Nonmeromorphic operator product expansion and C_2-cofiniteness for a family of $\mathcal{W}$-algebras*

Nils Carqueville

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- C_2-cofiniteness and rationality

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Vertex Operator Algebras

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Frenkel, Huang, Lepowsky 1989

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Important example: For a V-module W, the structure (W', Y') defined by $W' = \coprod_{h \in \mathbb{R}} W^*_{[h]}$ with the vertex operator

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given by the relation

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$\Longrightarrow \langle \psi'_m w', w\rangle = \langle w', \psi_{-m} w\rangle$ for primary fields $\sum_{m\in\mathbb{Z}} \psi_m x^{-m-\mathrm{wt}\psi}$

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Definition. Let (W_i, Y_i), (W_j, Y_j) and (W_k, Y_k) be (generalized) V-modules. A **(logarithmic) intertwining operator** of type $\binom{W_k}{W_i\ W_j}$ is a linear map

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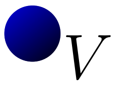
The dimensions of the spaces of all intertwining operators $\mathcal{Y}^k_{ij}$ are called the **fusion rules** N^k_{ij}.

Visualization



Visualization

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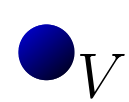

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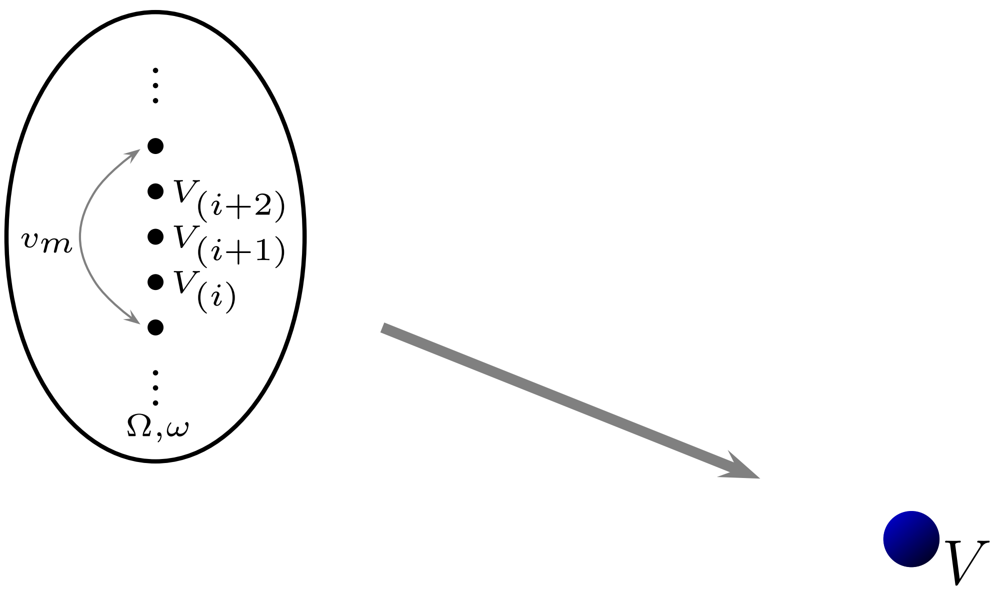

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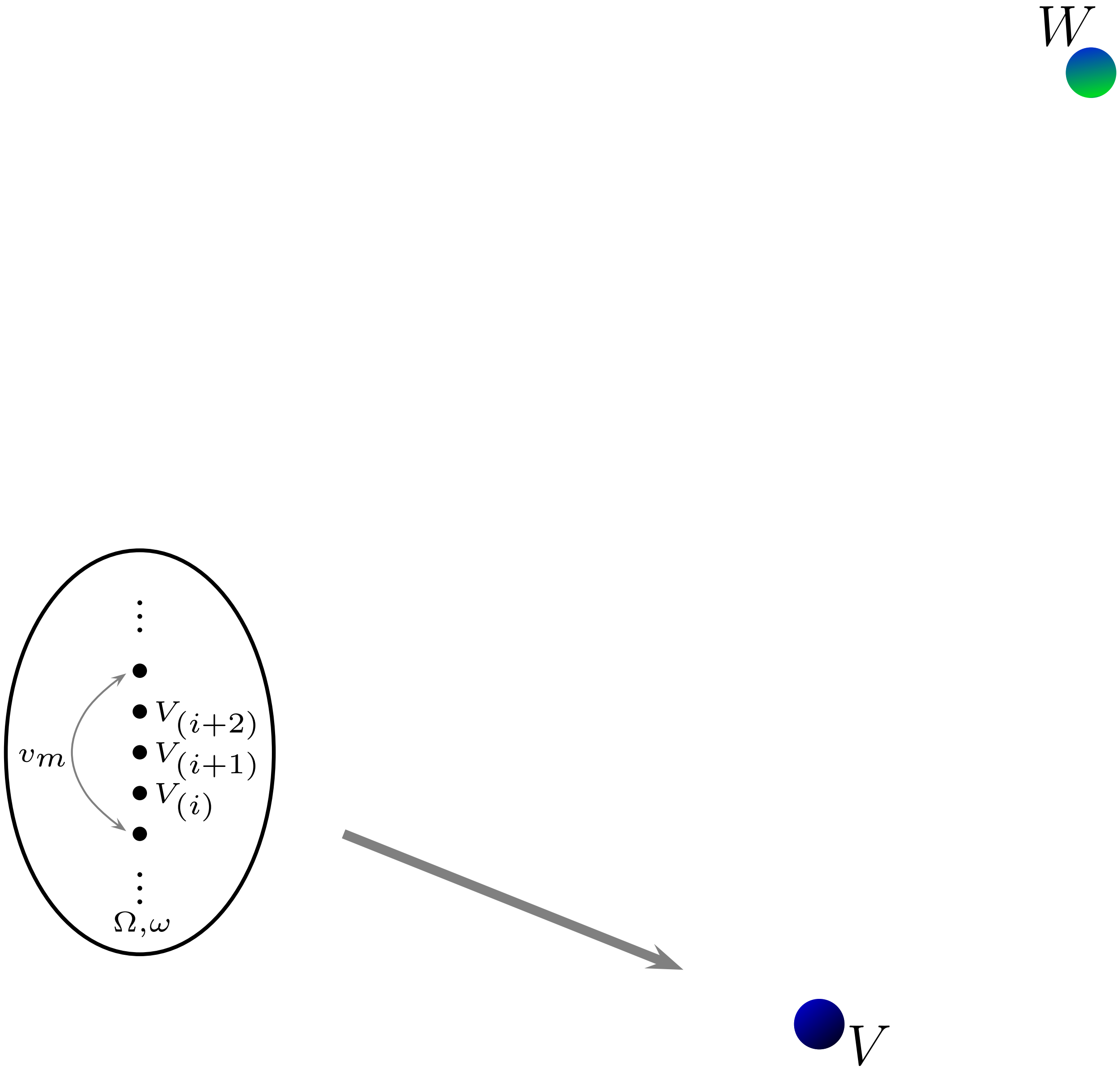

