Handout VII for the course SYMMETRIES IN PHYSICS — Michael Flohr

Multiplicities and Dimensions — 09. and 16. July 2010

THE FREUDENTHAL FORMULA

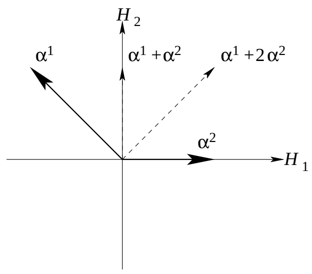
Let us consider a highest weight representation ρ of a Lie algebra $\mathfrak{g}$, whose highest weight is Λ. Let us denote the set of positive roots R^+. We have seen in the lecture how the weights of the irreducible representation ρ can be obtained with the help of the Dynkin labels associated to Λ and the simple positive roots α^i. Remember, that the Dynkin labels of the simple roots are given by rows of the Cartan matrix.

What could happen, however, is that a certain weight can be reached in various ways by applying lowering operators $E_{-\alpha^i}$ to the highest weight Λ. The question which then arises is whether these different ways correspond (partially) to a multiplicity of this weight or not. For example, in the case of $\mathfrak{g} = \mathfrak{su}(3)$, we learned that the weights in the inner region of the convex hull spanned by the images of the highest weight under the Weyl group have a multiplicity larger than one, increasing shell by shell until the hexagonal shell degenerates to a triangular shell, where the multiplicity then stays constant. How can we answer the question of multiplicity in general?

Fortunately, there exists an unfortunately complicated (and difficult to derive) formula to compute the multiplicities of any weight μ of a highest weight representation ρ, which is called *Freudenthal's formula*. We will demonstrate how it works at the example of $\mathfrak{g} = \mathfrak{so}(5)$.

$\mathfrak{so}(5)$. This algebra is of type B_2 and thus has rank two. However, the two simple roots have differing length. Thus, the Cartan matrix and the root diagram for the positive roots reads

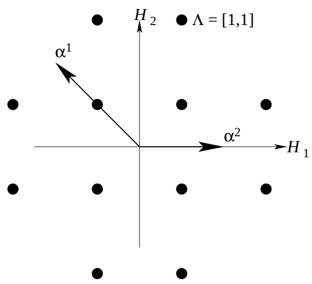
$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$



It is relatively easy to draw the weight diagram for the representation with highest weight Λ given as $[1, 1]$ in Dynkin labels. We leave it as an exercise to do the precise construction with the help of the master formula explicitly, and simply give the result

$$\begin{array}{ccccccc}
 & & & & \Lambda & & \\
 & & [2,-1] & \stackrel{-\alpha^2}{\longleftarrow} & [1,1] & & \\
 & & & \searrow & & \searrow^{-\alpha^1} & \\
[2,-3] & \longleftarrow & [1,-1] & \longleftarrow & [0,1] & \longleftarrow & [-1,3] \\
 & \searrow & & \searrow & & \searrow & \\
[1,-3] & \longleftarrow & [0,-1] & \longleftarrow & [-1,1] & \longleftarrow & [-2,3] \\
 & \searrow & & \searrow & & & \\
 & & [-1,-1] & \longleftarrow & [-2,1] & &
\end{array}$$

The resulting diagram looks like this, where each dot is the location of a weight contained in the representation.



Simply counting the dots, we see that this representation has $\dim \rho \geq 12$, but since we do not know the multiplicities of the four inner points, we cannot be sure what the exact dimension is.

Freudenthal's formula. This is a formula which works recursive. Given a highest weight Λ, the multiplicity of any weight μ contained in the representation ρ_Λ is given by the following expression

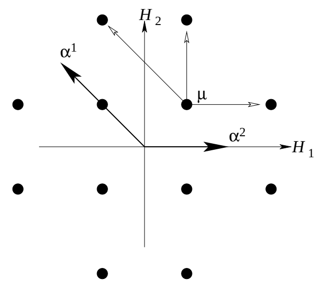
$$\operatorname{mult}(\mu) = \frac{\displaystyle\sum_{\alpha \in R^+} \sum_{k=1}^{\infty} 2 \operatorname{mult}(\mu + k\alpha)\,(\mu + k\alpha, \alpha)}{(\Lambda + \mu + 2\delta, \Lambda - \mu)} .$$

For better readability, we have denoted the scalar product $\nu \cdot \beta$ by (ν, β). Moreover, we introduced the so-called *Weyl vector*, defined as $2\delta = \sum_{\alpha \in R^+} \alpha$. Of course, as any linear combinations of positive roots, the Weyl vector has a unique linear decomposition into simple roots with non-negative integers, $2\delta = \sum_{i=1}^{\mathrm{rank}\,\mathfrak{g}} k_i \alpha^i$. In our example, the positive roots are $\alpha^1, \alpha^2, \alpha^1 + \alpha^2, \alpha^1 + 2\alpha^2$, and hence we find $2\delta = 3\alpha^1 + 4\alpha^2$.

