Mic$\hbar$ael Flohr
21. May 2010

IRREPS OF $\mathfrak{su}(2)$

In the last lecture, we constructed all finite-dimensional irreducible representations of $\mathfrak{su}(2)$. In contrast to the way one does this in quantum mechanics, we did not make use of the operator $\boldsymbol{J}^2$, since this is not an element of the Lie algebra. The only assumption we made is that the representation is finite-dimensional. In fact, if we are interested in unitary representations, one can show that these mus be finite dimensional for compact Lie groups and therefore also for their Lie algebras.

Highest weight construction. The Lie algebra $\mathfrak{su}(2)$ is three-dimensional and defined via the commutation relations $[J_j, J_k] = \mathrm{i}\epsilon_{jk}{}^l J_l$. In fact, this is the smallest non-trivial Lie algebra. Since there are no two generators which commute with each other, we can only diagonalize one generator at a time. Given a finite-dimensional representation ρ on a vector space V with dimension ℓ, we choose to diagonalize the linear operator $\rho(J_3)$. Since $\rho(J_3)$ is Hermitean, the eigenvalues are real. Since $\dim V < \infty$, we can choose the eigenstates of $\rho(J_3)$ as basis for V such that there must be a state $|j,x\rangle$ whose J_3-eigenvalue is maximal, $\rho(J_3)|j,x\rangle = j|j,x\rangle$ and $\rho(J_3)|j',x'\rangle = j'|j',x'\rangle$, $j' \leq j$ for all j'. Here, x denotes all further labels which might be necessary to specify states in V. The trick is that we start with a state with maximal eigenvalue which must exists, since V is finite dimensional.

The next trick is to redefine the other generators. Choosing $J^\pm = \frac{1}{\sqrt{2}}(J_1 \pm \mathrm{i}J_2)$, the commutation relations read $[J^+, J^-] = J_3$ and $[J_3, J^\pm] = \pm J^\pm$. In this basis for the generators, the action of $J^\pm$ on $|j',x\rangle$ is easy to compute: $\rho(J_3)(\rho(J^\pm)|j',x\rangle) = [\rho(J_3), \rho(J^\pm)]|j',x\rangle + \rho(J^\pm)\rho(J_3)|j',x\rangle = \rho(J^\pm)\rho(J_3)|j',x\rangle \pm \rho(J^\pm)|j',x\rangle = (j' \pm 1)(\rho(J^\pm)|j',x\rangle)$. We see that $J^\pm$ raise or lower the J_3 eigenvalue by plus or minus one, respectively.

The immediate consequence is that $\rho(J^+)|j,x\rangle = 0$ since j was by definition the maximal possible eigenvalue. This is why $|j,x\rangle$ is called a highest weight state and j its highest weight. Other states can be constructed by using $\rho(J^-)|j,x\rangle = N_{j,x}|j-1,x\rangle$. We have seen in the lecture that the normalization $N_{j,x} \equiv N_j$ is independent of x. Furthermore, we found that $\rho(J^+)|j-1,x\rangle = N_j|j,x\rangle$ and that $N_j^2 = j$. Finally, states $|j',x\rangle$ and $|j',y\rangle$ are orthogonal to each other for $x \neq y$. We can repeat this and find states $|j-2,x\rangle$, $|j-3,x\rangle$ and so on, in general $|j-k-1,x\rangle$ with normalization constants N_{j-k}, the latter turning out to be $N_{j-k}^2 = \frac{1}{2}(k+1)(2j-k)$. Renaming $k = j-m$, we find the well known formula

$$N_m = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}\,.$$

We can use the fact that V has to be finite dimensional once more. It means that we cannot lower the J_3 eigenvalue indefinitely. There must be a state $|j-h,x\rangle$ such that $\rho(J^-)|j-h,x\rangle = 0$, or equivalently, there must be an integer h such that the norm of $\rho(J^-)|j-h,x\rangle = 0$. Thus, we must have $N_{j-h} = \frac{1}{\sqrt{2}}\sqrt{(2j-h)(h+1)} = 0$. Since $h \geq 0$, the only solution is $h = 2j$ which implies that $j = h/2 \in \mathbb{Z}_+/2$. We also observe that all this is independent of x such that the representation is irreducible only, if there is only one highest weight state and thus no dependence on x at all. Thus, all finite-dimensional irreps are classified by just one number $j \in \mathbb{Z}_+/2$, the highest weight. The dimension of the irrep j is $\ell = 2j+1$.