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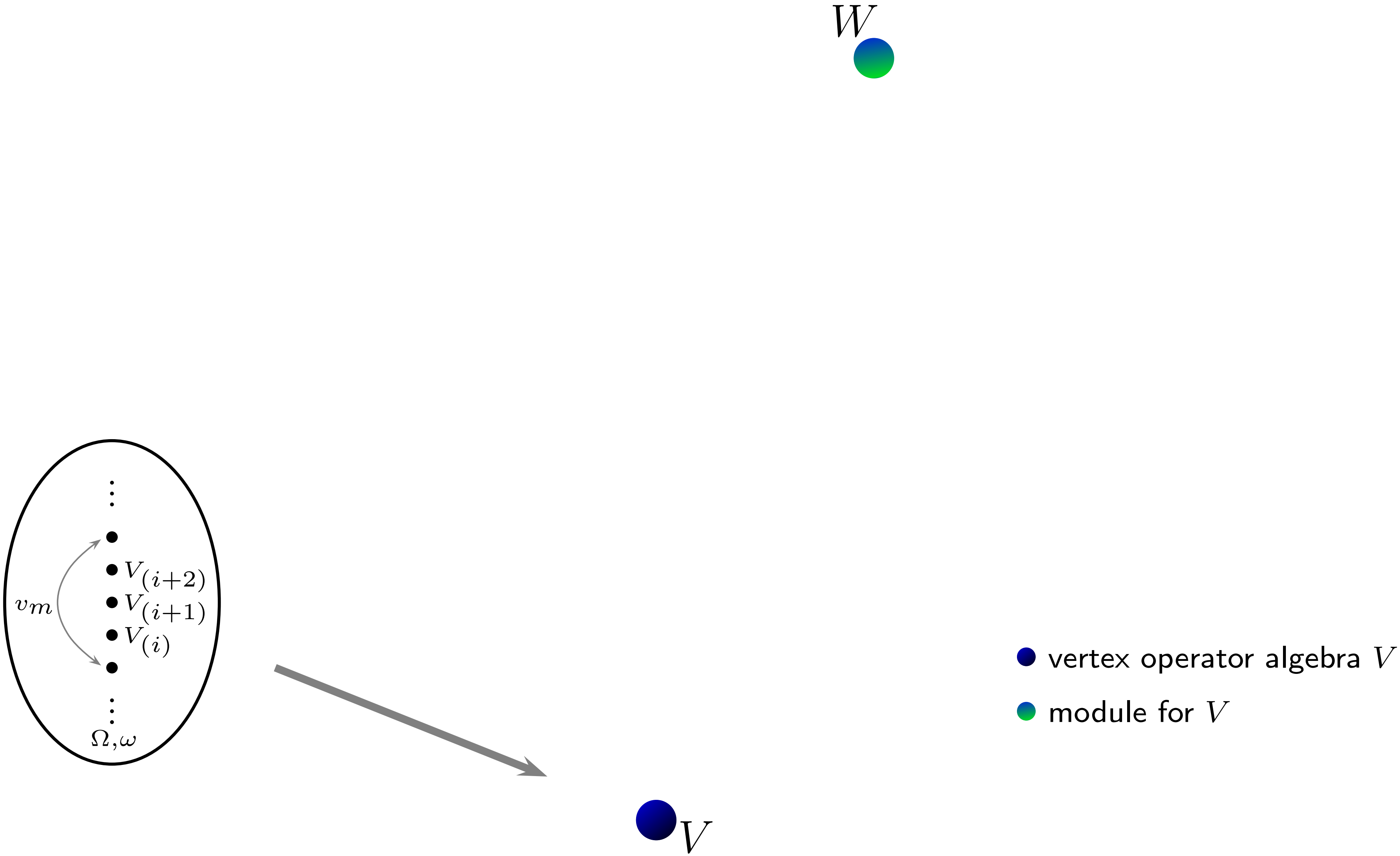

