Boundary States and Symplectic Fermions

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Abstract

We investigate the set of boundary states in the symplectic fermion description of the logarithmic conformal field theory with central charge c $= -2$. We show that the thus constructed states correspond exactly to those derived under the restrictions of the maximal chiral symmetry algebra for this model, $\mathcal{W}(2,3,3,3)$. This connects our previous work to the coherent state approach of Kawai and Wheater.

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1 Introduction

During the last 20 years, conformal field theory (CFT) in two dimensions [?] has become a very important tool in theoretical physics. Especially, two different directions are subject of current interest: The study of critical systems on surfaces involving boundaries led to a good knowledge of the so-called boundary CFTs (BCFT) [?, ?, ?, ?]. On the other hand, already in 1991, Saleur showed the existence of density fields with scaling dimension zero occurring in the treatment of dense polymers [?]. These fields may cause the existence of operators yielding logarithmically diverging correlation functions. The two subjects, BCFT and logarithmic conformal field theory (LCFT), enjoy increasing popularity in both condensed matter physics and string theory.

Even though there has been much progress in the field, LCFTs are not yet completely understood. However, it has been found that many properties of ordinary rational CFTs can be generalized to LCFT, such as characters, partition functions and fusion rules, see, e.g., [?, ?, ?, ?, ?, ?, ?, ?] and [?, ?] for some recent reviews. In ordinary CFTs, especially in unitary minimal models, the presence of a boundary is mathematically and physically described by a standard procedure introduced by Ishibashi [?] and Cardy [?] that allows to derive boundary states encoding the physical boundary conditions. Unfortunately, LCFTs involving a boundary happen to be more difficult to treat. There have been different approaches towards a consistent description of boundary LCFTs in terms of boundary states emerging first for two years ago [?, ?, ?, ?], see also [?, ?]. LCFT in the vicinity of a boundary is also dealt with in [?, ?]. All those works focus on the best understood example of a LCFT, the c $= -2$ realization with the maximally extended chiral symmetry algebra $\mathcal{W}(2, 3, 3, 3)$. The earlier results are different and partly contradictory. Most successful seem the ideas of Kawai and Wheater [?] using symplectic fermions and coherent states and of ourselves [?]. The concept of symplectic fermions was first introduced by Kausch [?] in order to describe the rational c $= -2$ (bulk) LCFT. In our own work, a general, very basic approach towards the derivation of boundary states in the case of the $\mathcal{W}$-algebra is presented that allows to handle complicated structures such as indecomposable representations in LCFTs.

This letter is positioned exactly at this point. We show that the two different symmetries – the symplectic fermions vs. $\mathcal{W}(2, 3, 3, 3)$-algebra – lead to the same set of boundary states. In particular, the former one, though extending the latter, implies no additional restrictions on the boundary states. By this, we can show that the coherent state approach is fully equivalent to ours yielding the same results. This corresponds to the presumption of Kawai [?] that the coherent states are indeed as good as taking the usual Ishibashi states.

The paper proceeds as follows: In section 2, a short introduction to the rational c $= -2$ LCFT is given both for the $\mathcal{W}(2, 3, 3, 3)$-algebra and in the symplectic fermion picture. Then, section 3 and 4 review the results of Kawai and Wheater and those deduced by us. In section 5, the boundary states for the symplectic fermion symmetry algebra are derived using the method of [?] and compared to both of the previous results. Finally, section 6 concludes the paper with a short discussion.

2 The model

The CFT realization at c = -2 is based on the extended chiral symmetry algebra $\mathcal{W}(2,3,3,3)$ consisting of the energy-momentum tensor $L(z)$ and a triplet of spin-3 fields $W^a(z)$. With the two quasi-primary normal-ordered fields $\Lambda =\,:L^2:\,-3/10\,\partial^2 L$ and $V^a =\,:LW^a:\,-3/14\,\partial^2 W^a$ the commutation relations for the corresponding modes read:

$$
\begin{aligned}
\big[L_m, L_n\big] &= (m-n)L_{m+n} - \frac{1}{6}\left(m^3 - m\right)\delta_{m+n,0}, \\
\big[L_m, W_n^a\big] &= (2m-n)W_{m+n}^a, \\
\big[W_m^a, W_n^b\big] &= \hat{g}^{ab}\left(2\,(m-n)\,\Lambda_{m+n} + \frac{1}{20}\,(m-n)\left(2m^2 + 2n^2 - mn - 8\right)L_{m+n}\right. \\
&\qquad\qquad \left. -\frac{1}{120}m\left(m^2 - 1\right)\left(m^2 - 4\right)\delta_{m+n,0}\right) \\
&\quad + \hat{f}_c^{ab}\left(\frac{5}{14}\left(2m^2 + 2n^2 - 3mn - 4\right)W_{m+n}^c + \frac{12}{5}V_{m+n}^c\right).
\end{aligned}
\tag{1}
$$

Here, $\hat{g}^{ab}$ is the metric and $\hat{f}_c^{ab}$ are the structure constants of $su(2)$. It is convenient to arrange the fields in a Cartan-Weyl basis W^0, $W^\pm$. In this framework, we have $\hat{g}^{00} = 1$, $\hat{g}^{+-} = \hat{g}^{-+} = 2$, $\hat{f}_\pm^{0\pm} = -\hat{f}_\pm^{\pm 0} = \pm 1$, and $\hat{f}_0^{+-} = -\hat{f}_0^{-+} = 2$.

