NOETHER THEOREM

In the last lecture, we derived the Noether theorem. It states that to any family of *differentiable* symmetry transformations g_s there exists a conservation law and a conserved quantity Q.

Angular Momentum I. Let us use the Noether theorem to find the conserved quantity of a physical system which is invariant under rotations around an axis $\vec{a}$. As a first step, verify that in this case $\frac{\mathrm{d}g_s}{\mathrm{d}s}(\vec{x}) = \vec{a} \times \vec{x}$.

Next, assume that the potential $V(\vec{x}) = V(|\vec{x}|)$ does only depend on the length of $\vec{x}$. Conclude that $F_s = 0$ in this case and find the Noether charge Q.

Angular Momentum II. Consider a Hamiltonian $H = \frac{1}{2m}\vec{p}^2 + V(|\vec{x}|)$. Verify that the Noether charge found above Poisson commutes with H.

GROUP MANIFOLDS

The notaion of group manifold will play an important role in the lecture course. We wish to study this object with the help of a simple example. To do so, let us consider the matrix Lie group $SU(2)$, i.e. the group of all 2×2 matrices M defined as

$$SU(2) := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : M^{-1} = M^\dagger \text{ and } \det M = +1 \text{ and } a, b, c, d \in \mathbb{C} \right\} .$$

Group. Show that the above defined set $SU(2)$ forms a group under matrix multiplication.

Manifold. An arbitrary element M of the group $SU(2)$ obviously is parametrized by four complex numbers a, b, c, d. However, not all possible quadruples of four complex numbers are allowed. Find the constraints on the parameters (a, b, c, d), such that $M = M(a, b, c, d) \in SU(2)$. It is helpful to consider the four complex parameters as eight real ones. But how many of the eight real parameters are really only needed? Which constraining equation must these satisfy in addition? Which manifold is thus spanned by the set of admitted parameters?

One more example. Consider now the matrix Lie group of rotations in $\mathbb{R}^3$. This group is defined as

$$SO(3) := \left\{ M = 3 \times 3 \text{ matrix} : M^{-1} = M^t \text{ and } \det M = +1 \text{ and } M_{ij} \in \mathbb{R} \right\} .$$

Therefore, we initially have nine real parameters. However, we (should) know that three-dimensional rotations are characterized by only three parameters, the *Euler angles*. Which manifold is spanned by the Euler angles (keep in mind the domains of definition)? Is there something noteworthy when you compare your results for the $SO(3)$ and the $SU(2)$ Lie groups?

Algebra (preparation). It is a by no means trivial task to determine the number of necessary parameters (i.e. the dimension) for such matrix Lie groups. This becomes much simpler, however, by considering infinitesimal small group elements, $M = \mathbb{1} + \delta M$. Then, we simply have that $M^{-1} = \mathbb{1} - \delta M$, which is very much easier to handle. In addition, make use of the relation

$$\det M = 1 + \mathrm{tr}\, \delta M ,$$

which will be proved in the lecture. Determine the defining constraints for the infinitesimal elements δM for $SU(2)$ or $SO(3)$, respectively, which follow from the one for the group elements M. Verify in this way your results on the *dimensions* of the groups $SU(2)$ and $SO(3)$.