Four-Point Functions in LCFT

Crossing Symmetry and SL(2,$\mathbb{C}$) covariance

Michael Flohr

Physics Institute
University of Bonn

Marco Krohn

Institute for Theoretical Physics
University of Hannover

Global Conformal Invariance

Ward Identities

- Correlation functions have to satisfy the *global conformal Ward identities*, i.e. for $m = -1, 0, 1$ we must have

$$\begin{aligned} 0 &= L_m \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle \\ &= \sum_{i=1}^{n} z_i^m \left[z_i \partial_i + (m+1)(h_i + \hat{\delta}_{h_i}) \right] \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle \ . \end{aligned}$$

Ward Identities

- Correlation functions have to satisfy the *global conformal Ward identities*, i.e. for $m = -1, 0, 1$ we must have

$$\begin{aligned} 0 &= L_m \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle \\ &= \sum_{i=1}^{n} z_i^m \left[z_i \partial_i + (m+1)(h_i + \hat{\delta}_{h_i}) \right] \langle \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle \,. \end{aligned}$$

- In case of rank $r > 1$ Jordan cells of indecomposable representations with respect to *Vir*, we have

$$\hat{\delta}_{h_i} \Psi_{(h_j;k_j)} = \begin{cases} \delta_{i,j} \Psi_{(h_j;k_j-1)} & \text{if} \quad 1 \leq k_j \leq r-1 \,, \\ 0 & \text{if} \quad k_j = 0 \,. \end{cases}$$

Ward Identities

- Correlation functions have to satisfy the *global conformal Ward identities*, i.e. for $m = -1, 0, 1$ we must have

$$\begin{aligned} 0 &= L_m \langle \Psi_1(z_1) \dots \Psi_n(z_n) \rangle \\ &= \sum_{i=1}^{n} z_i^m \left[z_i \partial_i + (m+1)(h_i + \hat{\delta}_{h_i}) \right] \langle \Psi_1(z_1) \dots \Psi_n(z_n) \rangle \ . \end{aligned}$$

- In case of rank $r > 1$ Jordan cells of indecomposable representations with respect to *Vir*, we have

$$\hat{\delta}_{h_i} \Psi_{(h_j;k_j)} = \begin{cases} \delta_{i,j} \Psi_{(h_j;k_j-1)} & \text{if} \quad 1 \le k_j \le r-1 \,, \\ 0 & \text{if} \quad k_j = 0 \,. \end{cases}$$

- Equivalently, $L_0|h;k\rangle = h|h;k\rangle + (1 - \delta_{k,0})|h;k-1\rangle$.

Recurrences of Inhomogeneities

- Ward identities become *inhomogeneous* in LCFT. The inhomogeneities are given by correlation functions with *total Jordan-level* $K = \sum_{i=1}^{n} k_i$ decreased by one,

$$\begin{aligned}
\left\langle \Psi_{(h_1;k_1)}(z_1)\ldots\Psi_{(h_n;k_n)}(z_n)\right\rangle &\equiv \langle k_1 k_2 \ldots k_n\rangle \,, \\
\frac{1}{(m+1)} L'_m \langle k_1 k_2 \ldots k_n\rangle &= -\, z_1^m \langle k_1 - 1, k_2 \ldots k_n\rangle \\
&\quad -\, z_2^m \langle k_1, k_2 - 1, k_3 \ldots k_n\rangle \\
&\quad -\, \ldots \\
&\quad -\, z_n^m \langle k_1 \ldots k_{n-1}, k_n - 1\rangle \,.
\end{aligned}$$

Recurrences of Inhomogeneities

- Ward identities become *inhomogeneous* in LCFT. The inhomogeneities are given by correlation functions with *total Jordan-level* $K = \sum_{i=1}^{n} k_i$ decreased by one,

$$\begin{aligned}
\left\langle \Psi_{(h_1;k_1)}(z_1)\ldots\Psi_{(h_n;k_n)}(z_n)\right\rangle &\equiv \langle k_1 k_2 \ldots k_n\rangle \,, \\
\frac{1}{(m+1)} L'_m \langle k_1 k_2 \ldots k_n\rangle &= -\, z_1^m \langle k_1 - 1, k_2 \ldots k_n\rangle \\
&\quad -\, z_2^m \langle k_1, k_2 - 1, k_3 \ldots k_n\rangle \\
&\quad -\, \ldots \\
&\quad -\, z_n^m \langle k_1 \ldots k_{n-1}, k_n - 1\rangle \,.
\end{aligned}$$

- We obtain a hierarchical scheme of solutions, starting with correlators of total Jordan-level $K = r - 1$.

Correlators

- Generic form of 1, 2 and 3pt functions for fields forming Jordan cells, pre-logarithmic fields and fermionic fields in arbitrary rank r LCFT uniquely fixed:

Correlators

- Generic form of 1, 2 and 3pt functions for fields forming Jordan cells, pre-logarithmic fields and fermionic fields in arbitrary rank r LCFT uniquely fixed:

$$\langle\Psi_{(h;k)}\rangle = \delta_{h,0}\delta_{k,r-1}\,,$$

$$\langle\Psi_{(h;k)}(z)\Psi_{(h';k')}(0)\rangle = \delta_{hh'}\sum_{j=r-1}^{k+k'} D_{(h;j)} \sum_{\substack{0\leq i\leq k,0\leq i'\leq k'\\ i+i'=k+k'-j}} \frac{(\partial_h)^i}{i!}\frac{(\partial_{h'})^{i'}}{i'!} z^{-h-h'}\,,$$

$$\langle\Psi_{(h_1;k_1)}(z_1)\Psi_{(h_2;k_2)}(z_2)\Psi_{(h_3;k_3)}(z_3)\rangle = \sum_{j=r-1}^{k_1+k_2+k_3} C_{(h_1h_2h_3;j)}$$

$$\times \sum_{\substack{0\leq i_l\leq k_l,l=1,2,3\\ i_1+i_2+i_3=k_1+k_2+k_3-j}} \frac{(\partial_{h_1})^{i_1}}{i_1!}\frac{(\partial_{h_2})^{i_2}}{i_2!}\frac{(\partial_{h_3})^{i_3}}{i_3!} \prod_{\substack{\sigma\in S_3\\ \sigma(1)<\sigma(2)}} (z_{\sigma(1)\sigma(2)})^{h_{\sigma(3)}-h_{\sigma(1)}-h_{\sigma(2)}}\,.$$

