ADJOINT REPRESENTATION OF THE LIE ALGEBRA

In the lecture, we introduced the adjoint representation ad of a Lie algebra $\mathfrak{g}$ on itself as a vector space. This representation has some remarkable and useful properties. We wish to study the adjoint representation in the example of the Lie algebra $\mathfrak{su}(2)$ in some detail. Remember the last tutorial, where the generators for $SU(2)$, the Pauli matrices, were introduced. We note here in advance, that the Pauli matrices are, more precisely, the generators of $SU(2)$ in the *fundamental* representation on a two-dimensional vector space.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Adjoint generators. The structure constants of $\mathfrak{su}(2)$ with respect to the above given generators are ${f_{jk}}^l = 2{\varepsilon_{jk}}^l$. Write down the generators T_j in the adjoint representation, for which we must have$(T_j)_k^{\ l} = -i{f_{jk}}^l$. Convince yourself that the T_j do indeed satisfy the correct algebra by computing $[T_j, T_k]$.

Algebra as vector space. The generators T_j span in a natural way the vector space of the algebra. Define a basis $|T_j\rangle = e_j$ associated with the generators, such that e_j is the j-th basis vector of a standard basis. Thus, e_j is the column vector, whose components are given by $(e_j)_k = \delta_{jk}$. Derive the action of the algebra on itself as vector space by computing $T_j|T_k\rangle = T_j \cdot e_k$. Express the result again in the basis $|T_j\rangle$, i.e. compute the coefficients a^l in $T_j|T_k\rangle = a^l|T_l\rangle$. Compare your result with the following definition of the action of the algebra on itself as vector space:

$$T_j|T_k\rangle = |[T_j, T_k]\rangle\,.$$

Adjoint representation of the group. With the adjoint representation of the algebra $\mathfrak{g}$ on itself as vector space at hand, it seems natural to also introduce a representation of the group G on the vector space of its algebra. To do so, we first introduce the operation of conjugation, $\Psi_g(h) = ghg^{-1}$ for $g, h \in G$. Obviously, Ψ_g is an automorphism of the group G for each $g \in G$. Choose for h an element close to the identity, i.e. $h = 1\!\!1 + \mathrm{d}u^a X_a$. Then you can easily read off, how the derivative of any Ψ_g looks like at the identity, i.e. at the point $h = 1\!\!1$. This derivative at the identity defines the adjoint representation of the group G on its algebra $\mathfrak{g}$. The derivative of Ψ_g is usually denoted Ad_g and is an automorphism of the algebra $\mathfrak{g}$ for each $g \in G$.

Our example again. With the notation from the last exercise sheet we have that $g = \exp(-\mathrm{i}\frac{\phi}{2}\vec{n}\cdot\vec{\sigma})$ is a generic element of the group $SU(2)$. Compute $\mathrm{Ad}_g(\sigma_j)$.

From the group to the algebra. It is now possible to go from the adjoint representation of the group to the adjoint representation of the algebra. All you have to do is to take the derivative of Ad_g at the point $g = 1\!\!1$. This defines the operation ad_X, where the generator X is precisely the one, which yields the group element g in the from $g(\lambda) = \exp(\lambda X)$. Remark: Obviously, this is not unique. This fixes X only up to a multiplicative constant. Show that with $\mathrm{Ad}_g(Y) = gYg^{-1}$ and $g = 1\!\!1 + \mathrm{d}u^a X_a$, $g^{-1} = 1\!\!1 - \mathrm{d}u^a X_a$, we have $\mathrm{ad}_X(Y) = [X, Y]$.

Killing form. In the lecture, we will introduce the Killing form $g_{ab} = \mathrm{tr}(T_a T_b)$. Compute the Killing form for $\mathfrak{su}(2)$ with the generators computed above in the adjoint representation. Then, diagonalize g_{ab}, i.e. bring the Killing form into the form $g_{ab} = k_a\delta_{ab}$. Finally, compute now the form $g'_{ab} = \mathrm{tr}(\sigma_a\sigma_b)$ associated to the fundamental representation of the algebra $\mathfrak{su}(2)$. Compare your result with what you obtained for the Killing form g_{ab}. Diagonalize g'_{ab} as well.