Four-Point Functions in LCFT

Surprises from SL(2,$\mathbb{C}$) covariance

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Motivation

- LCFT important for many applications such as
  - abelian sandpiles,
  - percolation,
  - Haldane-Rezayi fractional quantum Hall state,
  - disorder etc.
- Presumably LCFT will play a role in string theory, e.g.
  - D-brane recoil,
  - world-sheet formulation on AdS_3,
  - or, more generally, when non-compact CFTs arise.
- Subtleties in non-compact CFTs, e.g. Liouville theory:
  - non-uniqueness of fusion matrices,
  - non-trivial factorization properties of correlators,
  - difficulties in definition of consistent OPEs,
  - additional constraints for unitarity and locality.
- These subtleties are typical for LCFT!

Foundations: SL(2,$\mathbb{C}$) covariance

- Correlation functions have to satisfy the *global conformal Ward identities*, i.e. for $m = -1, 0, 1$ we must have

$$\begin{aligned} 0 &= L_m \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle \\ &= \sum_{i=1}^{n} z_i^m \left[z_i \partial_i + (m+1)(h_i + \hat{\delta}_{h_i}) \right] \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle \ . \end{aligned}$$

- In case of rank $r > 1$ Jordan cells of indecomposable representations with respect to *Vir*, we have

$$\hat{\delta}_{h_i} \Psi_{(h_j;k_j)} = \begin{cases} \delta_{i,j} \Psi_{(h_j;k_j-1)} & \text{if} \quad 1 \leq k_j \leq r-1 \,, \\ 0 & \text{if} \quad k_j = 0 \,. \end{cases}$$

- Equivalently, $L_0 |h;k\rangle = h|h;k\rangle + (1 - \delta_{k,0}) |h;k-1\rangle$.

Foundations: Recurrence

- Ward identities become *inhomogeneous* in LCFT. The inhomogeneities are given by correlation functions with *total Jordan-level* $K = \sum_{i=1}^{n} k_i$ decreased by one,

$$
\begin{aligned}
\left\langle \Psi_{(h_1;k_1)}(z_1) \ldots \Psi_{(h_n;k_n)}(z_n) \right\rangle &\equiv \langle k_1 k_2 \ldots k_n \rangle \;, \\
\frac{1}{(m+1)} L'_m \langle k_1 k_2 \ldots k_n \rangle &= - z_1^m \langle k_1 - 1, k_2 \ldots k_n \rangle \\
&\quad - z_2^m \langle k_1, k_2 - 1, k_3 \ldots k_n \rangle \\
&\quad - \ldots \\
&\quad - z_n^m \langle k_1 \ldots k_{n-1}, k_n - 1 \rangle \;.
\end{aligned}
$$

- We obtain a hierarchical scheme of solutions, starting with correlators of total Jordan-level $K = r - 1$.

Foundations: Correlators

- Generic form of 1-, 2- and 3-pt functions for fields forming Jordan cells, pre-logarithmic fields and fermionic fields in arbitrary rank r LCFT known:

$$\langle \Psi_{(h;k)} \rangle = \delta_{h,0}\delta_{k,r-1} \,,$$

$$\langle \Psi_{(h;k)}(z)\Psi_{(h';k')}(0)\rangle = \delta_{hh'} \sum_{j=r-1}^{k+k'} D_{(h;j)} \sum_{\substack{0\le i\le k,0\le i'\le k' \\ i+i'=k+k'-j}} \frac{(\partial_h)^i}{i!}\frac{(\partial_{h'})^{i'}}{i'!} z^{-h-h'} \,,$$

$$\langle \Psi_{(h_1;k_1)}(z_1)\Psi_{(h_2;k_2)}(z_2)\Psi_{(h_3;k_3)}(z_3)\rangle = \sum_{j=r-1}^{k_1+k_2+k_3} C_{(h_1h_2h_3;j)}$$

$$\times \sum_{\substack{0\le i_l\le k_l,l=1,2,3 \\ i_1+i_2+i_3=k_1+k_2+k_3-j}} \frac{(\partial_{h_1})^{i_1}}{i_1!}\frac{(\partial_{h_2})^{i_2}}{i_2!}\frac{(\partial_{h_3})^{i_3}}{i_3!} \prod_{\substack{\sigma\in S_3 \\ \sigma(1)<\sigma(2)}} \left(z_{\sigma(1)\sigma(2)}\right)^{h_{\sigma(3)}-h_{\sigma(1)}-h_{\sigma(2)}} \,.$$