Four-Point Functions

Beyond 3pt Functions

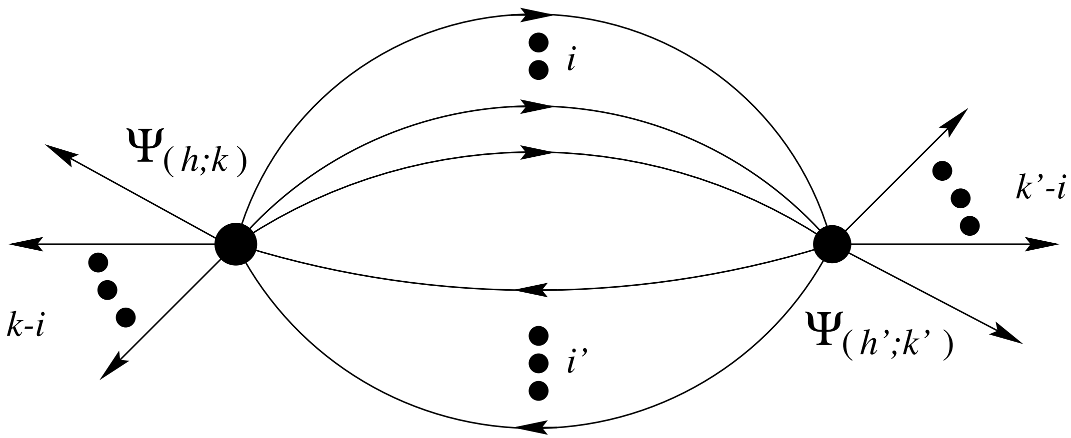
- To find a useful algorithm to fix the generic form of 4pt functions, visualize a logarithmic field $\Psi_{(h;k)}$ by a vertex with k outgoing lines.

Beyond 3pt Functions

- To find a useful algorithm to fix the generic form of 4pt functions, visualize a logarithmic field $\Psi_{(h;k)}$ by a vertex with k outgoing lines.
- Contractions of logarithmic fields give rise to logarithms in the correlators. The possible powers with which $\log(z_{ij})$ may occur, can be determined by graph combinatorics.

Beyond 3pt Functions

- To find a useful algorithm to fix the generic form of 4pt functions, visualize a logarithmic field $\Psi_{(h;k)}$ by a vertex with k outgoing lines.
- Contractions of logarithmic fields give rise to logarithms in the correlators. The possible powers with which $\log(z_{ij})$ may occur, can be determined by graph combinatorics.



Graphs

- Terms of generic form of n-pt function given by sum over all admissible graphs subject to the rules:

Graphs

- Terms of generic form of n-pt function given by sum over all admissible graphs subject to the rules:
  - Each k_{out}-vertex may receive $k'_{\text{in}} \leq (r-1)$ lines.

Graphs

- Terms of generic form of n-pt function given by sum over all admissible graphs subject to the rules:
  - Each k_{out}-vertex may receive $k'_{\text{in}} \leq (r-1)$ lines.
  - Vertices with $k_{\text{out}} = 0$ (proper primary fiels) do not receive any legs.

Graphs

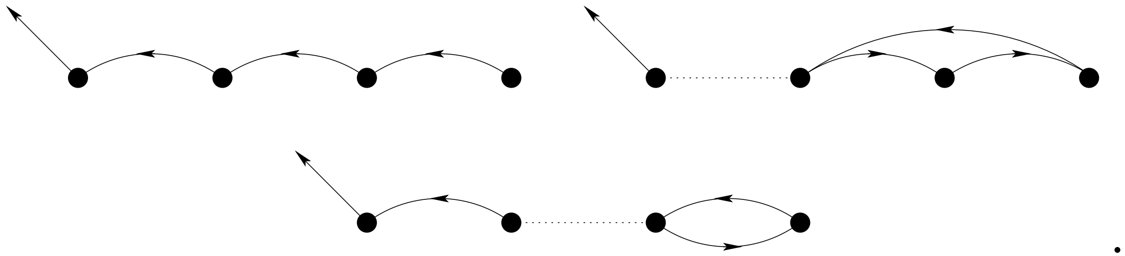
- Terms of generic form of n-pt function given by sum over all admissible graphs subject to the rules:
  - Each k_{out}-vertex may receive $k'_{\text{in}} \leq (r-1)$ lines.
  - Vertices with $k_{\text{out}} = 0$ (proper primary fiels) do not receive any legs.
  - Vertex i can receive legs from vertex j only for $j \neq i$.

Graphs

- Terms of generic form of n-pt function given by sum over all admissible graphs subject to the rules:
  - Each k_{out}-vertex may receive $k'_{\text{in}} \leq (r-1)$ lines.
  - Vertices with $k_{\text{out}} = 0$ (proper primary fiels) do not receive any legs.
  - Vertex i can receive legs from vertex j only for $j \neq i$.
  - Precisely $r-1$ lines in correlator remain open.

Graphs

- Terms of generic form of n-pt function given by sum over all admissible graphs subject to the rules:
  - Each k_{out}-vertex may receive $k'_{\text{in}} \leq (r-1)$ lines.
  - Vertices with $k_{\text{out}} = 0$ (proper primary fiels) do not receive any legs.
  - Vertex i can receive legs from vertex j only for $j \neq i$.
  - Precisely $r - 1$ lines in correlator remain open.
- **Example:** 4pt function for $r = 2$ and all fields logarithmic yields, upto permutations, the graphs



The Algorithm

- Linking numbers $A_{ij}(g)$ of given graph g yield *upper bounds* for power with which logarithms occur.

The Algorithm

- Linking numbers $A_{ij}(g)$ of given graph g yield *upper bounds* for power with which logarithms occur.
- **Recursive procedure:** start with all ways f_i to choose $r-1$ free legs, find at each level K' and for each configuration f_i all graphs, which connect the remaining $K - K' - (r-1)$ legs to vertices.

