A NON-TRIVIAL EXAMPLE

In the lecture, we first learned about the highest weight construction of the finite dimensional irreps of a semi-simple Lie algebra $\mathfrak{g}$ via the example of $\mathfrak{su}(2)$. Then, we started to generalize this to arbitrary Lie algebras. We demonstrated this with our second example, $\mathfrak{su}(3)$, the simplest Lie algebra of rank two. This example already covers all the properties and complications of the general case, with one exception: The roots of $\mathfrak{su}(3)$ all have the same length. In this tutorial, we wish to study a similarly simple example, again of rank two (so that we can make drawings of everything), the Lie algebra $\mathfrak{so}(5)$. The difference is, as we will see, that the roots cannot all have the same length.

Simple Roots. Let us start with betraying that the two positive, simple roots $\alpha^{(1)}$ and $\alpha^{(2)}$ enclose the angle $3\pi/4 = 135°$. It is very helpful to abbreviate the Euclidean scalar product of roots as follows: $(i,j) \equiv (\alpha^{(i)} \cdot \alpha^{(j)})$. Thus, we know that $\frac{(1,2)^2}{(1,1)(2,2)} = \frac{1}{2}$. Find with this peace of information the values of $(1,2)$, $(1,1)$ and $(2,2)$. In order to determine these uniquely, you must introduce an arbitrary ordering. Let us hence put $(1,1) \geq (2,2)$.

All Roots. Draw the root diagram. On the way, consider the following facts: The master formula tells that $2(i,j)/(i,i) = q - p$. Here, q denotes, how often we may subtract the root $\alpha^{(i)}$ from the root $\alpha^{(j)}$, and p denotes, how often we may add the root $\alpha^{(i)}$. The master formula thus encodes, how the root $\alpha^{(j)}$ behaves with respect to the $\mathfrak{su}(2)$-like sub-algebra $\mathfrak{g}_{\alpha^{(i)}}$, where $\mathfrak{g} = \mathfrak{so}(5)$. Unfortunately, we do not know p and q individually, but only their difference. However, we can consider the opposite case in an completely analogous manner, $2(j,i)/(j,j) = q' - p'$. Furthermore, the complete diagram must be symmetric under reflections in the planes orthogonal to the simple roots. This suffices to determine p, q, p' and q' and thus to draw the complete diagram.

Positive Roots. We defined simple roots as those positive roots, which cannot be written as sums of other positive roots. Start with your drawing of our root diagram and forget, for the moment, which roots we assumed to be the simple ones before. Choose an arbitrary basis for the roots, i.e. an arbitrary set of $\operatorname{rank}\mathfrak{g}$ linear independent roots, which we call α_i, $i = 1, \ldots, \operatorname{rank}\mathfrak{g}$. (We write this down in a generic way, but stick with our example $\mathfrak{g} = \mathfrak{so}(5)$ of rank two.) Write all the roots as linear combinations of the two roots you chose for the basis, i.e. $\beta = \sum_{i=1}^{\operatorname{rank}\mathfrak{g}} c_i\alpha_i$, where you have to take for β all the roots you have drawn into your diagram. We now simply define that $\beta > 0$, if the first non-vanishing coefficient $c_i > 0$, in the above defined ordering. Thus, $\beta > 0 \iff c_{i_0} > 0, i_0 = \min\{i : c_i \neq 0\}$. Mark the positive roots you found in that way. What are now the simple roots? Now choose a different basis and repeat the game. What do you notice?

Properties of simple Roots. Let α and β be two simple roots. Note that per definitionem simple roots are always positive roots. Show that $\alpha - \beta$ can never be a root. Show with this, and the master formula, that hence always $(\alpha \cdot \beta) \leq 0$.

Highest weight representations. The master formula suffices to reconstruct a representation completely, if one knows one weight to be a highest weight. This means that one cannot add any positive root to this highest weight. Check this by constructing in the case $\mathfrak{so}(5)$ the weight diagrams for a few representations. The highest weights (why these are in fact highest weights will be explained in the lecture) are either $\Lambda = \alpha^{(1)} + \alpha^{(2)}$, or $\Lambda = \frac{1}{2}\alpha^{(1)} + \alpha^{(2)}$ or finally $\Lambda = \alpha^{(1)} + 2\alpha^{(2)}$. Here, we choose the $\alpha^{(i)}$ as in the first exercise. Thus, we have in particular that $\alpha^{(1)}$ is the long root, and $\alpha^{(2)}$ is the short root.

Reminder: The master formula for a weight μ with respect to a simple root $\alpha^{(i)}$ reads

$$\frac{(\alpha^{(i)} \cdot \mu)}{(\alpha^{(i)} \cdot \alpha^{(i)})} = -\frac{1}{2}(p_\mu^{(i)} - q_\mu^{(i)}) \,.$$

Thus we obviously have for $\mu = \Lambda$ that all $p_\Lambda^{(i)} = 0$. The $q_\Lambda^{(i)}$ can hence be computed directly, as we have given the Λ as linear combinations of the simple roots.