The algebra yields a set of six representations that close under fusion. There are four ordinary highest weight representations: $\mathcal{V}_0$ is based on the vacuum state Ω with weight $h = 0$, $\mathcal{V}_{-1/8}$ emerges from the state μ with weight $h = -1/8$. Then, one has two doublet representations $\mathcal{V}_1$ based on the states $\phi^\pm$ and $\mathcal{V}_{3/8}$ built from $\nu^\pm$. Furthermore, two *indecomposable* or *generalized highest weight* representations $\mathcal{R}_0$ and $\mathcal{R}_1$ emerge. They base on the states ω and $\psi^\pm$, respectively. These states form rank-2 Jordan blocks in L_0 together with the states Ω and $\phi^\pm$. Thus, $\mathcal{V}_0$ and $\mathcal{V}_1$ are subrepresentations of $\mathcal{R}_0$ and $\mathcal{R}_1$, respectively. $\mathcal{R}_0$ also contains two subrepresentations of type $\mathcal{V}_1$ built on the states

$$
\begin{aligned}
\Psi_1^+ &= W_{-1}^+\omega, & \Psi_2^+ &= \left(W_{-1}^0 + \tfrac{1}{2}L_{-1}\right)\omega, \\
\Psi_1^- &= \left(-W_{-1}^0 + \tfrac{1}{2}L_{-1}\right)\omega, & \Psi_2^- &= W_{-1}^-\omega.
\end{aligned}
\tag{2}
$$

For the bulk states of the $\mathcal{R}_0$ and $\mathcal{R}_1$ we use the metric of [?] that reads:

$$
\begin{aligned}
\langle \Omega|\Omega\rangle &= 0, & \langle \Omega|\omega\rangle &= 1, & \langle \omega|\omega\rangle &= d, \\
\langle \phi^+|\phi^-\rangle &= 0, & \langle \phi^+|\psi^-\rangle &= -1, & \langle \psi^+|\psi^-\rangle &= -t,
\end{aligned}
\tag{3}
$$

where d and t are in principle arbitrary real numbers. The fusion rules for this model read:

$$
\begin{aligned}
\mathcal{V}_0 \times \Psi &= \Psi, & \mathcal{V}_{-1/8} \times \mathcal{V}_{-1/8} &= \mathcal{R}_0, & \mathcal{V}_{3/8} \times \mathcal{V}_1 &= \mathcal{V}_{-1/8}, \\
\mathcal{V}_1 \times \mathcal{V}_1 &= \mathcal{V}_0, & \mathcal{V}_{-1/8} \times \mathcal{V}_{3/8} &= \mathcal{R}_1, & \mathcal{V}_{-1/8} \times \mathcal{R}_m &= 2\,\mathcal{V}_{-1/8} + 2\,\mathcal{V}_{3/8}, \\
\mathcal{V}_1 \times \mathcal{R}_0 &= \mathcal{R}_1, & \mathcal{V}_{-1/8} \times \mathcal{V}_1 &= \mathcal{V}_{3/8}, & \mathcal{V}_{3/8} \times \mathcal{R}_m &= 2\,\mathcal{V}_{-1/8} + 2\,\mathcal{V}_{3/8}, \\
\mathcal{V}_1 \times \mathcal{R}_1 &= \mathcal{R}_0, & \mathcal{V}_{3/8} \times \mathcal{V}_{3/8} &= \mathcal{R}_0, & \mathcal{R}_m \times \mathcal{R}_n &= 2\,\mathcal{R}_0 + 2\,\mathcal{R}_1.
\end{aligned}
\tag{4}
$$

Here, m and n can take the values 0, 1. From (**??**) one reads off that $\{\mathcal{R}_0, \mathcal{R}_1, \mathcal{V}_{-1/8}, \mathcal{V}_{3/8}\}$ is a sub-group closed under fusion itself. The characters for the model are given by:

$$\chi_{\mathcal{V}_0}(q) = \frac{1}{2\eta(q)} \left(\Theta_{1,2}(q) + (\partial\Theta)_{1,2}(q)\right), \qquad \chi_{\mathcal{V}_{-1/8}}(q) = \frac{1}{\eta(q)}\Theta_{0,2}(q),$$

$$\chi_{\mathcal{V}_1}(q) = \frac{1}{2\eta(q)} \left(\Theta_{1,2}(q) - (\partial\Theta)_{1,2}(q)\right), \qquad \chi_{\mathcal{V}_{3/8}}(q) = \frac{1}{\eta(q)}\Theta_{2,2}(q), \qquad (5)$$

$$\chi_{\mathcal{R}}(q) \equiv \chi_{\mathcal{R}_0}(q) = \chi_{\mathcal{R}_1}(q) = \frac{2}{\eta(q)}\Theta_{1,2}(q).$$