The Algorithm

- Linking numbers $A_{ij}(g)$ of given graph g yield *upper bounds* for power with which logarithms occur.
- **Recursive procedure:** start with all ways f_i to choose $r-1$ free legs, find at each level K' and for each configuration f_i all graphs, which connect the remaining $K - K' - (r-1)$ legs to vertices.
- Write down corresponding monomial in $\log(z_{ij})$, multiplied with an as yet undetermined constant $C(g)$ for each graph g.

The Algorithm

- Linking numbers $A_{ij}(g)$ of given graph g yield *upper bounds* for power with which logarithms occur.
- **Recursive procedure:** start with all ways f_i to choose $r-1$ free legs, find at each level K' and for each configuration f_i all graphs, which connect the remaining $K - K' - (r-1)$ legs to vertices.
- Write down corresponding monomial in $\log(z_{ij})$, multiplied with an as yet undetermined constant $C(g)$ for each graph g.
- Determine some constants by imposing global conformal invariance.

The Algorithm

- Linking numbers $A_{ij}(g)$ of given graph g yield *upper bounds* for power with which logarithms occur.
- **Recursive procedure:** start with all ways f_i to choose $r-1$ free legs, find at each level K' and for each configuration f_i all graphs, which connect the remaining $K - K' - (r-1)$ legs to vertices.
- Write down corresponding monomial in $\log(z_{ij})$, multiplied with an as yet undetermined constant $C(g)$ for each graph g.
- Determine some constants by imposing global conformal invariance.
- Fix further constants by imposing admissible permutation symmetries among the Jordan-levels k_i.

Generic Form

- Generic form of the LCFT 4pt functions is

$$\langle k_1 k_2 k_3 k_4 \rangle \equiv \langle \Psi_{(h_1;k_1)}(z_1) \ldots \Psi_{(h_4;k_4)}(z_4) \rangle =$$

$$\prod_{i<j} (z_{ij})^{\mu_{ij}} \sum_{(k'_1,k'_2,k'_3,k'_4)} \left[\sum_{g \in G_{K-K'}} C(g) \left(\prod_{i<j} \log^{A_{ij}(g)}(z_{ij}) \right) \right] F_{k'_1 k'_2 k'_3 k'_4}(x) \,,$$

where

Generic Form

- Generic form of the LCFT 4pt functions is

$$\langle k_1 k_2 k_3 k_4 \rangle \equiv \langle \Psi_{(h_1;k_1)}(z_1) \ldots \Psi_{(h_4;k_4)}(z_4) \rangle =$$

$$\prod_{i<j} (z_{ij})^{\mu_{ij}} \sum_{(k'_1,k'_2,k'_3,k'_4)} \left[\sum_{g \in G_{K-K'}} C(g) \left(\prod_{i<j} \log^{A_{ij}(g)}(z_{ij}) \right) \right] F_{k'_1 k'_2 k'_3 k'_4}(x) \,,$$

where

  - $G_{K-K'}$ is set of graphs for $(k_1 - k'_1, \ldots, k_4 - k'_4)$,

Generic Form

- Generic form of the LCFT 4pt functions is

$$\langle k_1 k_2 k_3 k_4 \rangle \equiv \langle \Psi_{(h_1;k_1)}(z_1) \dots \Psi_{(h_4;k_4)}(z_4) \rangle =$$

$$\prod_{i<j} (z_{ij})^{\mu_{ij}} \sum_{(k'_1,k'_2,k'_3,k'_4)} \left[\sum_{g \in G_{K-K'}} C(g) \left(\prod_{i<j} \log^{A_{ij}(g)}(z_{ij}) \right) \right] F_{k'_1 k'_2 k'_3 k'_4}(x) \,,$$

where

- $G_{K-K'}$ is set of graphs for $(k_1 - k'_1, \dots, k_4 - k'_4)$,
- $A_{ij}(g)$ is linking number of vertices i, j of graph g,

Generic Form

- Generic form of the LCFT 4pt functions is

$$\langle k_1 k_2 k_3 k_4 \rangle \equiv \langle \Psi_{(h_1;k_1)}(z_1) \ldots \Psi_{(h_4;k_4)}(z_4) \rangle =$$

$$\prod_{i<j} (z_{ij})^{\mu_{ij}} \sum_{(k'_1,k'_2,k'_3,k'_4)} \left[\sum_{g \in G_{K-K'}} C(g) \left(\prod_{i<j} \log^{A_{ij}(g)}(z_{ij}) \right) \right] F_{k'_1 k'_2 k'_3 k'_4}(x) \,,$$

where

- $G_{K-K'}$ is set of graphs for $(k_1 - k'_1, \ldots, k_4 - k'_4)$,
- $A_{ij}(g)$ is linking number of vertices i, j of graph g,
- x is the crossing ratio $x = \frac{z_{12} z_{34}}{z_{14} z_{23}}$,

Generic Form

- Generic form of the LCFT 4pt functions is

$$\langle k_1 k_2 k_3 k_4 \rangle \equiv \langle \Psi_{(h_1;k_1)}(z_1) \ldots \Psi_{(h_4;k_4)}(z_4) \rangle =$$

$$\prod_{i<j} (z_{ij})^{\mu_{ij}} \sum_{(k'_1,k'_2,k'_3,k'_4)} \left[\sum_{g \in G_{K-K'}} C(g) \left(\prod_{i<j} \log^{A_{ij}(g)}(z_{ij}) \right) \right] F_{k'_1 k'_2 k'_3 k'_4}(x) \,,$$

where

- $G_{K-K'}$ is set of graphs for $(k_1 - k'_1, \ldots, k_4 - k'_4)$,
- $A_{ij}(g)$ is linking number of vertices i, j of graph g,
- x is the crossing ratio $x = \frac{z_{12} z_{34}}{z_{14} z_{23}}$,
- μ_{ij} is typically $\mu_{ij} = \frac{1}{3} \left(\sum_k h_k \right) - h_i - h_j$.

