FINITE GROUPS

Groups with a finite number of elements are called *finite groups*. The number of elements of a finite group G is called the *order* of the group G. We will collect some basic facts about them, which will be useful for the study of Lie algebras. Moreover, we see how everything works by doing some examples.

Example $\mathbb{Z}_3$. Write down the multiplication table for the group $\mathbb{Z}_3$. Is this group Abelian? Why does each element of G appear exactly once in each row and each column of the multicplication table? Find a one-dimensional representation of $\mathbb{Z}_3$.

Regular representation. The regular representation ρ_R of a finte group G is the representation of the group on itself. Now, representations of groups act linearly on vector spaces. Where does the vector space come into play? The trick goes as follows: We construct V by the map $g \mapsto |g\rangle$ for all $g \in G$ and declare the vectors $|g\rangle$ to form an orthonormal base of V, i.e. $V = \mathrm{span}\{|g\rangle : g \in G\}$ with $\langle g|g'\rangle = \delta_{gg'}$. Of course, $\mathrm{dim}V = \mathrm{order}G$. The *regular representation* is now defined by $\rho_R(g')|g\rangle = |g'g\rangle$. Check that this definition actually yields a representation. What is ρ_R for the group $\mathbb{Z}_3$?

Matrix elements. Given a representation ρ of a group G on a vector space V with orthonormal base $\{|i\rangle : i = 1, \ldots, \mathrm{dim}V\}$ with $\langle i|j\rangle = \delta_{ij}$, the matrix elements of the linear operators $\rho(g)$ are given by $(\rho(g))_{ij} = \langle i|\rho(g)|j\rangle$ for all $g \in G$. Check that the representationo ρ implements the group multiplication by matrix multiplication.

Irreducible representations. Since representations live on vector spaces, we have linearity. This means that we are free to choose the base in our vector space. Two representations ρ' and ρ are said to be *equivalent representations*, if there exists a similarity transformation S such that $\rho'(g) = S^{-1}\rho(g)S$ for all $g \in G$. Check that the multiplication rule is not changed by a similarity transformation.

A representation ρ is said to be a *unitary representation*, if $\rho(g)^\dagger = \rho^{-1}(g) = \rho(g^{-1})$ for all $g \in G$. A representation ρ on a vector space V is called a *reducible representation*, if there exists an *invariant subsapce* $W \subset V$ such that $\rho(g)w \in W$ for all $g \in G$ and for all $w \in W$. Then, there exists a projector Π_W onto W such that $\Pi_W\rho(g)\Pi_W = \rho(g)\Pi_W$ for all $g \in G$. Find an invariant subspace of the regular representation of $\mathbb{Z}_3$.

The representation defined by $\rho_{\mathrm{triv}}(g) = 1$ for all $g \in G$ is called the *trivial representation*. It is one-dimensional. Every group G has a trivial representation.

A representation ρ on a vector space V is called *irreducible*, if it is not reducible. Irreducible representations are often simply called *irreps*. Give an argument when a representation is completely reducible. This means that $V = V_1 \oplus V_2 \oplus \ldots \oplus V_k$ with ρ splitting into irreducible representations ρ_i acting on V_i. Thus, one equivalently says that a representation is completely reducible, if it can be decomposed into a direct sums of irreducible representations. Decompose ρ_R of $\mathbb{Z}_3$ by noting that all matrices can be simultaneously diagonalized, since $\mathbb{Z}_3$ is Abelian.

Example S_3. . A slightly more complicated and non-Abelian group is the symmetric group or permutation group S_3 on three elements. The order of S_3 is $|S_3| = 3! = 6$. Its elements are

$$e = (1)\,,\ \ a_1 = (123)\,,\ \ a_2 = (321) = a_1^{-1}\,,\ \ a_3 = (12)\,,\ \ a_4 = (23)\,,\ \ a_5 = (31)\,.$$

The notation of the group elements is in cylces, where $(i_1 \ldots i_k)$ means the permutation $\binom{12\ldots i_1\ldots i_2\ldots i_k\ldots n}{12\ldots i_2\ldots i_3\ldots i_1\ldots n}$. Give the full multiplication table of S_3. Can you identify an Abelian subgroup of S_3 ?