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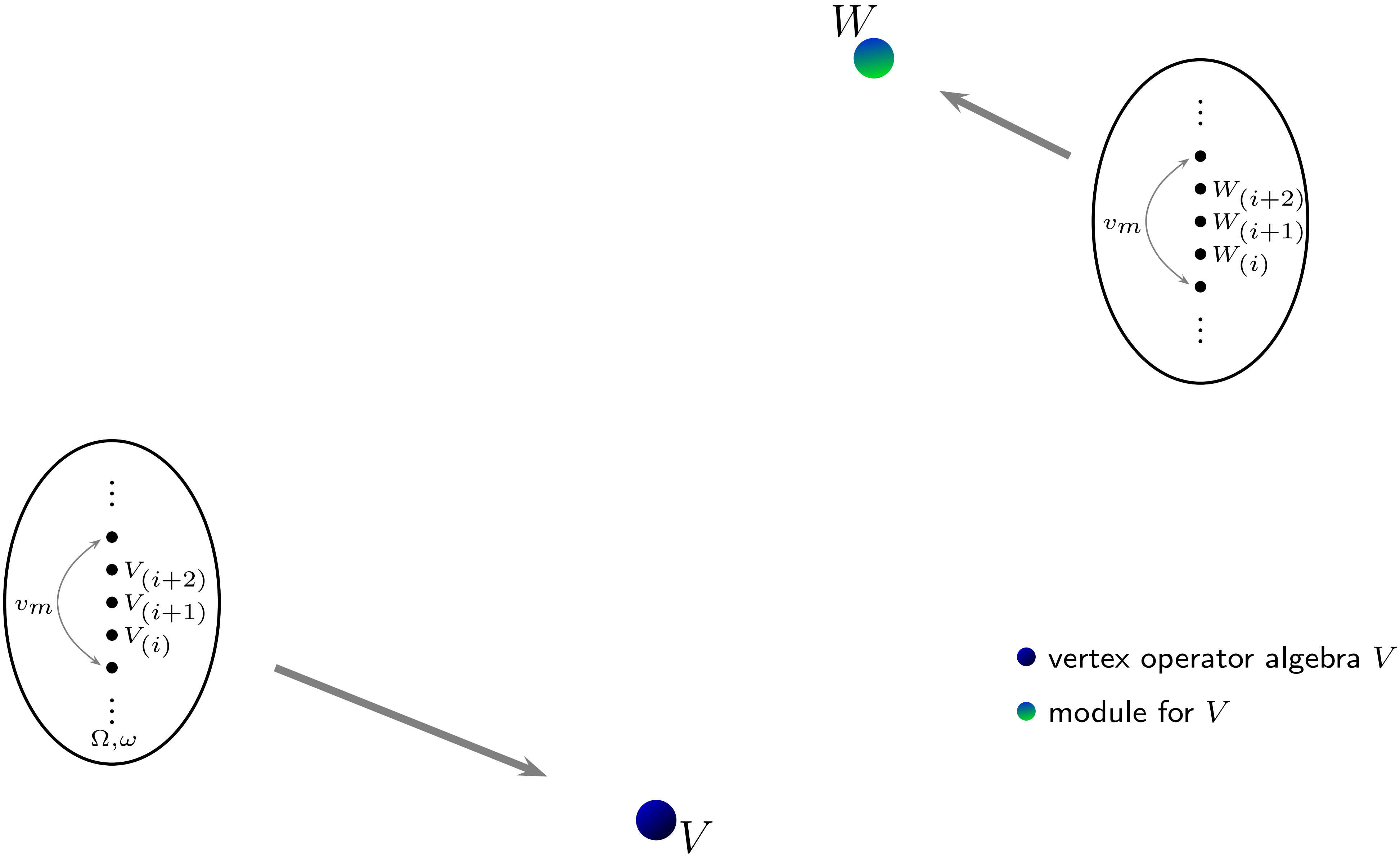

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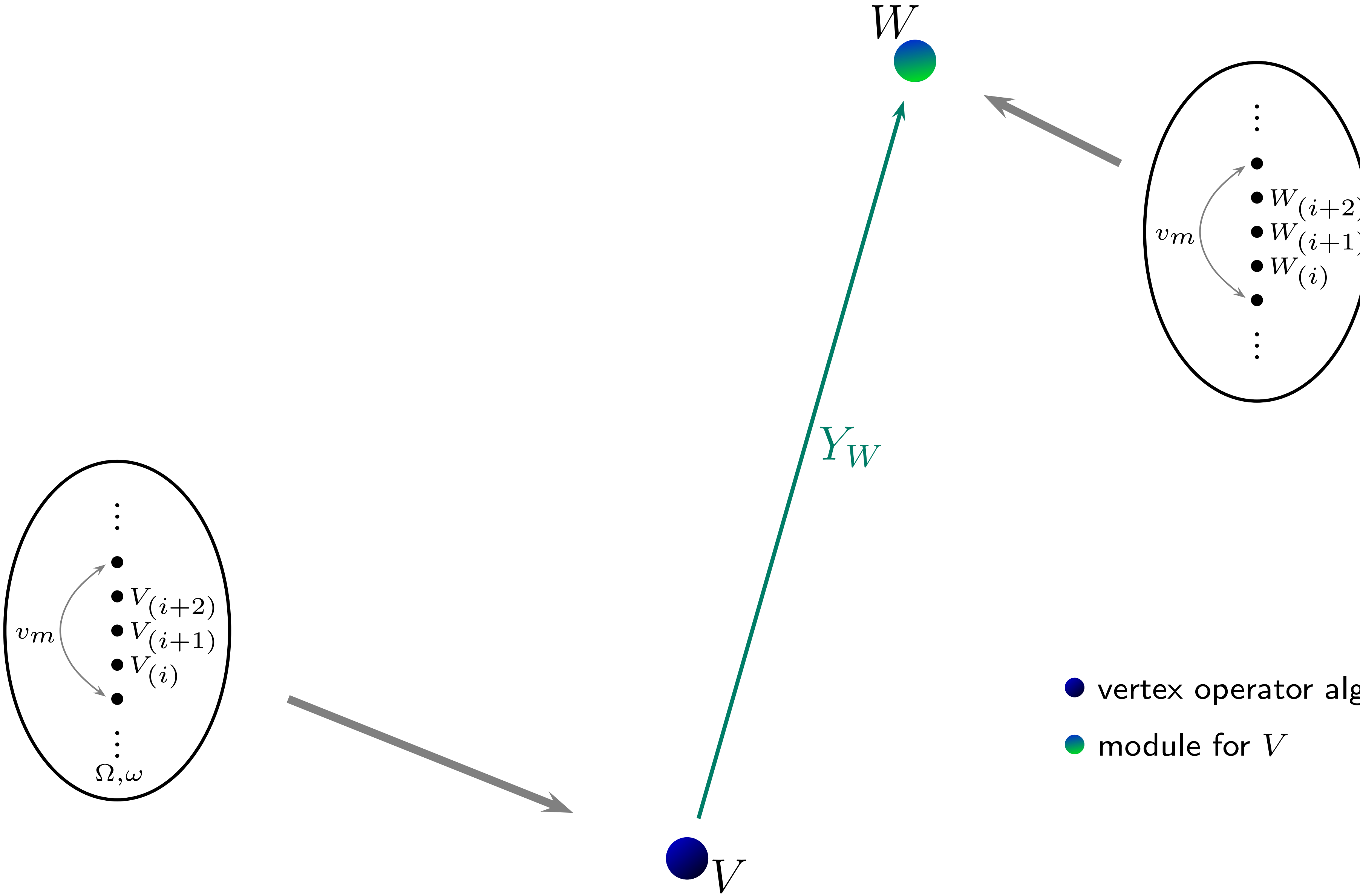