Simplification and Recursion

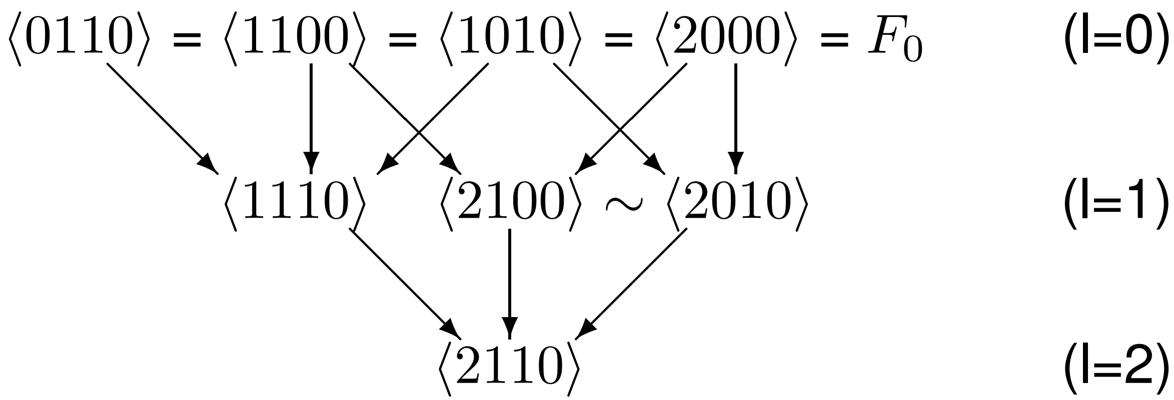
- Generic structure $\Longrightarrow$ set $h_i = 0$.
- L_{-1} is translation invariance $\Longrightarrow z_{ij}$.
- Initial conditions: $\langle k_1 k_2 k_3 k_4 \rangle = F_0(x)$ for $\sum_i k_i = r - 1$
- and $\langle k_1 k_2 k_3 k_4 \rangle = 0$ for $\sum_i k_i < r - 1$.

$$\begin{aligned} O_0 \langle \ldots \rangle &\equiv \sum_i z_i \partial_i \langle \ldots \rangle = -\sum_i \hat{\delta}_{h_i} \langle \ldots \rangle \\ O_1 \langle \ldots \rangle &\equiv \sum_i z_i^2 \partial_i \langle \ldots \rangle = -2 \sum_i z_i \hat{\delta}_{h_i} \langle \ldots \rangle \end{aligned}$$

Simplification and Recursion

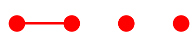
- Generic structure $\Longrightarrow$ set $h_i = 0$.
- L_{-1} is translation invariance $\Longrightarrow z_{ij}$.
- Initial conditions: $\langle k_1 k_2 k_3 k_4 \rangle = F_0(x)$ for $\sum_i k_i = r - 1$
- and $\langle k_1 k_2 k_3 k_4 \rangle = 0$ for $\sum_i k_i < r - 1$.

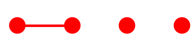
$$
\begin{aligned}
O_0 \langle \ldots \rangle &\equiv \sum_i z_i \partial_i \langle \ldots \rangle = -\sum_i \hat{\delta}_{h_i} \langle \ldots \rangle \\
O_1 \langle \ldots \rangle &\equiv \sum_i z_i^2 \partial_i \langle \ldots \rangle = -2 \sum_i z_i \hat{\delta}_{h_i} \langle \ldots \rangle
\end{aligned}
$$

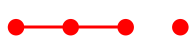
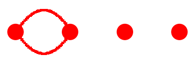


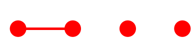
Rank $r = 2$

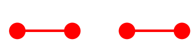
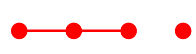
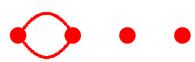
$$\langle 1000 \rangle = F_0$$

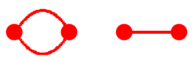
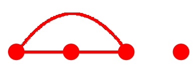
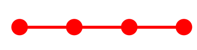
$\langle 1100 \rangle = \mathcal{P}_{S_2} \Big\{ \frac{1}{2} F_{1100} -$  $F_0 \Big\}$

$\langle 1110 \rangle = \mathcal{P}_{S_3} \Big\{ \frac{1}{6} F_{1110} + (\frac{1}{2} P_{(13)} - 1)$  $F_{0110} +$

$\Big[$  $- \frac{1}{2}$  $\Big] F_0 \Big\}$

$\langle 1111 \rangle = \mathcal{P}_{S_4} \Big\{ \frac{1}{24} F_{1111} + (\frac{1}{6} P_{(13)} - \frac{1}{3})$  $F_{0111} +$

$\Big[\frac{1}{2} (P_{(24)} - 1)$  $+ (1 - \frac{1}{2} P_{(14)})$  $- \frac{1}{4}$  $\Big] F_{0011} +$

$\Big[\frac{1}{2}$  $+ \frac{1}{3}$  $-$  $\Big] F_0 \Big\} .$

Redundancy

Interestingly, there remain *free constants*, when all four fields are logarithmic!