Standard notation. In order to label in which irrep we are, we denote the states by $|j,m\rangle$ where now j is the highest weight, and m the actual J_3 eigenvalue of the state, $\rho^{(j)}(J_3)|j,m\rangle = m|j,m\rangle$. We also made the irrep explicit in the notation for the linear operator, $\rho^{(j)}(J_3)$. Of course, $m \in \{j, j-1, \ldots, -j+1, -j\}$. The matrix elements of the linear operators representing the $\mathfrak{su}(2)$ algebra are now easy to find. Since $\rho(J^-)|j,j-k\rangle = N_{j-k}|j,j-k-1\rangle$ and $\rho(J^+)|j,j-k-1\rangle = N_{j-k}|j,j-k\rangle$, we find for the matrix elements $\left(\rho^{(j)}(J_a)\right)_{m'm} = \langle j,m'|\rho^{(j)}(J_a)|j,m\rangle$ the expressions

$$\langle j,m'|\rho^{(j)}(J_3)|j,m\rangle = m\,\delta_{m',m}\,, \quad \langle j,m'|\rho^{(j)}(J^\pm)|j,m'\rangle = \sqrt{(j\pm m+1)(j\mp m)/2}\,\delta_{m',m\pm 1}\,.$$

These results can easily be translated to the matrix elements for J_1 and J_2, since $J_1 = \frac{1}{\sqrt{2}}(J^+ + J^-)$ and $J_2 = \frac{\mathrm{i}}{\sqrt{2}}(J^+ - J^-)$. As examples, we give the matrices for $j = 1/2, 1$ and $3/2$. The spin $j = 1/2$ irrep is given by

$$\rho^{(1/2)}(J_1) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\sigma_1\,, \quad \rho^{(1/2)}(J_2) = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \frac{1}{2}\sigma_2\,, \quad \rho^{(1/2)}(J_3) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\sigma_3\,.$$

Indeed, these matrices are the Pauli matrices satisfying $\sigma_a\sigma_b = \delta_{ab} + \mathrm{i}\epsilon_{ab}{}^c\sigma_c$. This is the defining representation, since $\exp(\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\sigma})$ yields precisely all 2×2 matrices which are unitary and have determinant one. The spin $j=1$ irrep is given by the generators

$$\rho^{(1)}(J_1) = \frac{1}{\sqrt{2}}\begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix}\,, \quad \rho^{(1)}(J_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} & -\mathrm{i} & \\ \mathrm{i} & & -\mathrm{i} \\ & \mathrm{i} & \end{pmatrix}\,, \quad \rho^{(1)}(J_3) = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}\,.$$

This is indeed equivalent to the adjoint representation. To see this, one has to find a similarity transformation P such that $PT_aP^{-1} = \rho^{(1)}(J_a)$, where $(T_a)_b{}^c = -\epsilon_{ab}{}^c$ are the generators in the adjoint representation, $T_a = \mathrm{ad}(J_a)$. The similarity transformation which does the trick is

$$P = \begin{pmatrix} 1/2 & -\mathrm{i}/2 & 0 \\ 0 & 0 & -1/\sqrt{2} \\ -1/2 & -\mathrm{i}/2 & 0 \end{pmatrix}\,.$$

