IRREDUCIBLE REPRESENTATIONS OF A SEMI-SIMPLE LIE ALGEBRA

The method with which we found the finite-dimensional irreps of $\mathfrak{sl}(3,\mathbb{C})$, or $\mathfrak{su}(3)$, respectively, can immediately be generalized to any semi-simple Lie algebra. This yields a procedure in eight steps, which I will sketch here very briefly. The semi-simple Lie algebra is denoted by $\mathfrak{g}$.

[I] **Cartan subalgebra.** Find the maximal Abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

[II] **Cartan decomposition.** Perform the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in R} \mathfrak{g}_\alpha\right)$ for the adjoint representation, where the *root spaces* $\mathfrak{g}_\alpha$ are defined by the condition

$$\forall H \in \mathfrak{h}, \forall X \in \mathfrak{g}_\alpha : \mathrm{ad}(H)(X) = \alpha(H)\, X$$

for $\alpha \in R \subset \mathfrak{h}^*$, the set of the *roots* of $\mathfrak{g}$. We have:
(1) $\dim\mathfrak{g}_\alpha = 1$;
(2) $\mathrm{rank}\mathfrak{g} \equiv \mathrm{rank}\Lambda_R = \dim\mathfrak{h}$ with $\Lambda_R = \mathrm{span}_{\mathbb{Z}} R$ the *root lattice*;
(3) $\alpha \in R \iff -\alpha \in R$.

Let V be a finite-dimensional irrep of $\mathfrak{g}$. Perform the Cartan decomposition for V analogously, i.e. decompose $V = \bigoplus_{\alpha\in W(V)} V_\alpha$, where the *weight spaces* V_α are defined by the condition

$$\forall H \in \mathfrak{h}, \forall v \in V_\alpha : H(v) = \alpha(H)\, v$$

for $\alpha \in W(V) \subset \mathfrak{h}^*$, the set of the *weights* of the representation V. We have:
(1) $\dim V_\alpha = \mathrm{mult}(\alpha)$ in the representation V;
(2) the root spaces act on the V_α in such a way that $\mathfrak{g}_\beta : V_\alpha \to V_{\alpha+\beta}$ for all $\beta \in R$. Then, obviously, it is true that $\forall \alpha, \alpha' \in W(V) : \alpha - \alpha' \in \Lambda_R$.

[III] **Root subalgebras.** Find for each root α the corresponding subalgebra $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \cong \mathfrak{sl}(2,\mathbb{C})$. we have:
(1) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$, such that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$, $\dim[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = 1$;
(2) $[[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \mathfrak{g}_\alpha] \neq 0$, so that one can find generators, which satisfy the standard Lie brackets of $\mathfrak{sl}(2,\mathbb{C})$. In particular, there exists a $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ with $\alpha(H_\alpha) = 2$.

[IV] **Weight lattice.** Make use of the rather simple representation theory of the $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2,\mathbb{C})$ in order to construct the lattice $\Lambda_W = \{\beta \in \mathfrak{h}^* : \beta(H_\alpha) \in \mathbb{Z}\ \forall \alpha \in R\}$, since all eigen values of H_α have to be integers. Obviously, for any finite-dimensional irrep V is the set of weights $W(V) \subset \Lambda_W$. In particular, $R \subset \Lambda_W$, therefore $\Lambda_R \subset \Lambda_W$ is a sublattice with finite index.

[V] **Weyl group.** Use the fact that the weights of representations of $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2,\mathbb{C})$ possess a reflection symmetry by introducing the reflections W_α,

$$W_\alpha(\beta) = \beta - 2\frac{2\beta(H_\alpha)}{\alpha(H_\alpha)}\alpha = \beta - \beta(H_\alpha)\alpha\,,$$

which map the hyperplanes $\Omega_\alpha = \{\beta \in \mathfrak{h}^* : \langle H_\alpha, \beta\rangle = 0\}$ into themselves, and reflect the lines $\mathbb{C}\alpha$ into themselves, i.e. $W_\alpha(\alpha) = -\alpha$. The group $\mathfrak{W}$ generated from the W_α, $\alpha \in R$, is called *Weyl group*. In particular, one has that the set of weights of a representation is invariant under the Weyl group, i.e. $\mathfrak{W}(W(V)) = W(V)$.