Note, that the physical characters are only $\chi_{\mathcal{V}_{-1/8}}$, $\chi_{\mathcal{V}_{3/8}}$ and $\chi_{\mathcal{R}}$ forming a three-dimensional representation of the modular group that corresponds to the above-mentioned subgroup. Here, $\eta(q) = q^{1/24} \prod_{n \in N} (1 - q^n)$ is the Dedekind eta function and $\Theta_{r,2}(q)$ and $(\partial\Theta)_{1,2}(q)$ are the ordinary and affine Riemann-Jacobi theta functions:

$$\Theta_{r,k}(q) = \sum_{n \in \mathbb{Z}} q^{(2kn+r)^2/4k} \quad \text{and} \quad (\partial\Theta)_{r,k}(q) = \sum_{n \in \mathbb{Z}} (2kn + r)\, q^{(2kn+r)^2/4k}. \qquad (6)$$

In ordinary CFTs, the characters coincide with the torus amplitudes. Here, this is no longer the case: The torus amplitudes form a slightly larger, five-dimensional representation of the modular group. It reads:

$$\chi_{\mathcal{V}_0},\ \chi_{\mathcal{V}_{-1/8}},\ \chi_{\mathcal{V}_1},\ \chi_{\mathcal{V}_{3/8}},\ \text{and } \chi_{\widetilde{\mathcal{R}}}(q) \equiv \frac{2}{\eta(q)} \left(\Theta_{1,2}(q) + \mathrm{i}\alpha \log(q)\, (\partial\Theta)_{1,2}(q)\right). \qquad (7)$$

This representation was analyzed by Flohr [**?**]. There, the $\mathcal{S}$-matrix transforming the "characters" under $\tau \longrightarrow -1/\tau$ was constructed and it was shown that it yields the fusion rules (**??**) only in the limit $\alpha \longrightarrow 0$ under which the logarithmic term in (**??**) vanishes. However, in this limit, S became singular.

There exists an explicit Lagrangian formulation for the c $= -2$ LCFT based on two fermionic fields η and ξ of scaling dimension 1 and 0, respectively:

$$S = \frac{1}{\pi} \int \mathrm{d}^2z \left(\eta \bar{\partial}\xi + \bar{\eta}\partial\bar{\xi}\right). \qquad (8)$$

This is the fermionic ghost system at c $= -2$ with the operator product expansions

$$\eta(z)\xi(w) = \xi(z)\eta(w) = \frac{1}{z - w} + \dots . \qquad (9)$$

All other products are regular. Kausch [**?**] showed, that these two fields combine into a two-component symplectic fermion

$$\chi^+ \equiv \eta \quad \text{and} \quad \chi^- \equiv \partial\xi. \qquad (10)$$

The choice assures that χ^+ and χ^- have the same conformal weight $h = 1$. This description differs from the ghost system only by the treatment of the zero modes in χ^- and ξ. The fermion modes are defined by the usual power series expansion

$$\chi^\pm(z) = \sum_{m \in \mathbb{Z}+\lambda} \chi_m^\pm z^{-m-1}, \qquad (11)$$

where $\lambda = 0$ in the untwisted (bosonic) sector and $\lambda = \frac{1}{2}$ in the twisted (fermionic) sector. The modes satisfy the anticommutation relations

$$\left\{ \chi_m^\alpha, \chi_n^\beta \right\} = m \, \varepsilon^{\alpha\beta} \delta_{m+n,0} \,, \tag{12}$$

with the totally antisymmetric tensor $\varepsilon^{\pm\mp} = \pm 1$. The symplectic fermion decomposes the Virasoro modes and the modes of the three spin-3 fields $W(z)$ of $\mathcal{W}(2,3,3,3)$ [?, ?, ?]:

$$L_n = \frac{1}{2} \varepsilon_{\alpha\beta} \sum_{j \in \mathbb{Z}+\lambda} : \chi_j^\alpha \chi_{n-j}^\beta : + \frac{\lambda(\lambda-1)}{2} \delta_{n,0},$$

$$W_n^0 = -\frac{1}{2} \sum_{j \in \mathbb{Z}+\lambda} j \cdot \left\{ : \chi_{n-j}^+ \chi_j^- : + : \chi_{n-j}^- \chi_j^+ : \right\}, \tag{13}$$

$$W_n^\pm = \sum_{j \in \mathbb{Z}+\lambda} j \cdot \chi_{n-j}^\pm \chi_j^\pm.$$

The highest weight states become related to each other by introducing the fermion symmetry: In the twisted sector, the doublet states of weight $h = 3/8$ are connected to the singlet at weight $h = -1/8$ by $\nu^\alpha = \chi_{-1/2}^\alpha \mu$. The states of weight 0 in the untwisted sector are related by $\xi^\pm = -\chi_0^\pm \omega$, $\Omega = \chi_0^- \chi_0^+ \omega$. Furthermore, one finds $\phi^\alpha = \chi_{-1}^\alpha \Omega$ and $\psi^\alpha = \chi_{-1}^\alpha \omega$. Thus, this additional symmetry intertwines the representation $\mathcal{R}_0$ with $\mathcal{R}_1$ and $\mathcal{V}_{-1/8}$ with $\mathcal{V}_{3/8}$.

