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Integrable Models and Applications:

From Strings to Condensed Matter

RECENT DEVELOPMENTS IN LOGARITHMIC CONFORMAL FIELD THEORY

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and work in progress

THE MESSAGE

- Logarithmic Conformal Field Theory (**L**CFT)

  is Conformal Field Theory, where representations of the chiral symmetry algebra may be <u>indecomposable</u>.

- There are many Conformal Field Theories, where this is necessarily the case.

  - Ghost systems on Riemann surfaces.
    [M.Krohn, M.F.]
  - WZW models at level zero or at fractional level. $\widehat{SU(2)}_{-4/3}$
    [M.Gaberdiel; I.Kogan, A.Nichols, ...]
  - WZW models of supergroups. $GL(1,1)$
    [Rozansky, Saleur]
  - (WZW models of non-compact groups.) $SL(2,\mathbb{R})$
  - $c=0$ theories $osp(2n|2n)$, $CP(n|n)$
    [N.Read, H.Saleur; A.Ludwig, V.Gurarie, ...]
  - ...

INDECOMPOSABLE REPS.

- With respect to Vir:

Instead of $L_0 |h\rangle = h|h\rangle$

Highest weight rep.

$L_n |h\rangle = 0 \quad \forall n > 0$

Jordan blocks $\{ |h;0\rangle, \ldots, |h; r-1\rangle \}$

$$L_0 |h;k\rangle = h|h;k\rangle + |h;k-1\rangle \quad , k > 0$$
$$L_0 |h;0\rangle = h|h;0\rangle$$

$$L_n |h;k\rangle = 0 \quad \forall n > 0$$

- With respect to Current algebra or extended $\mathcal{W}$-algebra:

for example $\{ |h, q; k\rangle \}$

$$J^a_0 |h, q; k\rangle = q^a |h, q; k\rangle + |h, q; k-1\rangle$$

$$[J^a_m, J^b_n] = f^{ab}{}_c J^c_{m+n} + nk\delta^{ab}\delta_{m+n,0}$$

- With respect to both!

Today: only w.r.t. Vir

$$|h;k\rangle \longleftrightarrow \Phi_{(h,k)}(z)$$

ANALYTICAL APPROACH

- <u>Analytical:</u> Degenerate cases of differential equations to be satisfied by correlation functions:

<u>Example:</u> $c = -2$ (bc system with spin 1,0)

Consider admissible highest weight rep. with highest weight state

$$L_n|\mu\rangle = 0 \quad \forall n > 0, \quad L_0|\mu\rangle = -\frac{1}{8}|\mu\rangle$$

Consider 4-point function

$$\langle \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4)\rangle = (z_1-z_3)^{1/4}(z_2-z_4)^{1/4}[x(1-x)]^{1/4} F(x)$$

$$x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$$

The rep. generated by $|\mu\rangle$ contains nullstate:

$$|\chi^{(2)}_{-1/8}\rangle = (L_{-2} - 2L_{-1}^2)|\mu\rangle$$

$$\Rightarrow \quad x(1-x)\frac{\partial^2}{\partial x^2}F(x) + (1-2x)\frac{\partial}{\partial x}F(x) - \frac{1}{4}F(x) = 0$$

Indicial equation:

$$F(x) = x^\lambda\left(\sum_n a_n x^n\right) \quad \text{with} \quad \lambda(\lambda-1)+\lambda = \lambda^2 = 0$$

Analytical Approach (cont'd)

Solution:

$$\begin{cases} ① \quad F(x) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}, 1; x) = \sum_n \frac{(1/2)_n (1/2)_n}{(1)_n \, n!} x^n \\ \\ ② \quad F(x) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}, 1; 1-x) \end{cases}$$

$$= {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}, 1; x) \log(x)$$

$$+ \frac{\partial}{\partial \varepsilon} \, {}_3F_2(\tfrac{1}{2}+\varepsilon, \tfrac{1}{2}+\varepsilon, 1, 1+\varepsilon, 1+\varepsilon; x) \Big|_{\varepsilon = 0}$$

Crossing symmetry + conf. covariance $\Rightarrow$

$$② \rightsquigarrow \boxed{\mu(z)\mu(w) = (z-w)^{1/4} \left[\tilde{\mathbb{I}}(w) + \log(z-w) \, \mathbb{I} \right]}$$

$\uparrow$ log. partner of identity.

$$L_0 \begin{pmatrix} |0\rangle \\ |\tilde{0}\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} |0\rangle \\ |\tilde{0}\rangle \end{pmatrix} \quad \text{Jordan cell}$$

Thus, the OPE of the primary field $\mu(z)$ yields logarithmic fields.