[VI] **Killing form.** Define the Killing form $g(X,Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ as Scalar product on $\mathfrak{g}$, thus also on $\mathfrak{h} \subset \mathfrak{g}$, which naturally extends to a scalar product on $\mathfrak{h}^* \cong \mathfrak{h}$. The Weyl group is then nothing else than the orthogonal group, $\mathfrak{W} = \mathrm{O}(\Lambda_W)$, i.e. $g(W_\alpha(\beta), W_\alpha(\beta')) = g(\beta, \beta')$ for all $W_\alpha \in \mathfrak{W}$, $\beta, \beta' \in \Lambda_W \subset \mathfrak{h}^*$. With respect to this scalar product the line $\mathbb{C}\alpha$ and the hyperplane Ω_α are orthogonal, i.e. $\alpha \perp \Omega_\alpha$. The scalar product $g(\cdot,\cdot)$ is positive definite on $\mathfrak{h}$.

[VII] **Highest weights and highest weight vectors.** Choose a direction in $\mathfrak{h}^*$ by choosing a real linear function $\ell : \Lambda_R \to \mathbb{R}$, which divides the roots into two equally sized subsets $R = R^+ \cup R^-$. Here, $R^+ = \{\alpha \in R : \ell(\alpha) > 0\}$ is the set of *positive roots*, and analogously $R^- = \{\alpha \in R : \ell(\alpha) < 0\}$ is the set of *negative roots*. For a representation V of $\mathfrak{g}$ we call a vector $v \in V$, which is eigen vector to all $H \in \mathfrak{h}$, and which simultaneously is in the kernel of all root spaces of the positive roots, a *highest weight vector*, i.e. $v \in V$ is a highest weight vector

with *highest weight* or *dominant weight* $\alpha \iff H(v) = \alpha(H)\, v$ for all $H \in \mathfrak{h}$, and $\mathfrak{g}_\alpha(v) = 0$ for all $\alpha \in R^+$. We have:
(1) Any finite-dimensional representation V of $\mathfrak{g}$ possesses a highest weight vector;
(2) To any finite dimensional representation V of $\mathfrak{g}$ with highest weight vector $v \in V$ is the subrepresentation $W = \mathrm{span}\{v, \mathfrak{g}_\alpha(v), \mathfrak{g}_\alpha\mathfrak{g}_{\alpha'}(v), \ldots\,:\, \alpha, \alpha', \ldots \in R^-\} \subset V$ irreducible;
(3) Any finite-dimensional irrep V of $\mathfrak{g}$ has (up to normalization) a unique highest weight vector.

The so-called (positive) *primitive* or *simple roots* are those positive roots, which are not the sum of two other positive roots, i.e. $R_p^+ = \{\alpha \in R^+ : \alpha \neq \alpha' + \alpha'' \text{ for } \alpha', \alpha'' \in R^+\}$. Analogously one defines negative simple roots R_p^-. Then, the above definition of $W \subset V$ simplifies to $W = \mathrm{span}\{v, \mathfrak{g}_\alpha(v), \mathfrak{g}_\alpha\mathfrak{g}_{\alpha'}(v), \ldots\,:\, \alpha, \alpha', \ldots \in R_p^-\}$.

The *(closed) Weyl chamber* $\mathcal{W}$ is the region in $\mathfrak{h}^*$, within which all possible highest weights must reside. It is defined as $\mathcal{W} = \{\alpha \in \mathrm{span}_\mathbb{R} R : \alpha(H_\gamma) \geq 0\, \forall \gamma \in R^+\}$. An equivalent definition is as the closure of a connected component of the complement of the union of the hyperplanes Ω_α.

[IIX] Classification of irreps. Now, we have everything in place to completely describe all finite-dimensional irreps of a semi-simple Lie algebra $\mathfrak{g}$.
THEOREM: For any $\alpha \in \mathcal{W} \cap \Lambda_W$ there is exactly one finite-dimensional irrep Γ_α with α its highest weight. Let $\mathfrak{C}$ denote the closure of the open convex hull, whose vertices are given by the images of α under the action of the Weyl group $\mathfrak{W}$. Then, the set of weights of the irrep Γ_α are given by $W(\Gamma_\alpha) = \{\beta \in \Lambda_W \cap \mathfrak{C} : \beta - \alpha \in \Lambda_R\}$. Let the positive simple roots be labeled in an arbitrary manner as $\{\alpha_1, \ldots, \alpha_n\} = R_p^+$, $n = \mathrm{rank}\mathfrak{g}$. Then there exist weights $\omega_i \in \mathfrak{h}^*$, $1 \leq i \leq n$, such that $\omega_i(H_{\alpha_j}) = \delta_{ij}$. These weights are called *fundamental weights*. Each highest weight can be written in a unique way as linear combination $\alpha = a_1\omega_1 + \ldots + a_n\omega_n$, where all $a_i \in \mathbb{Z}_+$. Thus, often the notation $\Gamma_\alpha = \Gamma_{a_1\omega_1+\ldots+a_n\omega_n} = \Gamma_{a_1,\ldots,a_n}$ is used.