$$\begin{aligned}
\langle 1111\rangle \;=\;& F_{1111} + \mathcal{P}_{(1234)}\Big\{\big[(-\ell_{12}-\ell_{34}+\ell_{23}+\ell_{14})C_1 + (\ell_{13}+\ell_{24}-\ell_{12}-\ell_{34})C_2 \\
&\qquad - \ell_{14}+\ell_{34}-\ell_{13})\big]F_{0111}\Big\} \\
+\;& \mathcal{P}_{(12)(34)}\Big\{\big[(\ell_{13}^2+\ell_{24}^2-\ell_{14}^2-\ell_{23}^2+2(-\ell_{34}\ell_{24}-\ell_{12}\ell_{24}+\ell_{34}\ell_{14}+\ell_{13}\ell_{24} \\
&\quad -\ell_{13}\ell_{34}+\ell_{23}\ell_{34}+\ell_{12}\ell_{23}-\ell_{12}\ell_{13}-\ell_{23}\ell_{14}+\ell_{12}\ell_{14}))C_3 \\
&\; + (-(\ell_{23}+\ell_{14})^2+\ell_{23}\ell_{34}+\ell_{12}\ell_{14}-\ell_{13}\ell_{34}+\ell_{34}\ell_{14}+\ell_{13}\ell_{14} \\
&\quad -\ell_{34}\ell_{24}-\ell_{12}\ell_{13}-\ell_{12}\ell_{24}+\ell_{23}\ell_{24}+\ell_{23}\ell_{13}+\ell_{12}\ell_{23}+\ell_{24}\ell_{14}))C_4 \\
&\; -\ell_{34}^2-\ell_{23}^2-\ell_{14}^2+2\ell_{23}\ell_{34}+2\ell_{34}\ell_{14}-2\ell_{12}\ell_{34}-\ell_{23}\ell_{14}+\ell_{23}\ell_{24} \\
&\quad -\ell_{12}\ell_{13}+\ell_{12}\ell_{14}+\ell_{12}\ell_{23}-\ell_{12}\ell_{24}+\ell_{13}\ell_{14}+\ell_{13}\ell_{24})\big]F_{1100}\Big\} \\
+\;& \big[2(\ell_{12}\ell_{24}\ell_{14}-\ell_{23}\ell_{13}\ell_{14}+\ell_{23}\ell_{34}\ell_{24}-\ell_{24}\ell_{13}\ell_{34}-\ell_{23}\ell_{34}\ell_{14} \\
&\quad -\ell_{12}\ell_{23}\ell_{34}-\ell_{12}\ell_{34}\ell_{24}-\ell_{23}\ell_{13}\ell_{24}+\ell_{12}\ell_{23}\ell_{13}+\ell_{13}\ell_{34}\ell_{14} \\
&\quad -\ell_{13}\ell_{14}\ell_{24}-\ell_{23}\ell_{24}\ell_{14}-\ell_{12}\ell_{13}\ell_{24}-\ell_{12}\ell_{23}\ell_{14}-\ell_{12}\ell_{13}\ell_{34} \\
&\quad -\ell_{12}\ell_{34}\ell_{14}) \\
&\; + 2(\ell_{13}^2\ell_{24}+\ell_{12}^2\ell_{34}+\ell_{14}^2\ell_{23}+\ell_{23}^2\ell_{14}+\ell_{34}^2\ell_{12}+\ell_{24}^2\ell_{13})\big]F_0
\end{aligned}$$

Symmetries & Graphs

Additional Constants

$$\begin{aligned} O_0 \langle k_1 k_2 k_3 k_4 \rangle &= -\sum_i \hat{\delta}_{h_i} \langle k_1 k_2 k_3 k_4 \rangle \,, \\ O_1 \langle k_1 k_2 k_3 k_4 \rangle &= -2 \sum_i z_i \hat{\delta}_{h_i} \langle k_1 k_2 k_3 k_4 \rangle \,. \end{aligned}$$

Are there any $f \in \mathcal{F}_{\log}$ with $O_0 f = 0$ and $O_1 f = 0$?

Additional Constants

$$
\begin{aligned}
O_0 \langle k_1 k_2 k_3 k_4 \rangle &= -\sum_i \hat{\delta}_{h_i} \langle k_1 k_2 k_3 k_4 \rangle \ , \\
O_1 \langle k_1 k_2 k_3 k_4 \rangle &= -2 \sum_i z_i \hat{\delta}_{h_i} \langle k_1 k_2 k_3 k_4 \rangle \ .
\end{aligned}
$$

Are there any $f \in \mathcal{F}_{\log}$ with $O_0 f = 0$ and $O_1 f = 0$?

$$
\ker_{\mathcal{F}_{\log,g}} O = \left\{ \sum_{i=0}^{g} a_i {K_1}^i {K_2}^{g-i} : a_k \in \mathbb{R} \right\}
$$

$$
K_1 \equiv l_{12} + l_{34} - l_{13} - l_{24} = \log|x| - \log|1-x|
$$

$$
K_2 \equiv l_{12} + l_{34} - l_{14} - l_{23} = \log|x| \ .
$$

This means that all four fields have to be of logarithmic type. Symmetry considerations and combinatorical restrictions in fact constrain the number of additional constants further.

Additional Constants

$$O_0 \langle k_1 k_2 k_3 k_4 \rangle = -\sum_i \hat{\delta}_{h_i} \langle k_1 k_2 k_3 k_4 \rangle \,,$$
$$O_1 \langle k_1 k_2 k_3 k_4 \rangle = -2\sum_i z_i \hat{\delta}_{h_i} \langle k_1 k_2 k_3 k_4 \rangle \,.$$

Are there any $f \in \mathcal{F}_{\log}$ with $O_0 f = 0$ and $O_1 f = 0$?

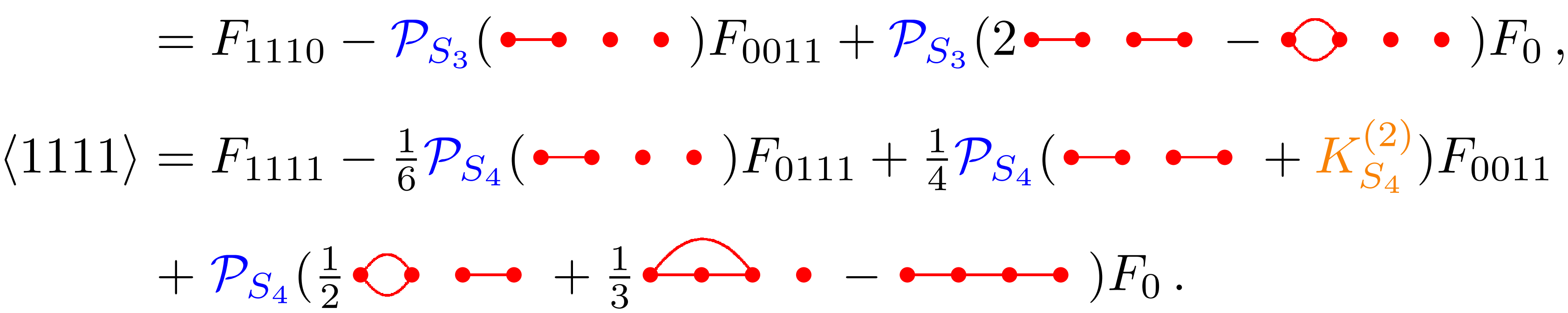
$$\ker_{\mathcal{F}_{\log,g}} O = \left\{ \sum_{i=0}^{g} a_i {K_1}^i {K_2}^{g-i} : a_k \in \mathbb{R} \right\}$$
$$K_1 \equiv l_{12} + l_{34} - l_{13} - l_{24} = \log|x| - \log|1-x|$$
$$K_2 \equiv l_{12} + l_{34} - l_{14} - l_{23} = \log|x| \,.$$

