THE CARTAN MATRIX

In the lecture, we learned that the simple roots suffice to reconstruct the Lie algebra completely. Let the Lie algebra $\mathfrak{g}$ have rank $\mathrm{rk}\,\mathfrak{g} = r$. Let us denote the simple roots with $\alpha^{(i)}$, $i = 1, \ldots, r$. We firs define an important obkect which contains all essential information about the Lie algebra, the Cartan matrix.

Roots as linear functionals. The following observation is very useful: Let $\mathcal{C}$ be the Cartan sub-algebra of $\mathfrak{g}$, spanned by generators H_i, $i = 1, \ldots, r$. In the eigenbasis of $\mathcal{C}$ we have for $E_\alpha \in \mathfrak{g}$ that $\mathrm{ad}(H_i)(E_\alpha) = \alpha_i E_\alpha$. Let now $H = \sum_i a_i H_i$. Then, $\mathrm{ad}(H)(E_\alpha) = \sum_i a_i \alpha_i E_\alpha = \alpha(H) E_\alpha$. Thus, $\alpha(H)$ is a scalar which depends linearly on H. Therefore, $\alpha \in \mathcal{C}^*$. Hence, the simple roots span $\mathcal{C}^*$. Show that we have for the Killing form $g(X, Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ that $g(H_i, H_j) = \sum_\alpha \alpha_i \alpha_j$, where the sum runs over all roots. Hint: $\mathfrak{g} = \mathcal{C} \oplus \bigoplus_\alpha \mathfrak{e}_\alpha$ with $\mathfrak{e}_\alpha$ the eigenspaces with respect to $\mathcal{C}$.

Killing form. Argue that for any root $\alpha \in \mathcal{C}^*$ one can find a generator $H_\alpha \in \mathcal{C}$ uniquely such that $g(H_\alpha, H_i) = \alpha_i$. Show with this result and linearity of the Killing form that $g(\alpha, \beta) \equiv g(H_\alpha, H_\beta)$ defines the Killing form on $\mathcal{C}^*$. It is useful to abbreviate the thus defines scalar product of simple roots as follows: $(i, j) \equiv g(\alpha^{(i)}, \alpha^{(j)})$. Remark: In contrast to the last tutorial, this is now all independent of the particular form of the Killing form.

Cartan matrix. One now defines the $r \times r$ matrix A with entries

$$A_{ij} = 2\frac{(i, j)}{(j, j)} .$$

Note that the matrix A is not symmetric. Which quantity already known to you is expressed by $A_{ij}A_{ji}$? Interpret the entries of the matrix with the help of the master formula. Show with this that for any root α one has

$$2\frac{g(\alpha, \alpha^{(j)})}{g(\alpha^{(j)}, \alpha^{(j)})} = q - p = \sum_i k_i A_{ij} .$$

where $k_i \in \mathbb{Z}$.

Examples. Consider the following Cartan matrices. Use a normalization, such that for α being the longest root we have $g(\alpha, \alpha) = 2$. Compute for these example all scalar products.

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} .$$

Construct the corresponding root diagrams, similar to last week's tutorial. Which Cartan matrix belongs to the example $\mathfrak{so}(5)$ of last week?

Weights. Consider now the general master formula for a weight Λ, i.e.

$$q - p = 2\frac{g(\Lambda, \alpha^{(j)})}{g(\alpha^{(j)}, \alpha^{(j)})} \equiv \Lambda_j .$$

What holds for Λ_j, if Λ is a highest weight? How can Λ be written as linear combination $\Lambda = \sum_i b_i \alpha^{(i)}$ of the simple roots? Solution: $b_i = \Lambda_j (A^{-1})_{ji}$. Interpret this result as follows: The roots determine the minimal distances between weights within a given representation. The weight diagram therefore is a subset of a lattice which is isomorphic to the root lattice. The weights themselves, for all representations together, compose a finer grained lattice, the weight lattice. Can you make the relation of these two lattices more precise?

The Dynkin labels

The integers Λ_j characterize a weight equally well as the coefficients b_i. These numbers Λ_j furthermore contain information, how often one can raise or lower in the direction of the simple root $\alpha^{(j)}$. In particular, if Λ is a highest weight, the Λ_j tell how often one can lower with root $\alpha^{(j)}$. These numbers are called Dynkin labels.

Again our examples. Consider the representations from last week's tutorial. Which Dynkin labels belong to these representations? Construct the representations once again and keep track how the Dynkin labels of each weight in the respective representation come about. Can you recoginize a simple rule? Hint: Look closely at the rows of the Cartan matrix.

Highest weights and simple roots. In the above examples, we assumed certain highest weights. From these, one can only descent. Check that the Dynkin labels of the other weights are obtained by successively subtracting the corresponding rows of the Cartan matrix. To find out how often you can descent from a weight in a certain direction, it helps to keep track of the p-values (i.e. of how often you can ascent from a weight). Obviously, for the highest weight $\Lambda = [\Lambda_1, \dots, \Lambda_r]$ we have $p = [0, \dots, 0]$. If you descent in an admissible direction, say $\alpha^{(i)}$, once, then you have on this next level $p = [0, \dots, 0, 1, 0, \dots, 0]$, where the 1 is on the i-th position. If you add these p-values to the Dynkin labels, then you can read off how often you can descent from a weight. Check this with the above examples. Do you recognize a symmetry on the Dynkin labels of the complete weight diagram?

A new example. Let us now consider $\mathfrak{su}(3)$. The corresponding Cartan matrix is the only symmetric among of the above mentioned ones. Construct the representation to $\Lambda = [1, 0]$, given by its Dynkin labels. Express Λ as linear combination of the simple roots. Can you give the Dynkin labels of the adjoint representation?