DYNKIN DIAGRAMS

If $\mathrm{rank}\mathfrak{g} > 2$, it is not very well possible to explicitly draw weight diagrams as we did for $\mathfrak{su}(3)$. Fortunately, there is a much more efficient way to graphically denote representations, which has been developed mainly by Dynkin. I will sketch here briefly, how all (semi-)simple Lie algebras can easily be classified with the help of a graphical notation, the so-called Dynkin diagrams, which encodes all the information on the Lie algebra. If one adds, in addition, the numbers $a_1, \ldots a_n$, $a_i = g(\alpha, \alpha_i)$, then the diagram also encodes all the information about the representations Γ_α, where I use the notation from [IIX].

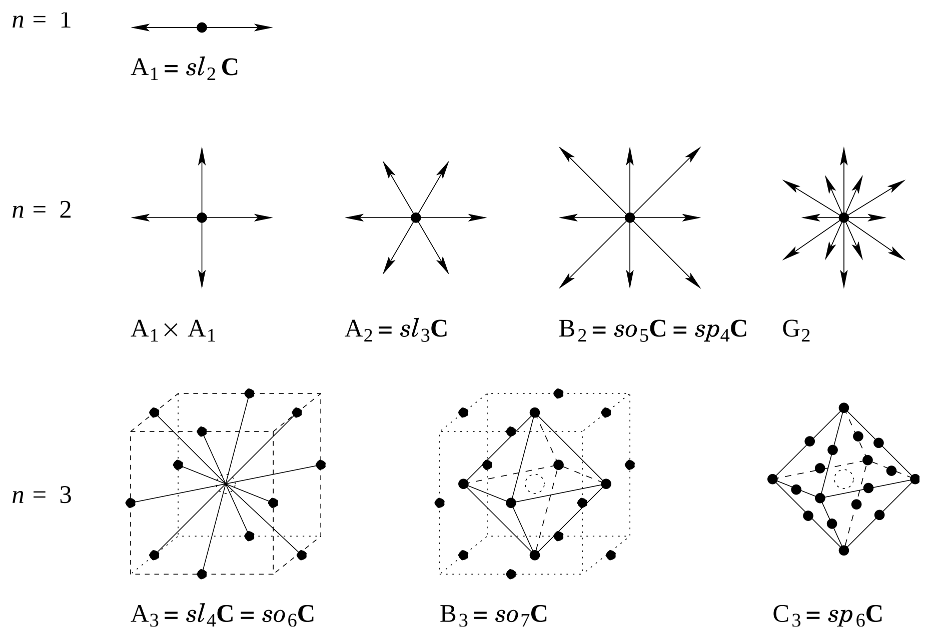
Root systems. Let $\mathfrak{g}$ be a semi-simple Lie algebra, $\mathfrak{h}$ its Cartan subalgebra, g its Killing form, etc. The Euclidian space $\mathbb{E} = \mathrm{span}_\mathbb{R} R$ is a real subvectorspace of $\mathfrak{h}^*$, on which g is positive definite. To characterize a Lie algebra, it suffices to classify the possible root systems $R \subset \mathbb{E}$ up to rotations and scalar multiplications. A root system has the properties:
[i] $|R| < \infty$, $\mathrm{span}_\mathbb{R} R = \mathbb{E}$;
[ii] $\alpha \in R \Longrightarrow -\alpha \in R$, and more strictly $\alpha \in R \Longrightarrow R \cap \{\mathbb{R}\alpha\} = \{\alpha, -\alpha\}$;
[iii] $\alpha \in R \Longrightarrow W_\alpha : R \to R$ with W_α the reflection in the $\alpha^\perp$-plane;
[iv] $\alpha, \beta \in R \Longrightarrow \eta_{\beta\alpha} = \beta(H_\alpha) \in \mathbb{Z}$. The quantity $\eta_{\beta\alpha}$ and the Weyl reflection W_α can be expressed via the Killing form,

$$\eta_{\beta\alpha} = 2\frac{g(\beta, \alpha)}{g(\alpha, \alpha)}\,, \quad W_\alpha(\beta) = \beta - \eta_{\beta\alpha}\alpha\,.$$

