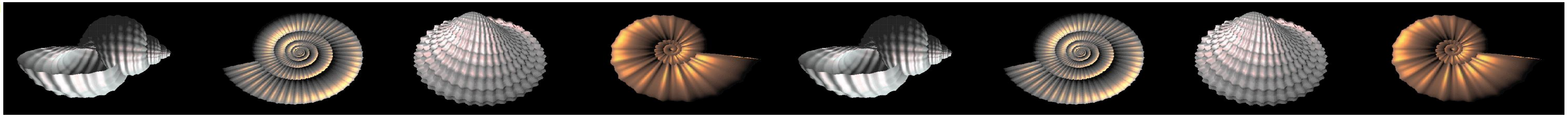

Logarithmic Conformal Field Theory
in a (Nut)Shell

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Why Mathematical Physics?

Identifies mathematical structures which describe reality.

- **Newton:** Everything is *matter* $\Longrightarrow$ Analysis.
  He assumed even light consists of particles.
- **Einstein:** Everything is *energy* $\Longrightarrow$ Geometry.
  We all know the famous $E = mc^2$.
- **Heisenberg:** Everything is *symmetry* $\Longrightarrow$ Algebra.
  Conservation laws, Noether theorem, selection rules, gauge groups, $\ldots$

Symmetries govern many aspects of modern theoretical physics.

- Natural question: What possible symmetries are there?
- More fundamental questions: What does it *mean* that Nature can be described by mathematical structures? Why is Nature so “symmetric”?

Why Symmetries?

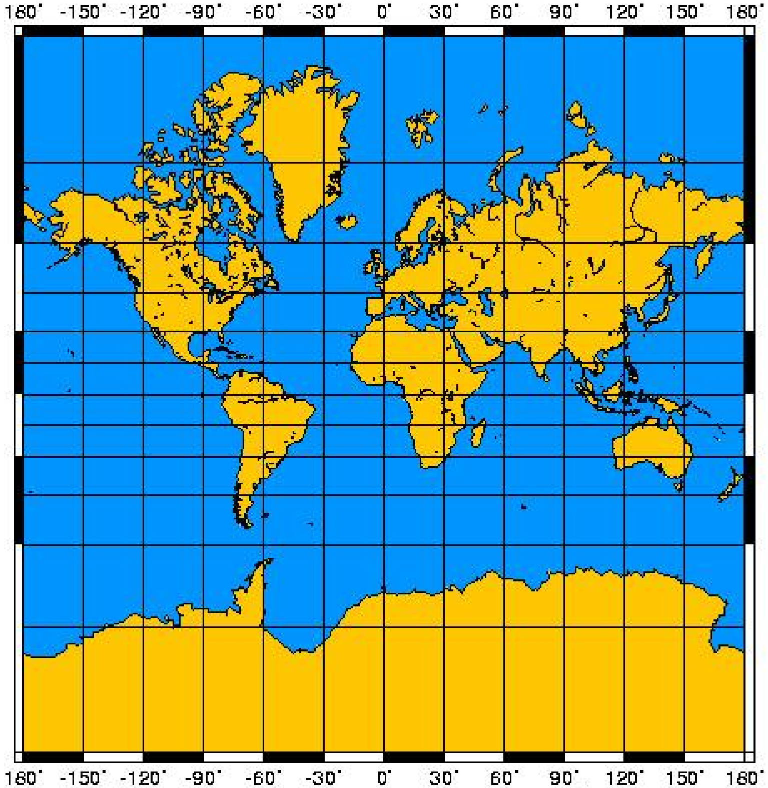
Experience shows that the laws of Nature are fixed by symmetries to a sometimes miraculous extent.