This means that all four fields have to be of logarithmic type. Symmetry considerations and combinatorical restrictions in fact constrain the number of additional constants further.

$$\mathrm{Ker}_{\langle 1111 \rangle} = \mathcal{P}_{S_4} \left\{ ({K_1}^2 - K_1 K_2 + {K_2}^2) F_{0011} \right\} .$$

- Now, the final results simply read

$$\langle 1110\rangle = F_{1110} - \mathcal{P}_{S_3}(\ell_{12})F_{0011} + \mathcal{P}_{S_3}(2\ell_{12}\ell_{23} - \ell_{12}^2)F_0$$



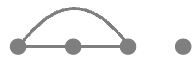
Graphical Solution

- Now, the final results simply read

$$\langle 1110 \rangle = F_{1110} - \mathcal{P}_{S_3}(\ell_{12}) F_{0011} + \mathcal{P}_{S_3}(2\ell_{12}\ell_{23} - \ell_{12}^2) F_0$$

$= F_{1110} - \mathcal{P}_{S_3}($  $) F_{0011} + \mathcal{P}_{S_3}(2$  $-$  $) F_0 \,,$

$\langle 1111 \rangle = F_{1111} - \frac{1}{6}\mathcal{P}_{S_4}($  $) F_{0111} + \frac{1}{4}\mathcal{P}_{S_4}($  $+ K_{S_4}^{(2)}) F_{0011}$

$+ \mathcal{P}_{S_4}(\frac{1}{2}$  $+ \frac{1}{3}$  $-$  $) F_0 \,.$

- Note the appearance of a term $K \in \ker L_m^{\text{offdiag}}$ in the kernel of the nilpotent part of the Virasoro generators:

$$\ker(L_m - L'_m) = \langle K_1 \equiv \log(x),\ K_2 \equiv -\log(1 - 1/x) \rangle$$
$$= \langle \ell_{12} + \ell_{34} - \ell_{14} - \ell_{23},\ \ell_{12} + \ell_{34} - \ell_{13} - \ell_{24} \rangle \;;$$
$$K_{S_4}^{(2)} = K_1^2 - K_1 K_2 + K_2^2 \,.$$

Rank $r = 3$

Again, if all $k_i > 0$, free constants remain:

$$\begin{aligned}
\langle 1111 \rangle \quad = \quad & F_{1111} \\
+ \quad & \mathcal{P}_{(1234)} \Big\{ \big[(\ell_{13} - \ell_{12} + \ell_{24} - \ell_{34}) C_1 \\
& \qquad + (\ell_{23} + \ell_{14} - \ell_{34} - \ell_{12}) C_2 - \ell_{14} + \ell_{24} - \ell_{12} \big] F_{0111} \Big\} \\
+ \quad & \Big[\big(\ell_{12}^2 + \ell_{24}^2 + \ell_{34}^2 + \ell_{13}^2 + 2(\ell_{12}\ell_{13} + \ell_{13}\ell_{24} - \ell_{13}\ell_{34} \\
& \quad - \ell_{34}\ell_{24} - \ell_{12}\ell_{24} + \ell_{12}\ell_{34}) \big) C_3 \\
& + \big(-2\ell_{13}\ell_{14} + \ell_{24}^2 + 2\ell_{23}\ell_{14} - 2\ell_{23}\ell_{24} + \ell_{23}^2 + 2\ell_{13}\ell_{24} \\
& \quad + \ell_{13}^2 - 2\ell_{23}\ell_{13} + \ell_{14}^2 - 2\ell_{24}\ell_{14} \big) C_4 \\
& + \big((\ell_{24} + \ell_{13})^2 \\
& \quad + \ell_{12}\ell_{14} - \ell_{23}\ell_{24} - \ell_{12}\ell_{24} - \ell_{24}\ell_{14} - \ell_{23}\ell_{13} + \ell_{34}\ell_{14} \\
& \quad - \ell_{13}\ell_{34} - \ell_{13}\ell_{14} + \ell_{23}\ell_{34} - \ell_{34}\ell_{24} + \ell_{12}\ell_{23} - \ell_{12}\ell_{13} \big) C_5 \\
& + 2(\ell_{13}\ell_{24} + \ell_{23}\ell_{14} + \ell_{12}\ell_{34}) \Big] F_0 \,.
\end{aligned}$$

Rank $r = 3$

Again, if all $k_i > 0$, free constants remain:

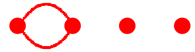
$$
\begin{aligned}
\langle 1111\rangle &= F_{1111} \\
&+ \mathcal{P}_{(1234)}\Big\{\big[(\ell_{13}-\ell_{12}+\ell_{24}-\ell_{34})C_1 \\
&\qquad + (\ell_{23}+\ell_{14}-\ell_{34}-\ell_{12})C_2 - \ell_{14}+\ell_{24}-\ell_{12}\big]F_{0111}\Big\} \\
&+ \Big[\big(\ell_{12}^2+\ell_{24}^2+\ell_{34}^2+\ell_{13}^2+2(\ell_{12}\ell_{13}+\ell_{13}\ell_{24}-\ell_{13}\ell_{34} \\
&\qquad -\ell_{34}\ell_{24}-\ell_{12}\ell_{24}+\ell_{12}\ell_{34})\big)C_3 \\
&\quad + \big(-2\ell_{13}\ell_{14}+\ell_{24}^2+2\ell_{23}\ell_{14}-2\ell_{23}\ell_{24}+\ell_{23}^2+2\ell_{13}\ell_{24} \\
&\qquad +\ell_{13}^2-2\ell_{23}\ell_{13}+\ell_{14}^2-2\ell_{24}\ell_{14}\big)C_4 \\
&\quad + \big((\ell_{24}+\ell_{13})^2 \\
&\qquad +\ell_{12}\ell_{14}-\ell_{23}\ell_{24}-\ell_{12}\ell_{24}-\ell_{24}\ell_{14}-\ell_{23}\ell_{13}+\ell_{34}\ell_{14} \\
&\qquad -\ell_{13}\ell_{34}-\ell_{13}\ell_{14}+\ell_{23}\ell_{34}-\ell_{34}\ell_{24}+\ell_{12}\ell_{23}-\ell_{12}\ell_{13}\big)C_5 \\
&\quad + 2(\ell_{13}\ell_{24}+\ell_{23}\ell_{14}+\ell_{12}\ell_{34})\Big]F_0\,.
\end{aligned}
$$