3   Approach 1: Coherent boundary states

Starting point for any derivation of boundary conditions is the absence of energy-momentum flow across the boundary and corresponding gluing conditions for the extended symmetry fields. On a cylinder, the boundary conditions are identified with an initial and final state of a propagating closed string: the *boundary states* $\left| B \right\rangle$. After radial ordering and in the framework of symplectic fermions this yields the following consistency equations:

$$\left(L_n - \overline{L}_{-n} \right) \left| B \right\rangle = 0, \tag{14}$$

$$\left(\chi_n^\pm - \mathrm{e}^{\pm \mathrm{i}\phi} \overline{\chi}_{-n}^\pm \right) \left| B \right\rangle = 0, \tag{15}$$

where ϕ is a phase that occurs in the gluing condition of χ and $\overline{\chi}$. The latter equation implies the first one due to (??). Kawai and Wheater showed that (??) is solved by the coherent states [?]

$$\left| B_{0\phi} \right\rangle = N \exp \left(\sum_{k>0} \frac{\mathrm{e}^{\mathrm{i}\phi}}{k} \chi_{-k}^- \overline{\chi}_{-k}^+ + \frac{\mathrm{e}^{-\mathrm{i}\phi}}{k} \overline{\chi}_{-k}^- \chi_{-k}^+ \right) \left| 0_\phi \right\rangle. \tag{16}$$

Here, N is a normalization factor and $\left| 0_\phi \right\rangle$ is a non-chiral ground state. The boundary states were designed in such a way that they are compatible with the $\mathcal{W}$-algebra and thus obey (??) and

$$\left(W_n^a + \overline{W}_{-n}^a \right) \left| B \right\rangle = 0. \tag{17}$$

4

This implies that the phase ϕ can only take the values $\phi = 0$ and $\phi = \pi$. Therefore, the non-chiral ground states are given by the "invariant vacua" $\left\{ \left(\Omega \otimes \overline{\Omega} \right), \left(\omega \otimes \overline{\omega} \right), \left(\mu \otimes \overline{\mu} \right) \right\}$. This yields six possible boundary states, denoted by $(+)$ if $\phi = 0$ and $(-)$ for $\phi = \pi$:

$$\left| B_{\Omega+} \right\rangle \equiv \left| B_{\Omega, \phi=0} \right\rangle, \ \left| B_{\Omega-} \right\rangle, \ \left| B_{\omega\pm} \right\rangle, \ \text{and} \ \left| B_{\mu\pm} \right\rangle. \tag{18}$$

The corresponding cylinder amplitudes are given by the natural pairings $\left\langle B \middle| \hat{q} \middle| C \right\rangle = \left\langle B \middle| q^{\mathcal{H}} \middle| C \right\rangle = \left\langle B \middle| (q^{1/2})^{(L_0 + \overline{L}_0 + 1/6)} \middle| C \right\rangle$. For the interesting (untwisted) sector, they are

$$
\begin{array}{c}
\begin{array}{cccc}
\left| B_{\Omega+} \right\rangle & \left| B_{\Omega-} \right\rangle & \left| B_{\omega+} \right\rangle & \left\langle B_{\omega-} \right|
\end{array} \\
\begin{array}{c}
\left\langle B_{\Omega+} \right| \\
\left\langle B_{\Omega-} \right| \\
\left\langle B_{\omega+} \right| \\
\left\langle B_{\omega-} \right|
\end{array}
\left(
\begin{array}{cccc}
0 & 0 & \eta(q)^2 & \Theta_{1,2}(q) \\
0 & 0 & \Theta_{1,2}(q) & \eta(q)^2 \\
\eta(q)^2 & \Theta_{1,2}(q) & d(d + \ln(q))\eta(q)^2 & d(d + \ln(q))\Theta_{1,2}(q) \\
\Theta_{1,2}(q) & \eta(q)^2 & d(d + \ln(q))\Theta_{1,2}(q) & d(d + \ln(q))\eta(q)^2
\end{array}
\right).
\end{array}
\tag{19}
$$

The different factors and signs in contrast to [?] arise due to our different normalization of the metric. To get rid of the unphysical terms proportional to $\log(q)\Theta_{1,2}(q)$, one of the states $\left| B_{\omega\pm} \right\rangle$ was discarded and the physical boundary conditions were derived with this reduced set. This was possible according to the $\mathbb{Z}_2$ symmetry $\phi \longrightarrow \phi + \pi \mod 2\pi$.

Candidates for the Ishibashi states were deduced by diagonalizing the cylinder amplitudes, i.e., $\left\langle i \middle| \hat{q} \middle| j \right\rangle = \delta_{ij} \chi_i(q)$. However, it was not possible to express the physical boundary states in terms of this basis. Kawai and Wheater proposed the following five states and five corresponding duals:

$$\left| V_0 \right\rangle = \tfrac{1}{2} \left| B_{\Omega+} \right\rangle + \tfrac{1}{2} \left| B_{\Omega-} \right\rangle, \qquad \left\langle V_0 \right| = -\tfrac{1}{2} \left\langle B_{\omega+} \right| - \tfrac{1}{2} \left\langle B_{\omega-} \right|,$$