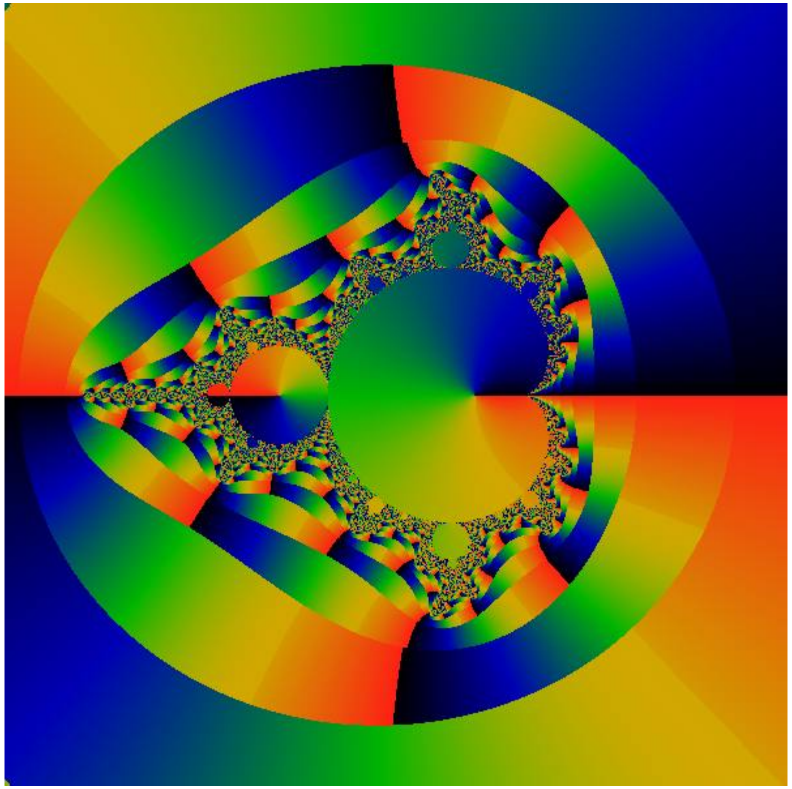
- Look at the spectrum of atoms in crystals.
  The discrete finite group of rigid symmetries of the crystal predicts which degeneracies are lifted.
- The known fundamental forces (except gravity) are described by gauge field theories.
  Quantum numbers appear as weights of representations of the gauge groups, which are all Lie groups such as $U(1)$, $SU(2)$ and $SU(3)$.
- Extended or composite objects possess even larger symmetries such as infinite-dimensional Lie algebras.

One symmetry is particularly interesting: symmetry under scaling (think of dimensional analysis!).

Conformal Symmetry

Suppose, a theory is invariant under **local** scale transformations: $g^{\mu\nu}(x) \mapsto \tilde{g}^{\mu\nu}(\tilde{x}) = \lambda(x)g^{\mu\nu}(x)$. Such maps locally conserve angles. That's why they are called **conformal**.





There is something very special about conformal maps in **two** dimensions . . .

Infinite Symmetry

In two dimensions, we can work on the complex plane $\mathbb{C}$: $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$. Any *holomorphic* map $z \mapsto z' = f(z)$, $\bar{\partial} f(z) = 0$, is conformal. Thus, two-dimensional conformally invariant theories have an infinite number of symmetries.

- My personal interest is in *quantum field theories*.

$$f(z) = z + \varepsilon(z) = z + \sum_{n\in\mathbb{Z}} \varepsilon_n z^n\,,$$

$$[L_\varepsilon, \Phi(z)] = \Phi(z + \varepsilon(z)) \quad \Longrightarrow$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}\,.$$

This is the **Virasoro algebra** of the generators of local conformal transformations. It is an example of an infinite-dimensional Lie algebra.

Where is this useful?

To classify conformally invariant theories means to study the *representation theory* of this algebra.

- Universality classes of two-dimensional statistical systems at their points of criticality are classified by the value of the central extension c.
- Critical exponents are given in terms of the scaling dimensions of primary fields, i.e. highest weights of irreps:

$$\Phi_h(f(z)) = \left(\frac{\partial f(z)}{\partial z}\right)^{-h} \Phi_h(z)\,, \qquad \lim_{z\to 0} \Phi_h(z)|0\rangle \equiv |h,c\rangle\,.$$

In particular $\Phi_h(\lambda z) = \lambda^{-h}\Phi_h(z)$.

- The two-dim. Ising model possesses *three* basic observables, the identity $\Phi_0 = \mathbb{I}$, the energy operator $\Phi_{1/2} = \epsilon$, and the spin field or order parameter $\Phi_{1/16} = \sigma$ or μ, respectively.

Minimal models

The simplest conformally invariant statistical field theories are classified very similar to spin in quantum mechanics: Put $L_z = L_0$, $L_\pm = L_{\pm 1}$ and the $\mathfrak{su}(2)$ algebra takes the form $[L_n, L_m] = (n-m)L_{n+m}$. The role of the Casimir $C = \vec{L}^2$ is roughly given by c.