Foundations: OPE

$$\Psi_{(h_1;k_1)}(z_1)\Psi_{(h_2;k_2)}(z_2) = \sum_{(h;k)} \Psi_{(h;k)}(z_2) \lim_{z_1\to z_2} \sum_{k'}$$

$$\langle\Psi_{(h_1;k_1)}(z_1)\Psi_{(h_2;k_2)}(z_2)\Psi_{(h;k')}(z_3)\rangle \left(\langle\Psi_{(h;\cdot)}(z_2)\Psi_{(h;\cdot)}(z_3)\rangle^{-1}\right)_{k',k} .$$

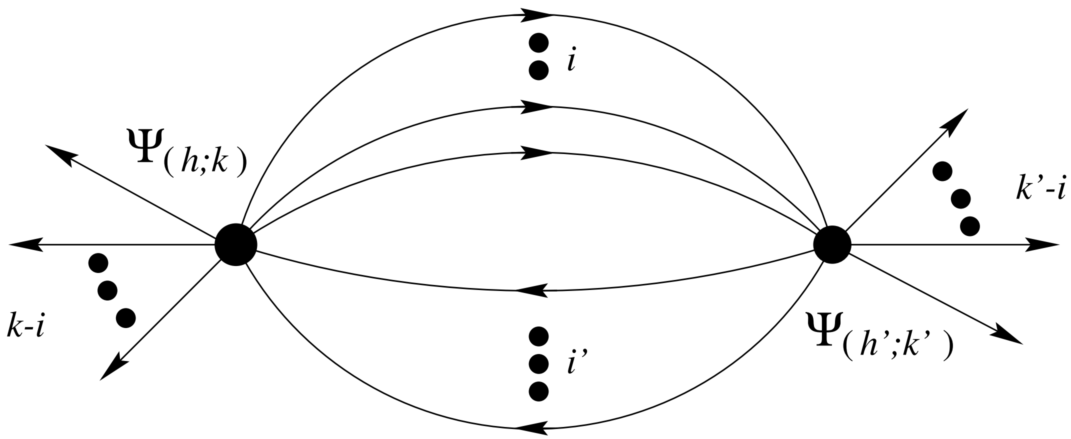
- Crucial role of zero modes worked out: all known LCFTs have realizations which include fermionic fields.
- Maximal power of logs bounded by zero mode content:

$$Z_*(\Psi_{(h;k)}) \leq Z_*(\Psi_{(h_1;k_1)}) + Z_*(\Psi_{(h_2;k_2)}) .$$

- Non-quasi-primary members of Jordan-cells: zero mode content yields BRST structure for correlators under action of Virasoro algebra.

n-pt Functions: Graphs

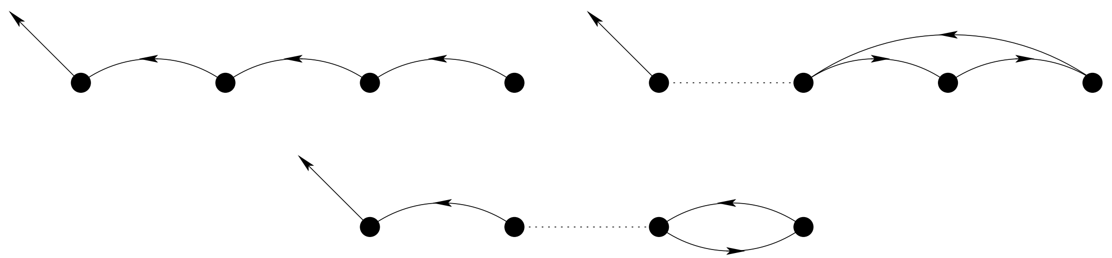
- To find a useful algorithm to fix the generic form of 4pt-functions, visualize a logarithmic field $\Psi_{(h;k)}$ by a vertex with k outgoing lines.