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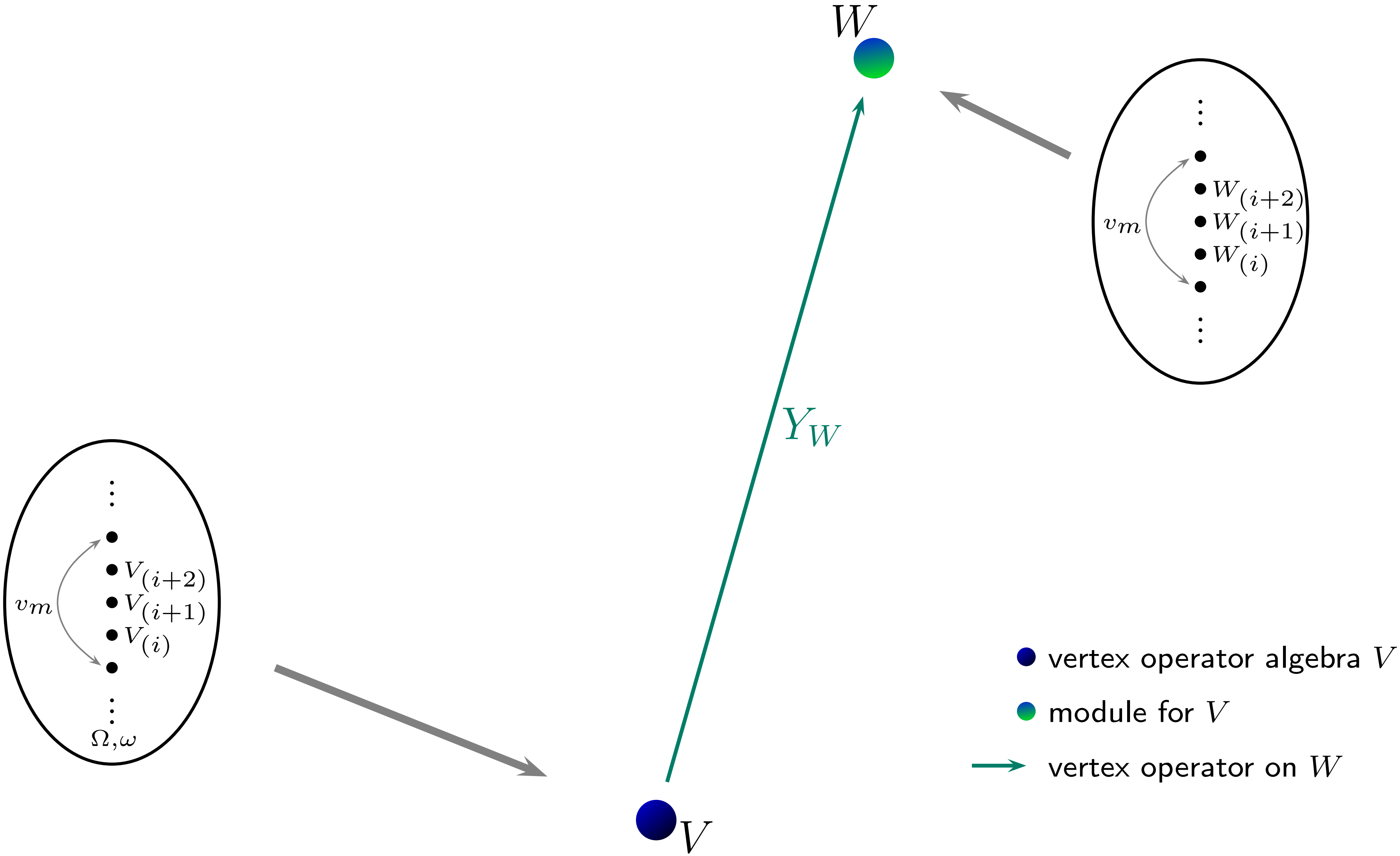