Finally, the spin $j = 3/2$ irrep reads

$$\rho^{(3/2)}(J_1) = \begin{pmatrix} & \sqrt{3/2} & & \\ \sqrt{3/2} & & 2 & \\ & 2 & & \sqrt{3/2} \\ & & \sqrt{3/2} & \end{pmatrix}\,, \quad \rho^{(3/2)}(J_2) = \begin{pmatrix} & -\sqrt{3/2}\mathrm{i} & & \\ \sqrt{3/2}\mathrm{i} & & -2\mathrm{i} & \\ & 2\mathrm{i} & & -\sqrt{3/2}\mathrm{i} \\ & & \sqrt{3/2}\mathrm{i} & \end{pmatrix}\,,$$

and $\rho^{(3/2)}(J_3) = \mathrm{diag}(3/2, 1/2, -1/2, -3/2)$. We note that an automatic consequence of our construction is that the states are orthonormal, i.e. $\langle j', m', x'|j, n, x\rangle = \delta_{j'j}\delta_{m'm}\delta_{x'x}$ where x denotes quantum numbers with respect to other possible observables.

Tensor Products

Classifying all irreps of a given Lie algebra $\mathfrak{g}$ is the first step to understand the representation theory of $\mathfrak{g}$. The next step is to study how an arbitrary representation decomposes into irreps. The most common reducible representations one encounters in physics are tensor products of irreps. We will see some of what goes on with tensor products in the well known example of angular momentum addition. The point is that a physical system might transform in such a way under a symmetry that it carries quantum numbers for different irreps of the symmetry algebra. For example, a particle with spin s and angular momentum ℓ can be described with a Hilbert space whose states have *independent* quantum numbers with respect to the irrep $\rho(\ell)$ and the irrep $\rho^{(s)}$ of $mathfraksu(2)$. The states can thus be denotes as $|\ell, m\rangle \otimes |s, m_s\rangle \equiv |\ell, m\rangle|s, m_s\rangle$ where it is customary to omit the tensor product symbol. Another common notation is $|\ell, m; s, m_s\rangle$.

Transformation properties. To understand how the Lie algebra acts on a tensor product, we have to change notation for this paragraph. We will denote the representations of the Lie group on vector space V and W by ρ^V and ρ^W, and in general group representations by ρ. The representations of the corresponding algebra $\mathfrak{g}$ are denoted by $\mathrm{d}\rho^V$, $\mathrm{d}\rho^W$ and $\mathrm{d}\rho$, respectively. This makes explicit that the linear operators $\mathrm{d}\rho(u^aX_a)$ can be thought of as the linear differentials of the linear operators $\rho(g)$, $g = \exp(\mathrm{i}u^aX_a)$.

The group acts in the following way on the vector space $V\times W$ with states $|v\rangle\otimes|w\rangle$:

$$\begin{aligned} \rho^{V\otimes W}(g)|v\rangle\otimes|w\rangle &= \sum_{v',w'} |v'\rangle\otimes|w'\rangle \left(\rho^{V\otimes W}(g)\right)_{(v'w')(vw)} \\ &= \left(\sum_{v'}|v'\rangle\left(\rho^V(g)\right)_{v'v}\right)\otimes\left(\sum_{w'}|w'\rangle\left(\rho^W(g)\right)_{w'w}\right)\,. \end{aligned}$$

This means nothing else than the statement that the factors of the tensor products states transform independently under the group action. Now, it is very easy to find how the algebra acts, since $\mathrm{d}\rho$ acts as derivation for ρ. Thus, we find

$$\begin{aligned} \left(\mathbb{1} + \mathrm{i}u^a\mathrm{d}\rho^{V\otimes W}(J_a)\right)|v\rangle\otimes|w\rangle &= \sum_{v',w'}|v'\rangle\otimes|w'\rangle\langle v'|\otimes\langle w'|\left(\mathbb{1} + \mathrm{i}u^a\mathrm{d}\rho^{V\otimes W}(J_a)\right)|v\rangle\otimes|w\rangle \\ &= \sum_{v',w'}|v'\rangle\otimes|w'\rangle\left(\delta_{v',v}\delta_{w',w} + \mathrm{i}u^a\left(\mathrm{d}\rho^{V\otimes W}(J_a)\right)_{(v'w')(vw)}\right) \\ &= \sum_{v',w'}|v'\rangle\otimes|w'\rangle\left(\delta_{v',v} + \mathrm{i}u^a\left(\mathrm{d}\rho^V(J_a)\right)_{v'v}\right)\left(\delta_{w',w} + \mathrm{i}u^a\left(\mathrm{d}\rho^W(J_a)\right)_{w'w}\right)\,. \end{aligned}$$

