Stochastic Löwner Evolution as an approach to Conformal Field Theory

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Die Hälfte dessen, was man schreibt, ist schädlich, die andere Hälfte unnütz.

Friedrich Dürrenmatt

Zusammenfassung

Das zentrale Thema dieser Arbeit ist die Beziehung zwischen zweidimensionaler Konformer Feldtheorie auf Gebieten mit Rand (bCFTs) und Schramm-Löwner Evolution (SLEs). Diese wird durch die Verbindung der beiden zum Skalengrenzwert von Modellen in der Statistischen Physik am kritischen Punkt motiviert. Der bCFT Ansatz zur Lösung der Modelle, der seit 25 Jahren genutzt wird, basiert auf der algebraischen Formulierung lokaler Objekte wie Feldern und deren Korrelationsfunktionen. Dahingegen beschreibt die in 1999 eingeführte SLE die physikalischen Eigenschaften von einem wahrscheinlichkeitstheoretischen Standpunkt aus, wobei der Schwerpunkt auf dem Studium von Maßen auf wachsenden Kurven, d. h. globalen Objekte wie Clustergrenzen, liegt.

Nach einer kurzen Motivation der Fragestellung, gefolgt von einer detailierteren Einführung in zweidimensionale bCFT und SLE, präsentieren wir unsere Forschungsergebnisse. Wir erweitern die Methode, SLE-Varianten von einem Maßwechsel des Einzel-SLE Maßes zu erhalten, und leiten so die allgemeinste SLE-Variante, die zu einer bCFT in Verbindung gebracht werden kann, her. Darüber hinaus interpretieren wir diesen Maßwechsel im Kontext der Physik und der Wahrscheinlichkeitstheorie. Zudem diskutieren wir die Bedeutung von Bulk-Feldern in der bCFT als Bulk-Kraftpunkte der SLE-Variante SLE$(\kappa, \vec{\rho})$.

Zusätzlich erforschen wir die Entwicklung der Randbedingungs-Felder, die Clustergrenzen, die von der SLE beschrieben werden können, erzeugen, in kurzer Distanz mit anderen Rand- oder Bulk-Feldern. Dabei leiten wir neue SLE Martingalen her, die in Verbindung mit der Existenz von Rand-Feldern mit verschwindendem Absteiger auf Level drei stehen. Wir motivieren dass das Distanz-Skalierungsverhalten dieser Martingalen, d. h. die Anpassung des Maßes, als die SLE Wahrscheinlichkeit, dass die SLE-Kurve dem Ort eines zweiten Feldes nahekommt, interpretiert werden kann.

Zuletzt erweitern wir die algebraische κ-Relation der erlaubten Varianzen in multipler SLE, die der Kommutatorrelationen der infinitesimalen Entwicklungsoperatoren entspringt, auf das verbundene Wachstum zweier SLE Kurven. Die Analyse legt direkt die Form der infinitesimalen Löwner Abbildung der verbundenen Prozesse nahe, die wahrscheinlich in Verbindung mit dem Wachstum von Kurven auf Grund der Existenz von Randbedingungs-Feldern mit verschwindendem Absteiger auf Level drei stehen.

Wir schließen mit einer Zusammenfassung unserer Resultate, ordnen unsere Arbeit in den Kontext vorhergehender Publikationen ein und geben einen Ausblick zu vielversprechenden neuen Methoden und offenen Fragestellungen.

Schlagworte: Schramm-Löwner Evolution, Konforme Feldtheorie mit Rand, Kritische Phänomene

Abstract

The main focus on this work lies on the relationship between two-dimensional boundary Conformal Field Theories (bCFTs) and Schramm-Löwner Evolutions (SLEs) as motivated by their connection to the scaling limit of Statistical Physics models at criticality. The bCFT approach used for the past 25 years is based on the algebraic formulation of local objects such as fields and their correlations in these models. Introduced in 1999, SLE describes the physical properties from a probabilistic point of view, studying measures on growing curves, i. e. global objects such as cluster interfaces.

After a short motivation of the topic, followed by a more detailed introduction to two-dimensional boundary Conformal Field Theory and Schramm-Löwner Evolution, we present the results of our original work. We extend the method of obtaining SLE variants from a change of measure of the single SLE to derive the most general bCFT model that can be related to SLE. Moreover, we interpret the change of the measure in the context of physics and Probability Theory. In addition, we discuss the meaning of bulk fields in bCFT as bulk force-points for the SLE variant SLE$(\kappa, \vec{\rho})$.

Furthermore, we investigate the short-distance expansion of the boundary condition changing fields, creating cluster interfaces that can be described by SLE, with other boundary or bulk fields. Thereby we derive new SLE martingales related to the existence of boundary fields with vanishing descendant on level three. We motivate that the short-distance scaling law of these martingales as adjustment of the measure can be interpreted as the SLE probability of curves coming close to the location of the second field.

Finally, we extend the algebraic κ-relation for the allowed variances in multiple SLE, arising due to the commutation requirement of the infinitesimal growth operators, to the joint growth of two SLE traces. The analysis straightforwardly suggests the form of the infinitesimal Löwner mapping of joint processes, presumably related to the growth of curves due to the existence of boundary condition changing fields with vanishing descendant on level three.

We conclude with a summary of our results, ranging our work into the context of preceding papers and giving an outlook to promising new methods and open questions.

Keywords: Schramm-Löwner Evolution, boundary Conformal Field Theory, Critical Phenomena
PACS: 11.25.Hf, 02.50.Ey

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1 Motivation and Introduction

1.1 Conformal Field Theory

Symmetry denotes that sort of concordance of several parts by which they integrate into a whole. Beauty is bound up with symmetry.

HERMANN WEYL

It often happens that the requirements of simplicity and beauty are the same, but where they clash the latter must take precedence.

PAUL DIRAC

The affection of the human mind to favor symmetric patterns and simple models has been a guide for theoretical physicists ever since. Symmetry provides us with constraints that enable us to solve the equations, e. g. of motion, that describe the model.

The assumption of symmetry under the change of reference frame for example is the main building block for Einstein's famous theory of gravity, whereas the electromagnetic, weak and strong force arise due to (internal) symmetries of elementary particles. In general, what one observes is that the higher the energy scale, the more symmetric the interactions among the elementary particles become. This hints at some broken symmetry which should be restored at infinite energy or vanishing masses, i. e. a *scale-invariant* theory.

Conformal invariance in arbitrary space-time dimensions is the direct extension of scale invariance and POINCARÉ invariance, i. e. the symmetry of MINKOWSKI space-time. Heuristically, this means the absence of any distinguishable scale such as a masses or characteristic length scales. In theories with nearest neighbor or local interactions, i. e. a theory where interactions only take place over infinitesimal space-time distances, the symmetry also has to be *local* in nature.

1.1.1 Conformal Symmetry in Two Dimensions

In *two dimensions*, the requirement of locality is especially restrictive. Splitting a conformal transformation into its generators, it is easy to see that a conformally invariant local theory in two dimensions has to be invariant under an infinite number of independent transformations, providing an infinite number of conservation laws. In some models, this allows to fix all free parameters, such that no experimental input is needed to compute all relevant data as in other quantum field theories. Therefore, two-dimensional conformal theories are often so-called integrable models, i. e. models that can be solved exactly. This intriguing feature is one of the reasons why we focus on *two-dimensional* conformally invariant theories in this thesis.

Admittedly, as we know from everyday observations, scale invariance is by no means an exact symmetry of nature. We are able to observe a number of finite characteristic length scales, over which interactions take place. However, for some models in two-dimensional *Statistical Physics*, there are so-called *critical* points in parameter space. At these points, the characteristic length scales diverge such that the observables exhibit statistically the same behavior at all scales – they are scale- and most times even conformally invariant*.

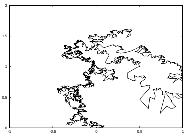
1.1.2 Statistical Physics, Field Theory and Lattice Models

Since the beginning of the last century, we know that the simple macroscopic physics, i. e. classical or "NEWTONian" physics, which we can observe in our every-day life, is not the final answer to the fundamental processes of nature. When it comes down to the behavior of the so-called elementary particles, we see that we can not determine their individual behavior exactly. Moreover, even if we were able to compute it for a single particle, we would certainly not be able to do this for the typical numbers we deal with in everyday life. Therefore, several ways to deal with that problem have been developed, one of which is *Statistical Physics*, providing a way to govern the *mean* or *expected* behavior of large collections of particles.

Accordingly, Statistical Physics specifies the macroscopic properties of systems with many degrees of freedom with the help of stochastic considerations on the microscopic details. It concentrates on the macroscopic *collective* phenomena which are largely independent of the microscopic details, allowing for a statistical description by only a few macroscopic quantities. In two dimensions, this is often done on a lattice whose nodes are the locations of the single particles, assuming only nearest-neighbor, i. e. local interactions. Considering two-dimensional models of this kind at their critical points and taking the so-called *continuum limit* of vanishing lattice spacing, these two-dimensional locally scale-invariant lattice models result in conformally invariant theories.

Besides Statistical Physics, another possible description for many-particle systems on a quantum level are *Field Theories*. Their formulation, in contrast to single particle theories such as *Quantum Mechanics*, includes annihilation and creation of particles as well as states with arbitrary excitation and interactions between them. In the continuum limit, both methods describing many *two-dimensional critical phenomena* – Statistical Physics and Field Theories with an underlying conformal symmetry (CFTs) – are conjectured to be equivalent [24, 25].

*There are some not physically relevant counterexamples that are scale but not conformally invariant [22], e. g. 6D non-critical self-dual string theory. However, e. g. for unitary compact conformal field theories or local Statistical Field Theories in two dimensions, scale invariance implies the tracelessness of the stress-energy tensor which by itself implies conformal invariance [23].



1.1.3 Phase Transitions, Critical Points and Universality

Macroscopically observable phase transitions occurring in critical two-dimensional models are among the most interesting phenomena when investigating collective phenomena in Statistical Physics. Small changes in the parameters (e. g. the temperature or an external magnetic field) result in qualitative macroscopic changes of the physical properties of the system: the observables exhibit a *power-law* behavior dependent on a so-called order parameter. This order parameter vanishes in one phase while it is non-zero in the other, e. g. the density when considering the liquid-gas transition in water or the magnetization in the Ising model. The power-law behavior on the order parameter is due to the scale-invariance of the system imposed by the divergence of the correlation length. The corresponding exponents are called *critical* exponents. The collection of the critical exponents of one model may also be characteristic for another model which is why the critical models can be assembled into *universality classes*, i. e. classes in which all models exhibit the same critical exponents. In the continuum limit of two-dimensional models, these universality classes are characterized by the central charge of the CFT, specifying all possible critical exponents consistent with the underlying symmetry algebra.

The fact that a small number of parameters governs that of the model allows for simple descriptions without going into too much details of interactions, e. g. lattice spin models such as the q-states Potts model or $O(n)$ models. Due to their relative simplicity, many of these models can also be solved exactly by other methods, determining the universal properties of their whole respective classes.

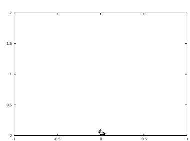
1.1.4 Physical Relevance of Two-Dimensional Conformal Models

The discussion of two-dimensional critical phenomena is by no means just of theoretical importance: there are experiments where the critical exponents have been measured. Typical experimental setups include thin magnetic films, chemisorption and physisorption as well as surface reconstruction (see e. g. [26] and references therein).

In particular, this means that effectively two-dimensional surfaces may be produced from specific molecular structures such as the tetragonal setup of the K_2NiF_4 family. Here, the magnetic ions form a square lattice in two dimensions.

Another possibility is adsorption on crystalline surfaces. Here, the atoms are bound either by the Van der Waals forces (with an interaction potential of about $\sim 10^{-2}$ eV "physisorption") or by chemical forces (~ 1 eV "chemisorption"). Near the critical temperature, there is little desorption such that the material can be thought of as a strictly two-dimensional system with constant coverage.

However, not only artificially constructed materials can be described by 2D Statistical Physics models, but also in nature, applications for models exhibiting a CFT as scaling limit at criticality can be found. For example, the Ising model, i. e. $c = 1/2$ at its critical point in 2D, has been successfully used as a model for the behavior of a number of biological systems, e. g. hemoglobin, systems of protomers [27] and de-



oxyribonucleic acid (DNA). Thereby, cooperative behavior is modeled by the nearest neighbor interactions of the ISING model [28]. In the case of hemoglobin for example, with its four binding sites for oxygen the probability to bind the next oxygen molecule increases with the number already attached. An analogous effect exists for the reaction rate of allosteric enzymes, as well as the denaturation curve, i. e. the fraction of broken bonds as a function of temperature, of DNA (cf. helix-coil transition [29]).