- Contractions of logarithmic fields give rise to logarithms in the correlators. The possible powers with which $\log(z_{ij})$ may occur, can be determined by graph combinatorics.

n-pt Functions: Graphs II

- Terms of generic form of n-pt function given by sum over all admissible graphs subject to the rules:
  - Each k_{out}-vertex may receive $k'_{\text{in}} \leq (r-1)$ lines.
  - Vertices with $k_{\text{out}} = 0$ (primary fiels) do not receive any legs.
  - Vertex i can receive legs from vertex j only for $j \neq i$.
  - Precisely $r-1$ lines in correlator remain open.
- **Example:** 4pt function for $r = 2$ and all fields logarithmic yields, upto permutations, the graphs



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4pt Functions: Algorithm

- Linking numbers $A_{ij}(g)$ of given graph g yield *upper bounds* for power with which logarithms occur.
- **Recursive procedure:** start with all ways f_i to choose $r-1$ free legs, find at each level K' and for each configuration f_i all graphs, which connect the remaining $K-K'-(r-1)$ legs to vertices.
- Write down corresponding monomial in $\log(z_{ij})$, multiplied with an as yet undetermined constant $C(g)$ for each graph g.
- Determine some constants by imposing global conformal invariance.
- Fix further constants by imposing admissible permutation symmetries.

4pt Functions: Generic Form

- Generic form of the LCFT 4pt functions $\langle k_1 k_2 k_3 k_4 \rangle \equiv \langle \Psi_{(h_1;k_1)}(z_1) \ldots \Psi_{(h_4;k_4)}(z_4) \rangle$ is

$$\langle k_1 k_2 k_3 k_4 \rangle = \prod_{i<j} (z_{ij})^{\mu_{ij}} \sum_{(k_1',k_2',k_3',k_4')} \Bigg[\sum_{g \in G_{K-K'}} C(g) \left(\prod_{i<j} \log^{A_{ij}(g)}(z_{ij}) \right) \Bigg] F_{k_1' k_2' k_3' k_4'}(x) \,,$$

where

- $G_{K-K'}$ is set of graphs for $(k_1 - k_1', \ldots, k_4 - k_4')$,
- $A_{ij}(g)$ is linking number of vertices i, j of graph g,
- x is the crossing ratio $x = \frac{z_{12} z_{34}}{z_{14} z_{23}}$,
- μ_{ij} is typically $\mu_{ij} = \frac{1}{3} \left(\sum_k h_k \right) - h_i - h_j$.

4pt Functions: $r = 2$

- The only direct dependence on the conformal weights is through the μ_{ij}. Put $h_1 = \ldots = h_4 = 0$ for simplicity.
- The generic form obeys some symmetry under permutations. Put $\ell_{ij} \equiv \log(z_{ij})$ and assume $i < j$ throughout.

$$
\begin{aligned}
\langle 1000 \rangle &= F_0 \,, \\
\langle 1100 \rangle &= F_{1100} - 2\ell_{12} F_0 \,, \\
\langle 1110 \rangle &= F_{1110} \\
&\quad + (\ell_{12} - \ell_{13} - \ell_{23}) F_{1100} + (\ell_{13} - \ell_{12} - \ell_{23}) F_{1010} \\
&\qquad - (\ell_{23} - \ell_{12} - \ell_{13}) F_{0110} \\
&\quad + (-\ell_{12}^2 - \ell_{13}^2 - \ell_{23}^2 + 2\ell_{12}\ell_{23} + 2\ell_{12}\ell_{13} + 2\ell_{23}\ell_{13}) F_0 \\
&= F_{1110} + \mathcal{P}_{(123)} \{ (\ell_{12} - \ell_{23} - \ell_{13}) F_{1100} \} \\
&\quad + \mathcal{P}_{(123)} \{ \ell_{12} (\ell_{12} - \ell_{23} - \ell_{13}) F_0 \} \,.
\end{aligned}
$$

4pt Functions: Symmetry

- Symmetry under permutations allows to write formulæ in more compact form.
- The permutation operators $\mathcal{P}$ run over all inequivalent permutations such that $i < j$ in all the z_{ij} and ℓ_{ij} involved.
- In the last example, $\mathcal{P}_{(123)} = (123) + (231) + (312)$ subject to the above rule.
- The ordering rule for ℓ_{ij} may be neglected, since in the full correlators, combined out of holomorphic and anti-holomorphic part in a single-valued way, only $\log|z_{ij}|^2$ will appear.

