MATRIX LIE GROUPS

Most of the Lie groups one encounters in physics are so-called matrix Lie groups. Hence, they are subgroups of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$. These two groups are nothing else than the invertible $n\times n$ matrices with coefficients out of $\mathbb{R}$ or $\mathbb{C}$. The subgroups which typically appear in physics are in general determined by quantities which are conserved under the symmetries defined by these subgroups. In this tutorial, we will learn a bit about the most important types of these subgroups, which are called the *classical Lie groups*.

$\boldsymbol{GL(n,\mathbb{K})}$**.** Show that GL_n (with both, real as well as complex coefficients) is indeed a Lie group by reasoning why the set of invertible $n\times n$ matrices forms a differentiable manifold. Hint: Why is the set of invertible matrices an *open* subset of the vector space of all $n\times n$ matrices? Show differentiability by considering matrix multiplication and inversion with the help of Cramer's rule. What is therefore the dimension of the group GL_n, obviously? How many connected components has $GL(n,\mathbb{R})$ (note that the group $GL(n,\mathbb{C})$ is simply connected)? How is the associated Lie algebra $\mathfrak{gl}_n$ defined?

Most other Lie groups can be described as subgroups of $GL(n,\mathbb{K})$, where either $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. There are two ways to define such subgroups. Either, one restricts the possible values of the coefficients by constraining equations. Or one defines subgroups by automorphisms of $V\cong\mathbb{K}^n$, which leave a given structure on $\mathbb{K}^n$ invariant.

$\boldsymbol{SL(n,\mathbb{K})}$**.** This group can be described as the subgroup of all matrices with determinant one. Which structure is left invariant by transformations on $\mathbb{K}^n$ given by such matrices? What must therefore the dimension of SL_n be? Which matrices do form the associated Lie algebra $\mathfrak{sl}(n,\mathbb{K})$ (make use of $\det e^A = e^{\mathrm{tr}A}$, a truly useful relation)?

$\boldsymbol{B_n}$ **and** $\boldsymbol{N_n}$**.** These are the upper triangular matrices and the upper triangular matrices with main diagonal elements all one, respectively. Show that these sets are indeed subgroups of $GL(n,\mathbb{K})$. Find there respective dimensions. The invariant structure is in this case not so easy to describe. It it the so-called *flag*, defined as a sequence $0=V_0\subset V_1\subset\ldots\subset V_{n-1}\subset V_n=\mathbb{K}^n$, where the subspaces $V_i=\mathrm{span}\{e_1,\ldots,e_i\}$ are stacked into each other like a Matryoshka. Can you visualize this (consider $n=2$ or $n=3$)? The group N_n has the additional property, that it operates as identity on the quotient spaces V_{i+1}/V_i. Of course, one can consider lower triangular matrices in just the completely analogous way.

Cartan Weyl basis. Recall that we decomposed each Lie algebra $\mathfrak{g}$ in its Cartan algebra $\mathfrak{h}$, which in the eigen basis consists entirely out of diagonal matrices, and in raising and lowering operators E_α and $E_{-\alpha}$, where the $\alpha>0$ are the positive roots. The subspace $\mathfrak{n}_+\subset\mathfrak{g}$ is the span of all E_α, and $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{n}_+\oplus\mathfrak{n}_-$. By construction the E_α are strict upper triangular matrices with vanishing diagonal elements. Show that these form a closed sub-algebra. Show further that the corresponding group is a subgroup of N_n, if $\dim\mathfrak{g}=n$. The Cartan Weyl basis hence implies a decomposition of a Lie group in diagonal matrices and upper and lower triangular matrices with main diagonal elements all being one.

STRUCTURES VIA FORMS

Most of the symmetries in physics can be understood with the help of bilinear or sesquilinear forms, which are defined on a given vector space, and which are left invariant under the symmetry operations. Such forms yield numbers from a pair of vectors and are therefore very suitable tools to construct observable quantities. A bilinear form is a bilinear map $Q:\mathbb{K}^n\times\mathbb{K}^n\to\mathbb{K}$. One may now wonder, which matrices A satisfy the equation $Q(Av,Aw)=Q(v,w)$ for all $v,w\in\mathbb{K}^n$. In the following, we discuss a few possible choices for Q. If such a from Q is given, we can represent it by a Cauchy matrix C, whose entries are scalar products $Q(e_i,e_j)$ for a canonical standard basis. Then, $Q(v,w)=v^t\,C\,w$, such that the set of matrices A, which leaves Q invariant, must obey the condition$A^t\,C\,A=C$.