1.1.5 The Continuum Limit: Conformal Field Theory

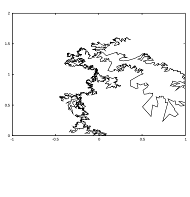
For the past 25 years, Conformal Field Theory (CFT) has been an intense field of research and most of the physically relevant lattice models have been conjectured to correspond to certain CFTs. In the fundamental paper by Belavin, Polyakov and Zamolodchikov [25] in 1984, CFT has been considered as a mathematical theory of its own. This has been followed quickly by many articles [30], expanding the theory and establishing the connection to critical phenomena in physics [31]. In addition to critical phenomena, CFTs have multiple other physical applications and many different formulations which we will briefly address in the following.

1.1.6 Different Approaches to CFT

As a two-dimensional field theory, CFT can either be result of the scaling limit of a conformally invariant $1+1$ dimensional Quantum or a $2+0$ dimensional Statistical Field Theory, meaning that it has either one time and one space dimension or just two space dimensions. Its basic objects are fields that represent particles such as free bosons or fermions, as well as energy or spin variables. The measurable quantities of the theory are the correlation functions of these fields, determining the transitions between different states.

In this thesis we will deal with two-dimensional CFTs containing a finite number of basic (or *primary*) fields, corresponding to highest weight states in the HILBERT space, transforming as irreducible representations of the underlying symmetry algebra. These are particularly well-understood, characterized by a rational central charge $c_{p,q} < 1$, parameterized by two coprime numbers $p, q \in \mathbb{N}$. For these models, there exist three major mathematical descriptions whose connection we want to explore in this thesis (for details see chapters 3 and 4).

Within the *algebraic* formulation of these CFT models, the focus lies on the stress energy tensor whose LAURENT series coefficients L_n are the infinitesimal generators of conformal transformations. These L_n form the so-called VIRASORO algebra which is an infinite-dimensional LIE algebra with a *central extension* $\hat{C}$. This central extension is proportional to the central charge $c_{p,q}$, fixing the universality class of the model. The finite number of primary fields $\phi_{h_{(r,s)}}$, specified by $r, s \in \mathbb{N}_0$ with $0 < r, s < q, p$, in each of these models correspond to irreducible highest weight representations $(h_{r,s}, c_{p,q})$ of the VIRASORO algebra. For each of the fields $\phi_{h_{(r,s)}}$ contained in a correlation function there exists a differential equation of order $r \cdot s$ which provides



a way to solve the correlation function without employing the path integral as in ordinary QFTs. This method became famous in the '90s within the "axiomatic" approach to rational conformal field theory [32, 33].

Another method is based on the bosonic representation of CFT. It is directly related to the COULOMB *Gas formalism* that emerged from the study of some 2D lattice models [34]. The basic fields are represented by so-called vertex operators that are exponentials of the free boson field, corresponding to representations of the VIRASORO algebra. This method has first been introduced by Richard Borcherds in 1986, for which he received the Fields Medal in 1998. Moreover, other vertex operator algebras have been proven useful in purely mathematical contexts such as the monstrous moonshine and the geometric Langlands correspondence.

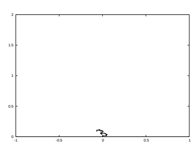
Both approaches provide information about the *local* order of the system and how it is transferred from one part of the domain to another, especially over long distances. This way, historically, CFT led to many non-trivial predictions about *local* observables, e. g. the density, the magnetization and their correlations. However, in most formulations, its axioms are based on physically-motivated principles of Quantum Field Theory which are often not well-defined[‡].

The third and most recent approach is *Stochastic* or SCHRAMM-LÖWNER *Evolution* [35] but might better be called Conformal Probability Theory. It brought substantial progress to the understanding of random fractal geometry in two dimensions and classical probabilistic objects. SLE describes the growth of random curves connected to the boundary of two-dimensional domains. Therefore it its a suitable model for domain interfaces in two-dimensional conformally invariant physics. Although directly connected to statistical models in the discrete case, SLE concentrates on *global* observables, neglecting the powerful algebraic structure describing the local observables. These have recently been included in one of its generalizations: Conformal Loop Ensembles (CLE) [36, 37], also describing local loops in addition to the random curves. The major advantage of SLE over other BCFT computations is not only *rigor* but also *precision*. The quantities investigated are exactly defined before being calculated or estimated which is often omitted by physicits working on the latter.

Therefore, SLE is well-defined and provides a framework in which the scaling limit of the underlying lattice models can be proven. SLE is especially interesting to study since it addresses the issue of extended random fractal shapes such as cluster boundaries in 2D by direct analysis in the continuum. It provides qualitatively new results, e. g. in the context of turbulence or spin glasses [38, 39], as well as extensions or proofs of earlier lattice results, such as the determination of new scaling exponents [40, 41, 42] and continuum limits such as Percolation [43, 44], the GAUSSian Free Field [45], Loop-Erased Random Walks and the Uniform Spanning Tree [46].

Of course, there are also other physical applications of CFT such as two-dimensional *Quantum Gravity*, *String Theory* or LIOUVILLE *Theory*. Quantum Gravity corre-

[‡] Local products of quantum fields are, in general, operator-valued distributions and may therefore be ill-defined.



sponds to a CFT coupled to gravity with a fluctuating metric, or, as a lattice theory, on a random dynamical lattice in contrast to the traditional regular lattices such as triangular, square or hexagonal lattice. Perturbative String theory can be viewed as a CFT by considering the space-time (target space) coordinates as fields on a two-dimensional world sheet. Consistency requires the central charge of the theory to vanish, i. e. the two contributions from e. g. bosonic matter and the ghost fields have to cancel, leading to the famous $d = 26$ target space dimensions for bosonic String Theory. In Liouville Theory, the metric is treated as a dynamic field, resulting in the Knizhnik-Polyakov-Zamolodchikov relation [47], yielding predictions for the dressed conformal dimensions of CFT in its relation to Quantum Gravity. There are also independent connections between SLE and Quantum Gravity [48], including various proofs of the KPZ relation (see e. g. [49, 50, 51]).

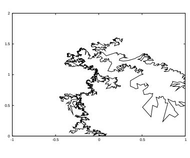
1.2 Stochastic Löwner Evolution

In critical two-dimensional Statistical Physics, great effort has gone into the study of shapes of clusters formed by objects in the same state. One mechanism to produce similar shapes are conformal mappings in the complex plane. This is particularly natural to do in the context of physics on two-dimensional simply connected domains with boundaries. Here, conformal mappings are completely determined by their boundary values and the order and location of their zeroes and poles. Therefore, the study of geometrical objects that are boundaries of such domains is closely related to the study of conformal mappings of them. The key point to this intriguing phenomenon is that in the complex plane, analysis and geometry are the same thing, connected by the Riemann mapping theorem: Conformal mappings provide unique one-to-one mappings of the interior of finite subdomains D of the complex plane into the interior of other simply connected subdomains D', while their boundaries are mapped onto each other. Therefore, interfaces connected to the boundary can be viewed as part of the boundary and hence be described by these mappings.

In 1923, Löwner [52] derived a method employing conformal mappings to construct deterministic line patterns which has been extended by Schramm in 1999 to random curves [35]. Later on, their method has quickly been picked up by the physics community to address shape problems in percolation, self-avoiding walks and other critical phenomena.

1.2.1 Löwner's Equation

While trying to prove the Bieberbach conjecture [53], Löwner studied the time-evolution of smooth curves growing from the boundary of a two-dimensional simply connected domain into its interior. As a side effect, he derived a differential equation for the time evolution of the mapping of the domain without the curve onto itself again. The singularities of this differential equation are fixed by a so-called driving process which is given by the images of the curve's tip on the boundary of the domain.



Henceforth, at time zero, the mapping is equal to the identity and the domain is mapped onto itself. At later times, the mapping cuts out the pre-images of the singularities by taking only their complement onto the full domain again. Therefore, the Löwner mapping is often called conformal slit map, describing a continuous curve in two dimensions as a growing slit via a one-dimensional driving process of the Löwner differential equation on the boundary of the full domain.

1.2.2 Stochastic Löwner Evolution

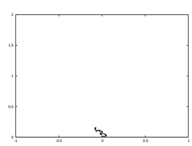
At first, the Löwner equation had been used to prove important results about analytic functions. However, its most intriguing feature had not been revealed for a long time: it is also suitable to study the geometry of boundaries of proper simply connected subdomains of the complex plane. These objects are also of interest in Statistical Physics, where conformally invariant crossing processes through a simply connected planar domain or cluster boundaries are investigated.

As processes evolving in time, such curves are continuous, conformally invariant and without memory. To describe their growth in time via the Löwner equation, it has been Oded Schramm's main result [35] that a continuous, conformally invariant stochastic driving function with stationary, independent increments has to be inserted into the Löwner equation. These conditions are only satisfied by Brownian motion of real positive variance κ. For $\kappa \leq 4$, the stochastic Löwner mapping describes simple curves, for $4 < \kappa < 8$, self-touching curves while for $\kappa \geq 8$, the pre-images of the Brownian motion form a space-filling object. The regions given or surrounded by these curves are geometrical objects formed in critical phenomena and other physical scale-invariant processes, e. g. the representatives of self-avoiding random walks for $\kappa = 8/3$ or the percolation clusters for $\kappa = 6$. By definition, these objects or, more precisely, their images, are *martingales*, i. e. their expectations with respect to the SLE measure are conserved in Löwner time.

Introduced as a mathematical theory, SLE has been developed in collaboration with Gerg Lawler and Wendelin Werner [40, 41, 42], who received the Fields medal in 2006 for "For his contribution to the development of Stochastic Löwner Evolution, the geometry of two-dimensional Brownian motion, and Conformal Field Theory.".

1.2.3 The Relation Between CFT and SLE

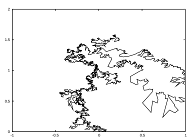
Over the past years, SLE has been successfully applied to the study of random curves that are interfaces connected to the boundary of two-dimensional simply connected bounded domains. Though the languages and basic concepts of SLE and bCFT are different, through their relation to Statistical Physics, it was straightforward to search for a connection between them. By now, some aspects of this relationship are understood quite well, mostly due to Bauer and Bernard [54, 55, 56, 57, 58, 59, 60], Cardy [61, 62] and Friedrich [63, 64]. Physically, the relationship can be motivated via the origin of random cluster interfaces defined between regions of different states



of the variables. On the boundary of the domain, these changes are imposed by inserting primary fields $\phi_{h_{(r,s)}}$ with $r \cdot s = 2$ that change the boundary conditions. From the algebraic approach to CFT, it is known that these lead to differential equations of order two for correlation functions containing these fields. Requiring that these correlation should be conserved in mean, i. e. martingales, imposes precisely the same differential equations on them, if a two-to-one relationship between the variance κ of the Brownian motion and its dual value $16/\kappa$ and the central charge $c = (3\kappa - 8)/6 - \kappa)/2\kappa$ of the bCFT holds. However, the relation to bCFT is far from being complete: up to now, only objects containing fields creating second-order differential equations in bCFTs with central charge $c \leq 1$ have received a direct meaning within SLE. However, these correspond only to two of the finitely many allowed representations for each distinct model. Our main motivation therefore is to identify *other* bCFT objects being relevant for SLE, investigating their properties in SLE.

Little progress in this direction has been achieved so far. An important step would be to identify SLE objects resulting from fields imposing differential equations of other order than two. A first step aiming at this has been accomplished within the Coulomb gas picture where other boundary fields receive a meaning as so-called *force-points*, modifying the dynamics of the SLE driving function, i. e. adding a drift term to the Brownian motion. But up to now, there is no process describing the growth of the interfaces created by other types of boundary condition changing operators. This open question has been the main impulse for the investigations done in this thesis, assembled under the topic "Stochastic Löwner Evolution as an approach to Conformal Field Theory".

The main results of our original work presented in this thesis provide substantial contributions to the correspondence of the two theories. First we define the most general SLE variant that can be connected to boundary CFT (section 5) within the current framework. We motivate that the connection between SLE variants and bCFT is solely build on the partition function of the underlying physical model. Second, we interpret SLE events corresponding to fusion in bCFT via considering the OPE of a curve-creating boundary field with another boundary or bulk field. This results in a boundary curve-creating field imposing a third-order differential equation (section 6) which is the first case in which a field of this type has been observed. Third, we derive the presumptive corresponding infinitesimal Löwner equation via Dubédat's commutation relation (section 7).



2 Outline

The focus of this work lies on extensions of the relationship between two mathematical methods describing the scaling limit of two-dimensional Statistical Physics models at criticality: boundary Conformal Field Theory (BCFT) and SCHRAMM-LÖWNER Evolution (SLE).

This thesis is organized as follows: Having provided a short motivation and introduction why and in which context the topic of our thesis is of interest, chapters 3 and 4 serve as a guide to the basic notions of Conformal Field Theory and Stochastic LÖWNER Evolution, respectively. They are aimed at an audience of theoretical physicists. We anticipate profound knowledge in field theory and basic knowledge of the special features of field theories with conformal symmetry. As Stochastic LÖWNER Evolution has only recently been applied to physical models, we will be more detailed in the introduction of this theory, providing supplementary definitions from Probability Theory in the appendix. However, both chapters mainly serve as a reminder fitted to the content of this thesis, not as a self-contained introduction. An expert reader, primarily interested in our original work, will probably skip this part and start directly with chapter 5.