$$\left| V_1 \right\rangle = \tfrac{1}{2} \left| B_{\Omega+} \right\rangle - \tfrac{1}{2} \left| B_{\Omega-} \right\rangle, \qquad \left\langle V_1 \right| = \tfrac{1}{2} \left\langle B_{\omega+} \right| - \tfrac{1}{2} \left\langle B_{\omega-} \right|,$$

$$\left| V_{-1/8} \right\rangle = \tfrac{1}{2} \left| B_{\mu+} \right\rangle + \tfrac{1}{2} \left| B_{\mu-} \right\rangle, \qquad \left\langle V_{-1/8} \right| = \tfrac{1}{2} \left\langle B_{\mu+} \right| + \tfrac{1}{2} \left\langle B_{\mu-} \right|, \tag{20}$$

$$\left| V_{3/8} \right\rangle = \tfrac{1}{2} \left| B_{\mu+} \right\rangle - \tfrac{1}{2} \left| B_{\mu-} \right\rangle, \qquad \left\langle V_{3/8} \right| = \tfrac{1}{2} \left\langle B_{\mu+} \right| - \tfrac{1}{2} \left\langle B_{\mu-} \right|,$$

$$\left| R \right\rangle = \sqrt{2} \left| B_{\Omega+} \right\rangle, \qquad \left\langle R \right| = -\sqrt{2} \left\langle B_{\omega-} \right|.$$

The (ket-)states form only a four-dimensional space. Especially, $\left| R \right\rangle$ is associated to the indecomposable representations but only built on the subrepresentations. It is evident that the states $\left| B_{\omega\pm} \right\rangle$ cannot obey equation (??) without further restrictions because they are based on the state $\left(\omega \otimes \overline{\omega} \right)$ which is obviously not a proper ground state:

$$\left[L_0 - \overline{L}_0 \right] \left(\omega \otimes \overline{\omega} \right) = \left(\Omega \otimes \overline{\omega} \right) - \left(\omega \otimes \overline{\Omega} \right) \neq 0, \tag{21}$$

unless the right-hand side state is discarded as in the unique local $c = -2$ LCFT [?]. There, a chiral and an anti-chiral version of the rational $c = -2$ LCFT are glued together to obtain a non-chiral theory. In order to keep locality of the correlators, certain states had to be divided out, namely the image of $(L_0 - \overline{L}_0)$. This was not mentioned by Kawai and Wheater. It is shown in the following that their considerations are indeed compatible with the result of [?] and lead to the same results if starting from the "vacua" of the complete chiral theory.

4 Approach 2: Boundary states for the $\mathcal{W}$-algebra

In [**?**] the span of boundary states under the constraints of the $\mathcal{W}(2,3,3,3)$-algebra was derived. This was done by inventing a straight-forward method that uses only basic properties of the theory and its representations. Due to that it was possible to keep especially the inner structure of the indecomposable representations $\mathcal{R}_0$ and $\mathcal{R}_1$ and their subrepresentations visible. This allowed to find relations between the derived states. Ten boundary states were identified:

The states $\big| V_{-1/8} \big\rangle$ and $\big| V_{3/8} \big\rangle$ corresponding to the admissible irreducible representations $\mathcal{V}_{-1/8}$ and $\mathcal{V}_{3/8}$ are the usual Ishibashi states for these modules.

For the indecomposable representations $\mathcal{R}_\lambda$, to stay close to the usual notions, the definition of the Ishibashi states was generalized. The two states

$$\big| R_\lambda \big\rangle = \sum_{l,m,n} \gamma^{\lambda\, l\, m}{}_n \big(\mathbb{1} \otimes \overline{\mathcal{U}} \big) \big| l, m \big\rangle \otimes \overline{\big| l, n \big\rangle}, \quad \lambda = 0,\, 1 \tag{22}$$

are called *generalized* Ishibashi states. Here, $\big\{ \big| l, m \big\rangle; \, l = h, h+1, \dots, \, m = 1, \dots \big\}$ is an arbitrary basis over the representation $\mathcal{R}_\lambda$ where l counts the levels beginning from the top-most, which is $h = 0$ in our case. The basis states on each level of the representation are counted by m. Similarly, $\big\{ \overline{\big| l, n \big\rangle} \big\}$ is the basis for the anti-holomorphic module $\overline{\mathcal{R}_\lambda}$. The matrix γ^λ was identified to be the inverse metric on $\mathcal{R}_\lambda$. In ordinary CFTs, these bases can be chosen orthonormal and then the result would coincide with the usual Ishibashi state. It was argued in [**?**] that this is not applicable here.

The Ishibashi states corresponding to the two subrepresentations $\mathcal{V}_0$ and $\mathcal{V}_1$ were derived with the help of an operator $\hat{\mathcal{N}} = \hat{\delta} + \hat{\bar{\delta}}$, where $\hat{\delta}$ is the off-diagonal part of L_0 that was considered to be in Jordan form. Since there are rank-2 Jordan cells at most, $\hat{\delta}^2 = 0$ and thus, $\hat{\mathcal{N}}^3 = 0$. It was argued that the states

$$\big| V_\lambda \big\rangle = \frac{1}{2} \hat{\mathcal{N}} \big| R_\lambda \big\rangle \tag{23}$$

do not vanish and fulfill (**??**) and (**??**), i.e., are properly defined boundary states. These are called *level-2* Ishibashi states and contain only contributions from the corresponding subrepresentations.