$c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$	$\vec{L}^2\vert\ell, m\rangle = \ell(\ell+1)\vert\ell, m\rangle$
$p, q \geq 1$ coprime	$\ell \in \mathbb{Z}_+$
$L_0\vert h, c\rangle = h\vert h, c\rangle$	$L_z\vert\ell, m\rangle = m\vert\ell, m\rangle$
$L_n\vert h, c\rangle = 0$ for $n > 0$	$L_+\vert\ell, \ell\rangle = 0$
$h_{r,s}(c) = \frac{(pr-qs)^2-(p-q)^2}{4pq}$	$h(\ell) = \ell$
$1 \leq r < q\,, 1 \leq s < p$	$\ell = \ell$
$\vert h; \{n\}, c\rangle = L_{-n_k} \ldots L_{-n_1}\vert h, c\rangle$	$\vert\ell, \ell - m\rangle = (L_-)^m\vert\ell, \ell\rangle$
$0 = \sum_{\vert\{n\}\vert=N} \beta^{\{n\}} L_{-\{n\}}\vert h, c\rangle$	$0 = (L_-)^{2\ell+1}\vert\ell, \ell\rangle$

Correlation functions

- Ultimately, one wants to compute expectation values of observables, $\langle 0|\Phi_{h_1}(z_1)\Phi_{h_2}(z_2)\dots\Phi_{h_n}(z_n)|0\rangle$.

$$
\begin{aligned}
\langle\Phi_{h_1}(z_1)\Phi_{h_2}(z_2)\rangle &= D_{h_1}(z_1-z_2)^{-h_1-h_2}\delta_{h_1,h_2}\,, \\
\langle\Phi_{h_1}(z_1)\Phi_{h_2}(z_2)\Phi_{h_3}(z_3)\rangle &= C_{h_1h_2h_3}(z_1-z_2)^{h_3-h_1-h_2} \\
&\times (z_1-z_3)^{h_2-h_1-h_3}(z_2-z_3)^{h_1-h_2-h_3}\,, \\
\langle\Phi_{h_1}(z_1)\Phi_{h_2}(z_2)\Phi_{h_3}(z_3)\Phi_{h_4}(z_4)\rangle &= \prod_{i<j}(z_i-z_j)^{\mu_{ij}}F^{(p)}_{h_1h_2h_3h_4}(x)\,,
\end{aligned}
$$

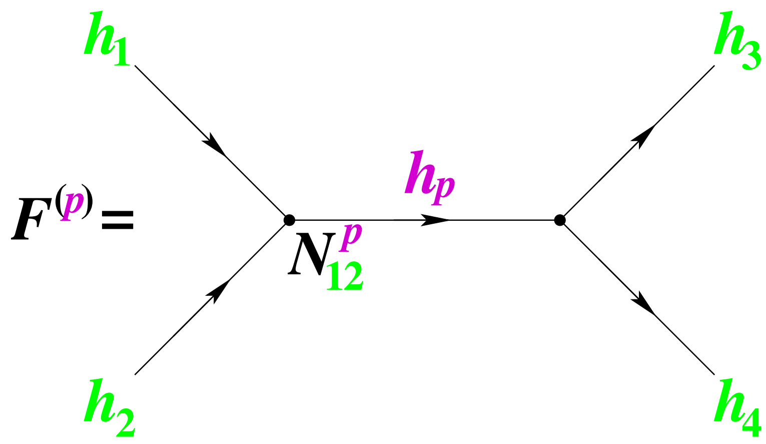
where $\sum_{j\neq i}\mu_{ij} = -2h_i$, $x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_2-z_3)}$ is the *crossing ratio*, and p labels the *conformal blocks*.

- During the last 20 years, a lot of technology has been developed to efficiently and *exactly* compute the $F^{(p)}(x)$ and higher n-point functions.

Representation theory

We are used to expect that (tensor) representations can be completely reduced into irreps. Coupling angular momentum in quantum mechanics amounts to

$$[\ell_1] \otimes [\ell_2] = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} [\ell] \,.$$



In conformal field theory, coupling fields works much the same,

$$[h_1, c] * [h_2, c] = \sum_h N_{h_1 h_2}^{\;\;h} [h, c] \,, \qquad N_{h_1 h_2}^{\;\;h} \in \mathbb{Z}_+ \,.$$

Classifying these so-called *fusion algebras* is a very important problem in conformal field theory, but . . .

A strange surprise

. . . but it may happen, that the fusion product of two irreps **cannot** again be decomposed into irreps!

- There exists a conformal field theory with $c = c_{2,1} = -2$. It contains an innocent and admissible irrep corresponding to a primary field μ with $h = h_{1,2} = -1/8$. However,

$$\langle\mu(\infty)\mu(1)\mu(x)\mu(0)\rangle =$$

$$[x(1-x)]^{1/4} \begin{cases} F^{(1)} = & {}_2F_1(\tfrac{1}{2},\tfrac{1}{2},1;x) \\ F^{(2)} = & \log(x)\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;x) \\ & + \partial_\epsilon\,{}_3F_2(\tfrac{1}{2}+\epsilon,\tfrac{1}{2}+\epsilon,1;1+\epsilon,1+\epsilon;x)\Big|_{\epsilon=0} \end{cases} .$$