In chapter 3, we briefly introduce the underlying algebraic structure of CFT, i. e. the VIRASORO algebra and its degenerate representations, as well as the implications on primary fields and their correlation functions. Furthermore, we put an emphasis on the fusion product of these representations and the resulting constraints on the operator product expansion. We conclude with a short introduction to the peculiarities of boundary CFT and provide some information on underlying Statistical Physics models.

In chapter 4, we start with LÖWNER's equation for general slit mapping. We then go on motivating Schramm's idea to use this equation to study conformally invariant curves exhibiting a MARKOV property, presenting the various variants of SLE. Furthermore, we review the connection to CFT and the scaling limit of Statistical Physics models. If more information is needed, some notions about Probability Theory can be found in appendix A.4. Otherwise we refer to standard texts listed in the bibliography on page 155.

Chapter 5 consists of unpublished materials extending preceding work [65, 61, 66, 60, 67, 68]. We derive the SLE variant connected to the most general partition function of a BCFT model. This is done by specifying the change of measure needed to go from single SLE to this most general physically relevant SLE variant which we call multiple SLE$(\kappa, \vec{\rho})$. In addition, we comment on how BCFT bulk fields should be treated in SLE, making a clear distinction between observables and bulk force-fields related to SLE$(\kappa, \vec{\rho})$ [68].

Chapter 6 is mainly based on the ideas published in [69] and [70]. As the theoretical background of these two papers is mostly the same, we only quote the content of the first paper [69] in full detail in part 6a. Therein, we present a physical interpretation of the probability of boundaries of single Stochastic Löwner Evolution hulls approaching marked points in the upper half-plane $\mathbb{H}$. In the perspective of boundary conformal field theory, boundaries of SLE hulls are created by boundary condition changing fields. We argue that the probability can be described via the short-distance behavior of these boundary and bulk fields in bCFT expectation values. Furthermore, we show that the resulting expectation values are also martingales in SLE.

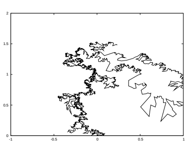
In part 6b, we include the recently gained insights presented in 5, applying them to the results of [69] and [70]. We investigate the short-distance behavior of an SLE curve-creating field and another bulk *force*-field in SLE$(\kappa, \vec{\rho})$, as well as another curve-creating boundary field in multiple SLE. We proceed in a similar way to part a of this chapter: We prove the martingale property the corresponding SLE partition function wherein we replace these field by their operator product expansion. We provide an interpretation for the resulting SLE objects in terms of a well known probability, i. e. the SLE curve corresponding to the boundary of the SLE hull hitting a disc around a point and the corresponding event.

Chapter 7 contains yet unpublished material [71]. We investigate the algebraic κ-relation in multiple SLE, extending it to the case of two traces joining. From this, we extract the infinitesimal Löwner equation for such a joint process. The results of this chapter are based on lengthy calculations whose details are given in appendix A.3.

In chapter 5, we provide an annotated outlook to other methods addressing the same question. We review selected preceding work [72, 73, 74, 75], concerning SLE on fractions of the upper half-plane and Lévy-type driving processes as well as our thoughts on the general nature of the problem faced. Additionally, we present yet another promising candidate for SLE type driving processes: fractional Brownian motion.

We conclude with chapter 9, sum up the results presented and provide an outlook to open questions.

We tried to formulate each of the chapters of our original work in a self-contained manner. Therefore, some of the introductions to chapters 5, 6 (a and b) and 7 may be redundant but with a focus on different aspects.



3 Conformal Field Theory

> *Die Sprache der Mathematik erweist sich als über alle Maßen effektiv, ein wunderbares Geschenk, das wir weder verstehen noch verdienen. Wir sollten dafür dankbar sein und hoffen, daß sie auch bei zukünftigen Forschungen ihre Gültigkeit behält und daß sie sich – in Freud und in Leid, zu unserem Vergnügen wie vielleicht auch zu unserer Verwirrung – auf viele Wissenszweige ausdehnt.*
>
> E. WIGNER

The scope of this thesis is the extension of the correspondence between Stochastic LÖWNER Evolution and the algebraic and COULOMB gas approach to boundary Conformal Field Theory. This connection has been established via algebraic constraints on correlation functions in BCFT that emerge when computing their variation in LÖWNER time, i. e. checking if they are martingales in the corresponding SLE. To provide a short review of these results which serves as a basis for our original work presented in chapters 5-7, we introduce the basic terms of the algebraic approach to BCFT and the COULOMB gas formalism in the following.

3.1 Mathematical Background

In this section we review some basics of the algebraic description of the so-called minimal models in Conformal Field Theory (CFT). These models are a special subset of rational CFTs and presumably the best known part of all CFT models. As motivated in the introduction, the underlying symmetry of a theory is most important to extract conserved quantities. This is why we concentrate on the underlying symmetry algebra and its standard representations. In quantum CFT, the underlying symmetry algebra is the VIRASORO algebra which is an infinite-dimensional LIE algebra with a central extension, extending the WITT algebra in the classical case. In application to CFT, its generators are the infinitely many infinitesimal parameters that are needed to specify conformal transformations.

3.1.1 Lie Algebras and Representations

A LIE *algebra* is a vector space $\mathfrak{g}$ together with a bilinear, antisymmetric mapping $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ that fulfils the JACOBI identity for all $x, y, z \in \mathfrak{g}$:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0\,. \tag{3.1}$$

A linear map $L : \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism of LIE *algebras* iff

$$[L(x), L(y)]_{\mathfrak{g}'} = L([x,y]_{\mathfrak{g}}) \ . \tag{3.2}$$

A *representation* of a LIE algebra $\mathfrak{g}$ over a vector space V is a LIE algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{End}(V)$ from $\mathfrak{g}$ to the LIE algebra of endomorphisms on V with the usual commutator as LIE bracket $[A, B] = AB - BA$.

To be precise, in the following we will have to deal with the so-called *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ that consists of polynomials in the basis vectors of a given vector space basis of $\mathfrak{g}$ modulo identification of $[x, y]$ with $xy - yx$. Hence, a representation of a LIE algebra can be thought of as a left $\mathcal{U}(\mathfrak{g})$ module with notation Xv for $X \in \mathcal{U}(\mathfrak{g})$ and $v \in V$. However, for the sake of simplicity, we will not distinguish between representations of the algebra and of its universal enveloping in the following.

The HILBERT space of states of any quantum theory is *projective*, e. g. states whose corresponding wave functions only differ by phases are identified with each other. If the underlying physical system admits a symmetry group, it induces automorphisms in the HILBERT space which yield *projective* representations of the group. In the case of connected topological symmetry groups with continuous representations, the set of central extensions gives the classes of projective representations. For any central extension, the unitary representations in the HILBERT space have to be calculated‡[76] (and references therein).

If G is a group and V a vector space over a field F, then a *projective representation* is a group homomorphism from G to $\mathfrak{Aut}(V)/F^*$ where F^* is the normal subgroup of $\mathfrak{Aut}(V)$ consisting of multiplications of elements of V by non-zero elements of F.

Projective representations can arise using the homomorphism between $GL(V)$ and $PGL(V)$ by taking the quotient by the subgroup F^*. However, the reverse operation can not be achieved trivially but needs the introduction of central extensions. From SCHUR's lemma, it follows that the irreducible representations of central extensions of G, and the projective representations of G, describe essentially the same in representation theory.

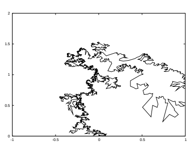
A *central extension* $\mathfrak{a}$ of a LIE algebra $\mathfrak{g}$ with vanishing commutator to another LIE algebra $\mathfrak{e}$, i. e. $\mathfrak{g} \cong \mathfrak{e}/\mathfrak{a}$ is an exact sequence of Lie algebra homomorphisms,

$$0 \to \mathfrak{a} \to \mathfrak{e} \to \mathfrak{g} \to 0\,, \tag{3.3}$$

such that $\mathfrak{a}$ is in the center of $\mathfrak{e}$. Simply speaking this means that the LIE bracket of any of the elements of $\mathfrak{a}$ and the central extension vanishes.

The example we will be using here is $\mathfrak{g} = \mathfrak{witt}$, $\mathfrak{e} = \mathfrak{vir}$ and $\mathfrak{e} = \mathrm{span}\{c\}$ where c is the central charge.

‡Admittedly, this is a rather general statement: most CFTs are non-unitary. However, in CFT, unitarity is not a necessary criterion for physical relevance as models such as the YANG-LEE Edge Singularity prove.



3.1.2 The Witt Algebra and its Representations

The WITT algebra $\mathfrak{witt}$ is spanned by infinitely many generators l_n, $n \in \mathbb{Z}$, for which we usually take a special representation in terms of complex vector fields $-z^{1+n}\partial_z$ with $z \in \mathbb{C}$. Hence, the inner product is given by

$$[l_n, l_m] = (n-m)l_{n+m}\,. \tag{3.4}$$

Heuristically, these generators can be thought of as the infinitesimal generators of local conformal transformations in a classical theory

$$z \mapsto z - \epsilon_n z^{1+n}\,. \tag{3.5}$$

with ϵ_n being one of the infinitely many infinitesimal parameters that are needed to specify a generic conformal transformation

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_n z^{1+n} \quad \text{usually normalized with } \epsilon_0 = 1\,. \tag{3.6}$$

As a symmetry algebra in classical field theory, the WITT algebra is employed in the investigation of conformally invariant systems with local interactions.

The effect of the transformation (3.6) on a spin- and dimensionless field, i. e. a bosonic field in two dimensions, on the plane, $\phi(z,\bar{z})$ illustrates how this representation of the WITT algebra arises:

$$\begin{aligned} \delta\phi(z,\bar{z}) &= -\epsilon(z)\partial_z\phi(z,\bar{z}) - \bar{\epsilon}(\bar{z})\partial_{\bar{z}}\phi(z,\bar{z}) \\ &= \sum_{n\in\mathbb{Z}} \left(\epsilon_n l_n + \bar{\epsilon}_n \bar{l}_n\right)\phi(z,\bar{z})\,, \end{aligned} \tag{3.7}$$

as can easily be deduced from (3.6).

3.1.3 (Anti-)Analytic Functions

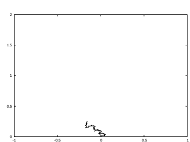
A complex function is said to be *analytic* on a region R if it is complex differentiable at every point in R. The terms *holomorphic* function, *differentiable* function, and *complex differentiable* function are sometimes used interchangeably with "analytic function". It satisfies

$$\partial_{\bar{z}} f(z) = 0\,, \tag{3.8}$$

which is the holomorphic CAUCHY-RIEMANN differential equation. Note that, obviously, (3.6) satisfies (3.8). Analogously, an antianalytic function satisfies

$$\partial_z f(\bar{z}) = 0\,. \tag{3.9}$$

Usually, we use the slightly less restrictive *meromorphic* functions, which are analytic on the whole domain except on a set of isolated points, which are poles for the functions. A possible representation of the conformal group in 2D is the set of



meromorphic functions with composition of function as group multiplication. It is infinite-dimensional since the infinite number of coefficients of the LAURENT series has to be specified to fix all functions in some neighborhood.

A function that is analytic on the whole complex plane is called an *entire* function. In regions where the first derivative is not zero, analytic functions are CONFORMAL in the sense that they preserve angles and the shape (but not size) of small figures.

In Conformal Field Theory, the measurable quantities, i. e. correlation functions or scattering amplitudes, are meromorphic functions of the positions of the fields involved. Note that in the following, we may be imprecise, using the term analytic instead of meromorphic, a habit acquired by many physicists in this field.

3.1.4 (Special) Conformal Transformations

While in physics, the term conformal transformation usually refers to (anti-) holomorphic mappings with non-vanishing derivative on subsets of $\mathbb{C}$, mathematicians use the same term for the *special* conformal transformations, defined on the whole complex plane. In addition to the properties of (anti-)holomorphic mappings mentioned above, they have to be invertible everywhere and map the full domain onto itself again. They are a representation of the special conformal group given by mappings (MÖBIUS transformations) of the form

$$f(z) = \frac{az+b}{cz+d}\,, \qquad \text{with} \qquad ac-bd=1\,, \quad a,b,c,d \in \mathbb{C}\,. \tag{3.10}$$

The (projective) special conformal group is isomorphic to $SL(2,\mathbb{C}) \cong SO(1,3)/\mathbb{Z}_2$. The conditions that reduce the number of conformal transformations to those fulfilling the conditions for special conformal transformations are due to the conditions that invertibility implies that there must not be any branch points or essential singularities present, i. e. the functions have to be fractions of polynomials. In addition, the polynomials have to be linear, i. e. the zeroes of the polynomials have to be distinct and of order one since otherwise the inverse of 0 is not defined and invertibility prohibits multiple wrappings of the image around 0. The determinant condition is due to invariance under overall scaling.