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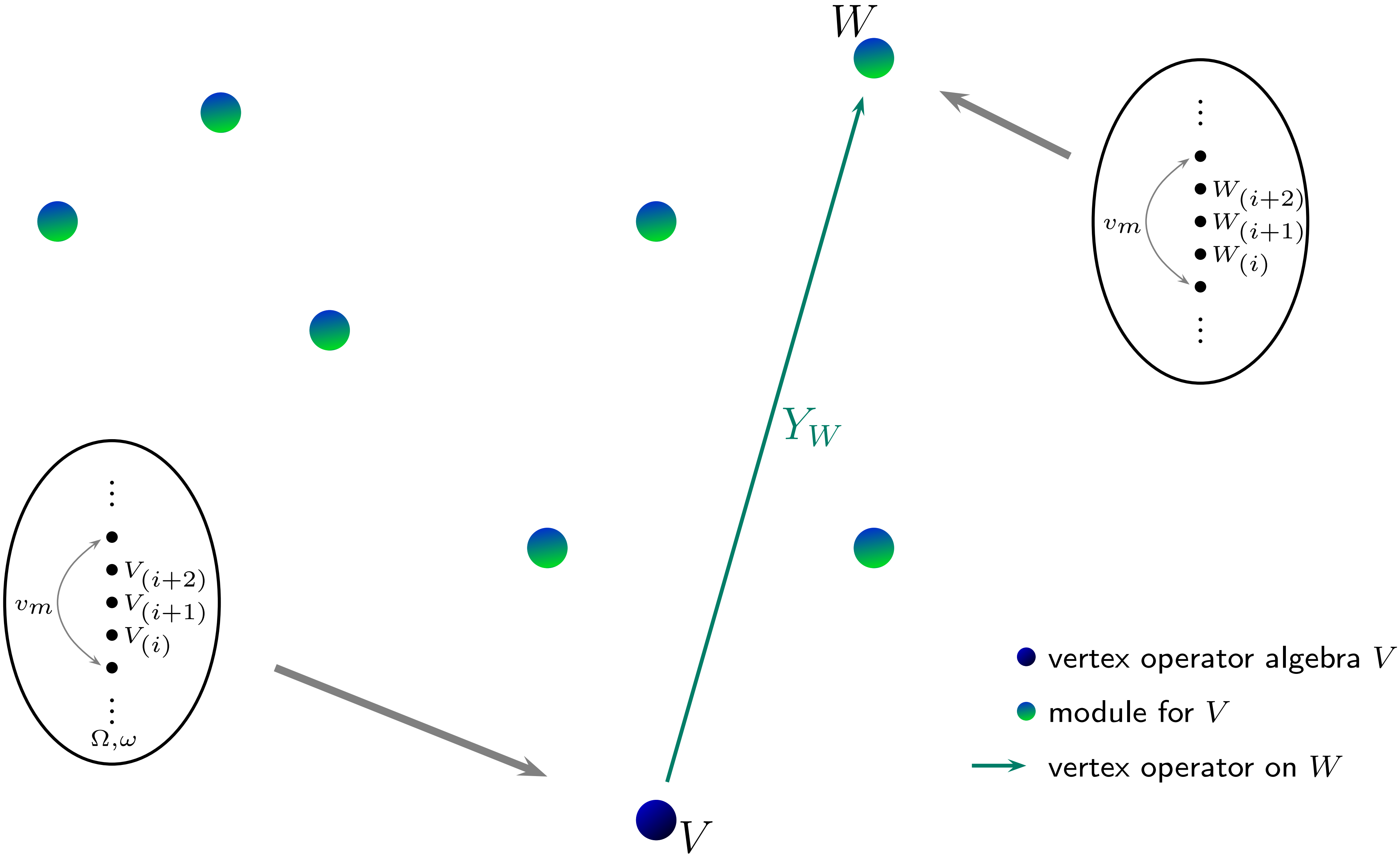

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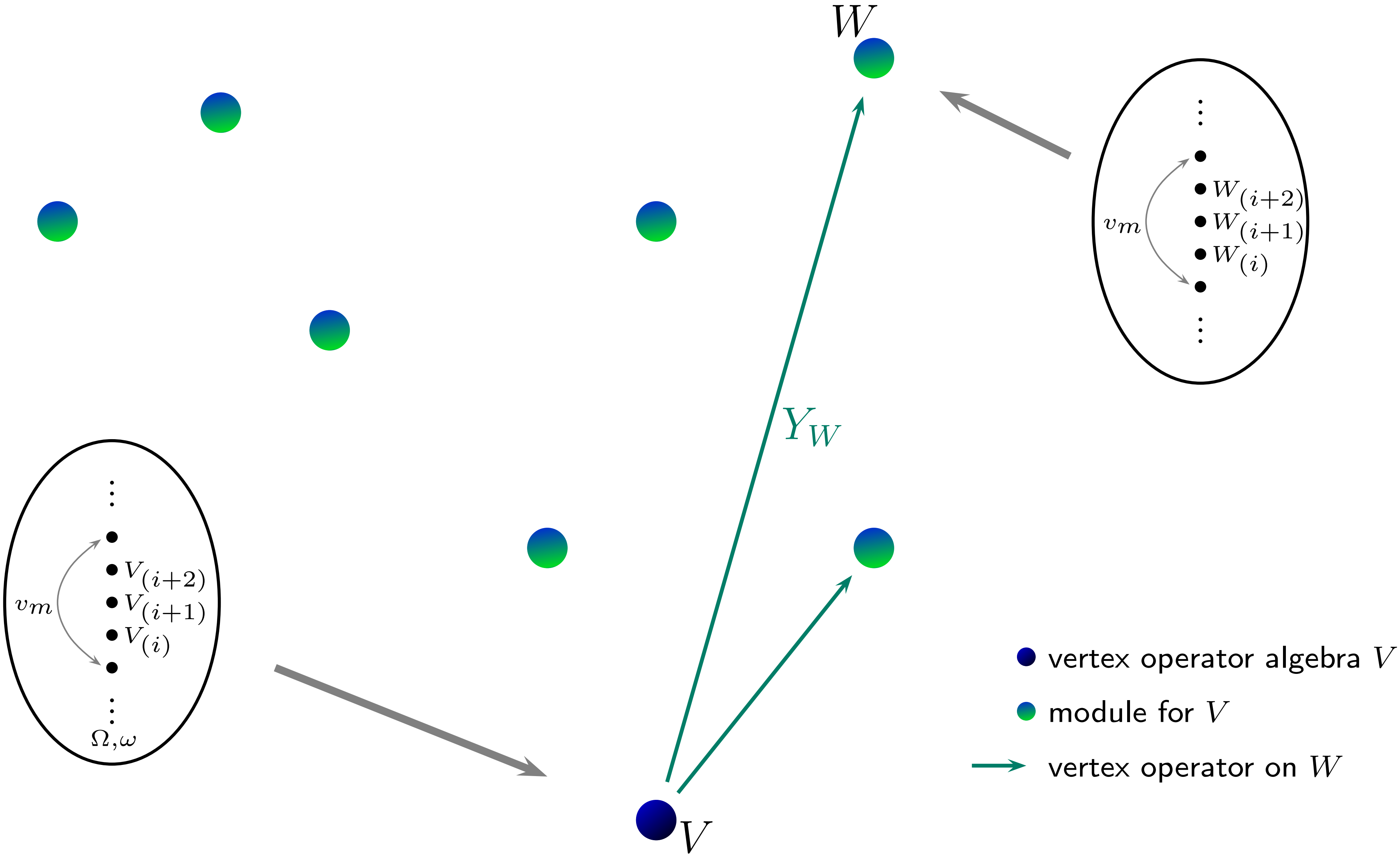

