Vertex Operator Algebra Approach to Logarithmic Conformal Field Theory

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Introduction and Synopsis

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- C_2-cofiniteness
- logarithmic mode algebras

Vertex Operator Algebras

Definition. A vertex operator algebra* is a $\mathbb{Z}$-graded $\mathbb{C}$-vector space

$$V = \coprod_{m\in\mathbb{Z}} V_{(m)} \quad \text{with} \quad \dim V_{(m)} < \infty \quad \text{for all } m \in \mathbb{Z}$$

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together with a linear vertex operator map

$$V \longrightarrow (\mathrm{End} V)[\![x, x^{-1}]\!] \,, \quad v \longmapsto Y(v,x) = \sum_{m\in\mathbb{Z}} v_m x^{-m-1} \,.$$

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$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)Y(v,x_2)-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1)$$
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$$[L_m,L_n]=(m-n)L_{m+n}+\frac{c}{12}(m^3-m)\delta_{m+n,0}\ ,$$

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(Logarithmic) Intertwining Operators

Definition. Let (W_i, Y_i), (W_j, Y_j) and (W_k, Y_k) be (generalized) V-modules. A **(logarithmic) intertwining operator** of type $\binom{W_k}{W_i\ W_j}$ is a linear map

$$\begin{aligned} W_i &\longrightarrow (\mathrm{Hom}(W_j, W_k))[\log x]\{x\}\ , \\ w_{(i)} &\longmapsto \mathcal{Y}_{ij}^k(w_{(i)}, x) = \sum_{m \in \mathbb{C}} \sum_{a \in \mathbb{N}} (w_{(i)})_{m,a}^{\mathcal{Y}} x^{-m-1} (\log x)^a\ . \end{aligned}$$

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The dimensions of the spaces of all intertwining operators $\mathcal{Y}^k_{ij}$ are called the **fusion rules** N^k_{ij}.

Nonmeromorphic Operator Product Expansion

Theorem. Let V be a vertex operator algebra. Given two logarithmic intertwining maps $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of type $\binom{W_4}{W_1\ M}$ and $\binom{M}{W_2\ W_3}$, there exists a logarithmic intertwining map $\mathcal{Y}$ of type $\binom{W_4}{W_1 \boxtimes_{P(z_1-z_2)} W_2\ W_3}$ such that

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Huang, Lepowsky 1995; Huang 1995, 2002; Buhl 2002; Huang, Lepowsky, Zhang 2003

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(3) Every object which is a finitely generated lower-truncated generalized V-module, except that it may have infinite-dimensional homogeneous subspaces, is an object in $\mathcal{C}$.

Triplet Algebras

An infinite family of **logarithmic conformal field theories** can be shown to satisfy the conditions: the **triplet algebras** $\{\mathcal{W}(2,(2p-1)^{\times 3})\}_{p\geq 2}{}^{\star}$ with central charge $c_{p,1} = 1 - 6(p-1)^2/p$.

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Definition. A **$\mathcal{W}$-algebra** of type $\mathcal{W}(2, h_1, \ldots, h_m)$ is a vertex operator algebra which has a minimal generating set consisting of the vacuum Ω, the conformal vector ω of weight 2 and m additional primary vectors W^i of weight h_i, $i \in \{1, \ldots, m\}$, with *all singular vectors divided out*.

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Definition. A **$\mathcal{W}$-algebra** of type $\mathcal{W}(2, h_1, \ldots, h_m)$ is a vertex operator algebra which has a minimal generating set consisting of the vacuum Ω, the conformal vector ω of weight 2 and m additional primary vectors W^i of weight h_i, $i \in \{1, \ldots, m\}$, with *all singular vectors divided out*.

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C_2-cofiniteness is easily proven for the first triplet algebra with $p = 2$, as all relevant commutators and singular vectors are explicitly known:*

$$\begin{aligned} N^{ab} = & W^a_{-3} W^b_{-3} \Omega - \delta_{ab} \left(\frac{8}{9} L^3_{-2} + \frac{19}{36} L^2_{-3} + \frac{14}{9} L_{-4} L_{-2} - \frac{16}{9} L_{-6} \right) \Omega \\ & + \mathrm{i}\varepsilon_{abc} \left(-2 W^c_{-4} L_{-2} + \frac{5}{4} W^c_{-6} \right) \Omega \,. \end{aligned}$$

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⋆Flohr 1995

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  - ▷ do a lot of careful calculations!

Main result: C_2-cofiniteness

Theorem. For all $p \in \mathbb{Z}_{\geq 2}$, the nonmeromorphic operator product expansion exists and is associative for the vertex operator algebra $\mathcal{W}(2,(2p-1)^{\times 3})$. Furthermore, all these vertex operator algebras are C_2-cofinite.

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