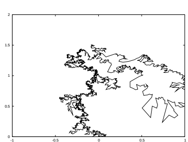
3.1.5 The Virasoro Algebra and its Representations

In this thesis, we analyze conformally invariant theories. Therefore we consider the central extension of the algebra of local conformal transformations, i. e. the WITT algebra $\mathfrak{witt}$, which is the VIRASORO algebra ($\mathfrak{vir}$).

Denoting the generators of $\mathfrak{vir}$ by L_n and $\hat{C}$ for its central extension, the commutator is given by

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{(n-1)n(n+1)}{12}\delta_{n+m,0}\hat{C}\,, \tag{3.11}$$

$$\left[L_n, \hat{C}\right] = 0\,. \tag{3.12}$$



There are four important subalgebras of $\mathfrak{vir}$: the nilpotent $\mathfrak{vir}_\pm$ that consist of the lowering and raising operators of the L_0 eigenvalue, respectively, and $\mathfrak{h}$ which is the commutative Cartan subalgebra,

$$\mathfrak{vir}_+ = \bigoplus_{n=1}^{\infty} \mathbb{C}L_n\,, \quad \mathfrak{vir}_- = \bigoplus_{n=-\infty}^{-1} \mathbb{C}L_n\,, \quad \mathfrak{h} = \mathbb{C}L_0 \oplus \mathbb{C}\hat{C}\,. \tag{3.13}$$

The fourth is given by the generators of the infinitesimal Möbius transformations L_{-1}, L_0 and L_1 which we will call $\mathfrak{vir}_g$. The group associated with $\mathfrak{vir}_g$ is $SL(2,\mathbb{C})$ which is the group of global conformal symmetry or Möbius invariance.

On the complex plane, L_{-1} and $\bar{L}_{-1}$ generate translations, $L_0 + \bar{L}_0$ and $\mathrm{i}(L_0 - \bar{L}_0)$ generate dilations and rotations while L_1 and $\bar{L}_1$ generate special conformal transformations.

A *positive energy representation* of $\mathfrak{vir}$ with central charge $c \in \mathbb{C}$ is a module V if $\hat{C}$ acts as $c\mathbb{1}$ on V and

$$V = \bigoplus_{m=0}^{\infty} V^{(m)}\,, \tag{3.14}$$

such that $\dim V^{(m)} < \infty$, $L_n V^{(m)} \subset V^{(m-n)}$ and L_0 diagonalizable on each $V^{(m)}$, e. g. excluding so-called logarithmic conformal field theories. Given such a positive energy representation, the contravariant representation V^* can be defined, again being a positive energy representation. Given this, we have a natural bilinear pairing

$$\langle \cdot, \cdot \rangle\,:\, V^* \times V \to \mathbb{C}\,, \tag{3.15}$$

since V^* is a direct sum of duals by construction. We will be interested in those representations whose grading corresponds to L_0 eigenvalues. Hence, if for some $h \in \mathbb{C}$, $V^{(m)}$ is the L_0 eigenspace of the L_0 eigenvalue $h+m$, we call the representation (h, c).

A vector $w \neq 0$ in a module of $\mathfrak{vir}$ is called *singular* or *null vector* if

$$L_n w = 0 \quad \forall n > 0\,, \tag{3.16}$$

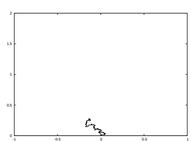
$$\hat{C} w = c w\,, \tag{3.17}$$

$$L_0 w = h w\,. \tag{3.18}$$

A *highest weight module* V is a module with $v_0 \in V$ iff for some $h, c \in \mathbb{C}$, v_0 is a singular vector and V is generated by v_0, i. e. $V = \mathcal{U}(\mathfrak{vir})v_0$. In this case, (h, c) is called *highest weight* of the *highest weight vector* v_0.

These highest weight representations can be decomposed in complete analogy to the singular (h, c) representations. Note that the subspaces $V^{(m)}$ are spanned by $L_{-n_1} \ldots L_{-n_k} v_0$ with $k \in \mathbb{N}$ and $n_1 \geq n_2 \geq \ldots \geq n_k > 0$ and $\sum_j n_j = m$. Hence, $\dim V^{(m)} \leq p(m)$ with

$$p(m) = \sharp \left\{ (\mu_1, \ldots, \mu_m) \in \mathbb{N}^m \,:\, \sum_{j=1}^{m} \mu_j = m \right\}\,. \tag{3.19}$$



A VERMA *module* $M_{h,c}$ is a special highest weight representation of $\mathfrak{vir}$. Let $v_{h,c} \in M_{h,c}$ be a highest weight vector and V a module of $\mathfrak{vir}$ containing a singular vector w of type (h, c). There exists a unique $\mathfrak{vir}$-homomorphism $\varphi : M_{h,c} \to V$ such that $\varphi(v_{h,c}) = w$. For VERMA modules $\dim V^{(m)} = p(m)$. $M_{h,c}$ is indecomposable but reducible and there exists a maximal subrepresentation $J_{h,c} \subsetneq M_{h,c}$ such that $V_{h,c} = M_{h,c}/J_{h,c}$ is the irreducible unique highest weight representation of highest weight (h, c). For every rational $c < 1$, these are the representations we are dealing with when speaking about the *rational minimal models* in CFT. Within these models, it can be shown that the central charge c and the highest weights h have to be parameterized by two coprime numbers $p, q \in \mathbb{N}$:

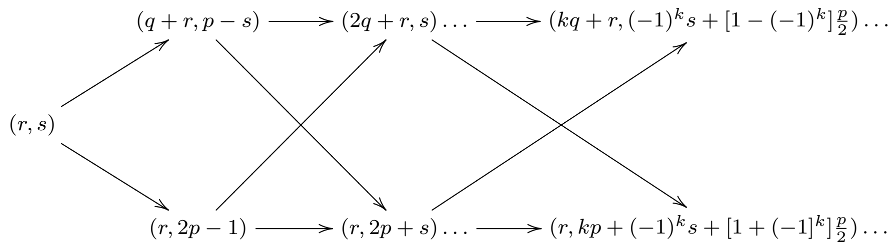
$$c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}\,, \tag{3.20}$$

$$h_{(r,s)}(c_{p,q}) = \frac{(pr-qs)^2 - (p-q)^2}{4pq} \quad \text{for} \quad 1 \leq r < q,\ 1 \leq < p\,. \tag{3.21}$$

The $(p-1) \times (q-1)$ table of weights is called KAČ table and the number $r \cdot s$ denotes the level. These representations are chosen such that there exist highest weight states which are special linear combinations of descendants of $v_{h,c}$ on level $m = r \cdot s$ such they and all their descendants are orthogonal to any other state of the rest. Therefore, the corresponding VERMA module is reducible and by symmetry properties of the KAČ table, it can be shown that an infinite number of mutually intersecting submodules has to be divided out to obtain $M_{h_{(r,s)},c_{p,q}}$ or, more precisely:

$$M_{h_{(r,s)}} = V_{h_{(r,s)}} - (V_{h_{(q+r,p-s)}} \cup V_{h_{(r,2p-s)}}) + (V_{h_{(2q+r,s)}} - V_{h_{(r,2p+s)}}) - \dots\,, \tag{3.22}$$

wherein we suppressed the second index $c_{p,q}$. The terms can be obtained according to the diagram:

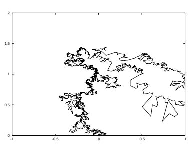


(3.23)

wherein the indices (m, n) denote the location in the KAČ table.

3.2 Conformal Field Theory

In all fundamental theories in physics, there is a search for symmetry, i. e. for those transformations that leave the system unchanged. It may be observable or only



intrinsic, continuous or discrete, but in all cases it leads to simplifications of the model, sometimes even allowing for exact solutions.

Conformal Field Theory, as its name indicates, is invariant under the so-called local conformal transformations that are, infinitesimally, described by the VIRASORO algebra. To see how the symmetry is manifested, we have to specify the coordinate system. Usually, the two basic coordinates are taken to be the complex variables $z = x + \mathrm{i}y$ and $\bar{z}$, rather than $x, y \in \mathbb{R}$. This way, the basic line element is $\mathrm{d}x\,\mathrm{d}y = \mathrm{d}z\,\mathrm{d}\bar{z}$ and it scales as

$$\mathrm{d}z\,\mathrm{d}\bar{z} \to \left|\frac{\partial f}{\partial z}\right|^2 \mathrm{d}z\mathrm{d}\bar{z} \tag{3.24}$$

under conformal transformations $f(z, \bar{z})$. As any such transformation satisfies the CAUCHY-RIEMANN equations

$$\frac{\partial}{\partial \bar{z}} f(z, \bar{z}) = \frac{\partial}{\partial z} f(z, \bar{z}) = 0\,, \tag{3.25}$$

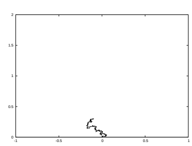
the analytic and antianalytic mappings can be regarded as independent. Their generators, $\{L_n\}$ and $\{\bar{L}_n\}$ (two copies of $\mathfrak{vir}$ generators), decouple and the two coordinates z and $\bar{z}$ can be treated as independent coordinates, which means that $\bar{z} \neq z^*$, i. e. $\bar{z}$ does not have to be treated as the complex conjugate of z. However, this also means that we are complexifying the physical space somehow artificially: $\mathbb{R}^2 \cong \mathbb{C} \to \mathbb{C}^2$. Different physical realities are therefore recovered by taking different 2D-sections of $\mathbb{C}^2$, e. g. EUCLIDean space: $\bar{z} = z^*$, or MINKOWSKI space: $\bar{z} = -z^*$. When performing computations in CFT, we regard the two parts of the theory as completely independent. Only upon extracting physical implications, we have to take the appropriate section.

For a physical interpretation of a quantum or classical theory, one is interested in observables. Despite the elements of the scattering matrix (S-matrix or, in CFT, the correlation functions) these are e. g. local charges and currents, as well as the local stress-energy tensor.

3.2.1 The Stress-Energy Tensor

In general, the stress-energy tensor is the conserved NOETHER current[†] associated to space-time translations, which is why it can be considered as the generator of arbitrary space-time translations in reverse. It is traceless for a special gauge (the addition of a pure divergence does not affect its definition). Also it can be shown that its components, T_{zz} and $T_{\bar{z}\bar{z}}$ are holomorphic and anti-holomorphic, respectively, which represents the fact that there are two independent copies of $\mathfrak{vir}$ generating the whole conformal algebra.

[†] More precisely, for non-classical but quantum systems, the current associated with the WARD-TAKAHASHI identity, which takes also care of the invariance of the functional measure of the path integral under the symmetry group transformations.



The stress energy tensor is the object in CFT that provides us with the necessary information about the central charge c that specifies the minimal model. Under conformal transformations, $T(z,\bar{z})$ does not simply reproduce itself modulo prefactors as primary fields do but exhibits an extra term:

$$T(z,\bar{z}) \to \tilde{T}(w,\bar{w}) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-2}\left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right)^{-2}\tilde{T}(z,\bar{z}) + \frac{c}{12}\mathcal{S}(z;w)\,, \tag{3.26}$$

where $\mathcal{S}(z;w)$ is the SCHWARZian derivative:

$$\mathcal{S}(z;w) = \frac{\partial_w^3 z}{\partial_w z} - \frac{3}{2}\left(\frac{\partial_w^2 z}{\partial_w z}\right)^2 . \tag{3.27}$$

Physically, the appearance of the central charge or *conformal anomaly* c is a “soft” breaking of conformal symmetry which may for example show up by introducing a macroscopic scale into the system, e. g. by mapping the theory onto a cylinder which would yield a CASIMIR energy term, i. e. the change in the vacuum (free) energy density due to the periodicity condition. It also appears when investigating finite size effects or curved manifolds as a proportionality factor of the curvature tensor which implies a macroscopic scale, too.

Additionally, $T(z,\bar{z})$ is the generator of conformal transformations. This can be seen when looking at its LAURENT expansion where the generators of the VIRASORO algebra appear as coefficients:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-(n+1)}L_n\,, \quad T(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-(n+1)}\bar{L}_n\,. \tag{3.28}$$

Note that $T(z,\bar{z})$ is conserved and traceless.

3.2.2 Primary and Descendant Fields

The basic objects in CFT besides the stress energy tensor or free bosons are the so-called *primary fields* $\phi(z,\bar{z})$ that satisfy the following commutation relations:

$$[L_n,\phi] = z^{n+1}\partial_z\phi + h(n+1)z^n\phi\,, \tag{3.29}$$

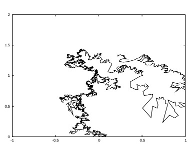
$$\left[\bar{L}_n,\phi\right] = \bar{z}^{n+1}\partial_{\bar{z}}\phi + h(n+1)\bar{z}^n\phi\,. \tag{3.30}$$

These fields correspond to highest weight states of weight $h,\bar{h}$ via

$$\lim_{z,\bar{z}\to 0}\phi(z,\bar{z})|0\rangle =: |h,\bar{h}\rangle\,. \tag{3.31}$$

Here, using radial ordering, $\langle 0|$ is the ”out” state at $z=0$ while $|0\rangle$ is the “in” state at $z=\infty$.