Primary $*$ Primary $=$ Log-field $+$ Primary $+ \cdots$

Irrep. $*$ Irrep. $=$ Indecomp. Rep. $+ \cdots$

Analytical Approach (cont'd)

Locality (single valuedness) $\Longrightarrow$

$$\langle \mu(\infty)\mu(1,1)\,\mu(x,\bar{x})\,\mu(0,0)\rangle =$$

$$[x(1-x)]^{1/4}\,[\bar{x}(1-\bar{x})]^{1/4}\left(F(x)\,\overline{F(1-x)} - \overline{F(x)}\,F(1-x)\right)$$

sign!

Typical feature of LCFTs:

The diagonal bilinear combination of conformal blocks is in general <u>not</u> a single valued solution!

$$\langle \phi_1(\infty)\,\phi_2(1,1)\,\phi_3(x,\bar{x})\,\phi_4(0,0)\rangle = \sum_{ij} U_{ij}\,F_i(x)\,F_j(\bar{x})$$

$$U_{ij} \neq \mathbb{1}$$

ALGEBRAIC APPROACH

Representation theoretic point of view for OPE or fusion:

"tensor product" of highest weight reps, which does not change central charge of chiral algebra.

Implicit definition of <u>fusion product</u> via

$$\oint_{(z_1,z_2)} dw\, w^{m+1} \langle v|\, T(w)\, \mu(z_1)\, \mu(z_2)\, |0\rangle$$

$$= \sum \langle v| \left(\Delta^{(1)}_{z_1,z_2}(L_m)\mu\right)(z_1) \left(\Delta^{(2)}_{z_1,z_2}(L_m)\mu\right)(z_2)|0\rangle$$

with <u>co-multiplication</u>

$$\Delta_{z_1 z_2}(L_m) = \sum \Delta^{(1)}_{z_1 z_2}(L_m) \otimes \Delta^{(2)}_{z_1 z_2}(L_m)$$

$$= \sum_{n=-1}^{m} \binom{m+1}{n+1} z_1^{m-n} (L_n \otimes \mathbb{1})$$

$$+ \sum_{l=-1}^{m} \binom{m+1}{l+1} z_2^{m-l} (\mathbb{1} \otimes L_l) \qquad \text{for } m \geq -1$$

ALGEBRAIC APPROACH (cont'd)

Depending on the choice of expansion, we have for $m \leq -2$:

$$\Delta_{z_1,z_2}(L_{-m}) = \sum_{n=-1}^{\infty} \binom{m+n-1}{n+1} (-)^{n+1} z_1^{-(m+n)} (L_n \otimes \mathbb{1}) + \sum_{l=m}^{\infty} \binom{l-2}{m-2} (-z_2)^{l-m} (\mathbb{1} \otimes L_{-l}) \qquad \text{for } m \geq 2$$

$$\tilde{\Delta}_{z_1,z_2}(L_{-m}) = \sum_{l=m}^{\infty} \binom{l-2}{m-2} (-z_1)^{l-m} (L_{-l} \otimes \mathbb{1}) + \sum_{n=-1}^{\infty} \binom{m+n-1}{n+1} (-)^{n+1} z_2^{-(m+n)} (\mathbb{1} \otimes L_n) \qquad \text{for } m \geq 2$$

Both co-multiplication formulae must lead to identical results in all correlation functions $\Longrightarrow$

<u>Fusion product</u> of reps. $\mathcal{H}_{\psi_1}, \mathcal{H}_{\psi_2}$.

$$\mathcal{H}_{\psi_1} * \mathcal{H}_{\psi_2} = \mathcal{H}_{\psi_1} \otimes \mathcal{H}_{\psi_2} \Big/ \left\{ \left(\Delta_{z_1,z_2}(L_m) - \tilde{\Delta}_{z_1,z_2}(L_m)\right)(\psi_1 \otimes \psi_2) \right\}$$

$\rightsquigarrow$ $(\mathcal{H}_{\psi_1} * \mathcal{H}_{\psi_2})$ is difficult to analyze.

It often suffices to analyse certain quotient spaces, in particular the <u>highest-weight space</u>

$$(\mathcal{H}_{\psi_1} * \mathcal{H}_{\psi_2})^{(0)} \equiv \psi_1 * \psi_2 = (\mathcal{H}_{\psi_1} * \mathcal{H}_{\psi_2}) \Big/ \mathcal{A}_{-}(\mathcal{H}_{\psi_1} * \mathcal{H}_{\psi_2})$$

$\mathcal{A}_{-}$ = algebra generated by L_{-n}, $n > 0$.