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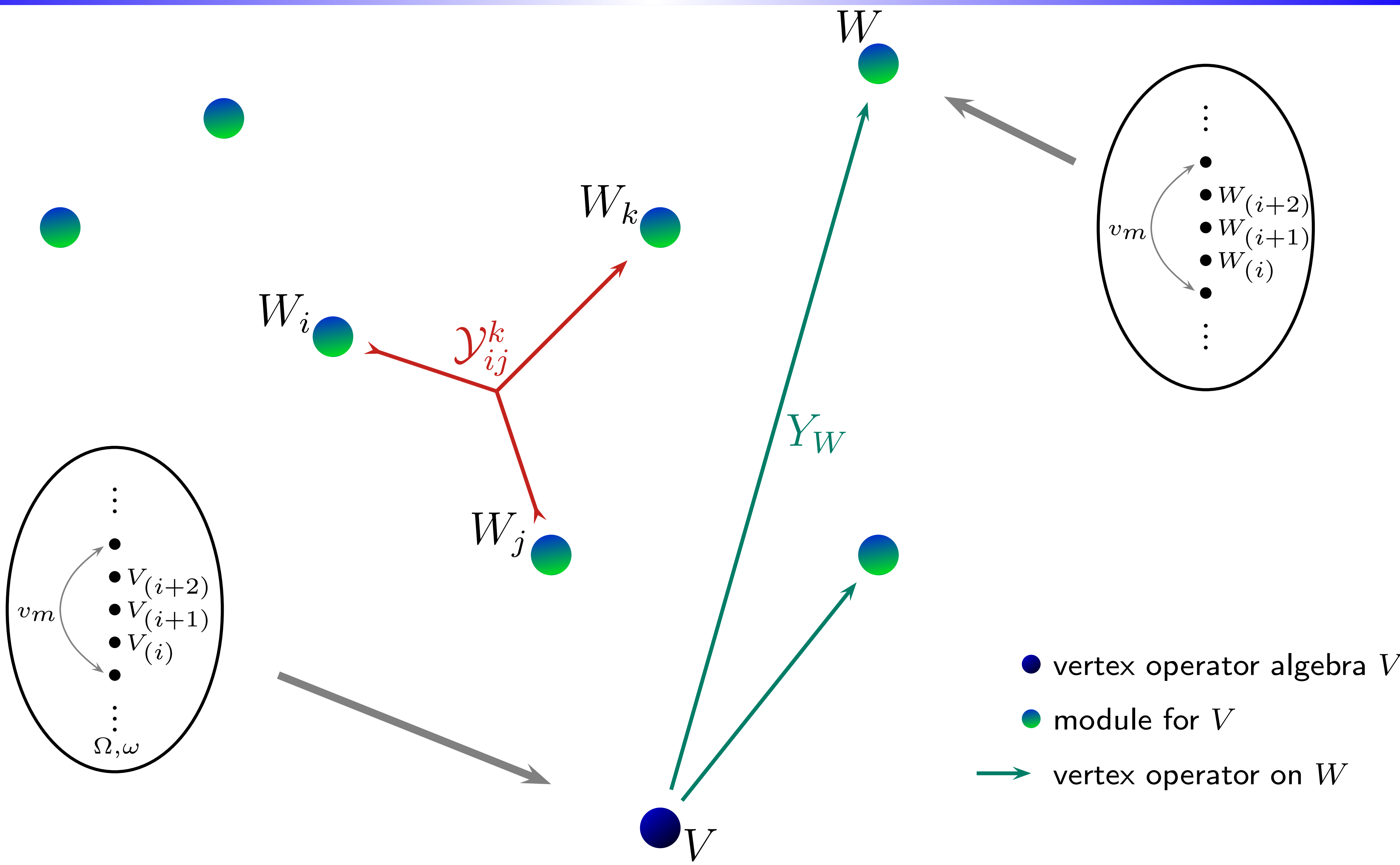

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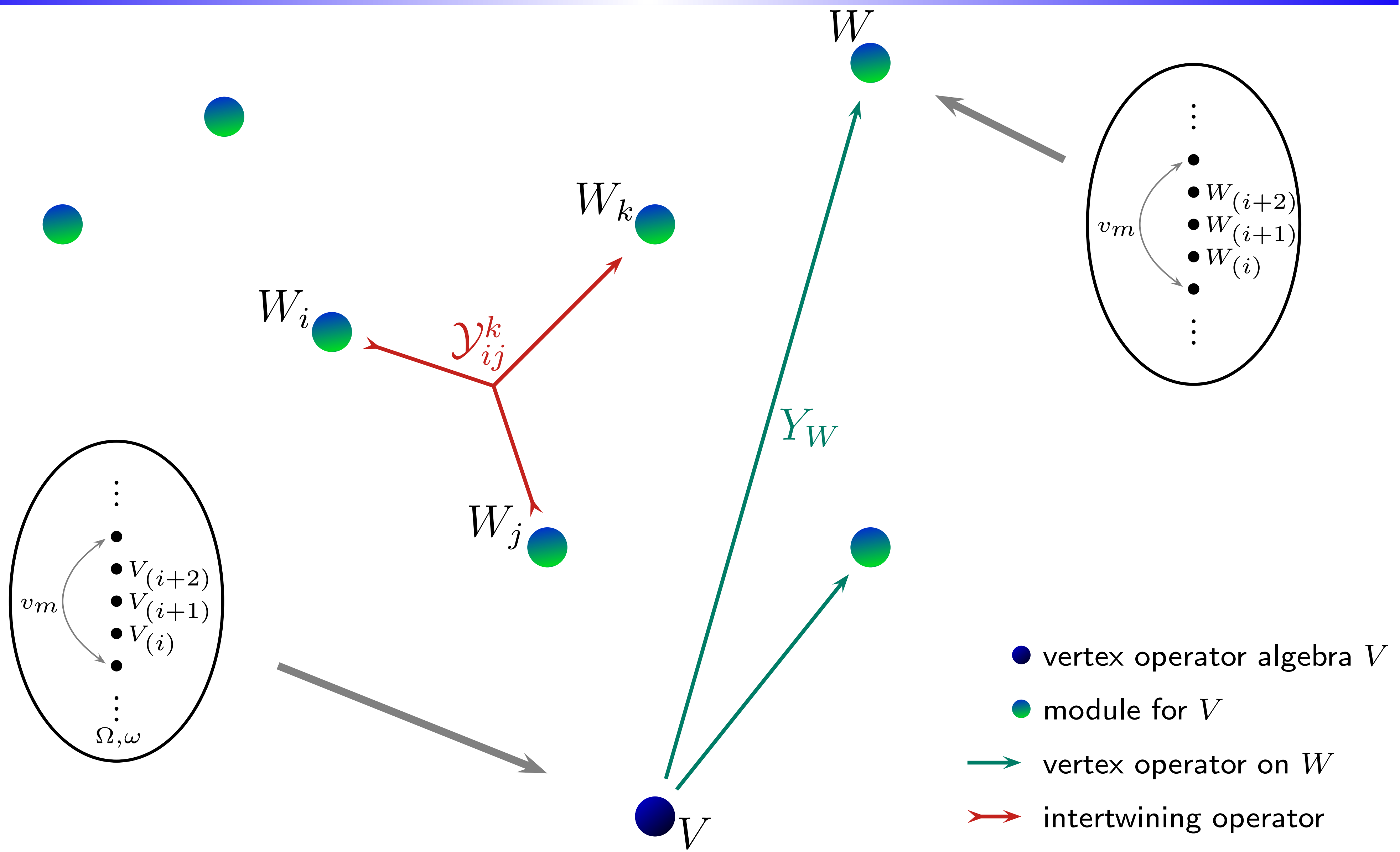

