{2,1})^m(v) = 0$, and let $\beta = \alpha + (m-1)(L_2 - L_1)$ and $v' \in V_\beta$. Per definitionem $E_{2,1}(v') = 0$, and there exists no $V_\gamma$ above the boundary of the cone, i.e. $E_{2,3}(v') = 0$ and $E_{1,3}(v') = 0$. Thus, $v'$ is also a highest weight vector. Everything we did so far for $\alpha$, can be repreated for $\beta$. At the end of the string of points $\mathfrak{g}_{\beta+k(L_3-L_1)}$ we find a $v''$, which again can be viewed as a highest weight vector, and which is annihilated by $E_{3,1}$ and $E_{2,1}$. All this can be visualized in the following two diagrams:

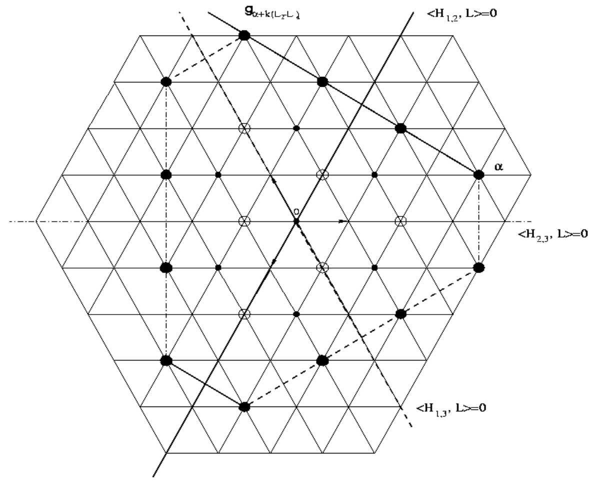



It follows, after performing this procedure successively for all the higehst weight vectors which one obtains step by step, that the eigenvalues, which occur in the decomposition of $V$, are bounded by a hexagon which is symmetric under reflections at the lines $\langle H_{i,j}, L\rangle = 0$, and which has a vertex at the point $\alpha$. Remark: The hexagon can degenerate to a triangle under certain circumstances, if two of the corners coincide pairwise. Put differently, the hexagon is the convex hull of the union of the images of $\alpha$ under the group of isometries of the plane, which is generated by reflections at the lines $\langle H_{i,j}, L\rangle = 0$. Since $\mathfrak{s}_{L_i-L_j} \cong \mathfrak{sl}(2,\mathbb{C})$, the eigenvalues $H_{i,j} \in \mathbb{Z}$ and therefore it is true that $\alpha \in \bigoplus_i \mathbb{Z}L_i = \Lambda_W$. $\Lambda_W$ is called the *weight lattice*. We obtain

PROPOSITION: All eigenvalues of an finite dimensional irrep $V$ of $\mathfrak{sl}(3,\mathbb{C})$ must lie on the lattice $\Lambda_W \subset \mathfrak{h}^*$, which is spanned by the $L_i$. Furthermore, all weights of the representation must be congruent modulo the lattice $\Lambda_R \subset \mathfrak{h}^*$, which is spanned by the roots $L_i - L_j$. □

Note that $\Lambda_W/\Lambda_R = \mathbb{Z}/3$ for $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$, while for $\mathfrak{sl}(2,\mathbb{C})$ we have $\Lambda_W/\Lambda_R = \mathbb{Z}/2$. This results in the following diagram:




One should keep in mind that the spaces $W_{\alpha,i,j} = \bigoplus_k \mathfrak{g}_{\alpha+k(L_i-L_j)}$ are not the only subspaces invariant under $\mathfrak{s}_{L_i-L_j}$. In fact, for a given $\beta \in \mathfrak{h}^*$, which is eigenvalue of the decomposition $V = \bigoplus_\alpha V_\alpha$, and for all $i \neq j$, the spaces $W_{\beta,i,j} = \bigoplus_k \mathfrak{g}_{\beta+k(L_i-L_j)}$ form a representation of $\mathfrak{s}_{L_i-L_j}$ which, however, is not necessarily irreducible. It follows from this fact that at least the number $k$, for wich $V_{\beta+k(L_i-L_j)} \neq 0$, form a gapless sequence of integers. Thus, all points inside the convex hull, which are elemets of $\Lambda_W/\Lambda_R$, also belong to the allowed eigenvalues. In the above diagram, I have marked these points by open circles. A highest weight representation can thus be understood diagrammatically in the following way: Choose within the lattice $\Lambda_W$ the highest weight, construct the convex hull and finally mark all points inside the convex hull which lie on the lattice $\Lambda_W/\Lambda_R$. Let me summarize: PROPOSITION: Let $V$ be an irrep of $\mathfrak{sl}(3,\mathbb{C})$. Then there exists an $\alpha \in \Lambda_W \subset \mathfrak{h}^*$, such that the set of weights, which occur in the decomposition $V = \bigoplus_\beta V_\beta$, is precisely the set of linear functionals congruent to $\alpha$ modulo $\Lambda_R$ and bounded by the convex hull, whose corners are given by the images of $\alpha$ under the action of the reflection group generated from reflections with respect to the lines $\langle H_{i,j}, L\rangle = 0$. $\square$