Condition [iv] is very restrictive, since it restricts the angle θ between to roots α, β to a very few possibilities. With $\cos\theta = g(\beta, \alpha)/\sqrt{g(\alpha, \alpha)g(\beta, \beta)}$, it follows that $\eta_{\beta\alpha} = 2\sqrt{g(\beta, \beta)/g(\alpha, \alpha)}\cos\theta \in \mathbb{Z}$, thus $4\cos^2\theta = \eta_{\alpha\beta}\eta_{\beta\alpha} \in \mathbb{Z}$. This leaves only the possibilities $4\cos^2\theta \in \{0, 1, 2, 3, 4\}$, where the last case $4\cos^2\theta = 4$ occurs only in the trivial setting $\beta = \pm\alpha$. Without loss of generality one can assume that $g(\beta, \beta) \geq g(\alpha, \alpha)$, or $|\eta_{\beta\alpha}| \geq |\eta_{\alpha\beta}|$, respectively. This leads to the following table of non-trivial possibilities:

$4\cos^2\theta$	3	2	1	0	1	2	3
$\cos\theta$	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$
θ	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$
$\eta_{\beta\alpha}$	3	2	1	0	-1	-2	-3
$\eta_{\alpha\beta}$	1	1	1	0	-1	-1	-1
$\sqrt{\frac{g(\beta,\beta)}{g(\alpha,\alpha)}}$	$\sqrt{3}$	$\sqrt{2}$	1	$*$	1	$\sqrt{2}$	$\sqrt{3}$

Let now $n = \dim_{\mathbb{R}}\mathbb{E} = \dim_{\mathbb{C}}\mathfrak{h} = \mathrm{rank}\mathfrak{g}$. Below, all root systems for $1 \leq n \leq 3$ are sketched:



With a suitable but otherwise arbitrary semi-ordering $\ell : \mathbb{E} \to \mathbb{R}$ we divide the roots into two halves, $R = R^+ \cup R^-$. The positive simple roots for the classical Lie algebras are given in terms of the basic weights L_i as follows:

$$R_p^+ = \begin{cases} \{L_i - L_{i+1} : i = 1, \ldots, n\} & \text{for} \quad \mathfrak{sl}(n+1, \mathbb{C}) & = & A_n\,, \\ \{L_i - L_{i+1} : i = 1, \ldots, n-1\} \cup \{L_n\} & \text{for} \quad \mathfrak{so}(2n+1, \mathbb{C}) & = & B_n\,, \\ \{L_i - L_{i+1} : i = 1, \ldots, n-1\} \cup \{2L_n\} & \text{for} \quad \mathfrak{sp}(2n, \mathbb{C}) & = & C_n\,, \\ \{L_i - L_{i+1} : i = 1, \ldots, n-1\} \cup \{L_{n-1} + L_n\} & \text{for} \quad \mathfrak{so}(2n, \mathbb{C}) & = & D_n\,. \end{cases}$$

The properties [i] to [iv] have immediate consequences, which must be satisfied by root systems R.

[v] For all $\alpha, \beta \in R$, $\beta \neq \pm\alpha$, the whole string $\{\beta - p\alpha, \beta - (p-1)\alpha, \ldots, \beta - \alpha, \beta, \beta + \alpha, \beta + 2\alpha, \ldots, \beta + q\alpha\} \subset R$ must belong to the root system. Since we must also have that $W_\alpha(\beta + q\alpha) = \beta - p\alpha = (\beta - \eta_{\beta\alpha}\alpha) - q\alpha$, it follows that $p = \eta_{\beta\alpha} + q$. This yields the restriction $p + q \leq 3$, $p - q = \eta_{\beta\alpha}$.

[vi] For all $\alpha, \beta \in R$, $\beta \neq \pm\alpha$, it follows with the help of the Killing form that

$$\begin{aligned} g(\beta, \alpha) > 0 \quad &\Longrightarrow \quad \alpha - \beta \in R\,, \\ g(\beta, \alpha) < 0 \quad &\Longrightarrow \quad \alpha + \beta \in R\,, \\ g(\beta, \alpha) = 0 \quad &\Longrightarrow \quad \alpha - \beta, \alpha + \beta \text{ either both } \in R \text{ or both } \notin R\,; \end{aligned}$$