- Compute action of L_0 on the space $\mu * \mu$ with co-multiplication and $L_{-1}^2 |\mu\rangle = \frac{1}{2} L_{-2} |\mu\rangle$

$$\begin{cases} L_0(|\mu\rangle \otimes |\mu\rangle) = (L_{-1}|\mu\rangle \otimes |\mu\rangle) - \frac{1}{4}(|\mu\rangle \otimes |\mu\rangle) \\ L_0(L_{-1}|\mu\rangle \otimes |\mu\rangle) = \frac{1}{4}(L_{-1}|\mu\rangle \otimes |\mu\rangle) - \frac{1}{16}(|\mu\rangle \otimes |\mu\rangle) \end{cases}$$

$$\Rightarrow L_0 = \begin{pmatrix} -\frac{1}{4} & 1 \\ -\frac{1}{16} & \frac{1}{4} \end{pmatrix} \qquad \det(L_0) = \operatorname{tr}(L_0) = 0$$

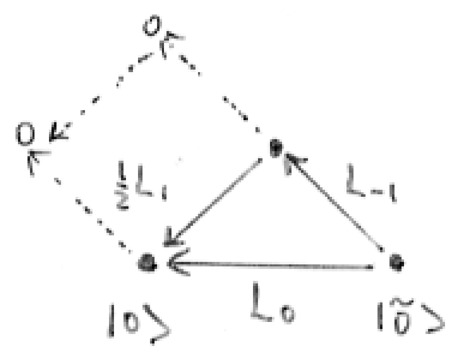
$$\Rightarrow L_0 \text{ conjugate to } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- Check compatibility with analytic approach

$$\boxed{\begin{aligned} |\tilde{0}\rangle &= (|\mu\rangle \otimes |\mu\rangle) \\ |0\rangle &= -\frac{1}{4}(|\mu\rangle \otimes |\mu\rangle) + (L_{-1}|\mu\rangle \otimes |\mu\rangle) \end{aligned}}$$

indeed behave as advertised.

$R_{1,1}$



h=1

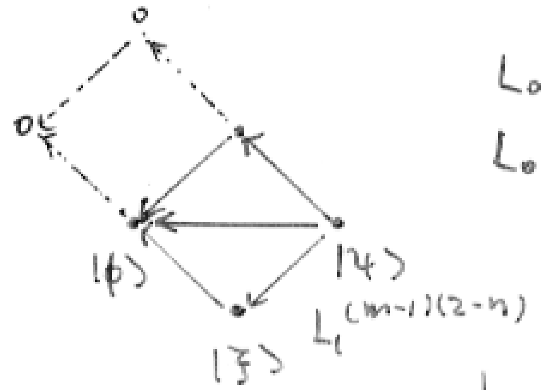
h=0

[Gaberdiel]

CHALLENGES I

- more complicated reducible but indecomposable reps.:

$R_{m,n}$



$$L_0|\psi\rangle = h_{m,n}|\psi\rangle + |\phi\rangle$$
$$L_0|\phi\rangle = h_{m,n}|\phi\rangle$$

$$L_0|\xi\rangle = h_{m-1,2-n}|\xi\rangle$$

  - $|\phi\rangle$ nullstate in V_ξ (irred. sub rep.)
  - $\exists$ "Jordan cells" with respect to L_{-n} instead of L_0
  - Jordan cell structures for chiral algebras (e.g. Current algebras $\longleftrightarrow$ KZ equations)

- Indecomposable reps. often <u>not</u> generated by highest weight state.

- No clear picture yet what types of generalized reps are possible

- Higher rank Jordan cells

ZHU'S ALGEBRA

$c=-2$ as Virasoro model is not rational.

Maximally extended chiral symmetry algebra:

$\mathcal{W}(2,3,3,3)$

$$[L_m, L_n] = (m-n)L_{m+n} - \binom{m+1}{3}\delta_{m+n,0}$$

$$[L_m, W^a_n] = (2m-n)W^a_{m+n}$$

$$[W^a_m, W^b_n] = g^{ab}\Big[2(m-n)\Lambda_{m+n} + \tfrac{1}{20}(m-n)(2m^2+2n^2-mn-8)L_{m+n} - \tfrac{1}{120}m(m^2-1)(m^2-4)\delta_{m+n,0}\Big] + f^{ab}{}_c\Big[\tfrac{5}{14}(2m^2+2n^2-3mn-4)W^c_{m+n} + \tfrac{12}{5}V^c_{m+n}\Big]$$

$\Lambda = \mathcal{N}(L,L) = :LL: - \frac{3}{10}\partial^2 L$

$V^a = \mathcal{N}(L,W^a) = :LW^a: - \frac{3}{14}\partial^2 W^a$

g^{ab} metric, $f^{ab}{}_c$ structure constants of $su(2)$.

$$\rightsquigarrow |N^a\rangle = \left(2L_{-3}W^a_{-3} - \tfrac{4}{3}L_{-2}W^a_{-4} + W^a_{-6}\right)|0\rangle$$

$$|N^{ab}\rangle = W^a_{-3}W^b_{-3}|0\rangle - f^{ab}{}_c\left(-2L_{-2}W^c_{-4} + \tfrac{5}{4}W^c_{-6}\right)|0\rangle - g^{ab}\left(\tfrac{8}{9}L^3_{-2} + \tfrac{19}{36}L^2_{-3} + \tfrac{14}{9}L_{-4}L_{-2} - \tfrac{19}{6}L_{-6}\right)|0\rangle$$

are non-trivial null states at level 6.