$\boldsymbol{Q}$ **symmetric, positive definite.** Argue that such a Q can be brought into the form of a Euclidean standard scalar product. Which matrices do leave the Euclidean length of a vector invariant? Which condition must these matrices fulfill? Which values can the determinant take? Which group do you obtain, if you require in addition that these matrices shall form a subgroup of $SL(n,\mathbb{K})$? Which dimensions do these groups have? In order to answer these questions explicitly, you should first argue that a symmetric, positive definite form Q can be brought into a form, whose Cauchy matrix is $C=\mathbb{1}$. How many connected components do these groups have? Which property must the matrices satisfy, which constitute the associated Lie algebras (use the mathematical convention for the generators of a Lie algebra)?

Q symmetric, indefinite but not degenerate. In this case the Cauchy matrix has no zero eigen values. Hence, it will possess k positive eigen values and l negative one with $k + l = n$. One calls (k, l) the signature of the form Q. The corresponding Lie groups are the groups$SO(k, l)$, one example being the Lorentz group $SO(1, 3)$. Why do these groups only make sense for $\mathbb{K} = \mathbb{R}$? Obviously we have $SO(k, l) \cong SO(l, k)$. If one adds reflections, one obtains the groups $O(k, l)$. Show that $SO(k, l)$ is not connected, if $(k, l) \neq (n, 0)$ or $(0, n)$. How many connected components are there? It suffices to demonstrate this with the example of the Lorentz group. Show in this way that $O(k, l)$ has in total four components and the discrete center $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Q skew symmetric, not degenerate. This case is important for studying Hamiltonian mechanics, which lives on symplectic manifolds. The group which leaves such a form invariant, is denoted $Sp(n, \mathbb{R})$, and is only defined for even n. Argue that for $n = 2m$, Q has a standard form with Cauchy matrix $C = \begin{pmatrix} 0 & 1\!\!1_m \\ -1\!\!1_m & 0 \end{pmatrix}$. Explain why the dimension is $\dim Sp(n, \mathbb{R}) = \frac{1}{2}n(n + 1)$. Hint: Let $A \in Sp(n, \mathbb{R})$. Such a matrix A is surely also an element of $GL(2m, \mathbb{R})$. The condition $A^t\, C\, A = C$ for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in block form with $a, b, c, d \in GL(m, \mathbb{R})$ yields the following constraint on the constituting blocks: $a^t c$ and $b^t d$ must be symmetric, $a^t d - c^t b = 1\!\!1_m$. Argue that the associated Lie algebra $\mathfrak{sp}(n, \mathbb{R})$ consists of the matrices $X \in \mathfrak{gl}(2m, \mathbb{R})$, which satisfy the condition $X^t\, C + C\, X = 0$. One can further show that all elements of $Sp(n, \mathbb{R})$ must have determinant one, and that $Sp(n, \mathbb{R})$ is connected. There is a completely analogous definition for sympelctic groups over complex vector spaces, $Sp(n, \mathbb{C})$, and their algebras.

Q Hermitean, positive definite. For complex vector spaces $V \cong \mathbb{C}^n$ oen can consider squilinear forms insstead of bilinear forms. Especially interesting are Hermitean forms, which obey $Q(\lambda v, \mu w) = \bar{\lambda} Q(v, w)\mu$ for all $v, w \in V$ and $\lambda, \mu \in \mathbb{C}$. Moreover, we have $Q(w, v) = \overline{Q(v, w)}$. Such a form is positive definite, if $Q(v, v) > 0$ for all $v \neq 0$. What is the condition for matrices A, which leave such a Hermitean form, given by its Cauchy matrix C, invariant? What does one therefore find, if Q can be brought into the standard form $C = 1\!\!1$? Which values can the determinant of A take? Which group does one obtain, if one requires $\det A = 1$? Which dimensions (over $\mathbb{R}$) do these groups have? Note that it does not make sense to define the dimensions of these groups over $\mathbb{C}$. In fact, these groups are real and not complex Lie groups! The reason for this is the fact that these are compact (why?). However, any compact complex Lie group must be Abelian.

Q Hermitean, indefinite, not degenerate. Similar to the case of the real bilinear forms one can consider indefinite Hermitean forms of signature (k, l). The corresponding groups are then denoted $U(k, l)$ and $SU(k, l)$, where the latter are the subgroups with elements whose determinant is one.