### EXPLICIT CONSTRUCTION OF IRREPS OF $\mathfrak{sl}(3,\mathbb{C})$

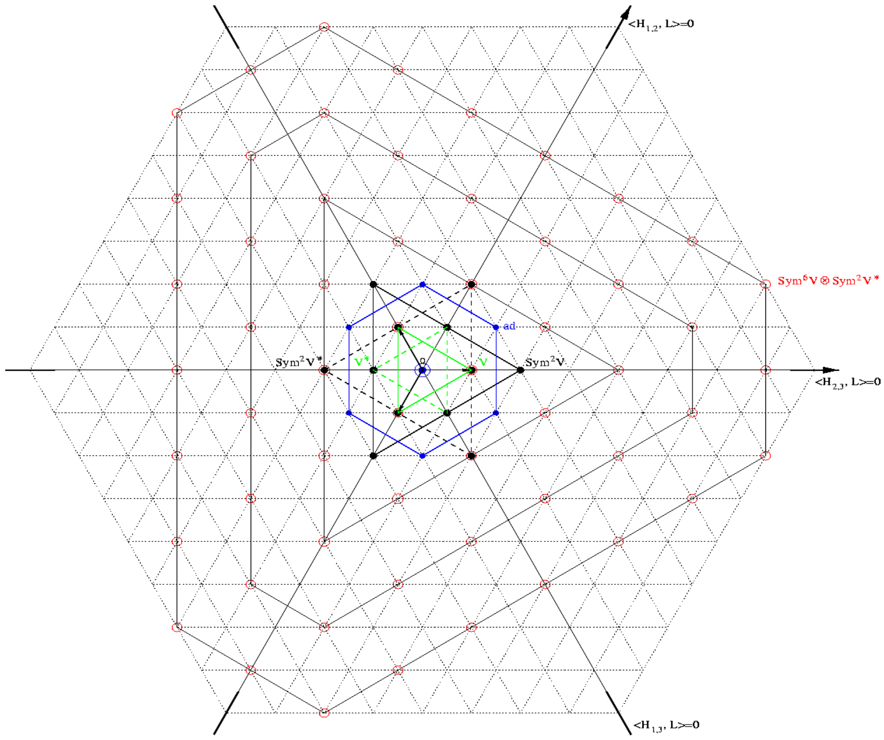
To gain a complete undersanding of representations of $\mathfrak{sl}(3,\mathbb{C})$, we first have to show the existence and uniqueness of irreps. It would be desirabel to have a similarly explicit construction as in the case of $\mathfrak{sl}(2,\mathbb{C})$, where every irrep can be written in terms of symmetric tensor products $\mathrm{Sym}^n V$ of the standard representation for some $n \in \mathbb{Z}_+$. Of course, for the proper analysis of tensor products we not only need the weights of the representations, but also the multiplicities with which they occur. (For $\mathfrak{su}(2)$, each rep. can be written as a tensor product of spin 1/2 reps.)

**Elementary examples.** The standard representation of $\mathfrak{sl}(3,\mathbb{C})$ is, of course, nothing else that $V \cong \mathbb{C}^3$. The eigenvectors of the action of $\mathfrak{h}$ are naturally the standard basis vectors $e_1, e_2, e_3$ with eigenvalues $L_1, L_2, L_3$.