Needed to ensure associativity of $\mathcal{W}$-algebra. [Kausch]

ZHU'S ALGEBRA (cont'd)

One considers the space

$$\mathcal{A}(H_0) = H_0 / \mathcal{O}(H_0)$$

where: H_0 = vacuum representation

$$\mathcal{O}(H_0) = \text{span}\left\{ \psi^{(N)} |\chi\rangle \;:\; N>0,\ \psi,\chi \in H_0 \right\}$$

with $$\psi^{(N)} |\chi\rangle = \sum_{n=0}^{h_\psi} \binom{h_\psi}{n} \psi_{-n-N} |\chi\rangle .$$

Zhu's work implies:

$$\mathcal{O}(H_0) = \text{span}\left\{ \psi^{(1)} |\chi\rangle \;:\; \psi,\chi \in H_0 \right\}$$

$$= \text{span}\left\{ \psi_{-h_\psi-1} |\chi\rangle \;:\; \psi,\chi \in H_0 \right\}$$

<u>C_2-condition:</u> $\dim \mathcal{A}(H_0) < \infty$ implies that the corresponding CFT has only finitely many HWRs

irred. HWRs of CFT $\xleftrightarrow{1:1}$ irreps. of $\mathcal{A}(H_0)$

$\mathcal{A}(H_0)$ associative algebra via $\psi *_L \chi \equiv \psi^{(0)} \chi$.

Remark: HWRs are reps. of CFT such that L_0 is bounded from below.

Not necessarily is the highest weight state a cyclic vector of the rep. !!

ZHU'S ALGEBRA (cont'd)

Our $c=-2$ example with $\mathcal{W}(2,3,3,3)$ symmetry yields:

$$\dim \mathcal{A}(H_0) = 11$$

$$\mathcal{A}(H_0) = \text{span}\left\{ L_{-2}^r |0\rangle,\ L_{-2}^s W_{-3}^a |0\rangle : \begin{smallmatrix} s=0,\dots,4 \\ r=0,1 \end{smallmatrix} \right\}$$

[Kausch]

$\mathcal{A}(H_0)$ is the algebra of zero modes of fields modulo the zero modes of null fields, since for a null field $\mathcal{N}(z)$

$$\langle \phi | \mathcal{N}(z) | \phi' \rangle = 0 \quad \forall \phi, \phi'$$

constraints from N^a and N^{ab} $\rightsquigarrow$

$$\left(W_0^a W_0^b - g^{ab} \tfrac{1}{9} L_0^2 (8L_0+1) - f^{ab}_c \tfrac{1}{5} (6L_0-1) W_0^c \right) |\phi\rangle = 0$$

$$\Longrightarrow \qquad L_0^2 (8L_0+1)(8L_0-3)(L_0-1)|\phi\rangle = 0$$

$\Rightarrow$ $h_\phi \in \{0, -1/8, 3/8, 1\}$ <u>and</u> $h=0$ may be indecomposable, since only $L_0^2 = 0$

Where does the 1:1 correspondence of the Zhu condition break down?

$\mathcal{A}(H_0)$ is <u>not</u> semi-simple!

CHALLENGES II

CFT <u>rational</u> $\Longleftrightarrow \begin{cases} \#\text{ irreps} < \infty \\ \text{rep} = \bigoplus \text{ irreps} \quad (\text{strict}) \end{cases}$

<u>But:</u> C_2-condition $\Rightarrow$ rationality
not true in the strict sense.

<u>Conjecture:</u> C_2-condition <u>and</u> semi-simplicity
$\Rightarrow$ strict rationality,
otherwise indecomposable reps.
may occur.

Proof?

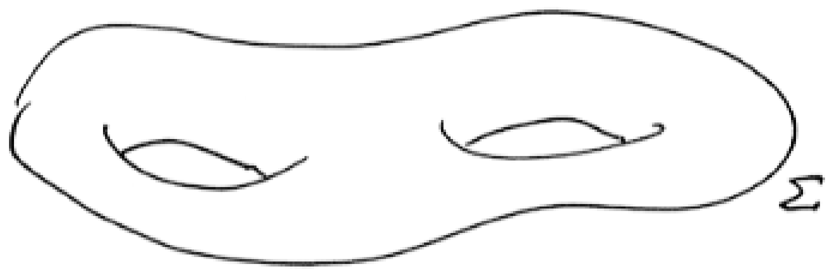
$c=-2$	yes ✓
$c=c_{1,q}$	strong indications (✓)

other LCFTs ?