Note that the analytic and antianalytic part of the primary fields decouple completely, enabling us to treat each sector separately. They become entangled again



when talking about boundary CFT. The notion "local" in this context requires trivial monodromy properties for fields $\phi(z, \bar{z})$ and has to be distinquished from the "local" meaning nearest neighbor interaction when talking about the underlying lattice models.

From any primary field we can get an infinite number of *descendant fields* via the action of the $\sum_{|\{k\}|=m} L_{-k_1} \cdots L_{-k_n}$, $k_1 \geq \ldots \geq k_n > 0$. It is said to be of level m if $m = |\{k\}| = \sum_i k_i$ and usually denoted by

$$L_{-k_j}\phi(z, \bar{z}) = \phi^{(-k_j)}(z, \bar{z}) = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{1}{(w-z)^{k_j - 1}} T(w)\phi(z, \bar{z}) \,. \tag{3.32}$$

Note that under conformal transformations, $z \mapsto w(z)$, $\bar{z} \mapsto \bar{w}(\bar{z})$, a primary field behaves as follows

$$\phi(z, \bar{z}) \to \tilde{\phi}(w, \bar{w}) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-h} \left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) \,, \tag{3.33}$$

whereas a descendant (or secondary) field in general exhibit additional terms. Primary fields transform covariantly with respect to the full local conformal algebra whereas *quasi-primary fields* with respect to the global conformal algebra only. Their eigenvalues with respect to L_0 and $\bar{L}_0$ are h and $\bar{h}$ (3.21), of which we can extract the basic quantum numbers

① scaling dimension: $\Delta = h + \bar{h}$,

② spin: $s = h - \bar{h}$.

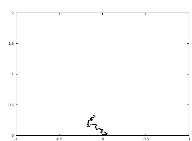
Note that in this thesis, we restrict ourselves to the so-called *diagonal* minimal models, i. e. those whose fields have vanishing spin quantum number and hence $h = \bar{h}$ if not explicitly said otherwise.

Via comparison with (3.26), we can deduce that the stress energy tensor is not a primary field. Fields that behave according to (3.26) are called *quasi primary* since for global conformal transformations, the deformation term, i. e. the SCHWARZIAN derivative, vanishes.

3.2.3 Differential Equations from Singular Vectors

The measurable objects in field theory are in general *correlation* (or GREEN'S) *functions* of fields (elements of the S-matrix), i. e. in minimal models correlation functions of primary fields and their descendants. If all of these objects are known, the theory is completely solvable since any scattering amplitude can be computed. In 2D minimal CFT models, there is an especially nice way to achieve this due to the presence of singular vectors such that we do not have to compute any path integrals††. Note

††Path integrals are, in general, not well-defined mathematical objects. There is some difficulty in defining a measure over the space of paths. In particular, the measure is concentrated on "fractal-like" distributional paths.



that due to global conformal invariance, we can fix three parameters (cf. (3.10)) and therefore the behavior of the one-, two- and three-point functions is known:

$$\begin{aligned} \langle \phi_{h_1}(z_1) \rangle &= 0\,, &(3.34)\\ \langle \phi_{h_1}(z_1)\phi_{h_2}(z_2) \rangle &= C_{12}(z_1 - z_2)^{h_1+h_2}\delta_{h_1,h_2}\,, &(3.35)\\ \langle \phi_{h_1}(z_1)\phi_{h_2}(z_2)\phi_{h_3}(z_3) \rangle &= g_{123}\prod_{i<j=1}^{3}(z_i - z_j)^{h_k - h_i - h_j}\epsilon_{ijk}\,. &(3.36) \end{aligned}$$

This can easily be shown via the transformation property of primary fields (3.33). Since due to the same arguments, we can choose a transformation which sends three coordinates in any correlation function to $0, 1, \infty$, four-point functions can be expressed in terms of the *anharmonic* or *cross ratio*:

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}\,, \tag{3.37}$$

which we will use in the following.

Within the minimal models, singular vectors (null states) that are special descendants of a highest weight vector of weight $h_{(r,s)}$ on level $m = r \cdot s$ occur. This means that there is a certain combination of descendants of a primary field on level m which vanishes under the action of all L_n with $n \geq -1$. These states are divided out to get the Verma module $V_{h,c}$ and thus any correlation function in a minimal model containing one of these states has to vanish:

$$\langle 0|\phi_{h_{(r_1,s_1)}}(z_1,\bar{z}_1)\cdots\sum_{|\{k\}|=m}L_{-\{k\}}\phi_{h_{(r_l,s_l)}}(z_l,\bar{z}_l)\cdots\phi_{h_{(r_n,s_n)}}(z_n,\bar{z}_n)|0\rangle = 0\,. \tag{3.38}$$

Using (3.32), we can pull the contour around the other points and get a homogeneous differential equation

$$\sum_{|\{k\}|=m}\mathcal{L}_{-\{k\}}\langle 0|\phi_{h_{(r_1,s_1)}}(z_1,\bar{z}_1)\cdots\phi_{h_{(r_l,s_l)}}(z_l,\bar{z}_l)\cdots\phi_{h_{(r_n,s_n)}}(z_n,\bar{z}_n)|0\rangle = 0 \tag{3.39}$$

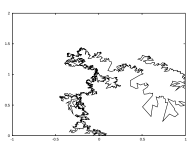
with

$$\mathcal{L}_{-k_j} = \sum_{i\neq l}^{n}\frac{(k-1)h_i}{(z_i - z_l)^k} + \frac{1}{(z_i - z_l)^k}\partial_{z_i}\,. \tag{3.40}$$

The same can be done for the antiholomorphic part.

Eq. (3.39) is the general form of a so-called *null vector differential equation* arising from singular vectors that are divided out of a Verma module.

With this procedure, we can fix the solutions for the four-point functions. However, a general CFT contains also more general $n > 4$ point functions. In this case, we have



to insert complete sets of intermediate states in the correlator to obtain four-point functions depending on the anharmonic ratios $x, \bar{x}$ only:

$$\langle \phi_{h_1}(z_1,\bar{z}_1)\cdots\phi_{h_4}(z_4,\bar{z}_4)\rangle = \prod_{i<j}(z_i-z_j)^{h-h_i-h_j}(\bar{z}_i-\bar{z}_j)^{\bar{h}-\bar{h}_i-\bar{h}_j}\mathfrak{R}(x,\bar{x})\,, \quad (3.41)$$

where $h = \sum_i h_i/3$. Now, in complete analogy to other Quantum Field Theories, either the s, the t or the u channel can propagate, giving the same result:

$$\mathfrak{R}_s(x,\bar{x}) = \sum_\gamma C_{12\gamma}C_{34\gamma}\mathfrak{F}_{12,34}(\gamma,x)\bar{\mathfrak{F}}_{12,34}(\gamma,\bar{x})\,, \quad (3.42)$$

$$\mathfrak{R}_t(x,\bar{x}) = \sum_\beta C_{31\beta}C_{24\beta}\mathfrak{F}_{31,24}(\beta,1/x)\bar{\mathfrak{F}}_{31,24}(\beta,1/\bar{x})\,, \quad (3.43)$$

$$\mathfrak{R}_u(x,\bar{x}) = \sum_\alpha C_{14\alpha}C_{23\alpha}\mathfrak{F}_{14,23}(\alpha,1-x)\bar{\mathfrak{F}}_{14,23}(\alpha,1-\bar{x})\,, \quad (3.44)$$

i. e. $\mathfrak{R}_s = \mathfrak{R}_t = \mathfrak{R}_u$, the *duality* condition (crossing symmetry)[‡]. The $\mathfrak{F}$s are called *conformal blocks* of the theory because any correlation function can be constructed from them [25]. They are meromorphic functions of the anharmonic ratios and possible point singularities at $x, \bar{x} = 0, 1, \infty$ and branch cuts joining these points. Their analytic continuation through these branch cuts provides a multivalued block function with nontrivial monodromy.

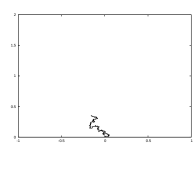
In order to decide which intermediate channels propagate when inserting the complete set of intermediate states, we have to introduce the concept of fusion and the operator product expansion.

3.2.4 Fusion and the Operator Product expansion

In the minimal models introduced in section 3.1.5 with central charge $c_{p,q}$ as specified in (3.20), the fusion product, i. e. the analog of the tensor product of representations without central extensions, closes, yielding only a finite number of representations $(h_{(r,s)}, c_{p,q})$ of highest weights $h_{(r,s)}$(3.20). In order to solve a general n point function of fields corresponding to representations in these models, we have to specify the small distance behavior of the fields $\phi_{h_{(r,s)}}$. Therefore we need to know the outcome of the so-called *fusion* product of two highest weight representations in the KAČ table (3.20):

$$(h_{(r,s)}, c_{p,q}) \times (h_{(k,l)}, c_{p,q}) = \sum_{\substack{m = k-r+1 \\ m-k+r-1 \text{ even}}}^{\max(k+r-1,2q-1-k-r)} \; \sum_{\substack{n = l-s+1 \\ n-l+s-1 \text{ even}}}^{\max(l+s-1,2p-1-l-s)} (h_{(m,n)}, c_{p,q})\,, \quad (3.45)$$

[‡]Duality on the sphere together with modular invariance on the torus which we will not introduce here implies consistency for any genus, i. e. provides enough constraints to build up all associative CFTs.



where m, n are incremented by 2. This leads us to the small distance product or *operator product expansion* (OPE) [77, 78, 79] for two primary fields:

$$\phi_{h_{(r,s)}}(z)\phi_{h_{(k,l)}}(w) = (z-w)^{\mu} \sum_{h_{(m,n)}} g_{(m,n)} \sum_{Y} (z-w)^{|Y|} \beta^{Y}_{(m,n)} L_{-Y} \phi_{h_{(m,n)}}(x)\,, \tag{3.46}$$

with $\mu = h_{(m,n)} - h_{(r,s)} - h_{(k,l)}$, $g_{(m,n)}$ the coefficient of the three point function involving $\phi_{(r,s)}(z), \phi_{(k,l)}$ and $\phi_{(m,n)}$, $2x = z + w$, $\beta^{Y}_{(m,n)}$ some suitable coefficients that can be computed from the differential equations of the correlation functions and $Y = \{t_j\}$ with $1 \leq t_1 \leq \ldots \leq t_u$ and $|Y| = \sum_j t_j$.

The OPE is defined for field in correlation functions if the distance between the two expanded fields is *small*. In more mathematical terms this means if we take a correlation function of $n+2$ fields, $A(x_i), B(x_j)$ and $C_k(x_k)$ for $k = 1, 2, \ldots, n$, the OPE of $A(x_i)$ with $B(x_j)$ converges if

$$|x_i - x_j| < \min_{k=1,2,\ldots,n} |x_i - x_k|\,. \tag{3.47}$$

This is often referred to as the *coordinate patch* in which the analytical object $\langle A(x_i)B(x_j)\prod_k C_k(x_k)\rangle$ is defined.

Note that for a generic (non-rational) value of c, the fusion product (3.45) would generate an infinite number of conformal families. However, minimal models are build in such a way, that the Kač table is periodic: $h_{(r,s)} = h_{(r+q,s+p)}$ and symmetric $h_{(r,s)} = h_{(q-r,p-s)}$. Together with (3.20), we can deduce that

$$h_{(r,s)} + rs = h_{(q+r,p-s)} = h_{(q-r,p+s)}\,, \tag{3.48}$$

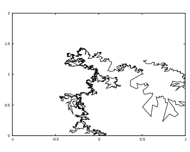
$$h_{(r,s)} + (q-r)(p-s) = h_{(r,2p-s)} = h_{(2q-r,s)}\,. \tag{3.49}$$

This means that the null vector at level $r \cdot s$ is itself the highest weight of a degenerate Verma module, again containing a null vector at level $(q-r)(p-s)$ etc. (3.23). This yields the truncation of the operator algebra which finally leads to a finite set of conformal families as we have introduced above. Due to (3.48) and (3.49) we always have only $(p-q)(q-1)/2$ distinct fields in a minimal model CFT. All in all, this leads us to the Hilbert space of states for a (diagonal)minimal model CFT:

$$\mathcal{H} = \bigoplus_{\substack{1 \leq r < q \\ 1 \leq s < p}} M(h_{(r,s)}, c_{p,q}) \otimes \overline{M}(h_{(r,s)}, c_{p,q})\,. \tag{3.50}$$

Minimal models are examples of rational CFTs, containing only a finite number of primary fields w.r.t. an algebra containing the Virasoro algebra as a sub-algebra. Those with $p - q = 1$ and $p, q \geq 3$ are the only unitary CFTs with $c < 1$ which means that they exhibit reflection positivity in correlation functions.

