TENSOR PRODUCTS

The Classification of all irreps of a Lie algebra $\mathfrak{g}$ is teh first step in understanding the representation theory of $\mathfrak{g}$. The next step consists in understanding, how arbitrary representations can be decomposed into irreps. The most common type of reducible representations in physics are tensor products of irreps. Some aspects of what happens with tensor products can already be inferred from the example of angular momentum addition in quantum mechanics. The key point is, that a physical system transforms in such a way under a symmetry, that it may possess quantum numbers for different irreps of the symmetry algebra. For instance, a particle with spin s and angular momentum ℓ can be described by a Hilbert space whose states have *independent* quantum numbers with respect to the irreps $\rho^{(\ell)}$ and $\rho^{(s)}$ of $\mathfrak{su}(2)$, respectively. The states can hence be given in the form $|\ell, m\rangle \otimes |s, m_s\rangle \equiv |\ell, m\rangle|s, m_s\rangle$, where it is conventional to omit the symbol for the tensor product. Another common notation is $|\ell, m; s, m_s\rangle$.

Transformation properties. In order to understand how a Lie algebra acts on a tensor product, we change our notation a little bit. We denote the representation of a Lie group on vector spaces V and W with ρ^V and ρ^W, and in general representation of the group with ρ. The representations of the corresponding Lie algebra $\mathfrak{g}$ are denoted correspondingly with $\mathrm{d}\rho^V$, $\mathrm{d}\rho^W$ and $\mathrm{d}\rho$, respectively. This emphasizes that the linear operators $\mathrm{d}\rho(u^a X_a)$ can be considered as the linear differentials of the linear operators $\rho(g)$, $g = \exp(\mathrm{i}u^a X_a)$.

Realize that the Lie group acts in a natural way as follows on the tensor product $V \otimes W$ with states $|v\rangle \otimes |w\rangle$, namely:

$$\begin{aligned}\rho^{V\otimes W}(g)|v\rangle \otimes |w\rangle &= \sum_{v',w'} |v'\rangle \otimes |w'\rangle \left(\rho^{V\otimes W}(g)\right)_{(v'w')(vw)} \\ &= \left(\sum_{v'} |v'\rangle \left(\rho^V(g)\right)_{v'v}\right) \otimes \left(\sum_{w'} |w'\rangle \left(\rho^W(g)\right)_{w'w}\right) .\end{aligned}$$

This means nothing else than that the factors of the tensor products transform independently under the group action.

Show now by expanding $\left(1\!\!1 + \mathrm{i}u^a \mathrm{d}\rho^{V\otimes W}(J_a)\right)|v\rangle \otimes |w\rangle$ to first order in u, that we have correspondingly for the action of the Lie algebra on $V \otimes W$ that

$$\left(\mathrm{d}\rho^{V\otimes W}(J_a)\right)_{(v'w')(vw)} = \left(\mathrm{d}\rho^V(J_a)\right)_{v'v}\delta_{w',w} + \delta_{v',v}\left(\mathrm{d}\rho^W(J_a)\right)_{w'w}$$

A shorter way to write this is $\mathrm{d}\rho^{V\otimes W}(J_a) = \mathrm{d}\rho^V(J_a) \otimes 1\!\!1_W + 1\!\!1_V \otimes \mathrm{d}\rho^W(J_a)$. Often it is rather cumbersome to explicitly refer to the tensor product and the various representations in the notation. Therefore, one also finds notations as

$$J_a\left(|v\rangle|w\rangle\right) = \left(J_a|v\rangle\right)|w\rangle + |v\rangle\left(J_a|w\rangle\right) .$$

The representation $\mathrm{d}\rho$ of the Lie algebra $\mathfrak{g}$ therefore acts as a derivation, i.e. it satisfies the Leibniz rule.

Decomposition of tensor products. One of the easier tasks with tensor products is, to determine the eigen values of generaotrs which can be diagonalized. Let $\mathfrak{g} = \mathfrak{su}(2)$ and let us choose in the representations $\rho^{(j_1)}$ and $\rho^{(j_2)}$ eigen bases to J_3. Show that the eigen values of J_3 in the tensor product representation simply add:

$$J_3\left(|j_1, m_1\rangle|j_2, m_2\rangle\right) = (m_1 + m_2)\left(|j_1, m_1\rangle|j_2, m_2\rangle\right) .$$

The knowledge of the action of the Lie algebra on the tensor product suffices completely, to decompose the tensor product representation in to irreps. All you have to do is to apply the highest weight construction to the states of the tensor product. Then, make use of the fact that $\mathrm{d}\rho$ acts like a derivation. Show this in the example $j_1 = 1$ and $j_2 = 1/2$, by starting with the (unique) highest weight state $|3/2, 3/2\rangle = |1, 1\rangle|1/2, 1/2\rangle$.