4pt Functions: Surprise

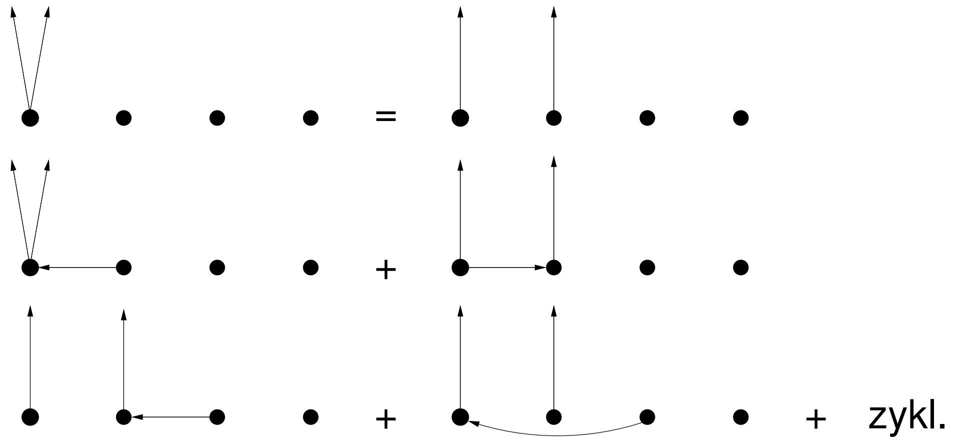
- Interestingly, there remain *free constants*, when all fields are logarithmic!

$$
\begin{aligned}
\langle 1111\rangle &= F_{1111} + \mathcal{P}_{(1234)}\big\{\big[(-\ell_{12}-\ell_{34}+\ell_{23}+\ell_{14})C_1 + (\ell_{13}+\ell_{24}-\ell_{12}-\ell_{34})C_2\\
&\qquad - \ell_{14}+\ell_{34}-\ell_{13})\big]F_{0111}\big\}\\
&+ \mathcal{P}_{(12)(34)}\big\{\big[(\ell_{13}^2+\ell_{24}^2-\ell_{14}^2-\ell_{23}^2+2(-\ell_{34}\ell_{24}-\ell_{12}\ell_{24}+\ell_{34}\ell_{14}+\ell_{13}\ell_{24}\\
&\qquad -\ell_{13}\ell_{34}+\ell_{23}\ell_{34}+\ell_{12}\ell_{23}-\ell_{12}\ell_{13}-\ell_{23}\ell_{14}+\ell_{12}\ell_{14}))C_3\\
&\quad +(-(\ell_{23}+\ell_{14})^2+\ell_{23}\ell_{34}+\ell_{12}\ell_{14}-\ell_{13}\ell_{34}+\ell_{34}\ell_{14}+\ell_{13}\ell_{14}\\
&\qquad -\ell_{34}\ell_{24}-\ell_{12}\ell_{13}-\ell_{12}\ell_{24}+\ell_{23}\ell_{24}+\ell_{23}\ell_{13}+\ell_{12}\ell_{23}+\ell_{24}\ell_{14}))C_4\\
&\quad -\ell_{34}^2-\ell_{23}^2-\ell_{14}^2+2\ell_{23}\ell_{34}+2\ell_{34}\ell_{14}-2\ell_{12}\ell_{34}-\ell_{23}\ell_{14}+\ell_{23}\ell_{24}\\
&\qquad -\ell_{12}\ell_{13}+\ell_{12}\ell_{14}+\ell_{12}\ell_{23}-\ell_{12}\ell_{24}+\ell_{13}\ell_{14}+\ell_{13}\ell_{24})\big]F_{1100}\big\}\\
&+ \big[2(\ell_{12}\ell_{24}\ell_{14}-\ell_{23}\ell_{13}\ell_{14}+\ell_{23}\ell_{34}\ell_{24}-\ell_{24}\ell_{13}\ell_{34}-\ell_{23}\ell_{34}\ell_{14}\\
&\qquad -\ell_{12}\ell_{23}\ell_{34}-\ell_{12}\ell_{34}\ell_{24}-\ell_{23}\ell_{13}\ell_{24}+\ell_{12}\ell_{23}\ell_{13}+\ell_{13}\ell_{34}\ell_{14}\\
&\qquad -\ell_{13}\ell_{14}\ell_{24}-\ell_{23}\ell_{24}\ell_{14}-\ell_{12}\ell_{13}\ell_{24}-\ell_{12}\ell_{23}\ell_{14}-\ell_{12}\ell_{13}\ell_{34}\\
&\qquad -\ell_{12}\ell_{34}\ell_{14})\\
&\quad +2(\ell_{13}^2\ell_{24}+\ell_{12}^2\ell_{34}+\ell_{14}^2\ell_{23}+\ell_{23}^2\ell_{14}+\ell_{34}^2\ell_{12}+\ell_{24}^2\ell_{13})\big]F_0
\end{aligned}
$$