Additionally we should stress that this is one of the basic assumptions made in CFT: the product of local quantum operators can always be expressed as a linear



combination of well-defined local operators. This expansion constitutes an operator algebra for the operators associated with the local fields. It is the basis for the non-perturbative solution of all correlation functions – without having to take advantage of a local action.

3.2.5 Boundary CFT

Up to now, we have only considered Conformal Field Theory on the full complex plane or other simply connected 2D domains without boundaries. However, in order to describe phenomena such as interfaces of spanning clusters etc., we have to include boundary conditions in our model [31, 80, 81, 82]. Therefore we will consider CFT on $\overline{\mathbb{H}} = \{z \in \mathbb{C} \cup \{\infty\} \,|\, \mathfrak{Im}(z) \geq 0\}$, which is one of the standard simply connected 2D domain with boundary $\partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$.

In CFT on the full complex plane, we can regard the holomorphic and antiholomorphic coordinates, z and $\bar{z}$, as independent variables since any conformal transformation factors into the two parts due to the Cauchy-Riemann differential equations. Any variation of a correlation function with respect to a conformal transformation given by $z \mapsto z + \epsilon(z)$ and $\bar{z} \mapsto \bar{z} + \bar{\epsilon}(\bar{z})$ can be written as

$$\delta_{\epsilon,\bar{\epsilon}}\langle X\rangle_{\mathbb{C}} = -\frac{1}{2\pi \mathrm{i}}\oint_{\mathcal{C}} \mathrm{d}z\,\epsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi \mathrm{i}}\oint_{\mathcal{C}} \mathrm{d}\bar{z}\,\bar{\epsilon}(\bar{z})\langle \bar{T}(\bar{z})X\rangle\,. \tag{3.51}$$

with the counterclockwise contour $\mathcal{C}$ including all the (anti-)holomorphic positions of the primary fields in X.

However, in boundary CFT on the upper half-plane $\overline{\mathbb{H}}$, we are confined to those conformal transformations that leave the real axis, i. e. the boundary, invariant. This gives us the constraint $\epsilon(x) = \bar{\epsilon}(x)$ and $T(x) = \bar{T}(x)$ for any $x \in \partial\mathbb{H}$. Physically, this means that in cartesian coordinates, $T_{xy} = 0$, which means that no momentum flows across the boundary. Taking a look at a correlation function trying to separate the holomorphic and antiholomorphic parts again, we see that we have a non-vanishing contribution from the boundary:

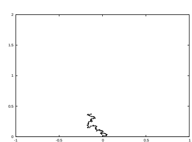
$$\delta_{\epsilon}\langle X\rangle_{\overline{\mathbb{H}}} = -\frac{1}{2\pi \mathrm{i}}\oint_{\mathcal{C}^+} \mathrm{d}z\,\epsilon(z)\langle T(z)X\rangle - \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} \mathrm{d}x\,\epsilon(x)\langle T(x)X\rangle\,, \tag{3.52}$$

$$\delta_{\bar{\epsilon}}\langle X\rangle_{\overline{\mathbb{H}}} = \frac{1}{2\pi \mathrm{i}}\oint_{\mathcal{C}^+} \mathrm{d}\bar{z}\,\bar{\epsilon}(\bar{z})\langle \bar{T}(\bar{z})X\rangle + \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} \mathrm{d}x\,\bar{\epsilon}(x)\langle \bar{T}(x)X\rangle\,. \tag{3.53}$$

where $\mathcal{C}^+$ indicates a counterclockwise contour in the upper half-plane $\overline{\mathbb{H}}$ (including the real axis) encircling all (anti-)holomorphic positions of the primary fields in X.

Obviously, both boundary terms give exactly the same contribution and the two terms are no longer independent. Setting $\bar{z} = z^*$, we can consider the antiholomorphic quantities, e. g. $\bar{T}(\bar{z})$, as being the analytic continuation of the holomorphic quantities, e. g. $T(z)$, in the lower half-plane ‡. This is why we arrive at only one set of

‡This can be proven via the extension of the Schwarz reflection principle to meromorphic functions [83].



VIRASORO generators. In this picture, the bulk fields depending on holomorphic and antiholomorphic coordinates become two separate fields, one being the "mirror-image" of the other in the lower half-plane $\phi(w,\bar{w})_{\overline{\mathbb{H}}} = \phi(w)_{\overline{\mathbb{H}}} \otimes \phi(\bar{w})_{\mathbb{C}\setminus\mathbb{H}} = \phi(w)_{\overline{\mathbb{H}}}\phi(w^*)_{\mathbb{C}\setminus\mathbb{H}}$. Effectively, we are using $2n$ holomorphic degrees of freedom in this picture instead of n holomorphic and n antiholomorphic ones with a boundary condition:

$$\delta_\epsilon \langle X \rangle_{\mathbb{C}_b} = -\frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}+} \mathrm{d}z\, \epsilon(z) \langle T(z) X \rangle + \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}-} \mathrm{d}z\, \epsilon(z) \langle \bar{T}(z) X \rangle \,. \tag{3.54}$$

This point of view suggests that by only considering conformal transformations $f(z)$, $f : D \subset \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ that preserve the boundary, we have to modify the behavior of the primary bulk and boundary fields under these transformations (cf. (3.33)):

$$\phi(z) \to \tilde{\phi}\left(f(z)\right) = \left(f'(z)\right)^{-h} \left(f^{*\prime}(z^*)\right)^{-\bar{h}} \phi(z)\,, \tag{3.55}$$

$$\psi(x) \to \tilde{\psi}\left(f(x)\right) = \left(f'(x)\right)^{-h} \psi(x)\,. \tag{3.56}$$

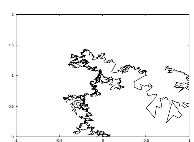
According to [84, 80, 85], changes of boundary conditions can be interpreted as due to insertions of boundary condition changing operators located on the boundary. In the diagonal minimal models which are considered in this thesis, these correspond to the usual representations of the KAČ-table, obeying the same fusion rules. Inserting such a boundary condition changing operator at the origin can be viewed as a vacuum state which is no longer annihilated by L_{-1} in radial quantization. This is equivalent to the action of a boundary operator $\phi_{h_{(r,s)}}(0)$ acting on the true vacuum $|0\rangle$. It corresponds to the highest weight state of the lowest value in the KAČ-table.

3.3 Lattice Models, CFT and Critical Curves

> *There is no such thing at the microscopic level as space or time or spacetime continuum.*
>
> J. A. WHEELER

There is a large variety of literature concerning the question of the relationship between discrete models of Statistical Physics and Field Theories in general. The central points of interests are the expectation values of observable quantities and the partition functions. In Field Theories, they are described via *path integral averages* (EUCLIDEAN or MINKOWSKIan space time) and the local LAGRANGEian $\mathcal{L}$ while in Statistical Physics, they are described via *thermal expectation values* (classical) or *quantum thermal averages* and the energies of the states or the quantum HAMILTONian. The relation between these two has been treated exhaustively in the literature [86, 87], mostly referring to the so-called ISING model which fits well into the best-known classes of Statistical Physics models as well as Conformal Field Theory.

Most of the properties of many 2D lattice statistical models such as Q-state POTTS models can be described in terms of random curves, emerging when boundary conditions are imposed, or closed loops only. Each of the curves and loops has a fixed statistical weight in the ensemble.



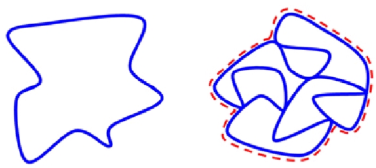


Figure 3.1: Dilute and dense phase of loops in the $O(n)$ model [88].

For a continuous description of the critical behavior of the $O(n)$ model by a local field theory, we need a description of the system in terms of local weights h invariant under $h \to h + 2\pi$ on the dual lattice (for details see section 3.3.2). In the continuum limit, this height function is coarse grained and becomes a fluctuating scalar field $h(z, \bar{z})$. The loops of the $O(n)$ model become the level lines of the field $h(z, \bar{z})$, invariant under $h(z, \bar{z}) \to h(z, \bar{z}) + \text{const}$. In the dense phase, it is believed to be a Gaussian free field, i. e. a massless free boson.Therefore, the relation between curves in critical Statistical Physics lattice models and operators of bCFT is most transparent in their representation by Gaussian (or free Bose) fields $\varphi(z, \bar{z})$ [89]. It has been shown [90, 91] that the rational CFTs with $c \leq 1$ can be represented as a theory of a free boson with Neumann or Dirichlet boundary conditions†. The different boundary conditions corerspond to the dilute and dense phase of the corresponding loop models (see figure 3.1). In the following, we introduce this so-called Coulomb Gas approach and motivate why it is the continuum limit of the $O(n)$ model on the honeycomb lattice (cf. section 3.3.2). Note that as in the case of any field theory, its derivation from the lattice model is not rigorous but a posteriori justified by the results.

3.3.1 The Coulomb Gas and Vertex Operator Algebra Description

The Gaussian free field or massless boson is specified by its action

$$\mathcal{S} = \frac{1}{8\pi} \int_D \mathrm{d}z\mathrm{d}\bar{z} (\nabla\varphi)^2 \,, \tag{3.57}$$

with propagator (or two-point function)

$$\langle \varphi(z, \bar{z}) \varphi(w, \bar{w}) \rangle = -\log |z - w|^2 \,. \tag{3.58}$$

Its general solution can be separated into its holomorphic and antiholomorphic part,

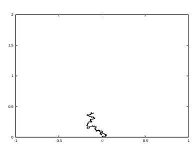
$$\varphi(z, \bar{z}) = \phi(z) + \bar{\phi}(\bar{z}) \,, \tag{3.59}$$

which has vanishing conformal dimension‡. The primary fields of the flat theory are $\partial_z \phi(z)$ and $\partial_{\bar{z}} \bar{\phi}(\bar{z})$ as well as the so-called vertex operators

$$V_\alpha(z) := :\exp(\mathrm{i}\alpha\phi(z)): \quad \text{and} \quad V_{\bar{\alpha}}(\bar{z}) := :\exp(\mathrm{i}\bar{\alpha}\bar{\phi}(\bar{z})):\,, \tag{3.60}$$

†Note that the Dirichlet boundary conditions are the key point of the mirror-charge method known from electrostatics.

‡This is precisely the reason why we are able to construct a variety of vertex operators $V_\alpha, V_{\bar{\alpha}}$ from it without introducing a scale.



of dimensions $h = 1$ or $\bar{h} = 1$ and $h = \alpha^2/2$ or $\bar{h} = \bar{\alpha}^2/2$, respectively.

Considering the action (3.57), we see that it is translation invariant. Without too much modification of the dynamics, we may therefore consider its quantized version on a cylinder of compactification radius R, with periodicity condition $\varphi = \varphi + 2\pi R$, ensuring the decoupling of the holomorphic and anti-holomorphic modes of the primary fields $\partial_z \phi(z)$ and $\partial_{\bar{z}} \bar{\phi}(\bar{z})$ §. This yields an additional term proportional to R which, itself, is related to the central charge $c_{p,q}$ of the underlying rational CFT via $R = \sqrt{2p/q}$:

$$\mathcal{S} = \frac{1}{8\pi} \int_D \mathrm{d}z\, \mathrm{d}\bar{z} \left[(\nabla\varphi)^2 + \mathrm{i}2\sqrt{2}\alpha_0 R \varphi \right] . \tag{3.61}$$

Note that the curvature R can be set to zero everywhere in the finite region of space, concentrating it at infinity which is why we can still consider the boson as a free boson only with a background charge term inserted at infinity. Calculating the stress-energy tensor, we can deduce that it is related to the central charge of the theory via

$$c = 1 - 24\alpha_0^2 . \tag{3.62}$$

For $\alpha_0 \neq 0$, the only primary fields are the vertex operators

$$V_\alpha(z) := \, : \exp(\mathrm{i}\alpha\phi(z)) : \quad \text{and} \quad \bar{V}_{\bar{\alpha}}(\bar{z}) := \, : \exp(\mathrm{i}\bar{\alpha}\bar{\phi}(\bar{z})) : , \tag{3.63}$$

From their OPE with the stress-energy tensor, we can deduce that their weights are modified from those of (3.60) to:

$$h = \alpha(\alpha - 2\alpha_0) \quad \text{and} \quad \bar{h} = \bar{\alpha}(\bar{\alpha} - 2\alpha_0) . \tag{3.64}$$