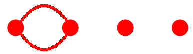
$$
\langle 1111\rangle = F_{1111} - \frac{1}{6}\mathcal{P}_{S_4}(\ell_{12})F_{0111} + \left\{\mathcal{P}_{S_4}\left[-\frac{1}{4}\ell_{34}^2+\frac{1}{2}\ell_{34}\ell_{24}\right]+K^{(2)}_{S_4}\right\}F_0\,.
$$

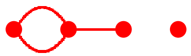
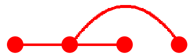
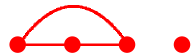
<2222> :: part I

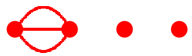
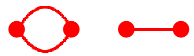
$\langle 2222 \rangle =$

$\mathcal{P}_{S_4}\Big\{ \frac{1}{24} F_{2222} + (\frac{1}{6} P_{(13)} - \frac{1}{3})$  $F_{1222} +$

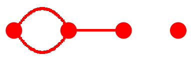
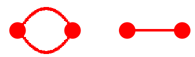
$\big[(\frac{1}{3} P_{(12)} + \frac{1}{6} P_{(14)})$  $- \frac{1}{3}$  $- \frac{1}{12} P_{(13)}$  $\big] F_{0222} +$

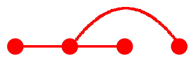
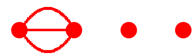
$\big[\frac{1}{2} P_{(24)}$  $+ (\frac{1}{2} P_{(23)} - P_{(24)})$  $+ \frac{1}{4} P_{(13)(24)}$  $\big] F_{1122} +$

$\big[(3P_{(34)} + 3 + P_{(14)})$  $- 5$  $- (\frac{13}{6} + \frac{5}{2} P_{(34)})$  $+$

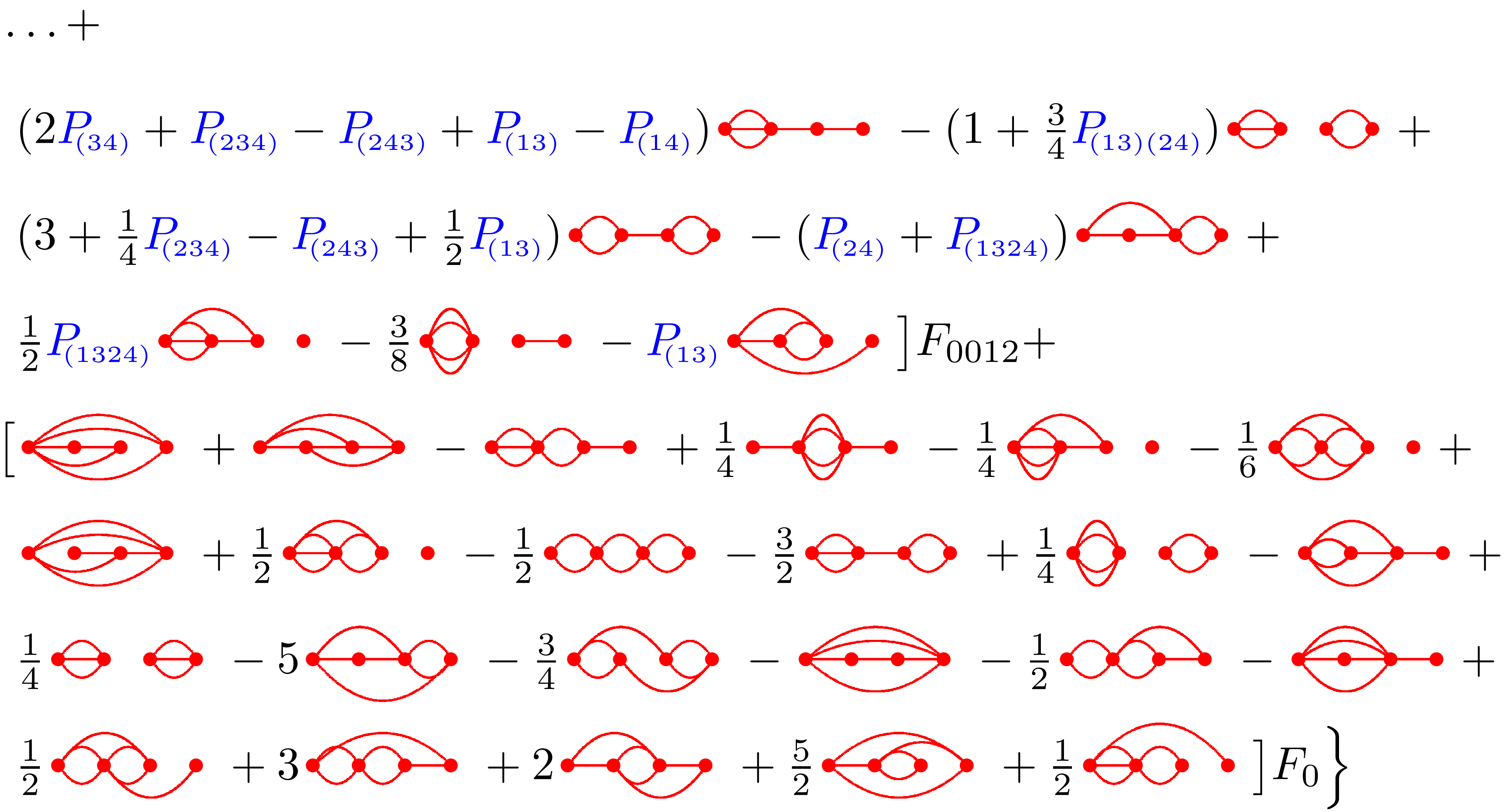
$(5 + 3P_{(24)})$  $-$  $- (3 + \frac{3}{2} P_{(14)})$  $) F_{1112} +$