In addition, two doublets of states were found that glue together the two different indecomposable representations $\mathcal{R}_0$ and $\mathcal{R}_1$ at the boundary. They were given in terms of operators $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}^\dagger$ that intertwine the two representations and have the following action on the (bulk) states:

$$\begin{aligned}
\hat{\mathcal{P}}_\pm^\dagger \big| \omega \big\rangle &= \big| \xi^\pm \big\rangle, & \hat{\mathcal{P}}_\pm^\dagger \big| \Omega \big\rangle &= 0, & \hat{\mathcal{P}}_+ \big| \psi^\pm \big\rangle &= -\big| \Psi_2^\pm \big\rangle, \\
\hat{\mathcal{P}}_\pm \big| \xi^\mp \big\rangle &= \pm \big| \Omega \big\rangle, & \hat{\mathcal{P}}_\pm \big| \phi^\pm \big\rangle &= 0, & \hat{\mathcal{P}}_- \big| \psi^\pm \big\rangle &= \big| \Psi_1^\pm \big\rangle.
\end{aligned} \tag{24}$$

This yields the so-called *mixed* Ishibashi states $\big| R_{01}^\pm \big\rangle$ and $\big| R_{10}^\pm \big\rangle$:

$$\begin{aligned}
\big| R_{01}^\pm \big\rangle &= \hat{\mathcal{P}}_\pm \big| R_1 \big\rangle = \hat{\overline{\mathcal{P}}}_\pm^\dagger \big| R_0 \big\rangle, & \big| V_0 \big\rangle &= \hat{\overline{\mathcal{P}}}_\mp \big| R_{01}^\pm \big\rangle = \hat{\mathcal{P}}_\mp \big| R_{10}^\pm \big\rangle, \\
\big| R_{10}^\pm \big\rangle &= \hat{\overline{\mathcal{P}}}_\pm \big| R_1 \big\rangle = \hat{\mathcal{P}}_\pm^\dagger \big| R_0 \big\rangle, & \big| V_1 \big\rangle &= \hat{\mathcal{P}}_\mp^\dagger \big| R_{01}^\pm \big\rangle = \hat{\overline{\mathcal{P}}}_\mp^\dagger \big| R_{10}^\pm \big\rangle.
\end{aligned} \tag{25}$$

These relations can be drawn schematically. It is not quite unexpected that there is a one-to-one correspondence to the embedding scheme of the local theory [**?**]: The states that are divided out there are due to (**??**) exactly those that do not contribute to the boundary states.
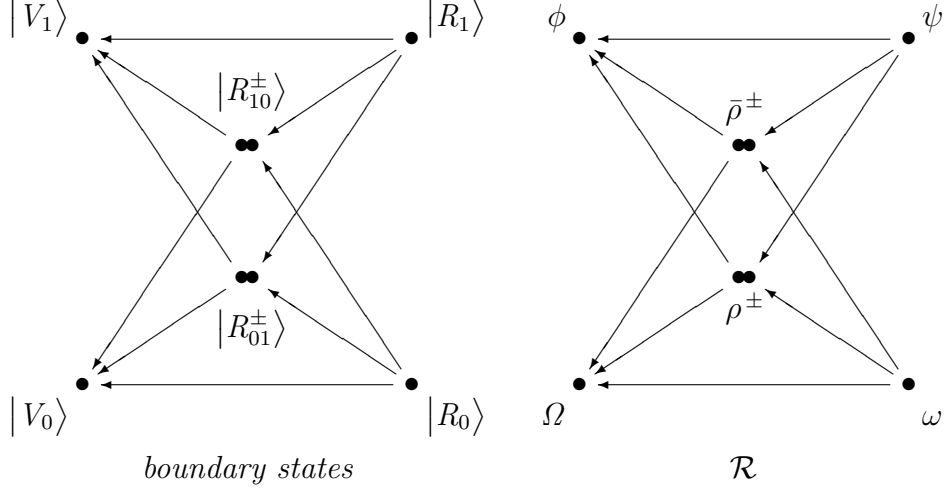

figure ??: boundary states vs. local theory

The lines in the left picture in figure **??** refer to the action of $\hat{\mathcal{N}}$, $\hat{\mathcal{P}}$, $\hat{\mathcal{P}}^{\dagger}$, $\hat{\bar{\mathcal{P}}}$, and $\hat{\bar{\mathcal{P}}}^{\dagger}$ while in the right one they denote the action of the (non-chiral) symmetry algebra.