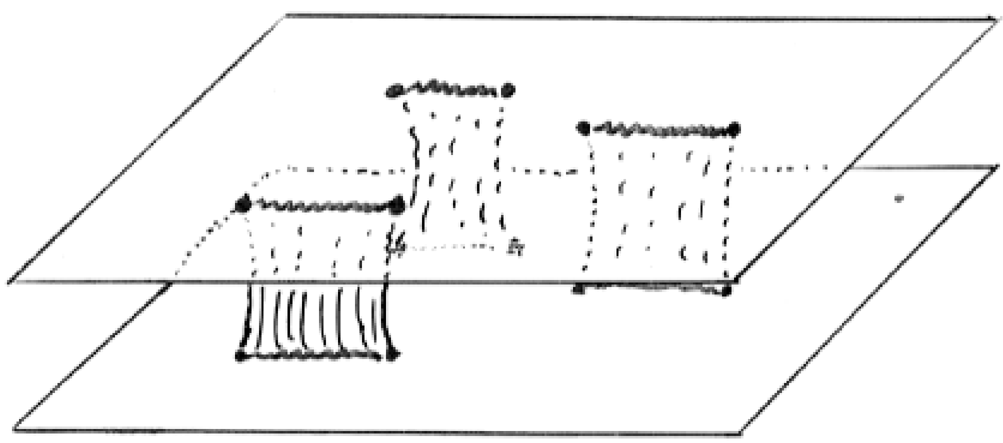
- Structure of Zhu's algebra, in particular in the non-semi-simple case
- Efficient methods to compute $\mathcal{A}(\mathcal{H}_0)$

MOTIVATION

CFT on a Riemann surface Σ



viewed as CFT on a branched covering of the (compactified) complex plane :



Include the branch points as suitable Vertex operators into the CFT [Knizhnik]

Branch points = Twist fields

MOTIVATION (cont'd)

Example: b-c system of weight $(j, 1-j)$

$$c(z)\, b(w) \sim \frac{1}{z-w}$$

$\rightsquigarrow$ CFT with central charge

$$c_{bc} = -2(1-6j+6j^2)$$

<u>$j=1$</u>: $c=-2$

simplest case: $\mathbb{Z}_2$-branch points

Branch field $\mu(z)$ of weight $h=-1/8$

$$\mu(z)\mu(w) \sim \begin{cases} (z-w)^{1/4}\, \mathbb{1} \\ (z-w)^{1/4}\, \tilde{\mathbb{I}}(w) + (z-w)^{1/4} \log(z-w)\, \mathbb{1} \end{cases}$$

Since

$$\langle \mu(\infty)\mu(1)\mu(x)\mu(0)\rangle$$

has two conformal blocks:

$$[x(1-x)]^{1/4} \begin{cases} {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}, 1 \mid x) \\ {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}, 1 \mid 1-x) \end{cases}$$

$$= \log(x)\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}, 1 \mid x) + \cdots$$

ZERO - MODES

$\tilde{\mathbb{I}}$ is not contained in the bc system!

$\langle \mathbb{I}\, \mathbb{I} \rangle = 0$

$\langle \mathbb{I}\, \tilde{\mathbb{I}}(z) \rangle = 1$

$\langle \tilde{\mathbb{I}}(z)\, \tilde{\mathbb{I}}(w) \rangle = -2 \log(z-w)$

$\langle 1 \rangle = 0$

$\langle b_0 \rangle = 0$

$\langle c_0 \rangle \neq 0$

$b(z) = \sum_n b_n z^{-n-1}$ $\qquad c(z) = \sum_n c_n z^{-n}$

$\rightsquigarrow \Theta^+(z)\, \Theta^-(w) = \log(z-w)$

$\Theta^\pm(z) = \xi^\pm + \Theta_0^\pm \log(z) + \sum_{n\neq 0} \Theta_n^\pm z^{-n}$

zero modes

$\{\xi^\pm, \Theta^\mp\} = 1$

$\Rightarrow \tilde{\mathbb{I}}(z) = \,:\Theta^+ \Theta^-:(z)$ $\qquad |0;1\rangle = \xi^+ \xi^- |0;0\rangle$

$c(z) \hat{=} \Theta^-(z) \Big|_{\Theta_0^- \equiv 0}$

$b(z) \hat{=} \partial_z \Theta^+(z)$

in sector built from $|in\rangle = |0\rangle$

$\langle out| = \langle \xi^+|$ $\qquad$ [Marco Krohn]

HIGHER RANK LCFTs

bc ghost systems for arbitrary spin $(j, 1-j)$
$j \in \mathbb{Z}$.

