MATRIX LIE GROUPS

Most Lie groups one ever encouters in physics are realized as matrix Lie groups and thus as subgroups of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$. This is the group of invertibel $n\times n$ matrices with coefficients in $\mathbb{R}$ or $\mathbb{C}$, respectively. This is a Lie group, since it forms as an open subset of the vector space of $n\times n$ matrices a manifold. Matrix multiplication is certainly a differentiable map, as is taking the inverse via Cramer's rule. The only condition defining the open subset is that the determinat must not be zero, which implies that $\dim_{\mathbb{K}}GL(n,\mathbb{K})=n^2$ is the same as the one of the vector space $M_n(\mathbb{K})$. However, $GL(n,\mathbb{R})$ is not connected, because we cannot move continuously from a matrix with determinant less than zero to one with determinant larger than zero. It is worth mentioning that $\mathfrak{gl}(n,\mathbb{K})$ is the vector space of all $n\times n$ matrices over the field $\mathbb{K}$, equipped with the standard commutator as Lie bracket.

We can describe most other Lie groups as subgroups of $GL(n,\mathbb{K})$ for either $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. There are two ways to do so. Firstly, one can give restricting equations to the coefficients of the matrices. Secondly, one can find subgroups of the automorphisms of $V\cong\mathbb{K}^n$, which conserve a given structure on $\mathbb{K}^n$. In the following, we give some examples for this:

$SL(n,\mathbb{K})$. This subgroup can be defined either as the subgroup of matrices with determinant one, or as the subgroup of transformations on $\mathbb{K}^n$, which conserve the volume element $\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\ldots\wedge\mathrm{d}x^n$. The condition $\det M=1$ yields a constraint on the n^2 coefficients such that $\dim_{\mathbb{K}}SL(n,\mathbb{K})=n^2-1$. Clearly, $\mathfrak{sl}(n,\mathbb{K})$ consists of all traceless matrices X in $\mathfrak{gl}(n,\mathbb{K})$, $\mathrm{tr}\,X=0$.

B_n and N_n. These are the upper triangular matrices and the upper triangular matrices with diagonal entries one, respectively. Clearly, these form subgroups of $GL(n,\mathbb{K})$ of dimensions $\dim_{\mathbb{K}}B_n=\frac{1}{2}n(n+1)$ and $\dim_{\mathbb{K}}N_n=\frac{1}{2}n(n-1)$. The structure they preserve is the so-called flag $0=V_0\subset V_1\subset\ldots\subset V_{n-1}\subset V_n=\mathbb{K}^n$, where $V_i=\mathrm{span}\{e_i,\ldots,e_n\}$. The group N_n has the additional property that it acts as identity on the quotients V_{i+1}/V_i. Of course, in the same way we can consider lower triangular matrices.

Cartan-Weyl basis. Remember that we decomposed any Lie algebra $\mathfrak{g}$ into its Cartan algebra $\mathfrak{h}$, which in the eigen basis consists only of diagonal matrices, and into the raising and lowering generators E_α and $E_{-\alpha}$ for $\alpha>0$ the positive roots. The subspace $\mathfrak{n}_+\subset\mathfrak{g}$ is the span of all the E_α, and $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{n}_+\oplus\mathfrak{n}_-$. Now, the E_α are strictly upper triangular matrices with diagonal entries all zero, and they form a closed subalgebra. The corresponding subgroup is a subgroup of N_n, if $\dim\mathfrak{g}=n$. The Cartan-Weyl basis thus implies a decomposition of a Lie group into diagonal matrices, and upper and lower triangular matrices with diagonal one.

STRUCTURES FROM FORMS

Most symmetries in physics can be understood in terms of bilinear or sesquilinear forms acting on the chosen vector space, which are left invariant under the symmetry operation. Such forms produce numbers from pairs of vectors, and thus they are good tools to produce observable quantities. A bilinear form is a bilinear map $Q:\mathbb{K}^n\times\mathbb{K}^n\to\mathbb{K}$. One may now ask, which matrices A fulfill $Q(Av,Aw)=Q(v,w)$ for all $v,w\in\mathbb{K}^n$. We will discuss several possibilities for Q. Given a form Q, we can realize it by a Cauchy matrix C whose entries are the scalar products $Q(e_i,e_j)$ for a canonical standard basis. We have $Q(v,w)=v^t\,C\,w$ such that the set of matrices A leaving Q invariant has to satisfy the condition $A^t\,C\,A=C$.