Here is an example of a unitary representation. Why must a non-Abelian group possess representations of dimension larger than one?

$$\rho(e) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \qquad \rho(a_1) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \qquad \rho(a_2) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},$$
$$\rho(a_3) = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \qquad \rho(a_4) = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \qquad \rho(a_4) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}.$$

Do you see a relationship between $\{e, a_1, a_2\}$ and $\{a_3, a_4, a_5\}$ respectively?

Theorem 1. Every rep. of a finite group is equivalent to a unitary rep.

Proof: We use an extremely powerful trick, namely averaging over all group elements. Let us define

$$S = \sum_{g\in G} \rho(g)^\dagger \rho(g) \,,$$

which is clearly a Hermitean positive definite operators. Thus, S can be diagonalized, $S = U^{-1}DU$ with $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots)$, and all its eigenvalues $\lambda_i \geq 0$. Why do we actually have the stronger result that $\lambda_i > 0$ for all i ? Thus, S has a square root $X = S^{1/2} = U^{-1}D^{1/2}U$ with $D^{1/2} = \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots)$. Since all $\lambda_i > 0$, X is invertible. Thus, the representation $\rho' = X\rho X^{-1}$ is unitary. Check this statement by computing $\rho'(g)^\dagger\rho'(g)$.

Theorem 2. Every rep. of a finite group is completely reducible.

Proof: Due to theorem 1, we only have to check this for unitary reps. Now, either the rep. is an irrep, then we are done. Or, the rep. is reducible. Thus, there exists a projector Π such that $\Pi\rho(g)\Pi = \rho(g)\Pi$ for all $g \in G$. Since $\Pi^\dagger = \Pi$, we have $(\Pi\rho(g)\Pi)^\dagger = \Pi\rho^\dagger(g)\Pi = \Pi\rho^\dagger(g)$. Now, ρ is unitary by assumption, so this implies $\Pi\rho(g^{-1})\Pi = \Pi\rho(g^{-1})$ for all $g \in G$. Since this is true for all g, this is equivalent to the statement $\Pi\rho(g)\Pi = \Pi\rho(g)$ for all $g \in G$. Use this last relation to compute $(1\!\!1 - \Pi)\rho(g)(1\!\!1 - \Pi)$. What property does $(1\!\!1 - \Pi)$ therefore possess? Complete the proof from there.

Subgroups and Cosets. We have seen that $\mathbb{Z}_3$ is a *subgroup* of S_3. Of course, there are always the trivial subgroups $\{e\}$ and G for any group G. Let $H \subset G$ be a non-trivial subgroup of G. Then we can define the *right-coset* of H, which is the set Hg for a fixed group element $g \in G$. Of course, one can analogously define left-cosets gH. In the following, a coset means a right-coset. Check that $\{a_3, a_4, a_5\}$ is a right-coset of $\mathbb{Z}_3$ in S_3. The number of elements of the coset is the order of the subgroup, i.e. $|Hg| = \mathrm{order} H$.

Every element g of G belongs to exactly one coset of a given subgroup H. It follows, that for finite groups G, the order of H must be a factor or divisor of the order of G.

A subgroup H of G is called *invariant* or *normal*, if $gH = Hg$ for all $g \in G$. This notation means more precisely the following: $\forall\, g \in G, h_1 \in H : \exists\, h_2 \in H : h_1 g = g h_2$ or $g h_2 g^{-1} = h_1$. Is $\mathbb{Z}_3$ a normal subgoup of S_3 ? What about the subgroup $\{e, a_4\}$ of S_3 ?

If H is an invariant subgroup of G, the so-called *coset space* G/H is again a group. The multiplication law on G implies the multiplication law on G/H. Check this. The group G/H is called the *factor group* of G by H. What is $S_3/\mathbb{Z}_3$?

The *center* of G is the always Abelian inavriant subgroup of G defined by $C(G) = \{c \in G : cg = gc\, \forall\, g \in G\}$. Note, however, that $C(G)$ might well be trivial.

Invariant subgroups were defined via the relation $gHg^{-1} = H$ for all $g \in G$. In a similar fashion, we can defined invariant subsets S by the relation $g^{-1}Sg = S$ for all $g \in G$. Such invariant sets are called *conjugacy classes*. They do not necessarily form groups. Find the conjugacy classes of S_3. It will turn out that the conjugacy classes of a finte group G are in one-to-one correspondence with the irreps of G. Moreover, every subgroup H of G, which is a union of conjugcy classes, is an invariant subgroup.