The non-vanishing natural pairings of the boundary states are given by

$$
\begin{aligned}
\big\langle V_{-1/8}\big|\hat{q}\big|V_{-1/8}\big\rangle &= \chi_{\mathcal{V}_{-1/8}}(q), & \big\langle V_{3/8}\big|\hat{q}\big|V_{3/8}\big\rangle &= \chi_{\mathcal{V}_{3/8}}(q), \\
\big\langle R_0\big|\hat{q}\big|R_0\big\rangle &= \chi_{\mathcal{R}}(q), & \big\langle R_1\big|\hat{q}\big|R_1\big\rangle &= \chi_{\mathcal{R}}(q).
\end{aligned}
\tag{26}
$$

These coincide with the physical characters forming the three-dimensional representation of the modular group. The torus amplitudes, on the other hand, are first seen with the help of additional, so-called *weak* boundary states $\big|X_\lambda\big\rangle$ and $\big|Y_\lambda\big\rangle$, $\lambda = 0, 1$, that obey

$$
\big|R_\lambda\big\rangle = \hat{\mathcal{N}}\big|X_\lambda\big\rangle + \big|Y_\lambda\big\rangle.
\tag{27}
$$

These states could be chosen uniquely in such a way that they serve as the duals to the null states $\big|V_\lambda\big\rangle$ obtaining

$$
\begin{aligned}
\big\langle X_\lambda\big|\hat{q}\big|V_\lambda\big\rangle &= \chi_{\mathcal{V}_\lambda}(q), & \big\langle X_\lambda\big|\hat{q}\big|R_\lambda\big\rangle &= \log(q)\cdot\chi_{\mathcal{V}_\lambda}(q), \\
\big\langle X_\lambda\big|\hat{q}\big|Y_\lambda\big\rangle &= 0, & \big\langle Y_\lambda\big|\hat{q}\big|R_\lambda\big\rangle &= \chi_{\mathcal{R}}(q) - 2\chi_{\mathcal{V}_\lambda}(q), \\
\big\langle R_\lambda\big|\hat{q}\big|R_\lambda\big\rangle &= \chi_{\mathcal{R}}(q).
\end{aligned}
\tag{28}
$$

Obviously, this does not exactly reproduce the elements of the five-dimensional representation given in (**??**) but rather linear combinations of them and the unphysical contribution $\log(q)\Theta_{1,2}(q)$. This has to be taken care of when calculating physical relevant boundary conditions with the help of Cardy's consistency equation.

5 With the general method

The method presented in [**?**] provides an efficient tool for the investigation of the boundary states under the restrictions of the symplectic fermion algebra. It bases on the general ansatz for a boundary state connecting a holomorphic and an anti-holomorphic representation $\mathcal{M}_h$ and $\overline{\mathcal{M}_{\bar{h}}}$ at the boundary

$$\big|B\big\rangle = \sum_{l,m,n} c^l_{mn}\big(\mathbb{1}\otimes\overline{\mathcal{U}}\big)\big|l,m\big\rangle\otimes\overline{\big|l,n\big\rangle}. \tag{29}$$

The task is to directly calculate the matrix c. This is done in an iterative procedure. Since the sum in (**??**) is infinite, the coefficients c^l_{mn} can only be derived up to any finite level $l = L$. The idea is that this provides the basis for the second step, the identification of the boundary states.

The boundary state consistency equation for this symmetry algebra is given by (**??**):

$$\big(\chi^\pm_m - \mathrm{e}^{\pm\mathrm{i}\phi}\overline{\chi}^\pm_{-m}\big)\big|B\big\rangle = 0, \tag{30}$$

where ϕ is the spin which can take the values $\phi = 0, \pi$ at the boundary, since we force $\big|B\big\rangle$ to be compatible with the $\mathcal{W}$-algebra. It is then clear that (**??**) and (**??**) are automatically satisfied once (**??**) is valid. This implies that the solutions are linear combinations of the boundary states of section **??**. The naturally arising question is, especially when comparing the results presented in the two previous sections, whether the fermion symmetry is more restrictive than the $\mathcal{W}$-algebra, i.e., if less states are found here than in the latter theory. The opposite is the case: Using the method of [**?**] we again find ten proper boundary states. Denoting the $\phi = 0$ case by the quantum number $(+)$ and $\phi = \pi$ by $(-)$ as in the previous discussion, these states are:

$$\begin{aligned}
\big|\Omega,\Omega;\pm\big\rangle &= \big|\Omega,\Omega\big\rangle \pm \big|\phi^+,\phi^-\big\rangle \mp \big|\phi^-,\phi^+\big\rangle + \dots, \\
\big|\Omega,\omega;\pm\big\rangle &= \big|\Omega,\omega\big\rangle + \big|\omega,\Omega\big\rangle \pm \big|\xi^+,\xi^-\big\rangle \mp \big|\xi^-,\xi^+\big\rangle + \dots, \\
\big|\Omega,\xi^a;\pm\big\rangle &= \big|\Omega,\xi^a\big\rangle \pm \big|\xi^a,\Omega\big\rangle + \dots, \quad a = +,-, \\
\big|\mu,\mu;\pm\big\rangle &= \big|\mu,\mu\big\rangle \pm \big|\nu^+,\nu^-\big\rangle \mp \big|\nu^-,\nu^+\big\rangle + \dots.
\end{aligned} \tag{31}$$

Here, $\big|m,n\big\rangle$ is used as a short-hand for $\big|m\big\rangle\otimes\overline{\big|n\big\rangle}$. This result may be compared to the one for the $\mathcal{W}$-algebra. We obtain the following identities

$$\begin{aligned}
\big|\Omega,\Omega;\pm\big\rangle &= \big|V_0\big\rangle \pm \big|V_1\big\rangle, & \big|\Omega,\omega;\pm\big\rangle &= \big(\big|R_0\big\rangle + d\big|V_0\big\rangle\big) \pm \big(\big|R_1\big\rangle - t\big|V_1\big\rangle\big), \\
\big|\Omega,\xi^a;\pm\big\rangle &= \big|R^a_{01}\big\rangle \pm \big|R^a_{10}\big\rangle, & \big|\mu,\mu;\pm\big\rangle &= \big|V_{3/8}\big\rangle \pm \big|V_{-1/8}\big\rangle.
\end{aligned} \tag{32}$$

This identification uses the fact that the boundary states fulfill (**??**) and (**??**). Thus, the first level contributions of (**??**) can be compared to the results of section **??** to gain the corresponding linear combinations of the states given there.