$\big[\frac{1}{2} (P_{(124)} - 11 - 9P_{(12)} - 7P_{(123)} - P_{(132)} - 3P_{(142)} - 7P_{(14)} - 7P_{(13)(24)} +$

$P_{(13)} - 8P_{(14)(23)})$  $+ (5 + \frac{3}{2} P_{(24)} + 3P_{(14)} + \frac{9}{2} P_{(13)(24)})$  $+$

$(6 + 7P_{(23)} + 5P_{(12)})$  $+ (\frac{3}{2} + P_{(13)}) + \frac{5}{4} P_{(13)(24)})$  $+ \ldots$

<2222> :: part VI


$\ldots +$
$(2P_{(34)} + P_{(234)} - P_{(243)} + P_{(13)} - P_{(14)})$ $- (1 + \frac{3}{4}P_{(13)(24)})$ $+$
$(3 + \frac{1}{4}P_{(234)} - P_{(243)} + \frac{1}{2}P_{(13)})$ $- (P_{(24)} + P_{(1324)})$ $+$
$\frac{1}{2}P_{(1324)}$ $- \frac{3}{8}$ $- P_{(13)}$ $]F_{0012}+$
$[$ $+$ $-$ $+ \frac{1}{4}$ $- \frac{1}{4}$ $- \frac{1}{6}$ $+$
$+ \frac{1}{2}$ $- \frac{1}{2}$ $- \frac{3}{2}$ $+ \frac{1}{4}$ $-$ $+$
$\frac{1}{4}$ $- 5$ $- \frac{3}{4}$ $-$ $- \frac{1}{2}$ $-$ $+$
$\frac{1}{2}$ $+ 3$ $+ 2$ $+ \frac{5}{2}$ $+ \frac{1}{2}$ $]F_0\}$

Towards $c = 0$

- That is all very elegant, very short, and very simple ;−)

Towards $c = 0$

- That is all very elegant, very short, and very simple ;-)
- Remember initial conditions:
  - $\langle k_1 k_2 k_3 k_4 \rangle = 0 \quad \forall\, k_1 + k_2 + k_3 + k_4 < r - 1$,
  - $\langle k_1 k_2 k_3 k_4 \rangle = \langle k'_1 k'_2 k'_3 k'_4 \rangle \quad \forall\, k_1 + k_2 + k_3 + k_4 = k'_1 + k'_2 + k'_3 + k'_4 = r - 1$.
  - All $\Psi_{(h;0)}$ are proper primaries.

Towards $c = 0$

- That is all very elegant, very short, and very simple ;-)
- Remember initial conditions:
  - $\langle k_1 k_2 k_3 k_4 \rangle = 0 \quad \forall\, k_1 + k_2 + k_3 + k_4 < r - 1,$
  - $\langle k_1 k_2 k_3 k_4 \rangle = \langle k_1' k_2' k_3' k_4' \rangle \quad \forall\, k_1 + k_2 + k_3 + k_4 = k_1' + k_2' + k_3' + k_4' = r - 1.$
  - All $\Psi_{(h;0)}$ are proper primaries.
- Generalize to pre-logarithmic fields.

Towards $c = 0$

- That is all very elegant, very short, and very simple ;-)
- Remember initial conditions:
  - $\langle k_1 k_2 k_3 k_4 \rangle = 0 \quad \forall\, k_1 + k_2 + k_3 + k_4 < r - 1,$
  - $\langle k_1 k_2 k_3 k_4 \rangle = \langle k_1' k_2' k_3' k_4' \rangle \quad \forall\, k_1 + k_2 + k_3 + k_4 = k_1' + k_2' + k_3' + k_4' = r - 1.$
  - All $\Psi_{(h;0)}$ are proper primaries.
- Generalize to pre-logarithmic fields.
- Generalize to case where not all Jordan cells have the same rank, since $c = 0$ seems to be that complicated [Kogan,Nichols; Gurarie,Ludwig].

Towards $c = 0$

- That is all very elegant, very short, and very simple ;-)
- Remember initial conditions:
  - $\langle k_1 k_2 k_3 k_4 \rangle = 0 \quad \forall\, k_1 + k_2 + k_3 + k_4 < r - 1,$
  - $\langle k_1 k_2 k_3 k_4 \rangle = \langle k_1' k_2' k_3' k_4' \rangle \quad \forall\, k_1 + k_2 + k_3 + k_4 = k_1' + k_2' + k_3' + k_4' = r - 1.$
  - All $\Psi_{(h;0)}$ are proper primaries.
- Generalize to pre-logarithmic fields.
- Generalize to case where not all Jordan cells have the same rank, since $c = 0$ seems to be that complicated [Kogan,Nichols; Gurarie,Ludwig].
- Finally, allow for mixing of Jordan cells of different sizes [Nagi; Rasmussen].

Towards $c = 0$

- That is all very elegant, very short, and very simple ;-)
- Remember initial conditions:
  - $\langle k_1 k_2 k_3 k_4 \rangle = 0 \quad \forall\, k_1 + k_2 + k_3 + k_4 < r - 1$,
  - $\langle k_1 k_2 k_3 k_4 \rangle = \langle k'_1 k'_2 k'_3 k'_4 \rangle \quad \forall\, k_1 + k_2 + k_3 + k_4 = k'_1 + k'_2 + k'_3 + k'_4 = r - 1$.
  - All $\Psi_{(h;0)}$ are proper primaries.
- Generalize to pre-logarithmic fields.
- Generalize to case where not all Jordan cells have the same rank, since $c = 0$ seems to be that complicated [Kogan,Nichols; Gurarie,Ludwig].
- Finally, allow for mixing of Jordan cells of different sizes [Nagi; Rasmussen].
- What is the structure of the $h = 0$ sector in $c = 0$ theories?