$$b(z) = \sum_n b_n z^{-n-j} \qquad c(z) = \sum_n c_n z^{-n-1+j}$$

There are $|j| + |1-j| = 2j-1$ zero modes
$(j > 0)$

$$b_{j-1}, b_{j-2}, \ldots, b_{1-j}$$

$\rightsquigarrow$ rank $2j$ LCFT:

$$b(z) \hat{=} \partial^{2j-1} \Theta^+(z) \qquad c(z) = \Theta^-(z) \Big|_{\Theta^-_{j-1} = \Theta^-_{j-2} = \ldots = \Theta^-_{1-j} = 0}$$

Example: $j = 2 \quad (c_{bc} = -26)$

$$\Theta^\pm(z) = \xi^\pm_{-1} z^2 + \xi^\pm_0 z + \xi^\pm_{+1}$$
$$+ \Theta^\pm_{-1} \tfrac{1}{2} z^2 (\log(z) - \tfrac{3}{2}) + \Theta^\pm_0 z(\log(z) - 1) + \Theta^\pm_{+1} \log(z)$$
$$+ \sum_{|n|>1} \Theta^\pm_n z^{-n+1}$$

$$\rightsquigarrow \quad \tilde{\mathbb{I}}^{(3)} \simeq :\Theta^+ \Theta^- \partial\Theta^+ \partial\Theta^- \partial^2\Theta^+ \partial^2\Theta^- :(z)$$
$$\tilde{\mathbb{I}}^{(2)} \simeq :\Theta^+ \Theta^- \partial^2\Theta^+ \partial^2\Theta^- :(z)$$
$$\tilde{\mathbb{I}}^{(1)} \simeq :\partial\Theta^- \partial\Theta^+ :(z)$$
$$\mathbb{I}^{(0)} \simeq \mathbb{1}$$

[Marco Krohn, M.F.]

/ich

HIGHER RANK LCFTs (cont'd)

Problem: Consistent definition of energy momentum tensor.

b.c ghost system:

$$T^{bc} = -j :b(\partial c): + (1-j) :(\partial b)c:$$

solution: [Fjelstad, Fuchs, et.al]
add zero modes by deforming T:

$$T^{\log}(z) = T^{bc}(z) + A\theta_1^{-}\partial b(z) + B\theta_0^{-} z^{-1}\partial(z^2 b(z)) + A\theta_{-1}^{-} z^{-2}\partial(z^4 b(z))$$

New Problem: $L_0^{\log}$ acts inconsistently on $\mathcal{H}$

$$L_0^{\log} |\xi_{-1}^{+}\rangle = 0$$

solution: full CFT = $\mathrm{CFT}^{(1)} \otimes \mathrm{CFT}^{(2)}$
since we live on double covering of cplx. plane.

$$T^{\log} = T^{(1)\log} + T^{(2)\log}, \quad [T^{(1)}, T^{(2)}] = 0$$

with the crucial identifications

$$\theta_i^{\pm(1)} = -\theta_i^{\mp(2)} \qquad i = -1, 0, 1$$

$$\xi_i^{\pm(1)} = \xi_i^{\mp(1)}$$

[Krohn, MF.]

CHALLENGES III

- Ghost systems on $\mathbb{Z}_n$-symmetric Riemann surfaces $\longleftrightarrow$ holomorphic j-differentials.

- Is there a geometrical meaning of this kind for other LCFTs?

  - $c_{p,1} = 1 - 6\frac{(p-1)^2}{p}$ models
  - WZW models $\widehat{su(N)}_0$, $\widehat{su(2)}_{-4/3}$, $\widehat{GL(1,1)}$, ...

- Can one generalize this to arbitrary Riemann surfaces?

  - $\mathbb{Z}_n$ case "easy", since all monodromies can be diagonalized simultaneously.
  - logarithms indicate degenerate regions of moduli space. (pinched homology cycles)

- Rep. theory of higher rank LCFTs.

- Meaning of additional zero modes seem to shine through to other sheets ...

GENERAL CASE

General case: rank r Jordan cells spaned by $\{ |h;0\rangle, |h;1\rangle, \ldots, |h;r-1\rangle \}$.

$$L_m \langle \Phi_{(h_1,k_1)}(z_1) \ldots \Phi_{(h_N,k_N)}(z_N) \rangle = 0$$

$m = -1, 0, 1$.

$$L_m = \sum_i z_i^m \left[z_i \partial_i + (m+1)(h_i + \delta_{h_i}) \right]$$

$$\delta_{h_i} \Phi_{(h_j,k_j)} = \delta_{ij} \Phi_{(h_j,k_j-1)} \qquad k_j \geq 1$$

$$\delta_{h_i} \Phi_{(h_j,0)} = 0.$$

These modified*) Ward-identities allow us to compute 1-point, 2-point, and 3-point functions upto constants.

$$\langle \Phi_{(h,k)} \rangle \propto \delta_{h,0}\, \delta_{k,r-1}$$

*) Homogeneous Equations $\rightsquigarrow$ Inhomogeneous Equations!