Q symmetric, positive definite. In this case, Q can be brought into the Euclidean standard form of a scalar product. Thus, we seek matrices, which leave the Euclidean legnth of vectors invariant. These are the orthogonal transformations $O(n,\mathbb{K})$. If we restrict to subgroups of $SL(n,\mathbb{K})$, then we obtain the special orthogonal group $SO(n,\mathbb{K})$, which are the rotations, but not the reflections. The determinant of an orthogonal transformation must square to one, so it can only be ±1. Thus, $O(n,\mathbb{K})$ and $SO(n,\mathbb{K})$ have the same dimension. A symmetric, positive definite form Q can always be brought into a form, where its Cauchy matrix $C=\mathbb{1}$. Then, the condition reads $A^t\,A=\mathbb{1}$ which implies that not all matrix elements are independet. In fact, $A^t\,A$ is a symmetric matrix which yields $\frac{1}{2}n(n+1)$ independent equations. Thus, we find $\dim_{\mathbb{K}}=\frac{1}{2}n(n-1)$. One important point is that $O(n,\mathbb{R})$ is not connected. In fact, $SO(n,\mathbb{R})=O(n,\mathbb{R})/\mathbb{Z}_2$. The algebras $\mathfrak{so}(n,\mathbb{K})=\mathfrak{o}(n,\mathbb{K})$ are given by the skew-symmetric matrices $X\in\mathfrak{gl}(n,\mathbb{K})$, $X^t=-X$, when using the mathematical convention for defining the generators, since $\exp(X)^t=\exp(X^t)=\exp(X)^{-1}=\exp(-X)$.

***Q* symmetric, indefinite but non-degenerate.** In this case, the Cauchy matrix still has no zero eigenvalues. In fact, it will have k positive eigen values and l negative eigen values, $k + l = n$. One calls (k, l) the signature of the form Q. This leads to the groups $SO(k, l)$, an example being the Lorentz group $SO(1, 3)$. Note that this makes only sense for $\mathbb{K} = \mathbb{R}$, sine there are no symmetric, non-degenerate bilinear forms on complex vector spaces. Obviously, $SO(k, l) \cong SO(l, k)$. Of course, we can also include reflections to define $O(k, l)$. Note that $SO(k, l)$ is not connected if $(k, l) \neq (n, 0)$ or $(0, n)$. As in the case of the Lorentz group, one has to components, and $O(k, l)$ has in total four components with a discrete center $\mathbb{Z}_2 \times \mathbb{Z}_2$.

***Q* skew-symmetric, non-degenerate.** This case is interesting for the study of Hamiltonian mechanics, which lives on symplectic manifolds. The group which leaves such a form invariant, is called $Sp(n, \mathbb{R})$, and is defined for even n only. The standard form of Q is for $n = 2m$ given by the Cauchy matrix $C = \begin{pmatrix} 0 & 1\!\!1_m \\ -1\!\!1_m & 0 \end{pmatrix}$. Furthermore, $\dim Sp(n, \mathbb{R}) = \frac{1}{2}n(n+1)$. Let $A \in Sp(n, \mathbb{R})$. The matrix A is clearly also an element of $GL(2m, \mathbb{R})$. The condition $A^t\, C\, A = C$ for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in block form with $a, b, c, d \in GL(m, \mathbb{R})$ yields the following constraints for the block constituents: $a^t c$ and $b^t d$ must be symmetric, $a^t d - c^t b = 1\!\!1_m$. The corresponding Lie algebra $\mathfrak{sp}(n, \mathbb{R})$ consists of the matrices $X \in \mathfrak{gl}(2m, \mathbb{R})$ which satisfy $X^t\, C + C\, X = 0$. One can further show that all elements of $Sp(n, \mathbb{R})$ have determinate one, and that $Sp(n, \mathbb{R})$ is connected. There exists completely analogous definitions for sthe symplectic group over complex vector spaces, $Sp(n, \mathbb{C})$ and its algebra.

***Q* Hermitean, positive definite.** For complex vector spaces $V \cong \mathbb{C}^n$, one can consider sesqui-linear forms instead of bilinear forms. Of particular interest are Hermitean forms satisfying $Q(\lambda v, \mu w) = \bar{\lambda} Q(v, w)\mu$ for all $v, w \in V$ and $\lambda, \mu \in \mathbb{C}$. Furthermore, $Q(w, v) = \overline{Q(v, w)}$. It is positive definite, if $Q(v, v) > 0$ for all $v \neq 0$. The condition for matrices A leaving a Hermitean form, given via its Cauchy matrix C, invariant, reads $A^\dagger\, C\, A = C$, where $A^\dagger = \overline{A}^t$. If Q can be brought to standard form $C = 1\!\!1$, the conditions transforms to $A^\dagger = A^{-1}$, which defines the group $U(n, \mathbb{C}) = U(n)$ of unitary matrices. Note that only $|\det A| = 1$ can be fixed by this condition, so the determinant can be any phase. Restricting to $\det A = 1$ defines the group $SU(n)$ of special unitary transformations. Theses groups have dimensions $\dim_{\mathbb{R}} U(n) = n^2$ since we have n^2 conditions from the relation $A^\dagger = A^{-1}$ for $2n^2$ real parameters. Thus, with the additional condition $\det A = 1$, we find $\dim_{\mathbb{R}} SU(n) = n^2 - 1$. Note that it does not make sense to define the complex dimension for unitary groups, and in fact, unitary Lie groups are not complex Lie groups. The reason for this is burried in the fact that unitary groups are compact, but that any compact complex Lie group must be Abelian. One nice thing one can show is the relation $U(n) = O(2n, \mathbb{R}) \cap Sp(2n, \mathbb{R})$.