To first order in u we thus get what we expect of a derivation:

$$\left(\mathrm{d}\rho^{V\otimes W}(J_a)\right)_{(v'w')(vw)} = \left(\mathrm{d}\rho^{V}(J_a)\right)_{v'v}\delta_{w',w} + \delta_{v',v}\left(\mathrm{d}\rho^{W}(J_a)\right)_{w'w}$$

or simply $\mathrm{d}\rho^{V\otimes W}(J_a) = \mathrm{d}\rho^{V}(J_a)\otimes 1\!\!1_W + 1\!\!1_V \otimes \mathrm{d}\rho^{W}(J_a)$. It is often quite cumbersome to keep track of the different representations and the explicit notion of the tensor products. Thus, the reader will often find shorter notations such as

$$J_a\left(|v\rangle|w\rangle\right) = \left(J_a|v\rangle\right)|w\rangle + |v\rangle\left(J_a|w\rangle\right) .$$

One of the easier things to work out with tensor products are the eigenvalues of the generators which can be diagonalized. We chose to diagonalize to J_3 and the eigenvalues of this generator simply add up:

$$J_3\left(|j_1,m_1\rangle|j_2,m_2\rangle\right) = (m_1+m_2)\left(|j_1,m_1\rangle|j_2,m_2\rangle\right) .$$

The specific way how the Lie algebra acts on a tensor product is all one needs to decompose the tensor representation into irreps by applying the highest weight construction to the tensor states and use the derivation property of the representation. That is exactly the procedure one goes through in quantum mechanics when decomposing, for instance, the tensor product of a $j=1$ and a $j=1/2$ representation, starting with the (unique) highest weight state $|3/2,3/2\rangle = |1,1\rangle|1/2,1/2\rangle$. This is left as an exercise.

Tensor operators. It might be helpful at this stage to repeat some stuff from quantum mechanics like tensor operators and the Wigner-Eckart theorem. A tensor operator $\mathcal{O}^{(r)}$ of rank r is simply an operator which transforms in the spin r irreducible representation, i.e.

$$[\rho^{(r)}(J_a), \mathcal{O}^{(r)}_m] = \sum_{m'} \mathcal{O}^{(r)}_{m'}\left(\rho^{(r)}(J_a)\right)_{m'm} .$$

Note that we now go back to use the symbol ρ for a representation of the Lie algebra, instead of $\mathrm{d}\rho$. Of course, a tensor operator has components since otherwise it could not possibly transform according to the spin r representation. A brief example might help, a particle in a spherically symmetric potential. The angular momentum is given by $L_a = \epsilon_a{}^{bc} r_b p_c$. The operators L_a form a representation of the Lie algebra $\mathfrak{su}(2)$. Now, the position operator r_b is related to a rank one tensor operator (i.e. a tensor operator transforming in the spin one irrep), because it transforms under the adjoint representation:

$$[\rho(J_a), r_b] = \epsilon_a{}^{cd}[r_c p_d, r_b] = -\mathrm{i}\epsilon_a{}^{cd} r_c \delta_{b,d} = -\mathrm{i}\epsilon_a{}^{cd} r_c = r_c (T_a)^c{}_b = r_c\,\mathrm{ad}(J_a)^c{}_b .$$