Computing the multiplicity. Continuing with our example, we are interested in the multiplicity of the first inner weight below the highest weight. By symmetry, all other inner points will have the same multiplicity. Expressed in the simple roots, we find $\Lambda = \frac{3}{2}\alpha^1 + 2\alpha^2$, and for the weight μ, chosen as just described, we find $\mu = \frac{1}{2}\alpha^1 + \alpha^2$. We already know that $2\delta = 3\alpha^1 + 4\alpha^2$ such that we can compute the weights appearing in the denominator easily: $\Lambda + \mu + 2\delta = 5\alpha^1 + 7\alpha^2$ and $\Lambda - \mu = \alpha^1 + \alpha^2$. Thus, we find

$$\begin{aligned} (\Lambda + \mu + 2\delta, \Lambda - \mu) &= (5\alpha^1 + 7\alpha^2, \alpha^1 + \alpha^2) \\ &= 5(\alpha^1, \alpha^1) + 5(\alpha^1, \alpha^2) + 7(\alpha^1, \alpha^2) + 7(\alpha^2, \alpha^2) \\ &= 5 \cdot 2 + 12 \cdot (-1) + 7 \cdot 1 \;=\; 5\,. \end{aligned}$$

So, we have computed the denominator. To tackle the numerator, one should keep in mind that the multiplicity of a weight not belonging to ρ_Λ is actually zero. This ensures that the sum is always finite. Looking at our specific example, we see that from μ we can reach three other weights by adding positive roots.



Fortunately, all these three weights lie on the border of the representation's weight diagram, so they all have multiplicity one. Thus, in the numerator, all the multiplicities are one and we have to compute the following scalar products:

$$\begin{aligned} \mu + \alpha^1 &= \tfrac{3}{2}\alpha^1 + \alpha^2 &\implies& (\tfrac{3}{2}\alpha^1 + \alpha^2, \alpha^1) &=& 2\,, \\ \mu + \alpha^2 &= \tfrac{1}{2}\alpha^1 + 2\alpha^2 &\implies& (\tfrac{1}{2}\alpha^1 + 2\alpha^2, \alpha^2) &=& \tfrac{3}{2}\,, \\ \mu + \alpha^1 + \alpha^2 &= \tfrac{3}{2}\alpha^1 + 2\alpha^2 &\implies& (\tfrac{3}{2}\alpha^1 + 2\alpha^2, \alpha^1 + \alpha^2) &=& \tfrac{3}{2}\,. \end{aligned}$$

So, Freudenthal tells us that $\mathrm{mult}(\mu) = \frac{1}{5}(2 + \frac{3}{2} + \frac{3}{2}) \cdot 2 = \frac{10}{5} = 2$.

Weyl dimension formula. Often, one is not interested in the individual multiplicities of the weights, but only in the overall dimension of a representation ρ_Λ to a highest weight Λ. Then, there is a simpler way to obtain this. Let us denote the Dynkin labels of the highest weight Λ by $[\Lambda_1, \Lambda_2, \ldots, \Lambda_r]$ for a given Lie algebra $\mathfrak{g}$ of rank r. Let us further denote the decompositions of positive roots $\alpha > 0$ into linear combinations of simple roots α^i, $i = 1, \ldots r$, with non-negative integers k^i_α, such that $\alpha = \sum_i k^i_\alpha \alpha^i$ for all $\alpha \in R^+$. Then the *Weyl dimension formula* reads

$$\dim \rho_\Lambda = \prod_{\alpha \in R^+} \frac{\sum\limits_i k^i_\alpha (\Lambda_i + 1)(\alpha^i, \alpha^i)}{\sum\limits_i k^i_\alpha (\alpha^i, \alpha^i)}\,.$$

This formula is a special case of the Weyl character formula. It is typically not too difficult to obtain all positive roots, such that one can tabulate all the k^i_α, In our example, we find for the four positive roots the $k_\alpha = (k^1_\alpha, k^2_\alpha)$ as

$$\alpha^1 : k_\alpha = (1,0)\,,\;\; \alpha^2 : k_\alpha = (0,1)\,,\;\; \alpha^1 + \alpha^2 : k_\alpha = (1,1)\,,\;\; \alpha^1 + 2\alpha^2 : k_\alpha = (1,2)\,.$$

Plugging this into the Weyl dimension formula we find a simple result for the dimension of any $\mathfrak{so}(5)$ highest weight representation ρ_Λ with Dynkin labels $[\Lambda_1, \Lambda_2]$, namely

$$\dim \rho_\Lambda = \frac{1}{6}(\Lambda_1 + 1)(\Lambda_2 + 1)(2\Lambda_1 + \Lambda_2 + 3)(\Lambda_1 + \Lambda_2 + 2)\,,$$

whose derivation from the general formula we leave as an exercise. Indeed, we find $\dim \rho_{[1,1]} = 16$ which is correct, since the inner four of the twelve weights all have multiplicity two. As an exercise, compute the dimension formula for highest weight irreps of $\mathfrak{su}(3)$. Note that this is also a rank two algebra, so that each irrep has again two Dynkin labels. Finally, you can find in the same way the dimensions of the irreps of our other rank two example, the exceptional algebra G_2.