4pt Functions: $r = 3$

- Next trivial case: Jordan cells of rank $r = 3$. Each Jordan level $0 \leq k_i \leq 2$.
- Graph combinatorics gets more involved.

$$
\begin{aligned}
\langle 2000 \rangle = \langle 1100 \rangle &= F_0 \,, \\
\langle 2100 \rangle &= F_{2100} - 2\ell_{12} F_0 \,, \\
\langle 1110 \rangle &= F_{1110} - (\ell_{12} + \ell_{23} + \ell_{13}) F_0 \\
&= F_{1110} - \mathcal{P}_{(123)} \{\ell_{12} F_0\} \,,
\end{aligned}
$$



4pt Functions: $r = 3$ contd

- Again, if all $k_i > 0$, free constants remain:

$$
\begin{aligned}
\langle 1111\rangle &= F_{1111} \\
&+ \mathcal{P}_{(1234)}\Big\{\big[(\ell_{13}-\ell_{12}+\ell_{24}-\ell_{34})C_1 \\
&\qquad +(\ell_{23}+\ell_{14}-\ell_{34}-\ell_{12})C_2-\ell_{14}+\ell_{24}-\ell_{12}\big]F_{0111}\Big\} \\
&+ \Big[\big(\ell_{12}^2+\ell_{24}^2+\ell_{34}^2+\ell_{13}^2+2(\ell_{12}\ell_{13}+\ell_{13}\ell_{24}-\ell_{13}\ell_{34} \\
&\qquad -\ell_{34}\ell_{24}-\ell_{12}\ell_{24}+\ell_{12}\ell_{34})\big)C_3 \\
&\quad +\big(-2\ell_{13}\ell_{14}+\ell_{24}^2+2\ell_{23}\ell_{14}-2\ell_{23}\ell_{24}+\ell_{23}^2+2\ell_{13}\ell_{24} \\
&\qquad +\ell_{13}^2-2\ell_{23}\ell_{13}+\ell_{14}^2-2\ell_{24}\ell_{14}\big)C_4 \\
&\quad +\big((\ell_{24}+\ell_{13})^2 \\
&\qquad +\ell_{12}\ell_{14}-\ell_{23}\ell_{24}-\ell_{12}\ell_{24}-\ell_{24}\ell_{14}-\ell_{23}\ell_{13}+\ell_{34}\ell_{14} \\
&\qquad -\ell_{13}\ell_{34}-\ell_{13}\ell_{14}+\ell_{23}\ell_{34}-\ell_{34}\ell_{24}+\ell_{12}\ell_{23}-\ell_{12}\ell_{13}\big)C_5 \\
&\quad +2(\ell_{13}\ell_{24}+\ell_{23}\ell_{14}+\ell_{12}\ell_{34})\Big]F_0\,.
\end{aligned}
$$

4pt Functions: To Do

- Further examples . . . need bigger transparencies ;-)
- **Problem:** Computational complexity grows heavily with rank r and total Jordan level K. Already the generic solution for $r = 2$ and $K_{\text{max}} = 4(r-1) = 4$ needs a computer program.
- **Solution:** MAPLE package, written by Marco Krohn, almost finished. Need to make implementation of algorithm more efficient. So far, $K > 2(r-1)$ for $r > 3$ still too complex.
- Permutation symmetry for the highest degree polynomial in the ℓ_{ij}, appearing in front of $F_0(x)$, is not obvious and difficult to find.

Outlook

- **Questions:**
  - Need to understand origin of additional free constants.
  - Include explicit crossing symmetry. Should decrease number of different functions $F_{k'_1,k'_2,k'_3,k'_4}(x)$, in particular for cases where several conformal weights are equal, $h_i = h_j$.
  - Need to generalize to $c = 0$ LCFTs important for percolation and for disorder. Problem: the naive vacuum representation is trivial.
  - Adapt algorithm to include pre-logarithmic fields: Skip the rule that primary vertices do not receive legs.

Summary

- We found a method to fix the generic form of 4-pt and n-pt functions in arbitrary rank LCFT.
- Already the form of 4-pt functions, as determined by global conformal invariance, is much more complicated than in the ordinary case.
- There seems to exist additional degrees of freedom not present in ordinary CFT.
- We showed a few examples of non-trivial solutions. Already the solution for $r = 2$ and $K = 4$, is new and generalizes known expressions.
- ...