Note however, that r_b does not transform in the canonical way, since the representation matrices for the adjoint representation have not the standard form given above. If we have in general an operator $\mathcal{O}_b$, such that $[\rho(J_a), \mathcal{O}_b] = \sum_{b'} \mathcal{O}_{b'}\left(\rho(J_a)\right)_{b'b}$ with ρ being equivalent to a spin r irrep, then we can find a matrix S such that $S\rho(J_a)S^{-1} = \rho^{(r)}(J_a)$. We can then use this matrix S to redefine the tensor operator, $\mathcal{O}^{(r)}_m = \mathcal{O}_b(S^{-1})^b{}_m$. This redefined operator now transforms precisely in the irrep $\rho^{(r)}$, i.e. $[\rho^{(r)}(J_a), \mathcal{O}^{(r)}_m] = [S\rho(J_a)S^{-1}, (\mathcal{O}S^{-1})_m] = \mathcal{O}_{b'}(S^{-1})_m{}^{b'} S_{b'}^{c'}(\rho(J_a))_{c'd'}(S^{-1})_{m'}^{d'} = \mathcal{O}^{(r)}_{m'}(\rho^{(r)}(J_a))_{m'm}$. It is often not necessary to find S explicitly. If we find a linear combinations of the components $\mathcal{O}_b$ that is an eigenstate of J_3 with eigenvalue r', then we can take this as a component of $\mathcal{O}^{(r)}$ and construct the other components by applying $J\pm$. For the position operator, this is easy. We know that $[\rho(J_3), r_3] = 0$, therefore r_3 can be identified with the component $r^{(1)}_0$. We find the other two components by simply computing $[\rho^{(1)}(J^\pm), r^{(1)}_0] = r^{(1)}_{\pm 1} = \mp(r_1 \pm \mathrm{i}r_2)/\sqrt{2}$.

Wigner-Eckart theorem. Tensor operators have the great advantage that their matrix elements are determined by the $\mathfrak{su}(2)$ symmetry up to a constant which is independent of the symmetry (usually this constant is determined by the dynamics of the physical system under consideration). When a tensor operators $\mathcal{O}^{(r)}_k$ acts on a state $|j,m\rangle$, the whole object transforms in the tensor representation $\rho^{(r)\otimes(j)}$. Let us denote the coefficients of a base change from the basis $\{|r,k\rangle|j,m\rangle : k=-r,\dots,r\,,\ m=-j,\dots,j\}$ to the basis $\{|J,M\rangle : J=|r-j|,\dots r+j\,,\ M=-J,\dots J\}$ for the decomposition $(r)\otimes(j) = \bigoplus_{J=|r-j|}^{r+j}(J)$ by $\langle J,M|r,k;j,m\rangle$, the *Clebsh-Gordan coefficients*. These coefficients are entirely determined by the $\mathfrak{su}(2)$ structure, and can be obtained by applying the highest weight construction to both bases, using the derivation property of the tensor representation. In essence, they are determined up to some overall normalization and signs by the two recursion relations

$$\begin{aligned}\sqrt{(j\mp m)(j\pm m+1)}\langle j_1,m_1;j_2,m_2|j,m\pm 1\rangle &= \sqrt{(j_1\mp m_1+1)(j_1\pm m_1)}\langle j_1,m_1\mp 1;j_2,m_2|j,m\rangle \\ &+ \sqrt{(j_2\mp m_2+1)(j_2\pm m_2)}\langle j_1,m_1,j_2,m_2\mp 1|j,m\rangle ,\end{aligned}$$

subject to to the condition $m_1 + m_2 = m \pm 1$. Note that we have used here the inverse base change, just because we like to make things irritating for the reader ;-). Once we know these, the matrix elements of tensor operators have the simple form

$$\langle J, m', x' | \mathcal{O}_k^{(r)} | j, m, x \rangle = \delta_{m', k+m} \langle J, k+m | r, k; j, m \rangle \, \langle J, x' \| \mathcal{O}^{(r)} \| j, x \rangle \,,$$

where $\langle J, x' \| \mathcal{O}^{(r)} \| j, x \rangle$ is called the *reduced matrix element* of the tensor operator. It only depends on the irreps involved, and any remaining dynamical degrees of freedom, denoted here x' and x, but not on the components, i.e. the magnetic quantum numbers. Thus it depends neither on the inner structure of the involved irrpes, nor the particular states in them. This statement is known as the Wigner-Eckart theorem. It is valid for any Lie algebra, as we will see in due course, but here we have repeated it in the form well known from the theory of angular momentum and spin in quantum mechanics.