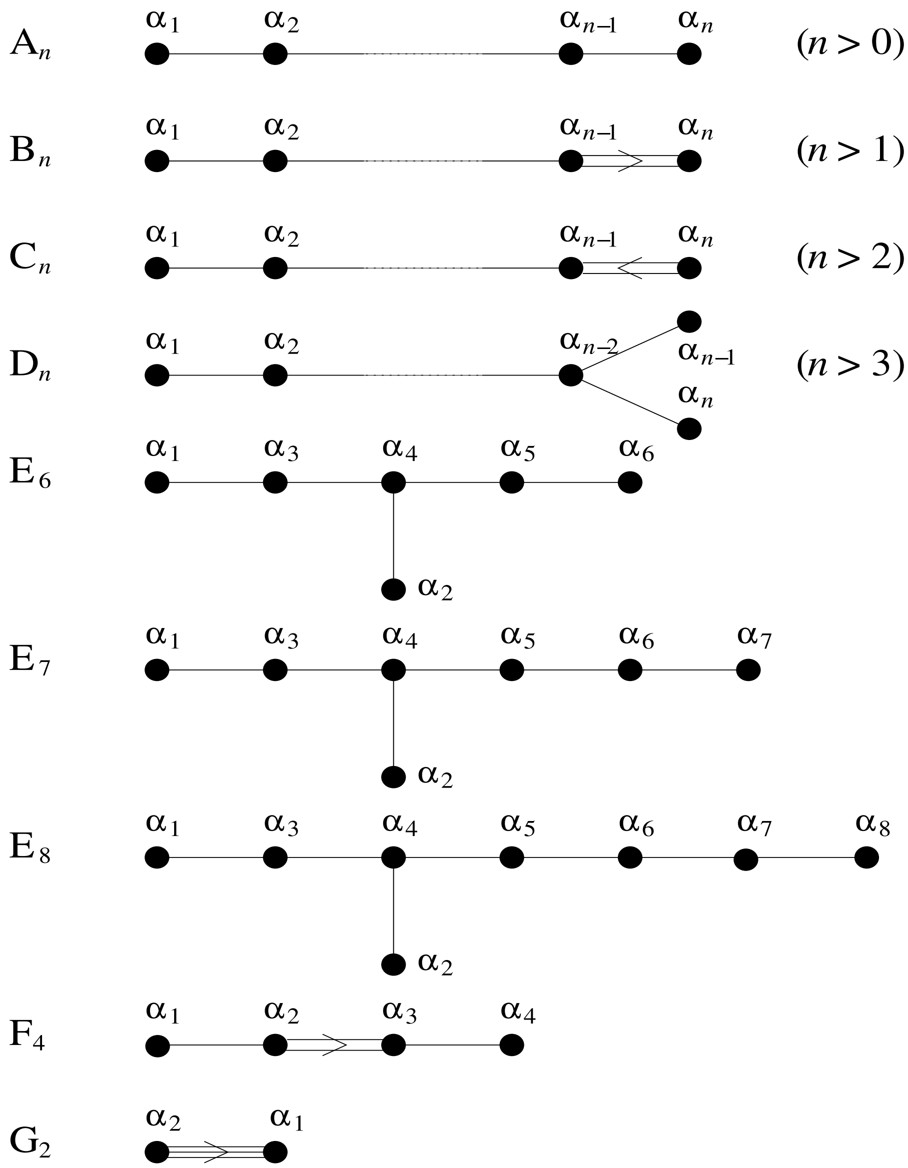
[vii] If $\alpha \neq \beta \in R_p^+$ are simple positive roots, then $\alpha - \beta \notin R$, $\beta - \alpha \notin R$ cannot be roots;

[iix] If $\alpha \neq \beta \in R_p^+$ are simple positive roots, then the angle between them cannot be sharp, i.e. $\cos\theta = \sqrt{g(\alpha, \alpha)/g(\beta, \beta)}\eta_{\beta\alpha}/2 \leq 0$;

[ix] The simple positive roots are linearly independent;

[x] $|R_p^+| = n = \mathrm{rank}\mathfrak{g}$, such that each $\alpha \in R^+$ has a unique decomposition $\alpha = a_1\alpha_1 + \ldots + a_n\alpha_n$, where $\alpha_i \in R_p^+$ and $a_i \in \mathbb{Z}_+$.