To show that the result (**??**) of Kawai and Wheater is compatible to ours one has to keep in mind that the coherent states obey the consistency equation (**??**), and hence (**??**) and (**??**). Therefore, they can be expressed in terms of the states (**??**). Indeed, we find

$$\big|B_{\Omega\pm}\big\rangle = \big|\Omega,\Omega;\pm\big\rangle \quad\text{and}\quad \big|B_{\mu\pm}\big\rangle = \big|\mu,\mu;\pm\big\rangle, \tag{33}$$

up to possible additional contributions from null-states and the different normalization. It seems contradictory that here, no boundary state based on $(\omega \otimes \overline{\omega})$ is found. But reviewing [?] as quoted in section ?? these are the states $|B_{\omega\pm}\rangle$ (or rather $\langle B_{\omega\pm}|$) which occur only as the duals to $|B_{\Omega\pm}\rangle$ in the Ishibashi states. This is remarkable, since in our framework the only states having such logarithmic contributions, i.e., $(\omega \otimes \overline{\omega})$-like terms, are $|X_\lambda\rangle$ that we used in precisely the same manner. This suggests, that the coherent states based on $(\omega \otimes \overline{\omega})$ are related to $|X_\lambda\rangle$ in the same way as above:

$$|\omega, \omega; \pm\rangle = |X_0\rangle \pm |X_1\rangle. \tag{34}$$

Observe the fact that these states do not exactly correspond to $|B_{\omega\pm}\rangle$ due to the connection to the local theory as discussed above.

The generic procedure of [?] yields a much bigger collection of states in comparison to Kawai and Wheater. Especially, the mixed boundary states were not discussed by them and the Ishibashi boundary state for the module $\mathcal{R}$ was obtained by the identification $2\mathcal{V}_0 + 2\mathcal{V}_1 \equiv \mathcal{R}$. Presumably therefore and by referring to the local theory by setting $(\Omega \otimes \overline{\omega}) - (\omega \otimes \overline{\Omega})$ to zero, their physical boundary conditions differ from the set of Ishibashi states.

Indeed, we find that the coherent state method produces exactly the same amount of states when starting from the "invariant vacua" that we have:

$$\left\{ (\Omega \otimes \overline{\Omega}), (\Omega \otimes \overline{\omega}) + (\omega \otimes \overline{\Omega}), (\Omega \otimes \overline{\xi}^a) - \mathrm{e}^{\mathrm{i}\phi}(\xi^a \otimes \overline{\Omega}), (\mu \otimes \overline{\mu}) \right\}. \tag{35}$$

The symplectic fermions decompose the L_0 operator in such a way that

$$\chi_0^\pm \omega = -\xi^\pm \quad \text{and} \quad \chi_0^\pm \chi_0^\mp \omega = \mp \Omega. \tag{36}$$

With respect to (??) and (??) this suggest that the intertwining operators $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}^\dagger$ and the corresponding boundary states $|R_{01}^a\rangle$ and $|R_{10}^a\rangle$ might be closely related to the fermionic zero modes.

6 Discussion

We worked out the space of boundary states in the rational LCFT with central charge c $= -2$ under the restrictions of the symplectic fermion symmetry. It turned out that these states coincide with the solution we presented in [?]. In particular, this implies that the symplectic fermion algebra gives no additional constraints on the boundary states in comparison to the $\mathcal{W}(2,3,3,3)$-algebra of the rational c $= -2$ LCFT. This is interesting because the latter one is embedded in the former. One might guess that the boundary state consistency equation for the symplectic fermion symmetry is more restrictive than the one for the $\mathcal{W}$-algebra. On the other hand, already in [?] we noticed the close relation between the derivation of boundary states and the construction of a local theory (see fig. ??). At least for c $= -2$ the latter one is uniquely defined which would suggest, that there exists exactly one consistent solution for the set of boundary states.

To construct the boundary states, we used the same method that we presented in [?] for the $\mathcal{W}$-algebra case. This shows that this method really yields a general prescription for

the treatment of boundary states and is easily adoptable to different frameworks (like the symplectic fermions in this case). Thus, it seems natural that the presented results generalize to more complicated theories. For the coherent states this was already pointed out by Kawai [**?**].

We compared the results to the coherent state solution of Kawai and Wheater and were able to show that both approaches are equivalent, leading to exactly the same set of states. However, our results differ in some crucial aspects compared to [**?**]: They had to divide out the image of $(L_0 - \overline{L}_0)$ by hand while in our prescription this is implicitly included.

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