Since the eigenvalues of the dual of a representation of a Lie algebra are simply the negative eigenvalues of the original representation, we immediately obtain that $V^* \cong \mathbb{C}^3$ with the canonical dual standard basis $e_1^*, e_2^*, e_3^*$ with eigenvalues $-L_1, -L_2, -L_3$. By the way, the representations $V$ and $V^*$ are mapped into each other by the automorphism $X \mapsto -{}^tX$ of $\mathfrak{sl}(3,\mathbb{C})$. Furthermore, $V^*$ is isomorphic to $\bigwedge^2 V$, whose weights are simply the pairwise sums of different weights of $V$. Conversely, $V \cong \bigwedge^2 V^*$.

We already know the adjoint representation. It has, in total, eight weights, namely $L_i - L_j$ for $i \neq j$, and in addition the weight zero with multiplicity two (since $\dim\mathfrak{h} = 2$).

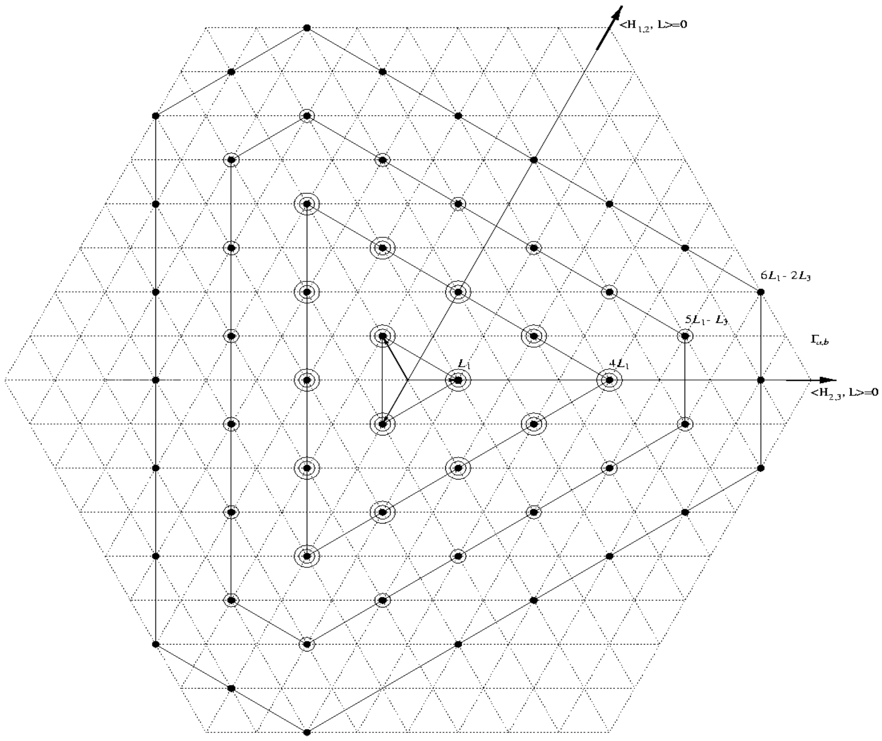
Next, we consider $\mathrm{Sym}^2 V$, $\mathrm{Sym}^2 V^*$ and $V \otimes V^*$. Weights of symmetric tensor products are given by the sums of the weights of the original representations. Therefore, $\mathrm{Sym}^2 V^*$ possesses the weights $\{-2L_k, -L_i - L_j : 0 < i < j\} = \{-2L_i - 2L_j, L_k : 0 < i < j\}$. The weights of the tensor product $V \otimes V^*$ are the sums of the individual weights (quantum numbers of tensor product representations add!), which results in the set $\{L_i - L_j\}$. This set contains the element zero three times. So, $V \otimes V^*$ is not irreducibel, but a direct sum of the adjoint and the trivial representation. The weight vectors are $e_i \otimes e_j^*$, where the three weight vectors $e_i \otimes e_i^*$ belong to the threefold degenerate weight zero. More generally, it is true for each faithful representation $W$ that the tensor product $W \otimes W^*$ contains the adjoint representation.




**Irreducible representations.** It follows from our considerations so far on the weights of representations of $\mathfrak{sl}(3,\mathbb{C})$, that each highest weight vector must lie in the $(\frac{1}{6})$-plane defined by the inequalities $\langle H_{1,2}, L\rangle \geq 0$ and $\langle H_{2,3}, L\rangle \geq 0$.