Dynkin diagrams. Label the positive simple roots in an arbitrary manner, $R_p^+ = \{\alpha_1, \ldots, \alpha_n\}$. It follows from [iix] that $\alpha_i, \alpha_j \in R_p^+$ can only from the angles $\theta \in \{\pi/2, 2\pi/3, 3\pi/4, 5\pi/6\}$. Correspondingly, η_{α_i,α_j} takes the values $\{0, -1, -2, -3\}$. Draw a graph with one node for each α_i, and with exactly $\eta_{\alpha_i,\alpha_j}\eta_{\alpha_j,\alpha_i}$ lines linking the nodes α_i and α_j. To make it even more beautiful, draw an arrow on the linking lines from the longer root to the shorter one, if $g(\alpha_i, \alpha_i) \neq g(\alpha_j, \alpha_j)$. One can prove that only the connected graphs listed below correspond to irreducible root systems which satisfy the properties [i] to [iv] (and therefore also [v] to [x]). These are the *Dynkin diagrams* of the semi-simple Lie algebras. This classifies all semis-simple Lie algebras! Furthermore, any irrep $\Gamma_\alpha = \Gamma_{a_1,a_2,\ldots,a_n}$ can be completely characterized by a Dynkin diagram by simply denoting the number a_i near the node α_i. These coefficients a_i were obtained by introducing the fundamental weights ω_i with $g(\omega_i, \alpha_j) = \delta_{ij}$, such that obviously $a_i = g(\alpha, \alpha_i)$. Indeed, any irrep, i.e. its weight diagram including all multiplicities, can be reconstructed from the Dynkin diagram of the underlying Lie algebra together with the weight coefficients a_i. The Dynkin diagram contains, for instance, all values of the so-called *Cartan matrix* $n_{i,j} \equiv \eta_{\alpha_i,\alpha_j}$. In the diagrams below, the labeling goes from left to right following the lists for $R_p^+ = \{\alpha_1, \ldots, \alpha_n\}$ given earlier in the text for the classical groups, and further below for the exceptional ones.



Finally, I make some comments regarding the restrictions concerning the minimal rank for Lie algebras in the series A, B, C, D. These restrictions avoid that the same graph appears multiple times in different series.
For $n = 1$ we find $B_1 = C_1 = A_1$, which corresponds to the isomorphies $\mathfrak{so}(3,C) \cong \mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})$. All these Dynkin diagrams consist of just one single node. The case $D_1 = \mathfrak{so}(2,\mathbb{C})$ must be excluded, because this Lie algebra is not semi-simple.
For $n = 2$ we find $D_2 = A_1 \times A_1$ corresponding to the isomorphy $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$. The Dynkin diagrams consist out of two disjunct nodes without a joining line. Further, we find $C_2 = B_2$ corresponding to the isomorphy $\mathfrak{sp}(4,\mathbb{C}) \cong \mathfrak{so}(5,\mathbb{C})$. The associated Dynkin diagrams are equal, since the direction of the arrow on the linking line is irrelevant in the case of just two nodes.
For $n = 3$ we finally find $D_3 = A_3$ corresponding to the isomorphy $\mathfrak{so}(6,\mathbb{C}) \cong \mathfrak{sl}(4,\mathbb{C})$.
If one wishes, one can successively eliminate nodes from right to left to formally obtain the equivalences $E_5 = D_5$, $E_4 = A_4$, $E_3 = A_2 \times A_1$, $E_2 = A_1 \times A_1$ and $E_1 = A_1$.

The root systems for the exceptional Lie algebras read as follows:

$$R_p^+ = \begin{cases} \{L_1, -\frac{3}{2}L_1 + \frac{\sqrt{3}}{2}L_2\} & \text{für} \quad G_2\,, \\ \{L_2 - L_3, L_3 - L_4, L_4, \frac{1}{2}(L_1 - L_2 - L_3 - L_4)\} & \text{für} \quad F_4\,, \\ \{\frac{1}{2}(L_1 - L_2 - L_3 - L_4 - L_5 + \sqrt{3}L_6), L_1 + L_2,\} \cup \{L_{i+1} - L_i : i = 1, \ldots, 4\} & \text{für} \quad E_6\,, \\ \{\frac{1}{2}(L_1 - L_2 - \ldots - L_6 + \sqrt{2}L_7), L_1 + L_2\} \cup \{L_{i+1} - L_i : i = 1, \ldots, 5\} & \text{für} \quad E_7\,, \\ \{\frac{1}{2}(L_1 - L_2 - \ldots - L_7 + L_8), L_1 + L_2\} \cup \{L_{i+1} - L_i : i = 1, \ldots, 6\} & \text{für} \quad E_8\,. \end{cases}$$