Tensor operators. From quantum mechanics, we do know some things such as tensor operators and the Wigner Eckart theorem, which we wish to recapitulate briefly. Hopefully, this makes a few things more clear.. A tensor operator $\mathcal{O}^{(r)}$ of rank r simply is an operatos which transforms in the spin r irrep, i.e.

$$[\rho^{(r)}(J_a), \mathcal{O}^{(r)}_m] = \sum_{m'} \mathcal{O}^{(r)}_{m'}\left(\rho^{(r)}(J_a)\right)_{m'm} .$$

From here on, we again use the symbol ρ for the representation of the Lie algebra, instead of $\mathrm{d}\rho$. Furthermore, we consider the particular example $\mathfrak{g} = \mathfrak{su}(2)$. Of course, a tensor operator has components, as otherwise it could never transform in the spin r representation for $r > 0$. Consider as an example a particle in a radially symmetric potential. Angular momentum is given as $L_a = \epsilon_a{}^{bc} r_b p_c$. The operators L_a form a representation of the Lie algebra $\mathfrak{su}(2)$. The coordinate operator r_b essentially is a rank one tensor operator (i.e. a tensor operator, which transforms in the spin one irrpep), as it transforms under the adjoint representation:

$$[\rho(J_a), r_b] = \epsilon_a{}^{cd}[r_c p_d, r_b] = -\mathrm{i}\epsilon_a{}^{cd} r_c \delta_{b,d} = -\mathrm{i}\epsilon_a{}^{cb} r_c = r_c (T_a)^c{}_b = r_c\,\mathrm{ad}(J_a)^c{}_b\,.$$

Note that r_b does not transform in the canonical way, as the representation matrices of the adjoint representation do not have the standard form of the spin one irrep, which was given in the lecture. If we have a generic operator $\mathcal{O}_b$, such that $[\rho(J_a), \mathcal{O}_b] = \sum_{b'} \mathcal{O}_{b'}\,(\rho(J_a))_{b'b}$ whth ρ equivalent to a spin r irrep, then we can find a matrix S, such that $S\rho(J_a)S^{-1} = \rho^{(r)}(J_a)$. We then can use this matrix S to redefine the tensor operator, $\mathcal{O}^{(r)}_m = \mathcal{O}_b (S^{-1})^b{}_m$. The such redefined operator now transforms precisely in the irrep $\rho^{(r)}$, i.e.

$$[\rho^{(r)}(J_a), \mathcal{O}^{(r)}_m] = [S\rho(J_a)S^{-1}, (\mathcal{O}S^{-1})_m] = \mathcal{O}_{b'}(S^{-1})_m{}^{b'} S_{b'}^{c'}(\rho(J_a))_{c'd'}(S^{-1})_{m'}{}^{d'} = \mathcal{O}^{(r)}_{m'}(\rho^{(r)}(J_a))_{m'm}\,.$$

Often it is not necessary to compute S explicitly. If we can find a linear combination of the components $\mathcal{O}_b$, which is an eigen state to J_3 with eigen value r', then we can use this component as component of $\mathcal{O}^{(r)}$ and construct the remaining components simply by applying $J^{\pm}$. For the coordinate operator, this is very simple. Realize that $[\rho(J_3), r_3] = 0$. Identify r_3 with the component $r^{(1)}_0$. Find the remaining two components by computation of $[\rho^{(1)}(J^{\pm}), r^{(1)}_0] = r^{(1)}_{\pm 1}$. Give, for this example, the matrix S explicitly.

Wigner Eckart theorem. Tensor operators have the great advantage that their matrix elements are fixed by the symmetry, here $\mathfrak{su}(2)$, up to a constant, which is independent of the symmetry (this constant is typically determined by the dynamics of the physical system under consideration). If a tensor operator $\mathcal{O}^{(r)}_k$ acts of a state $|j,m\rangle$, the whole object transorms in the tensor representation $\rho^{(r)\otimes(j)}$. Let us denote the coefficients of the base change from the basis $\{|r,k\rangle|j,m\rangle : k = -r,\ldots,r\,,\ m = -j,\ldots,j\}$ to the basis $\{|J,M\rangle : J = |r-j|,\ldots r+j\,,\ M = -J,\ldots J\}$ for the decomposition $(r)\otimes(j) = \bigoplus_{J=|r-j|}^{r+j}(J)$ by $\langle J,M|r,k;j,m\rangle$, the *Clebsh Gordan coefficients*. These coefficients are completely fixed by the $\mathfrak{su}(2)$ structure, and can easily be computed by applying the highest weight construction to both sides and using the derivation property of the tensor representation. In essence, this results in the two following recurrence relations, which fix the coefficients up to a common normalization and a few signs:

$$\begin{aligned}\sqrt{(j\mp m)(j\pm m+1)}\langle j_1,m_1;j_2,m_2|j,m\pm 1\rangle &= \sqrt{(j_1\mp m_1+1)(j_1\pm m_1)}\langle j_1,m_1\mp 1;j_2,m_2|j,m\rangle\\ &+ \sqrt{(j_2\mp m_2+1)(j_2\pm m_2)}\langle j_1,m_1,j_2,m_2\mp 1|j,m\rangle\,,\end{aligned}$$

where one has to observe the condition $m_1 + m_2 = m \pm 1$. Note that we used the inverse base change here just in order to increase the general bewilderment ;-) . Compute with all this the Clebsh Gordon coefficients for $j_1 = 1$ and $j_2 = 1/2$.

If these coefficients are known (once and for all), we can write the matrix elements of tensor operators in a much simpler form:

$$\langle J,m',x'|\mathcal{O}^{(r)}_k|j,m,x\rangle = \delta_{m',k+m}\langle J,k+m|r,k;j,m\rangle\,\langle J,x'\|\mathcal{O}^{(r)}\|j,x\rangle\,,$$

where $\langle J,x'\|\mathcal{O}^{(r)}\|j,x\rangle$ is called the *reduced matrix element* of the tensor operatod. It only depends on the contributing irrpes and, possibly, further dynamical degrees of freedom, which we denoted here with x' and x. However, it does not depend on the components, i.e. the magnetic quantum numbers, at all. Therefore, the reduced matrix element depends neither on the inner structure of the contributing irreps, nor on the specific states in the irreps. This statement is known as the Wigner Eckart theorem. Actually, it holds for any Lie algebra $\mathfrak{g}$, not only for the example $\mathfrak{su}(2)$, which we devoured here in length.