Tutorial X to the lecture SYMMETRIES IN PHYSICS — Michael Flohr
Dynkin Diagrams — 12. July 2010

CLASSIFICATION OF COXETER GRAPHS

In the lecture, we did not have time to prove the theorem on admissible Coxeter graphs. These are precisely the graphs, out of which root systems can be constructed in a unique way, which belong to Lie algebras. A root system to a graph with r nodes essentially is a set of r linearly independent vectors which possess certain properties as explained in the lecture, such that they can represent the r simple roots of a Lie algebra of rank r. The proof argues in a nice way with properties of graphs and the corresponding properties of systems of unit vectors. A few steps of this proof shall be performed in this tutorial. We hope that this may enhance the feeling for beauty and elegance of this mathematical subject.

Let R_p^+ be the set of simple roots of a Lie algebra. We know that the elements of this set have the following properties:

[A] The $\alpha \in R_p^+$ are linearly independent vectors.

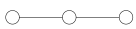
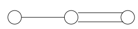
[B] If $\alpha, \beta \in R_p^+$ with $\alpha \neq \beta$, then $2g(\alpha, \beta)/g(\alpha, \alpha) \in \mathbb{Z}_-$ is a non-positive integer.

[C] The root system R_p^+ is indecomposable.

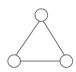
Π-Systems. A set of vectors satisfying **[A]**, **[B]** and **[C]** is called a Π-system.

The last condition **[C]** is necessary to guarantee that a root system, which satisfies **[A]** and **[B]**, belong to a simple Lie algebra. A root system R_p^+ is called decomposable, if we can spit it into to mutually orthogonal sub-systems. Otherwise, it is called indecomposable. Decomposable root systems belong to Lie algebras, which are direct sums of simple Lie algebras, which therefore are semi.simple. The Classification of the simple (and hence of the semi-simple) Lie algebras is therefore achieved when one succeeds to classify all Π-systems. To anyΠ-system there belongs in a unique way a Coxeter graph. Condition **[C]** tells that the Coxeter graph cannot be divided into to parts without cutting (at least) one line.

Lemma 1. Show that only the two following systems with three nodes are Π-systems:

Show then that the following three systems with three nodes violate condition **[A]**, since the three vectors lie in a plane:

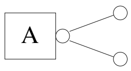


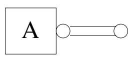
Sub-graphs. Argue that any connected sub-graph of a Coxeter graph is again a Coxeter graph. Therefore, any indecomposable sub-system of a Π-system is again a Π-system. Any three connected vectors of an arbitrary Π-system must hence yield one of the two allowed forms of Lemma 1. Draw with this the only one Π-system which contains a triple line.

Lemma 2. Show the following: If a Π-system contains two vectors which are connected by a single line, then one obtains a new admissible Π-system by replacing this line including its to end nodes by one single node. Thus, the graph is reduced by one node and one line.

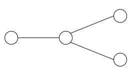
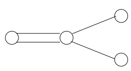
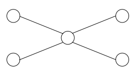
Corollaries. Lemma 2 has two important corollaries. Argue that the shrinking method described in Lemma 2 excludes the following cases: No Π-system can have more than one double line, and no Π-system contains a closed loop.

Lemma 3. Let A be an admissible sub-graph. Show that, if the first of the following graphs is a Π-system, then so is the second:



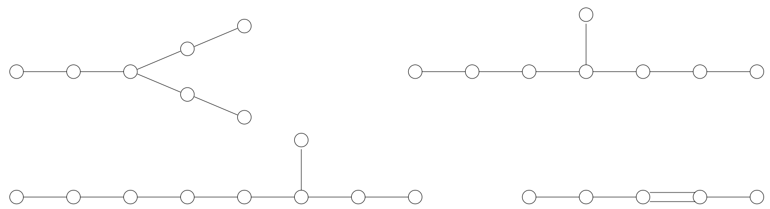


More Corollaries. Argue with Lemma 3 and Lemma 1, which of the following (sub-)graphs represent allowed branchings of an admissible Coxeter graph.

Conclude then that no Π-system can contain two or more branchings.

Exceptional Cases. So far we have analyzed general properties of Π-systems, considered as graphs. In order to conclude the classification, one has to consider a few exceptional cases, which cannot be excluded in this simple manner. These are:



Each node j stands for a vector α_j. Assume now that all vectors α_j have the same length (except in the last graph, there the vectors right from the double line have a length differing by a factor of $\sqrt{2}$ or $1/\sqrt{2}$). Find now positive integers $\mu_j \in \mathbb{Z}_+$, such that

$$\left(\sum_j \mu_j \alpha_j\right)^2 = 0\,.$$

Hint: It is helpful to enumerate the nodes. In the last case there are two solutions, depending on whether the vectors right from the double line are the longer or the shorter ones.

The Classification. With our results we easily can draw a list of all still admissible graphs. Convince yourself that the following list is complete and that indeed any of these graphs is a Π-system. In order to also display the information that vectors may have different lengths in the graphs, one proceeds as follows: Our considerations so far show that vectors can only be of differing lengths, if they are connected by a multiple line. Such multiple lines now get an arrow which shows from the longer to the shorter vector. (Crib: The arrow can be interpreted as the symbol for "greater than" or "less than" in the resulting inequality on the lengths of the roots.