Of course, what we have just said equally applies to products of tensor operators. Such products $\mathcal{O}_k^{(r)} \mathcal{O}_{k'}^{(r'o)}$ simply transform in the tensor representation $(r) \otimes (r')$ and can thus be decomposed into a sum of tensor operators by again the highest weight procedure. Also, the J_3 eigenvalues again simply add up, i.e. we get nothing else than $[\rho^{(r)\otimes(r')}(J_3), \mathcal{O}_k^{(r)} \mathcal{O}_{k'}^{(r'o)}] = (k+k') \mathcal{O}_k^{(r)} \mathcal{O}_{k'}^{(r'o)}$. More generally, the action of the generators J_a of the Lie algebra $\mathfrak{su}(2)$ on a product of tensor operators is given as for tensor products of vector spaces, such that,

$$\begin{aligned}
[\rho^{(r)\otimes(r')}(J_a), \mathcal{O}_k^{(r)} \mathcal{O}_{k'}^{(r')}] &= [\rho^{(r)}(J_a), \mathcal{O}_k^{(r)}] \, \mathcal{O}_{k'}^{(r')} + \mathcal{O}_k^{(r)} \, [\rho^{(r')}(J_a), \mathcal{O}_{k'}^{(r')}] \\
&= \mathcal{O}_l^{(r)} \mathcal{O}_{k'}^{(r')} \left(\rho^{(r)}(J_a) \right)_{lk} + \mathcal{O}_k^{(r)} \mathcal{O}_{l'}^{(r')} \left(\rho^{(r')}(J_a) \right)_{l'k'} \,.
\end{aligned}$$

ADDENDUM TO HANDOUT III

We have seen in handout III that the tangent space at any point g of a Lie group G carries the structure of a Lie algebra. In particular, the tangent space at the identity element, T_eG, carries this structure. The abstract Lie algebra associated to a Lie group is therefore given by the identification $\mathfrak{g} \cong T_eG$. Thus, the real dimension of $\mathfrak{g}$ is equal to the dimension d of the manifold G. Equivalently, $\mathfrak{g}$ can be identified with the space of left- or right-invariant vector fields, and one may interchange the identifications freely.

Universal covering group. One important issue has to be clarified here. The Lie algebra carries almost all of the information on the Lie group manifold. The only information about a finite-dimensional Lie group G which is lost when the linearized and purely *local* information encoded in its Lie algebra $\mathfrak{g}$ is considered, are properties of topological and entirely global nature. More precisely, the Lie algebra cannot contain any information that depends either on the set $\pi_0(G)$ of different connected components of the group manifold, or on the fundamental group $\pi_1(G)$. In particular, for any simple compact real Lie algebra $\mathfrak{g}$, there is a *unique* compact simple Lie group $\tilde{G}$, for which the Lie algebra of invariant vector fields is isomorphic to $\mathfrak{g}$, and such that $\tilde{G}$ is *connected* and *simply connected*. This means that $\tilde{G}$ has trivial groups $\pi_0(\tilde{G}) = 0 = \pi_1(\tilde{G})$. Moreover, one can show that for any connected Lie group G with the same Lie algebra $\mathfrak{g}$ there is a surjective Lie group homomorphism $\varphi : \tilde{G} \longrightarrow G$ such that the kernel of φ is a subgroup of the center of $\tilde{G}$. This subgroup is then isomorphic to $\pi_1(G)$. The center of a group G is the subgroup consisting of all those elements $h \in G$ which commute with all the elements $g \in G$, i.e. $C(G) = \{h \in G : hg = gh \, \forall g \in G\}$. For this reason $\tilde{G}$ is also called the *universal covering group* associated with the Lie algebra $\mathfrak{g}$.