Theorem 3. If $\rho_1(g)A = A\rho_2(g)$ for all $g \in G$, where ρ_1 and ρ_2 are inequivalent irreps, then $A = 0$.

Proof: If there is a vector $|\mu\rangle$ with $|\mu\rangle = 0$, then there exists a non-zero projector Π onto the subspace that is annihilated by A. This subspace is invariant under ρ_2 because $A\rho_2(g)\Pi = \rho_1 A\Pi = 0$ for all $g \in G$. But since ρ_2 is an irrep, Π mus project onto the whole space and A must vanish. If A annihilates one state, it annihilates all states. In a similar fashion, one shows the analogous statement, if there is a bra $\langle\nu|$ with $\langle\nu|A = 0$. If no state is annihilated by A to either side, A must be an invertible square matrix. But then, $A^{-1}\rho_1(g)A = \rho_2(g)$ for all $g \in G$ which simply says that ρ_1 and ρ_2 are equivalent reps. contradicting the assumption.

Theorem 4. If $\rho(g)A = A\rho(g)$ for all $g \in G$, where ρ is a finite-dimensional irrep, then $A \propto 1\!\!1$.

Remark: In the proof of the this theorem, the assumption that the representation is finite dimensional is curcial. Both these theorems form what is known as *Schur's Lemma*. Of course, theorem 4 is valid for the case where ρ_1 is equivalent to ρ_2. Due to a simple base change, we can then always manage to have $\rho_1 = \rho_2$, as assumed in the theorem. An important side effect of the theorem is that the form of the basis states of an irrep is essentially fixed. One sees this be rewriting theorem 4 in the form $\forall\, g \in G : A^{-1}\rho(g)A = \rho(g) \implies A \propto 1\!\!1$. Thus, once the form of ρ is fixed, there are no non-trivial similarity transformations anymore. The only unitary transformations we still can make is to multiply all states by the same phase factor.

Proof: We use the fact that ρ is finite-dimensional, because we use the fact that any finite-dimensional matrix A has at least one eigenvalue. Indeed, the characteristic equation $\det(A - \lambda 1\!\!1) = 0$ has at least one root. Let $|\mu\rangle$ be an eigenvector to this one eigenvalue. We then have $\rho(g)(A - \lambda 1\!\!1) = (A - \lambda 1\!\!1)\rho(g)$ for all $g \in G$ and $(A - \lambda 1\!\!1)|\mu\rangle = 0$. We can then apply the argument of the proof of theorem 3 to the matrix $(A - \lambda 1\!\!1)$ to conclude that $(A - \lambda 1\!\!1) = 0$.

Orthogonality relations. Irreps have some very remarkable properties. Since irreps are more or less uniquely determined, we can introduce a label α for inequivalent irreps. Thus, $|\alpha, i\rangle$ is the i-th state of an orhtonormal basis in the vector space V_α of the irrep ρ_α, where ρ_α is chosen unitary and in a canonical form, i.e. all occurences of $\rho_\alpha(g)$ shall be represented by the same matrix. Let us define the quantities

$$A_{jk}^{\alpha\beta} = \sum_{g\in G} \rho_\alpha(g^{-1})|\alpha, j\rangle\langle\beta, k|\rho_\beta(g)\,.$$

Note, that again we average over the whole group. Check the intertwining property

$$\rho_\alpha(g')A_{jk}^{\alpha\beta} = A_{jk}^{\alpha\beta}\rho_\beta(g')\,,$$

which holds for all $g' \in G$. Now, Schur's Lemma tells us immediately the following: If $\alpha \neq \beta$, then ρ_α is inequivalent to ρ_β and thus $A = 0$. If $\alpha = \beta$, however, $A \propto 1\!\!1$. Thus, we find $A_{jk}^{\alpha\beta} = \delta_{\alpha\beta}\lambda_{jk}^{\alpha}1\!\!1$. Let us determine the constants λ_{jk}^{α}. For this, we compute the trace of $A_{jk}^{\alpha\beta}$ on the Hilbert space in two ways:

$$\mathrm{tr}A_{jk}^{\alpha\beta} = \delta_{\alpha\beta}\mathrm{tr}(\lambda_{jk}^{\alpha}1\!\!1) = \delta_{\alpha\beta}\lambda_{jk}^{\alpha}\mathrm{tr}1\!\!1 = \delta_{\alpha\beta}\lambda_{jk}^{\alpha}n_\alpha$$

with $n_\alpha = \mathrm{dim}\rho_\alpha = \mathrm{dim}V_\alpha$. Now we compute the trace again, but pluging in the definition of the $A_{jk}^{\alpha\beta}$ first:

$$\mathrm{tr}A_{jk}^{\alpha\beta} = \mathrm{tr}\left(\sum_{g\in G}\rho_\alpha(g^{-1})|\alpha, j\rangle\langle\beta, k|\rho_\beta(g)\right) = \delta_{\alpha\beta}\sum_{g\in G}\langle\alpha, k|\rho_\alpha(g)\rho_\alpha(g^{-1})|\alpha, j\rangle = N\delta_{\alpha\beta}\delta_{jk}$$

with $N = \mathrm{orderG}$. In order to derive this, we have used the cyclic property of the trace. Thus, we arrive at

$$\lambda_{jk}^{\alpha} = \frac{N}{n_\alpha}\delta_{jk}\,, \quad \sum_{g\in G}\rho_\alpha(g^{-1})|\alpha, j\rangle\langle\beta, k|\rho_\beta(g) = \frac{N}{n_\alpha}\delta_{\alpha\beta}\delta_{jk}1\!\!1\,.$$

If we multiply the last result in such a way that we get matrix elements, we find the remarkable formula

$$\begin{aligned} \frac{n_\alpha}{N}\sum_{g\in G}\langle\alpha, i|\rho_\alpha(g^{-1})|\alpha, j\rangle\langle\beta, k|\rho_\beta(g)|\beta, l\rangle &= \sum_{g\in G}\frac{n_\alpha}{N}\left(\rho_\alpha(g^{-1})\right)_{ij}\left(\rho_\beta(g)\right)_{kl} \\ &= \sum_{g\in G}\frac{n_\alpha}{N}\left(\rho_\alpha(g)\right)_{ji}^{*}\left(\rho_\beta(g)\right)_{kl} = \delta_{\alpha\beta}\delta_{jk}\delta_{im}\,. \end{aligned}$$

Note, that we have used unitarity here. What this formula tells is that the normalized matrix elements

$$\sqrt{\frac{n_\alpha}{N}}\left(\rho_\alpha(g)\right)_{ij}$$

of inequivlanet unitary irreps are orthonormal functions of the group elements. Moreover, the matrix elements are not only orthonormal and thus linearly independent, but they form a complete set of functions of the group elements. Let $f(g)$ be an arbitrary function of group elements $g \in G$. Defining $f(g) = \langle f|g\rangle = \langle f|\rho_R(g)|e\rangle$ with the help of the regular representation ρ_R and $\langle f| = \sum_{g'\in G} f(g')\langle g'|$, we find

$$f(g) = \sum_{g'\in G} f(g')\langle g'|\rho_R(g)|e\rangle = \sum_{g'\in G} f(g')\left(\rho_R(g)\right)_{g'e}\,.$$

Since ρ_R is completely reducible, it can be decomposed into a linear combination of matrix elements of the unitary irreps. Thus, we have proven:

Theorem 5. The matrix elements of the unitary irreps of G are a complete orthonormal set for the vector space of the regular representation, or alternatively, for functions of the group elements $g \in G$.

A very important corollary of all this stuff, in particular of the orthogonality relation, is the small formula

$$N = \sum_\alpha n_\alpha^2\,.$$

Since the $n_\alpha \geq 1$, we see that a finite group admits only a finite number of inequivalent unitary irreps.