THE ASSUMPTIONS

$$L_0 |h;k\rangle = h|h;k\rangle + (1-\delta_{k,0})|h;k-1\rangle$$
$$L_n |h;k\rangle = 0 \quad \forall n>0 \qquad k=0,\dots,r-1$$

$\Rightarrow$ Jordan cell of rank $r = r(h)$,
all logarithmic partners are quasiprimary,
state $|h;k\rangle$ is said to have J-level k,
generalized highest weight states.

$\rightsquigarrow$ Basic assumption:
Primary fields from Jordan cells have standard OPEs, i.e.

$$\Phi_{(h,0)}(z)\,\Phi_{(h',0)}(w) = \sum_{h''} C^{h''}_{hh'} \cdot (z-w)^{h''-h-h'}\,\Phi_{(h'',0)}(w)$$

Primary $*$ Primary $= \sum$ Primaries
Irrep $*$ Irrep $= \sum$ Irreps

Note: We already know:
$\exists$ Primary fields (pre-logarithmic fields), whose OPEs contain logarithmic operators.
Assumption: pre-logarithmic fields are not part of Jordan cell.

$$\langle \Phi_{(h,\ell)}(z)\, \Phi_{(h',\ell')}(z') \rangle =$$

$$\left[\sum_{j=r-1}^{\ell+\ell'} D_{(h;j)} \sum_{\substack{i=0 \\ i+i'=\ell+\ell'-j}}^{\ell} \sum_{i'=0}^{\ell'} \frac{1}{i!\,i'!} (\partial_h)^i (\partial_{h'})^{i'} (z-z')^{-h-h'} \right] \delta_{hh'}$$

$$\langle \Phi_{(h_1,\ell_1)}(z_1)\, \Phi_{(h_2,\ell_2)}(z_2)\, \Phi_{(h_3,\ell_3)}(z_3) \rangle =$$

$$\left[\sum_{j=r-1}^{\ell_1+\ell_2+\ell_3} C_{(h_1,h_2,h_3;j)} \sum_{\substack{i_1=0 \\ i_1+i_2+i_3=\ell_1+\ell_2+\ell_3-j}}^{\ell_1} \sum_{i_2=0}^{\ell_2} \sum_{i_3=0}^{\ell_3} \frac{1}{i_1!\,i_2!\,i_3!} (\partial_{h_1})^{i_1} (\partial_{h_2})^{i_2} (\partial_{h_3})^{i_3} \right.$$

$$\left. (z_{12})^{h_3-h_1-h_2} (z_{13})^{h_2-h_1-h_3} (z_{23})^{h_1-h_2-h_3} \right].$$

Defines $(r \times r)$ - Matrices

$$\left(D_{h,h'}\right)_{\ell,\ell'} \equiv \delta_{hh'} \langle \Phi_{(h,\ell)}(z_1) \Phi_{(h',\ell')}(z_3) \rangle$$

$$\left(C_{h_1,h_2,h_3[\ell_1]}\right)_{\ell_2\ell_3} \equiv \langle \Phi_{(h_1,\ell_1)}(z_1)\, \Phi_{(h_2,\ell_2)}(z_2)\, \Phi_{(h_3,\ell_3)}(z_3) \rangle$$

$$\Rightarrow \boxed{\begin{aligned} &\text{OPE:} \quad \Phi_{(h_1,\ell_1)}(z_1)\, \Phi_{(h_2,\ell_2)}(z_2) = \lim_{z_1 \to z_2} \\ &\sum_{h_3} \sum_{\ell} \left(C_{h_1,h_2,h_3[\ell_1]}\right)_{\ell_2,\ell} \left(D_{h_3,h_3}^{-1}\right)_{\ell,\ell_3} \Phi_{(h_3,\ell_3)}(z_2) \\ &\qquad + \cdots \end{aligned}}$$

ZERO MODE CONTENT

Schematic OPE:

$$\Psi_{(h_1;k_1)}(z)\,\Psi_{(h_2;k_2)}(w) = \sum_h \sum_{k=0}^{r(h)-1} C^{(h;k)}_{(h_1,k_1)(h_2;k_2)}$$

$$\times (z-w)^{h-h_1-h_2}\, f^{k}_{k_1 k_2}(z-w)\, \Psi_{(h;k)}$$

Unfortunately, there is no bound

$$k \leq k_1 + k_2$$

(e.g. prelogarithmic fields: $k_1 = k_2 = 0$, $k \leq 1$)

However, in all explicitly known examples, we can define a zero mode content

z_0 = total # of zero modes $\mathcal{I}_\alpha$ with $\mathcal{I}_\alpha|0\rangle \neq 0$

$\Rightarrow$ bound

$$z_0(\Psi_{(h;k)}) \leq z_0(\Psi_{(h_1;k_1)}) + z_0(\Psi_{(h_2;k_2)})$$

In fermionic theories, we also have

z_+, z_- with $z_+ + z_- = z_0$

$z_\pm$ = total # of fermionic zero modes $\mathcal{I}^\pm_\alpha$ with $\mathcal{I}^\pm_\alpha|0\rangle \neq 0$.