Meromorphic Operator Product Expansion

An essential notion in field theories are **correlation functions** like

$$\langle w', Y(v_1, x_1)Y(v_2, x_2) \dots Y(v_n, x_n)w \rangle \; .$$

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Nonmeromorphic Operator Product Expansion

Theorem. Given two logarithmic intertwining maps $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of type $\binom{W_4}{W_1\ M}$ and $\binom{M}{W_2\ W_3}$, there exists a logarithmic intertwining map $\mathcal{Y}$ of type $\binom{W_4}{W_1 \boxtimes_{P(z_1-z_2)} W_2\ W_3}$ such that

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(1) All generalized V-modules W in $\mathrm{ob}\,\mathcal{C}$ are **C_1-cofinite**, i.e. $\dim(W/C_1(W)) < \infty$ with $C_1(W) = \mathrm{span}\{u_{-1}w \,|\, u \in \coprod_{m>0} V_{(m)},\ w \in W\}$.

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Many well-understood, *rational* vertex operator algebras satisfy the conditions of the theorem, e.g. the *minimal Virasoro models* and those associated to *Kac-Moody algebras*.

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Using further results* on vertex operator algebras, instead of studying C_1-cofiniteness of *all* modules, one "only" needs to check **C_2-cofiniteness** of the triplet algebras $V_{2p-1} = \mathcal{W}(2,(2p-1)^{\times 3})$ themselves:

$$\dim\left(V_{2p-1}/C_2(V_{2p-1})\right) < \infty \quad \text{with} \quad C_2(V_{2p-1}) = \operatorname{span}\left\{u_{-2}v \,\middle|\, u,v \in V_{2p-1}\right\}$$

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$$[L_m, L_n] = (m-n)L_{m+n} - \frac{1}{6}\left(m^3 - m\right)\delta_{m+n,0} \ ,$$

$$[L_m, W^a_n] = (2m-n)W^a_{m+n} \ ,$$

$$\left[W^a_m, W^b_n\right] = \delta_{ab}\Big(2(m-n)\Lambda_{m+n} + \frac{1}{20}(m-n)\big(2m^2+2n^2-mn-8\big)L_{m+n}$$
$$-\frac{1}{120}m(m^2-1)(m^2-4)\delta_{m+n,0}\Big)$$
$$+\,\mathrm{i}\varepsilon_{abc}\left(\frac{5}{14}\left(2m^2+2n^2-3mn-4\right)W^c_{m+n} + \frac{12}{15}V^c_{m+n}\right)$$

Triplet Algebras

and the singular vectors

$$N^{ab} = W^a_{-3} W^b_{-3} \Omega - \delta_{ab} \left(\frac{8}{9} L^3_{-2} + \frac{19}{36} L^2_{-3} + \frac{14}{9} L_{-4} L_{-2} - \frac{16}{9} L_{-6} \right) \Omega + \mathrm{i} \varepsilon_{abc} \left(-2 W^c_{-4} L_{-2} + \frac{5}{4} W^c_{-6} \right) \Omega$$

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Proposition. The vertex operator algebra $\mathcal{W}(2, 3^{\times 3})$ is C_2-cofinite and the nonmeromorphic operator product expansion exists.

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1^{st} *step*: Analyze the **characters***

$$\chi_{V_{2p-1}}(q) = \mathrm{tr}_{V_{2p-1}} q^{L_0 - c_{p,1}/24} = \frac{q^{-1/24}}{\varphi(q)} \sum_{n \in \mathbb{Z}} (2n+1) q^{(2np+p-1)^2/(4p)}$$

in detail to obtain information on singular vectors.

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Triplet Algebras

For example:

$$q^{c_{2,1}/24}\,\chi_{V_3}(q) = 1 + q^2 + 4q^3 + 5q^4 + 8q^5 + 10q^6 + 16q^7 + 22q^8 + \ldots$$

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Proposition. For all $p \in \mathbb{Z}_+$, the triplet algebra $\mathcal{W}(2,(2p-1)^{\times 3})$ has six singular vectors at level $4p-2$

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$$q^{c_{3,1}/24}\,\chi_{V_5}(q) = 1 + q^2 + q^3 + 2q^4 + 5q^5 + 7q^6 + 10q^7 + 13q^8 + 20q^9 + 27q^{10} + 38q^{11} + 51q^{12} + 69q^{13} + \dots$$

$$q^{c_{3,1}/24}\,\tilde{\chi}_{V_5}(q) = 1 + q^2 + q^3 + 2q^4 + 5q^5 + 7q^6 + 10q^7 + 13q^8 + 20q^9 + 33q^{10} + 50q^{11} + 75q^{12} + 105q^{13} + \dots$$