Characters. Given a rep. ρ of G on a vector space V, we have matrix elements $(\rho(g))_{ij} = \langle i|\rho(g)|j\rangle$. With this, one can define a function $\chi_\rho : G \longrightarrow \mathbb{C}$ by taking the trace of the matrix $\rho(g)$, i.e.

$$\chi_\rho(g) = \mathrm{tr}_V\, \rho(g) = \sum_i \langle i|\rho(g)|i\rangle = \sum_i (\rho(g))_{ii}\,.$$

This function is called the *character* of the representation ρ. Note, that this time we sum over the base of the representation space, not over the group elements. It follows from the cyclic property of the trace that characters of equivalent reps. are identical. Moreover, $\chi_{\rho_\alpha} \neq \chi_{\rho_\beta}$ for inequivalent reps. ρ_α and ρ_β. In fact, the characters are orthonormal. To see this, start from the formula

$$\sum_{g\in G} \frac{1}{N} \left(\rho_\alpha(g)\right)^*_{jk} \left(\rho_\beta(g)\right)_{lm} = \frac{1}{n_\alpha}\delta_{\alpha\beta}\delta_{jl}\delta_{km}\,,$$

which we have shown above, and sum over $j = k$ and $l = m$ in order to take the trace. This yields

$$\frac{1}{N}\sum_{g\in G} \chi_{\rho_\alpha}(g)^* \chi_{\rho_\beta}(g) = \delta_{\alpha\beta}\,. \tag{$*$}$$

Why are characters constant on conjugacy classes? Compute the character for the two-dimensional representation of S_3 given above. What does $\chi_\rho(e)$ tell you?

One can show that the characters form a complete orthonormal basis for functions which are constant on conjugacy classes. This means in particular, that there are precisley as many inequivalent unitary irreps as there are conjugacy classes. To show this, recall that any such function can be expanded in terms of the matrix elements of the irreps,

$$f(g_1) = \sum_{\alpha,j,k} C^\alpha_{j,k} \left(\rho_\alpha(g_1)\right)_{jk}\,.$$

Since f shall be constant on conjugacy classes, we can write it as

$$f(g_1) = \frac{1}{N}\sum_{g\in G} f(g^{-1}g_1 g) = \frac{1}{N}\sum_{g,\alpha,j,k} C^\alpha_{j,k}\left(\rho_\alpha(g^{-1}g_1g)\right)_{jk} = \frac{1}{N}\sum_{\substack{\alpha,j,k\\ g,l.m}} C^\alpha_{j,k}\left(\rho_\alpha(g^{-1})\right)_{jl}\left(\rho_\alpha(g_1)\right)_{lm}\left(\rho_\alpha(g)\right)_{mk}\,.$$

In this last formula, we can perform the sum over the group elements explicitly making use of the orthogonality relation. We therefore find

$$f(g_1) = \sum_{\substack{\alpha,j,k\\ l,m}} \frac{1}{n_\alpha} C^\alpha_{j,k}\left(\rho_\alpha(g_1)\right)_{lm}\delta_{jk}\delta_{lm} = \sum_{\alpha,j,l}\frac{1}{n_\alpha}C^\alpha_{j,j}\left(\rho_\alpha(g)\right)_{ll} = \sum_{\alpha,j}\frac{1}{n_\alpha}C^\alpha_{j,j}\chi_{\rho_\alpha}(g_1)\,.$$

Let us label the conjugacy classes by c and let k_c be the number of elements of the conjugacy class g_c. Let us define further a matrix M with matrix elements

$$M_{c\alpha} = \sqrt{\frac{k_c}{N}}\chi_{\rho_\alpha}(k_c)\,.$$

The orthonormality relation $(*)$ can then be written as $M^\dagger M = 1\!\!1$. But M is a square matrix, since the number of inequivalent unitary irreps is equal to the number of conjugacy classes. Thus, we also have $MM^\dagger = 1\!\!1$, or

$$\sum_\alpha \chi_{\rho_\alpha}(g_c)^*\chi_{\rho_\alpha}(g_{c'}) = \frac{N}{k_c}\delta_{cc'}\,.$$

This all has some interesting consequences: Let ρ be any rep. (not necessarily irreducible). In its completely reduced form, each irrep will occur an integer number of times, $m_\alpha \geq 0$, called the *multiplicity* of ρ_α in $\rho = \bigoplus_\alpha m_\alpha\rho_\alpha$. We have from the orthogonality relation that

$$\frac{1}{N}\sum_{g\in G}\chi_{\rho_\alpha}(g)^*\chi_\rho(g) = m_\alpha\,.$$

Compute the character of the regular representation. What follows for the decomposition of the regular rep. into irreps? Play around with the characters of some representations of S_3.