Thus, a highest weight vector necessarily has the form $v = (a+b)L_1 + bL_2 = aL_1 - bL_3$ for two non-negative integers $a, b \in \mathbb{Z}_+$. Hence, our construction scheme yields the theorem:
THEOREM: For each pair of non-negative integers $a, b$ there exists a unique finite dimensional irrep $\Gamma_{a,b}$ of $\mathfrak{sl}(3,\mathbb{C})$ with higehst weight vector $v = aL_1 - bL_3$. $\square$
One can show that $\Gamma_{a,b} \subset \mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$. More precisely, one has that $\Gamma_{a,b} = \mathrm{Ker}(\imath_{a,b})$ with $\imath_{a,b} : \mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^* \to \mathrm{Sym}^{a-1} V \otimes \mathrm{Sym}^{b-1} V^*$ a map which performs the contraction $(v_1 \cdot \ldots \cdot v_a) \otimes (v_1^* \cdot \ldots \cdot v_b^*) \mapsto \sum_{i,j} \langle v_i, v_j^* \rangle (v_1 \cdot \ldots \cdot v_{i-1} \cdot v_{i+1} \cdot \ldots \cdot v_a) \otimes (v_1^* \cdot \ldots \cdot v_{j-1}^* \cdot v_{j+1}^* \cdot \ldots \cdot v_b^*)$.

I collected all the examples considered so far in the above diagram. $V$ and $V^*$ are green, their symmetric tensor products are black. The adjoint representation is blue, and the tensor product $\mathrm{Sym}^6 V \otimes \mathrm{Sym}^2 V^*$ is red. The weight diagram of $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$, or equivalently of $\Gamma_{a,b}$, is shown for the example $\mathrm{Sym}^6 V \otimes \mathrm{Sym}^2 V^*$. The occuring highest weights are, from outer to inner ones, $6L_1 - 2L_3$, $5L_1 - L_3$, $4L_1$ and finally $L_1$. In general, one has for the case $a \geq b$ firstly a sequence of concentric (not necessarily regular) hexagons with corner points $(a-i)L_1 - (b-i)L_3$, $i = 0, 1, \ldots b-1$, followed by a sequence of trianlges (after the smaller edge of the hexagons has shrunken to zero) with corner points $(a-b-3j)L_1$, $j = 0, 1, \ldots, [\frac{1}{3}(a-b)]$. The last expression means the Gauss bracket. The multiplicities of the hexagons $H_i$ and trianlges $T_j$ (note that $T_0 = H_b$) are for $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ as follows: $\mathrm{mult}(H_i) = \frac{1}{2}(i+1)(i+2)$ and $\mathrm{mult}(T_j) = \frac{1}{2}(b+1)(b+2)$. The multiplicities for $\Gamma_{a,b}$, on the other hand, are $\mathrm{mult}(H_i) = (i+1)$ and $\mathrm{mult}(T_j) = b$. Put differently, the multiplicities for $\Gamma_{a,b}$ increase from outer to inner hulls, starting with one and increasing by one until the sequence of triangles is reached after $b-1$ steps. From then on, the multiplicity stays constant for all triangles at the value $b$. In the following diagram, the multiplicities for the representation $\Gamma_{6,2}$ are denoted by concentric circles. Obviously, this is not a very effective way, to visualize representations graphically, and indeed, there are much better methods . . .




The decomposition of an arbitrary representation $U$ into irreps $\Gamma_{a,b}$ can be found in the following way:
(1) Write down the decomposition of $U$ into eigenvalues, i.e. weights $U = \bigoplus_\beta U_\beta$.
(2) Find the weight $\alpha = aL_1 - bL_3$ occuring in this decomposition, for which $\ell(\alpha)$ becomes maximal.
(3) Thus, $U$ must contain a copy of the irrep $\Gamma_\alpha = \Gamma_{a,b}$, i.e. $U \cong \Gamma_\alpha \oplus U'$ for a $U'$. Since the weight diagram of $\Gamma_\alpha$ is known, you can obtain the weight diagram of $U'$ from it.
(4) Repeat the above procedure for $U'$.
As an exercise, you might try to decompose in this way the tensor product $U = \Gamma_{2,1} \otimes V$ into irreps. The solution is $U \cong \Gamma_{3,1} \oplus \Gamma_{1,2} \oplus \Gamma_{2,0}$. With the map $\imath_{a,b}$ described somewhere above, one can show more generally that for $b \leq a$

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^* = \bigoplus_{i=1}^{b} \Gamma_{a-i,b-i} \,.$$

In particular, we have thus that $\Gamma_{n,0} = \mathrm{Sym}^n V$ and $\Gamma_{0,n} = \mathrm{Sym}^n V^*$.