One can show that this **implies** $[-\frac{1}{8},-2]*[-\frac{1}{8},-2] = [\tilde{0},-2]$, where $L_0|\tilde{h},c\rangle = h|\tilde{h},c\rangle + |h,c\rangle$ spans a *Jordan cell*. Thus, the representation on the rhs is **indecomposable**.

$$\tilde{\Phi}_{\tilde{h}}(\lambda z) = \lambda^{-h}\left(\tilde{\Phi}_{\tilde{h}}(z) - \log(\lambda)\Phi_h(z)\right).$$

Logarithmic CFT

- Indecomposable representations are at the heart of logarithmic conformal field theory.
- Correlation functions have to satisfy the *global conformal Ward identities*, i.e. for $m = -1, 0, 1$ we must have

$$\begin{aligned} 0 &= L_m \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle \\ &= \sum_{i=1}^{n} z_i^m \left[z_i \partial_i + (m+1)(h_i + \hat{\delta}_{h_i}) \right] \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle \ . \end{aligned}$$

- In case of rank $r > 1$ Jordan cells of indecomposable representations with respect to *Vir*, we have

$$\hat{\delta}_{h_i} \Psi_{(h_j;k_j)} = \begin{cases} \delta_{i,j} \Psi_{(h_j;k_j-1)} & \text{if} \quad 1 \leq k_j \leq r-1 \,, \\ 0 & \text{if} \quad k_j = 0 \,. \end{cases}$$

Logarithmic CFT

- Although logarithms break scale invariance, correlators can still be invariant under global conformal maps.
- Generic form of 1-, 2- and 3-pt functions for fields forming Jordan cells in arbitrary rank r LCFT is known:

$$\langle\Psi_{(h;k)}\rangle = \delta_{h,0}\delta_{k,r-1}\,,$$

$$\langle\Psi_{(h;k)}(z)\Psi_{(h';k')}(0)\rangle = \delta_{hh'}\sum_{j=r-1}^{k+k'} D_{(h;j)} \sum_{\substack{0\leq i\leq k,0\leq i'\leq k'\\ i+i'=k+k'-j}} \frac{(\partial_h)^i}{i!}\frac{(\partial_{h'})^{i'}}{i'!} z^{-h-h'}\,,$$

$$\langle\Psi_{(h_1;k_1)}(z_1)\Psi_{(h_2;k_2)}(z_2)\Psi_{(h_3;k_3)}(z_3)\rangle = \sum_{j=r-1}^{k_1+k_2+k_3} C_{(h_1h_2h_3;j)}$$

$$\times \sum_{\substack{0\leq i_l\leq k_l,l=1,2,3\\ i_1+i_2+i_3=k_1+k_2+k_3-j}} \frac{(\partial_{h_1})^{i_1}}{i_1!}\frac{(\partial_{h_2})^{i_2}}{i_2!}\frac{(\partial_{h_3})^{i_3}}{i_3!} \prod_{\substack{\sigma\in S_3\\ \sigma(1)<\sigma(2)}} \left(z_{\sigma(1)\sigma(2)}\right)^{h_{\sigma(3)}-h_{\sigma(1)}-h_{\sigma(2)}}\,.$$

Some Achievements

- LCFT on a torus and other non-trivial Riemann surfaces $\Longrightarrow$ modular invariants and characters of indecomposable representations $\Longrightarrow$ $N_{ij}^{\ \ k}$.
- Null vectors in indecomposable representations $\Longrightarrow$ exploiting local conformal symmetry to *exactly* compute correlators in LCFT $\Longrightarrow$ C_{ijk} and $F^{(p)}(x)$.
- Classification of LCFTs similar to the minimal models $\Longrightarrow$ identifying theories of potential interest in physics $\Longrightarrow$ LCFTs as limits of sequences of ordinary CFTs.
- LCFT on surfaces with boundaries,
  LCFT wrt extended chiral algebras,
  LCFT and vertex operator algebras,
  LCFT and modular differential eqn,
  ...

Motivation

- LCFT important for many applications such as
  - abelian sandpiles,
  - percolation and disorder,
  - Haldane-Rezayi fractional quantum Hall state,
  - mathematics (e.g. alternating sign matrices).
- Presumably LCFT will play a role in string theory, e.g.
  - *D*-brane recoil,
  - world-sheet formulation on AdS_3,
  - or, more generally, when non-compact CFTs arise.
- Subtleties in non-compact CFTs, e.g. Liouville theory:
  - non-uniqueness of fusion matrices $N_{ij}^{\ \ k}$,
  - non-trivial factorisation properties of correlators into $F^{(p)}$,
  - difficulties in definition of consistent OPEs via C_{ijk},
  - additional constraints for unitarity and locality: $h, c \leq 0$.
- These subtleties are typical for LCFT!