This is defined such that the usual (r, s) labelling of weights in CFT is still valid. The corresponding charge $\alpha_{(r,s)}$ can be expressed by α_+ and α_-, satisfying $\alpha_+ + \alpha_- = 2\alpha_0$ and $\alpha_+ \cdot \alpha_- = -1$:

$$\alpha_{r,s} = \alpha_0 - \frac{1}{2}(r\alpha_+ s\alpha_-) . \tag{3.65}$$

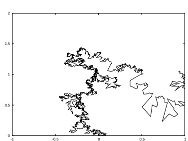
The so-called background charge α_0 [92, 90, 91] ensures zero curvature (everywhere except at infinity) and hence correlation functions of vertex operators have to satisfy a charge neutrality condition

$$\sum_k \alpha_k = 2\alpha_0 . \tag{3.66}$$

To ensure the charge neutrality condition, non-local screening charge operators have to be inserted. These operators only change the charge but not the conformal properties of the correlator:

$$Q_\pm(z, \bar{z}) = \oint \mathrm{d}z\, \mathrm{d}\bar{z}\, V_\pm(z, \bar{z}) , \tag{3.67}$$

§ $\varphi(z, \bar{z}) = \varphi_0 - \frac{\mathrm{i}}{4\pi g}\pi_0 \log(z\bar{z}) + \frac{\mathrm{i}}{\sqrt{4\pi g}} \sum_{n\neq 0} \frac{1}{n}(a_n z^{-2} + \bar{a}_n \bar{z}^{-n})$



with $\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1}$ which add $\alpha_\pm$ to the charge balance. With the help of the two point function of the free boson (3.58), the correlation function of the vertex operators can be computed as follows:

$$\langle V_{\alpha_1}(z_1, \bar{z}_1) \ldots V_{\alpha_n}(z_n, \bar{z}_n) \rangle = \prod_{i<j} |z_i - z_j|^{4\alpha_i \alpha_j} \,. \tag{3.68}$$

Remark 1. As is easy to see from the action (3.57), the potential of the COULOMB gas is related to the LAPLACEian of the GAUSSian Free Field $h(z, \bar{z})$. The level lines of the GAUSSian Free Field are related to the loops of the COULOMB gas. It can be regarded as the two-dimensional analog to BROWNian motion: similar to BM being the scaling limit of simple random walks and other 1D systems, the GFF is the scaling limit of several discrete models for random surfaces, i. e. height models. Taking its discrete LAPLACEian, these become random loop models, e. g. the $O(n)$ model.

3.3.2 O(n)-model

The $O(n)$-model can either be formulated as a model of classical spins $\vec{\sigma}_i$ on the vertices of a lattice

$$H = -J \sum_{\langle i,j \rangle} \langle \vec{\sigma}_i, \vec{\sigma}_j \rangle \,, \tag{3.69}$$

or via critical curves or loops on the honeycomb lattice (for a part of the parameter range, i. e. $|n| \leq 2$). In this case, the partition function can be obtained by randomly assigning orientations to the loops and summing over all possible configurations. Attaching local weights $\exp(\pm \mathrm{i} \frac{e_0 \pi}{6})$ to each lattice site, depending on whether the graph makes a left or right turn at it, the weight of an oriented closed loop is obviously $\exp(\pm \mathrm{i} e_0 \pi)$ and the sum over the orientations gives $n = 2 \cos \pi e_0$. On the dual lattice, these local weights form a configuration where neighboring sites never differ by more than 1. Therefore it is easy to see that in the continuum limit, the model on the dual lattice corresponds to a GAUSSIAN free field while the loops on the honeycomb lattice form its level lines.

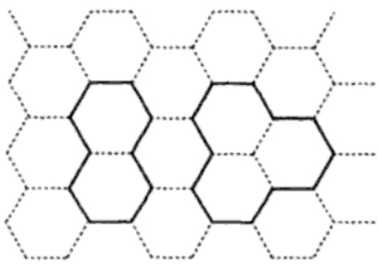
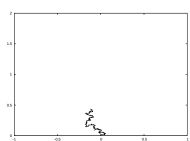


Figure 3.2: Loops on the honeycomb lattice [23].

Choosing $e_0 \in [-1, 1]$, we get the dense phase for positive and the dilute phase for negative e_0. The two phases correspond to DIRICHLET and NEUMANN boundary conditions in BCFT; the external perimeter of loops (dashed curve in figure 3.1) in the dense phase is a always dilute loop which will become important when speaking about the duality in SLE later on.

The $O(n)$ models exhibits a critical point in the range $-2 \leq n \leq 2$, but according



to the Mermin-Wagner theorem¶, they can not undergo a phase transition at non-vanishing temperature in two dimensions for $n > 2$. The critical point is at $\beta J_c = (s + \sqrt{2-n})^{-1/2}$.

They are connected to CFT via $e_0 = g - 1$ for $g \in [0, 2]$ and the relation

$$n = -2\cos\pi g\,, \quad c(n) = 1 - 6\frac{(g-1)^2}{g}\,. \tag{3.70}$$

Hence, in the continuum limit, the $O(n)$ model can describe rational CFTs with central charges $-2 \leq c \leq 1$.

n	Model
$n = 0$	Self avoiding walks
$n = 1$	Ising model [93, 94, 95]
$n = 2$	XY-model at criticality (Kosterlitz-Thouless point)
$n = 3$	(classical) Heisenberg model (superfluid helium, liquid crystals)
$n \to \infty$	spherical model (exactly solvable)

Table 3.1: Models described by the $O(n)$ model

3.3.3 Q-states Potts model

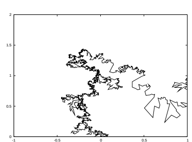
The Q-states Potts model describes integer valued spins θ_j on lattice vertices with nearest neighbor interactions and is therefore defined via the Hamiltonian

$$H_p = -J_p \sum_{\langle i,j\rangle} \delta(\theta_i, \theta_j)\,, \tag{3.71}$$

where θ_n can take one of $q \in \mathbb{N}$ different states, e. g. $\theta = 0, 1, \ldots, Q-1$.

There are two distinct phases in the Q-states Potts model: a disordered high-temperature phase and an ordered low-temperature phase. The transition between these two phases is of first order for $Q > 4$ and of second order for $Q \leq 4$. The critical point is at $K_c = J/T_c = \ln(\sqrt{Q} + 1)$.

¶Continuous symmetries cannot be spontaneously broken at finite temperature in one and two dimensions. Otherwise the Goldstone bosons would exhibit an IR divergent correlation function.



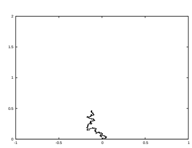
n	Model
$Q \to 1$	Percolation
$Q = 2$	Ising model (take $\sigma = 2(\theta - 1/2)$)
$Q = 3$	Heisenberg model (superfluid helium, liquid crystals)

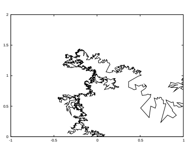
Table 3.2: Models described by the Q-states Potts model

The Q-state Potts model is connected to the unitary minimal models via

$$c = 2\left(1 - \frac{6}{(k+1)(k+2)}\right) \quad \leftrightarrow \quad \sqrt{Q} = 2\cos\frac{\pi}{k+2}\,. \tag{3.72}$$

Remark 2. Note that the Q-state Potts model can be generalized to $Q \in \mathbb{R}$ via the loop/cluster formulation and its relation to the six-vertex model. It can be viewed as the lattice version of the Dotsenko-Fateev twisted bosonic theory and has therefore been very important for issues in CFT and integrable systems. In this representation, it is defined non-locally, which makes a relation to CFT quite tricky while it can be related to SLE in a natural way.





4 Stochastic Löwner Evolution

A mathematician is a blind man in a dark room looking for a black cat which isn't there.

CHARLES R. DARWIN

4.1 Mathematical Background

The dynamics of the Stochastic LÖWNER Evolution are encoded in a differential equation (LÖWNER's equation) for a conformal mapping of a subset of the upper half-plane, i. e. the upper half-plane minus a continuous curve connected to the boundary, onto the upper half-plane again. In order to define SLE, we need some useful theorems and other definitions.

4.1.1 Standard Domains

As defined in the previous chapter, *conformal mappings* f from $D \subset \mathbb{C}$ to $D' \subset \mathbb{C}$ are bi-holomorphic mappings, i. e. holomorphic mappings whose inverse exists and is also holomorphic. If we restrict ourselves to $f(D) \subset D'$, f is also surjective and therefore f^{-1} is also (anti-)analytic. Such domains D, D' are said to be *conformally equivalent.*

There are a couple of standard domains for simply connected open domains of the complex plane $\mathbb{C}$:

① the unit disk

$$\mathbb{D} := \{z \in \mathbb{C} \, : \, |z| < 1\} \,, \tag{4.1}$$

② the upper half-plane

$$\mathbb{H} := \{z \in \mathbb{C} \, : \, |z| > 0\} \,. \tag{4.2}$$

In the following, we will introduce three theorems about mappings between subsets of two dimensional complex domains.

Theorem 1 (RIEMANN Mapping Theorem)**.** Let $D \subset \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be a non-empty, open, simply connected set such that its complement contains at least two points. For $z_0 \in D$ there exists a unique analytic bijection $f \, : \, D \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) \in \mathbb{R}^+$.

Note that this conversely shows that by fixing three real parameters, the mapping between two conformally invariant domains is completely determined. In CFT this is usually done by fixing the image of three points, $0, 1, \infty$, or by the so-called hydrodynamic normalization where $f(z) = z + \frac{\mathrm{cap}}{z} + \mathcal{O}(z^{-2})$ around infinity.

Usually, we will denote the subsets of $\mathbb{H}$ that satisfy the condition for the RIEMANN mapping theorem by *hulls* where $K \subset \overline{\mathbb{H}}$ is a hull if it is bounded, compact and $\mathbb{H} \backslash K$ is simply connected.

4.1.2 Fractal and Topological Dimension

> *Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.*
>
> BENOÎT MANDELBROT [96]

From a naive point of view, the dimension of a set is just the number of independent parameters needed to describe an element of the set, e. g. two coordinates for a point on the plane. This notion of dimension is often called *topological* dimension.

However, when considering complex more objects, e. g. with fractal features, one observes that the topological notion of dimension fails to fully grasp the nature of the object. This lead to the definition of *fractal* measures, extending the topological dimension to e. g. fractal dimensions like the HAUSDORFF dimension. Famous examples for this are CANTOR sets with $\dim_{\mathrm{top}} = 0$ vs. $\dim_{\mathcal{H}} = \ln \frac{2}{3}$ or the PEANO curve with $\dim_{\mathrm{top}} = 1$ vs. $\dim_{\mathcal{H}} = 2$.

One of the solution to this problem is the box-counting (or MINKOWSKI- BOULIGAND) dimension which corresponds to counting the squares of graph paper needed to cover an object as their size shrinks to zero. Another method, heuristically based on the number of balls N of radius r needed to cover the set completely, has been invented by Felix Hausdorff. Roughly speaking, if $N(r) \propto r^{-d}$ for $r \to 0$, the set has dimension d.

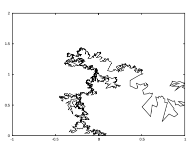
Note that for well-behaved fractal sets considered in this thesis, the different dimensions always yield the same values[†].

Definition 1 (HAUSDORFF measure)**.** If $V \in \mathbb{R}^d$ and $\alpha, \epsilon > 0$, let the HAUSDORFF measure be defined as

$$\mathcal{H}^{\alpha}_{\epsilon}(V) = \inf \sum_{n=1}^{\infty} [\mathrm{diam}\ (U_n)]^{\alpha} \tag{4.3}$$

where the infimum is over all countable collections of sets $U_1, U_2, \ldots$ with $V \subset \bigcup_n U_n$ and $\mathrm{diam}\ (U_n) < \epsilon$ for all $n \in \mathbb{N}$.

[†]An example of a fractal set can be found on the backcover of this thesis. Courtesy of Jock Cooper [97].



Definition 2 (Hausdorff α-measure)**.** The Hausdorff α-measure is defined by

$$\mathcal{H}^\alpha(V) = \lim_{\epsilon \to 0} \mathcal{H}^\alpha_\epsilon(V)\,. \tag{4.4}$$

Definition 3 (Hausdorff dimension)**.** The Hausdorff dimension of V is defined by

$$\begin{aligned} \dim_{\mathcal{H}}(V) &= \inf\{\alpha : \mathcal{H}^\alpha(V) = 0\} & (4.5)\\ &= \sup\{\alpha : \mathcal{H}^\alpha(V) = \infty\}\,. & (4.6) \end{aligned}$$

The Hausdorff dimension is closely related to the so-called Hurst exponent which is much easier to obtain experimentally:

$$H = 2 - \dim_{\mathcal{H}}\,. \tag{4.7}$$

The Hurst exponent splits stochastic processes on the plane into two classes:

① $0 < H < 0.5$: *anti-persistent*, i. e. negative correlations between its non-overlapping increments. After a period of decreases, a period of increases tends to show up. This corresponds to high fractal dimensions, i. e. a noisy profile – curves are highly filling up the plane.

② $0.5 < H < 1$: *persistent*, i. e. positive correlations between its non-overlapping increments. If the curve has been increasing for a period, it is expected to continue increasing for a period.