[Prelogarithmic (twist) fields: $z_+, z_- \in \mathbb{Q}$
$z_+ + z_- \in \mathbb{Z}$.]

ZERO MODE CONTENT (cont'd)

Example $c=-2$ LCFT : is rank 2 LCFT.

- $Z_0(\mathbb{I}) = 0$ $\qquad$ $Z_0(\tilde{\mathbb{I}}) = Z_0(\xi^+\xi^-|0\rangle) = 2$

  Primary $\qquad$ Log. Field

  Jordan cell

  $Z_+(\mathbb{I}) = Z_-(\mathbb{I}) = 0$ $\qquad$ $Z_+(\tilde{\mathbb{I}}) = 1 = Z_-(\tilde{\mathbb{I}})$

- $Z_0(|\xi^+\rangle) = Z_0(|\xi^-\rangle) = 1$

  $Z_+(|\xi^+\rangle) = Z_-(|\xi^-\rangle) = 1$

  $Z_-(|\xi^+\rangle) = Z_+(|\xi^-\rangle) = 0$

  $\langle\xi^+\rangle = \langle\xi^-\rangle = 0, \; \langle\xi^-|\xi^+\rangle = 1$

  Fermionic Fields $\theta^\pm(z)$

- $Z_0(|\mu\rangle) = 1 = Z_0(|\sigma\rangle)$ $\qquad$ $h(\sigma) = 3/8$, $h(\mu) = -1/8$

  $Z_+(|\mu\rangle) = Z_-(|\mu\rangle) = \frac{1}{2}$

  $\begin{cases} Z_+(|\sigma\rangle) = -\frac{1}{2} \text{ oder } 3/2 \\ Z_-(|\sigma\rangle) = 3/2 \text{ oder } -1/2 \end{cases}$

  Twist fields.

ZERO MODE CONTENT & CORRELATORS

$$G = \langle \prod_i \Phi_{(h_i,k_i)}(z_i) \prod_j \Theta_{(h_j,k_j^+,k_j^-)}(w_j) \prod_\ell \mu_{\alpha_\ell}(u_\ell) \rangle \neq 0 \,?$$

Jordan — Fermionic — Twist

$$= \langle \prod_i \Phi_{(h_i,k_i)} \prod_j \Theta_{(h_j,k_j^+,k_j^-)} \prod_\ell \mu_{\alpha_\ell} \rangle \quad \text{in short}$$

$$(*) \begin{cases} \sum_i z_0(\Phi_i) + \sum_j z_0(\Theta_j) + \sum_\ell z_0(\mu_\ell) \in \mathbb{Z}_{\geq 2(r-1)} \\ \sum_j z_+(\Theta_j) = \sum_j z_-(\Theta_j) \in \mathbb{Z} \\ \sum_\ell z_+(\mu_\ell) \in \mathbb{Z} \text{ and } \sum_\ell z_-(\mu_\ell) \in \mathbb{Z} \end{cases}$$

Necessary conditions for $G \neq 0$.

Non-quasi-primary fields : o.k. if

$L_n G = \sum_i G_i'$ for $n \in \{-1, 0, 1\}$ such that

G_i' do not satisfy $(*)$

ZERO MODE STRUCTURE

Example: rank $r = 4$ (z_+, z_-)

$$
\begin{array}{ccccccc}
 & & & (3,3) & & & \\
 & & (3,2) & & (2,3) & & \\
 & (3,1) & & (2,2) & & (1,3) & \\
(3,0) & & (2,1) & & (1,2) & & (0,3) \\
 & (2,0) & & (1,1) & & (0,2) & \\
 & & (1,0) & & (0,1) & & \\
 & & & (0,0) & & &
\end{array}
$$

Action of Virasoro algebra:

$$L_n : (k, l) \longrightarrow (k', l') \quad \text{with } k+l \geq k'+l'$$

Non-quasi-primary fields:

If $L_n : (k, l) \longrightarrow (k', l')$ with $k+l \not\equiv k'+l' \bmod 2$

for example: $(k, l-1)$
$(k-1, l)$

then this will not have an effect in correlation fct's.

$\rightsquigarrow$ BRST-like structure [S. Rouhani et al.]

CHALLENGES IV

- Hybrid fields?
  Fermionic fields $\in$ Jordan cells
  Twist fields $\in$ Jordan cells

- Para fermionic zero mode structure

- Indecomposable structures other than L_0 - Jordan cells

- General case of non-quasi-primary logarithmic partner fields.

- General case of mixed correlation functions

- Locality (in particular with non-chiral fields such as fermionic fields $\Theta^{\pm}(z)$)