Proposition. For all $p \in \mathbb{Z}_+$, the triplet algebra $\mathcal{W}(2,(2p-1)^{\times 3})$ has six singular vectors at level $4p-2$ of the form

$$N^{ab} = W^a_{-2p+1} W^b_{-2p+1}\Omega + \delta_{ab}\big(\text{Virasoro polynomial}\big)\Omega + \varepsilon_{abc}\big(\text{Virasoro-}W^c_m\text{ polynomial}\big)\Omega\ .$$

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$$
\begin{aligned}
\mathcal{N}\left(\phi_j, \partial^n \phi_i\right) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{2(h_i + h_j + n - 1)}{r}^{-1} \binom{2h_i + n - 1}{r} \\
\cdot \partial^r N^{(h_i+n+r)} \left(\phi_j, \partial^{n-r} \phi_i\right) \\
- (-1)^n \sum_{\{k \,|\, h(ijk) \geq 1\}} C_{ij}^{k} \binom{h(ijk) + n - 1}{n} \\
\cdot \binom{2(h_i + h_j + n - 1)}{n}^{-1} \binom{2h_i + n - 1}{h(ijk) + n} \binom{\sigma(ijk) - 1}{h(ijk) - 1}^{-1} \\
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\end{aligned}
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$$2\Delta-4)\left(\ldots N^{(6)}\left(T, N^{(4)}\left(T, N^{(2)}(T,T)\right)\right)\ldots\right)$$

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$$2\Delta-4)\left(\ldots N^{(6)}\left(T, \sum_{n\in\mathbb{Z}} x^{-n-6}\left\{\sum_{k_2=-\infty}^{3} L_{k_2+n}\left[\sum_{k_1=-\infty}^{1} L_{k_1-k_2}L_{-k_1} + \sum_{k_1=2}^{\infty} L_{-k_1}L_{k_1-\ldots}\right.\right.\right.$$

$$+ \sum_{k_2=4}^{\infty}\left[\sum_{k_1=-\infty}^{1} L_{k_1-k_2}L_{-k_1} + \sum_{k_1=2}^{\infty} L_{-k_1}L_{k_1-k_2}\right] L_{k_2+n}\left.\left.\left.\right\}\right)\right)\ldots\right)$$

$$2\Delta-4)\left(\ldots N^{(8)}\left(T, \sum_{n\in\mathbb{Z}} x^{-n-8}\left\{\sum_{k_3=-\infty}^{5} L_{k_3+n}\left[\sum_{k_2=-\infty}^{3} L_{k_2-k_3}\left(\sum_{k_1=-\infty}^{1} L_{k_1-k_2}L_{-k_1}\right.\right.\right.\right.\right.$$

$$\left.+ \sum_{k_1=2}^{\infty} L_{-k_1}L_{k_1-k_2}\right) + \sum_{k_2=4}^{\infty}\left(\sum_{k_1=-\infty}^{1} L_{k_1-k_2}L_{-k_1} + \sum_{k_1=2}^{\infty} L_{-k_1}L_{k_1-k_2}\right) L_{k_2-\ldots}$$

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$$\left. + \sum \left(\sum^{1} L_{k_1-k_2}L_{-k_1} + \sum^{\infty} L_{-k_1}L_{k_1-k_2}\right) L_{k_2-k_3}\right] L_{k_3+n}\left.\left.\left.\right\}\right)\ldots\right)\ldots$$

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$$\mathcal{N}(W^a, W^a) - \frac{1}{2} C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a} 2\Delta \cdot \Omega \times \Omega^{-2\Delta-1} \ldots$$

$$\times (\xi_1 L_{-5} L_{-2}^{\Delta-2} \times \xi_2 L_{-4} L_{-3} L_{-2}^{\Delta-3} \times \xi_3 L_{-3}^{3} L_{-2}^{\Delta-3}) \ldots (\Delta-1) \ldots (2\Delta-1)(-n-2\Delta+1) \ldots \Omega,$$

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Proposition. For all $p \in \mathbb{Z}_+$, the triplet algebra $\mathcal{W}(2,(2p-1)^{\times 3})$ has six singular vectors at level $4p-2$ of the form

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***Theorem*. For all $p \in \mathbb{Z}_{\geq 2}$, the nonmeromorphic operator product expansion exists and is associative for the vertex operator algebra $\mathcal{W}(2,(2p-1)^{\times 3})$. Furthermore, all these vertex operator algebras are C_2-cofinite.**

C_2-cofiniteness

$$\dim\left(V/C_2(V)\right) < \infty \quad \text{with} \quad C_2(V) = \operatorname{span}\left\{ u_{-2}v \;\middle|\; u, v \in V \right\}$$

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- ☐ finite fusion rules⋆
- ☐ finitely many inequivalent irreducible modules⋆
- ☐ every weak module is a direct sum of generalized eigenspaces of L_0⋆

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It is desirable that all modules can be concisely organized.

C_2-cofiniteness and Rationality

A vertex operator algebra V is called **rational** if

(i) there are only finitely many **irreducible** V-modules;

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This conjecture is wrong, the triplet algebras present counter-examples.

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(ii) any V-module is completely reducible;

(iii) all *fusion rules* for V-modules are finite, i.e. $N_{ij}^k < \infty$.

Conjecture: C_2-cofiniteness and rationality are equivalent.

This conjecture is wrong, the triplet algebras present counter-examples.

New conjecture: C_2-cofiniteness is equivalent to "rationality" in the sense that a finite set $\mathcal{S}$ of (generalized) modules *closes under fusion*, i.e. for $W_i, W_j \in \mathcal{S}$, $N_{ij}^k = 0$ if $W_k \notin \mathcal{S}$.

Conclusion

Theorem. For all $p \in \mathbb{Z}_{\geq 2}$, the nonmeromorphic operator product expansion exists and is associative for the vertex operator algebra $\mathcal{W}(2,(2p-1)^{\times 3})$. Furthermore, all these vertex operator algebras are C_2-cofinite.

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- Many logarithmic conformal field theories have very nice properties!

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