***Q* Hermitean, indefinite, non-degenerate.** In a similar way as we did for the real bilinear forms, we can define indefinite Hermitean forms of signature (k, l). The corresponding groups are then denoted $U(k, l)$ with $SU(k, l)$ the subgroup of elements with determinant one.

***Q* quaternionic Hermitean.** Finally, one can consider vector spaces over the division algebra $\mathbb{H}$, the quaternions. It is possible to view $\mathbb{H} = \mathbb{C} \oplus \mathrm{j}\mathbb{C} \cong \mathbb{C}^2$, where for $v = v_1 + \mathrm{j}v_2$, we have $v \cdot \mathrm{j} = -\bar{v}_2 + \mathrm{j}\bar{v}_1$. Note that we can realize quaternion multiplication by action of matrices on $\mathbb{C}^2$, and indeed, the Pauli matrices will do the job just fine, implementing the three quaternionic elements $\mathrm{i}, \mathrm{j}, \mathrm{k}$ with the algebra $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$, and $\mathrm{i} \cdot \mathrm{j} = -\mathrm{j} \cdot \mathrm{i} = \mathrm{k}$ and its cyclic perumtations. Care has to be taken with the order, since multiplication in the quaternions is not commutative. Similarly, vector spaces are defined as $V \cong \mathbb{H}^n = \mathbb{C}^n + j\mathbb{C}^n = \mathbb{C}^{2n}$. Thus, viewing vectors $v = v_1 + \mathrm{j}v_2$ as two-component vectors $\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)$ with $v_1, v_2 \in \mathbb{C}^n$, the action of j is given by $\mathrm{j} : \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \bar{v}_1 \\ \bar{v}_2 \end{smallmatrix}\right)$, where $I = 1\!\!1_n$. The involved matrix is just the standard form for a symplectic bilinear form, often called J. Then, we can define $GL(n, \mathbb{H})$ as the group of matrices $A \in GL(2n, \mathbb{C})$ which satisfy $AJ = J\overline{A}$. Those matrices A with real determinant one form the subgroup $SL(n, \mathbb{H})$. A Hermitean form on $\mathbb{H}^n$, also called a symplectic scalar product, is an $\mathbb{R}$-bilinear form that satisfies $Q(\lambda v, \mu w) = \bar{\lambda} Q(v, w)\mu$ for $v, w \in \mathbb{H}^n$ and $\lambda, \mu \in \mathbb{H}$, and $Q(w, v) = \overline{Q(v, w)}$. The conjugate of a quaternion $\lambda = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ is given by $\bar{\lambda} = a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}$. The standard Hermitean form on $\mathbb{H}^n$ is simply $Q(v, w) = \sum_i \bar{v}_i w_i$. The group of matrices leaving this form invariant is called the compact symplectic group $Sp(n, \mathbb{H}) = Sp(n)$ or alternatively $U(n, \mathbb{H})$. One can show that $Sp(n) = U(2n) \cap Sp(2n, \mathbb{C})$. We conclude by observing that if Q is an indefinite non-degenerate quaternionic Hermitean form with signature (k, l), the corresponding group leaving it invariant is called $U(k, l, \mathbb{H})$. If Q is skew-symmetric, $Q(w, v) = -\overline{Q(v, w)}$, the corresponding group is denoted $U^*(n, \mathbb{H})$.

***Q* degenerate.** One small remark concerning the case that Q is degenerate: Any transformation preserving Q will map the kernel of Q into itself, where $\ker(Q) = \{v \in V : Q(v, w) = 0\ \forall w \in V\}$. Thus, the group of matrices leaving Q invariant is the group of matrices which preserve the subspace $\ker(Q)$ and leaving the induced non-degenerate form $\tilde{Q}$ on the quotient space $V/\ker(Q)$ invariant.