4.2 Stochastic Löwner Evolution

In this thesis, we are dealing with the dynamics of probability measures $\mu_{D;a,b}$ on growing critical curves in compact two-dimensional domains D, starting and ending on points a, b on the boundary ∂D, which are known to be continuous and non self-crossing. In 1923, Löwner [52] derived an efficient method via encoding the one-dimensional curves $\gamma_{(0,t]}$ in two-dimensional mappings g_t from a subset of the domain $D \backslash K_t$ onto itself again. Their behavior is governed by one-dimensional so-called driving functions X_t, i. e. the images of the curve's tip γ_t at times $t \in \mathbb{R}^+$.

Definition 4. A mapping $g_t : D \to D'$ with $D, D' \subset \mathbb{C}$ is called *hydrodynamically normalized* if

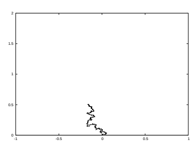
$$\lim_{z \to \infty} g_t(z) - z = 0\,. \tag{4.8}$$

Definition 5. The *half-plane capacity* hcap of a curve $\gamma : [0, \infty) \to \mathbb{H}$ is defined as

$$\mathrm{hcap}_\infty(\gamma_{(0,t)}) := \lim_{z \to \infty} \frac{2}{z}(g_t(z) - z)\,, \tag{4.9}$$

such that in its *natural parameterization* we have

$$\mathrm{hcap}_\infty(\gamma_{(0,t)}) = t\,. \tag{4.10}$$



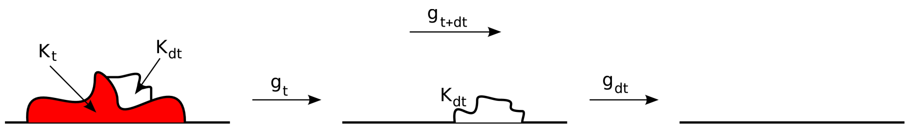


Figure 4.1: The LÖWNER map g_t maps $\mathbb{H}_t = \mathbb{H}/K_t$ onto $\mathbb{H}$. For subsequent time intervals $(0,t)$ (t,s) for $0 < t < s$, the composite map g_{s-t} is identical in distribution to $g_s(g_t^{-1})$.

4.2.1 The Löwner Mapping

The original context in which Löwner derived his famous equation was the BIEBERBACH conjecture stating that $|a_n| \leq n$ for the coefficients in the TAYLOR expansion $f(z) = \sum_{n\geq 0} a_n z^n$. In order to proof the conjecture, Löwner considered growing parameterized conformal maps to a standard domain. In our context, his method provides an indirect access to growing shapes in 2D by means of a time-dependent conformal transformation $g_t(z)$.

Theorem 2. Let $\gamma : [0,\infty) \to \mathbb{H}$ be a continuous parameterization of a curve. This curve should start at $\gamma(0) \in \mathbb{R}$ and may be self-touching but not self-crossing. For $t \in [0,\infty)$, H_t is the unbounded connected component of $\mathbb{H}\backslash\gamma_{(0,t)}$ and for $s > t$, we have $H_s \subsetneq H_t$, meaning that the tip of the curve can not be cut from infinity and the parameterization is never constant. In the following, we will denote the LÖWNER hulls by $K_t := \gamma_t \cup \mathbb{H}\backslash H_t$.

Defining $g_t : H_t \to \mathbb{H}$, we have a family of hydrodynamically normalized conformal maps $(g_t)_{t\in[0,\infty)}$ that encodes the curve γ. The image of the tip of the curve γ_t gets mapped onto $X_t \in \mathbb{R}$: $\lim_{z\to\gamma(t)} g_t(z) = X_t$, defining a continuous process in LÖWNER time. The family of curves and hence $\gamma_{(0,t)}$ can be recovered from X_t via the so-called LÖWNER equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) = \frac{\mathrm{hcap}}{g_t(z) - X_t}\,, \quad g_0(z) = z\,, \tag{4.11}$$

for all $z \in H_t$. X_t is called the driving process of the LÖWNER equation.

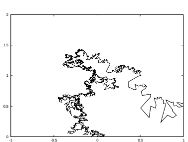
In the standard time parameterization, hcap $= 2$.
The mapping g_T is well defined for all times $t < T(z)$, where

$$T(z) := \sup\{t \geq 0 \,:\, \min_{s\in[0,t]} |g_s(z) - X_s| > 0\}\,. \tag{4.12}$$

Therefore, the LÖWNER hulls can also be defined as

$$K_t = \{z \in \mathbb{H} \,:\, T(z) \leq t\}\,. \tag{4.13}$$

The LÖWNER equation describes the growth of a curve γ_t or a whole shape, e. g. its hull K_t which is the subdomain disconnected from infinity by γ_t, including the path itself. The topological properties and the shape of the curves is now encoded in



the real, time-dependent function X_t, living on the real axis. Since the domain $\mathbb{H}/K_t$ must be simply connected for the RIEMANN theorem to apply, the curve cannot cross itself or the real axis. Whenever it touches itself or the real axis, the enclosed part (i. e. the hull) will be excluded from the domain. The properties of the curves are inherited by the driving function X_t: a continuous curve yields a continuous X_t while discontinuities result in branching curves.

4.2.2 A Stochastic Driving Parameter

> *In fact, all epistemological value of the theory of probability is based on this: that large-scale random phenomena in their collective action create strict, non-random regularity.*
>
> B. V. GNEDENKO and A. N. KOLMOGOROV [98]

In 1999, Oded Schramm [35] investigated the measure on critical curves described by the Loop Erased Random Walk (LERW), exhibiting conformal invariance and the

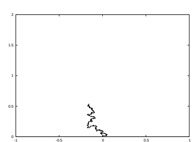
Definition 6 ((Domain) MARKOV Property)**.** If $\gamma_{(0,\infty)}$ is the random curved picked from the measure $\mu_{\mathbb{H};0,\infty}$, i. e. the probability measure on the curve in the domain $\mathbb{H}$ from the boundary point 0 to another point ∞, and parameterized by the capacity of the hull it generates, then conditioned on the law for $\gamma|_{[0,t]}$, the law of $\gamma_{[t,\infty]}$ is $\mu_{H_t;\gamma(t),\infty}$.

Remark 3. Note that the domain MARKOV property, in contrast to the continuum limit needed for conformal invariance, can easily be satisfied on the lattice, too. There are some attempts to define SLE on the lattice which gives a quite promising perspective on proofs for the continuum limit of critical lattices models in two dimensions[99].

Remark 4. From a physicists point of view, it can be considered as the mathematical manifestation of locality. Therefore it is not surprising at all that almost all models of Statistical Physics in two dimensions at criticality satisfy the MARKOV property.

Oded Schramm was well aware of LÖWNER's work on the slit-mapping, describing continuous curves connected to the boundary of bounded simply connected two-dimensional domains. This approach, however, had only been derived for deterministic curves. Therefore he posed the question how to transfer the properties of the measure on the stochastic curves in the LERW, i. e. the MARKOV property and conformal invariance, to the measure on the driving function to obtain the latter.

Theorem 3 (Stochastic LÖWNER Evolution)**.** The only consistent driving process for the LÖWNER mapping g_t to describe conformally invariant curves on $\mathbb{H}$, starting at 0, aiming at ∞, exhibiting the MARKOV property has to be continuous with stationary, independent increments. Therefore it has to be one-dimensional BROWNIAN motion $\xi_t = \sqrt{\kappa} B_t$ with $\kappa \in \mathbb{R}^+$.



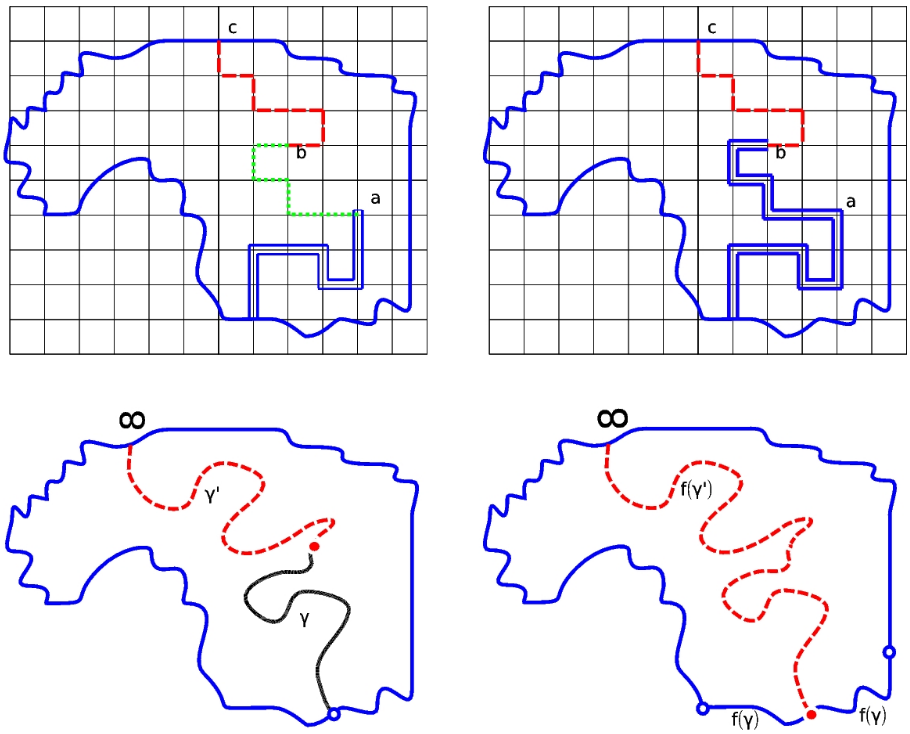
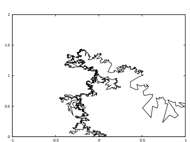


Figure 4.2: Illustration of the domain MARKOV property in the discrete and continuous case. Fix the curve $\gamma_{(a,c)}$ of a possible interface $\gamma_{(a,b)}$ up to a point c. Then considering either the conditional distribution for the rest of the interface $\gamma_{(a,c)}$ or cutting the domain along $\gamma_{(a,c)}$ and considering the distribution of $\gamma_{(c,b)}$ in the cut domain $D\backslash\gamma_{(a,c)}$ gives the same in law: $\mu_{D;a,c}\big|_{\gamma(a,c)} \triangleq \mu_{D\backslash\gamma_{(a,c)};c,b}$. Note that this coincides with the explanation in the text for $a = 0$, $c = \gamma_t$ and $b = \infty$. In the continuous case, we have applied the mapping $g_t(z)$ to map $\gamma(t)$ onto ξ_t on the boundary while the domain gets cut along $\gamma_{(0,t]}$ which is then mapped to the right and left of ξ_t.



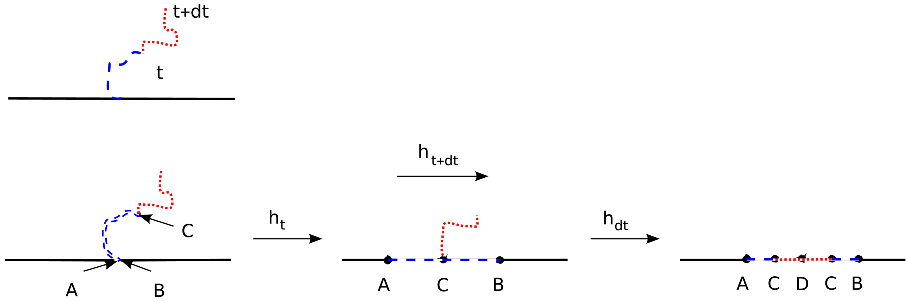


Figure 4.3: The map SLE curve γ_t at time t and $t + \mathrm{d}t$, the action of h_t and $h_{t+\mathrm{d}t}$.

Proof. The proof can be motivated as follows: first define $h_t := g_t - \xi_t$, Then let the curve γ grow from the origin to the tip γ_t such that h_t is the uniformizing map back to the origin. Now let an additional curve segment $\mathrm{d}\gamma$ grow in time $\mathrm{d}t$ which is subsequently absorbed by $h_{t+\mathrm{d}t}$. The Markov property tells us that the distribution of $\mathrm{d}\gamma$ from the origin to $\gamma_{\mathrm{d}t}$ is the same as the distribution of $d\gamma$ grown from t to $t + \mathrm{d}t$ conditioned on γ grown in time 0 to t. Absorbing the segment $d\gamma$ from the origin to $\gamma_{\mathrm{d}t}$ by means of $h_{\mathrm{d}t}$ is the same as $h_{t+\mathrm{d}t} \circ h_t^{-1}$ – both absorb the initial curve $\gamma + d\gamma$ completely. Denoting identical distributions or measures by $\triangleq$, we infer that $h_{\mathrm{d}t}(z) \triangleq h_{t+\mathrm{d}t}(h_t^{-1}(z))$. Using the asymptotic form of $h_t(z) \sim z - \xi_t - \frac{2t}{z}$, we obtain $\xi_{t+\mathrm{d}t} \triangleq \xi_{\mathrm{d}t}$ which means *stationary increments* for the stochastic process. additionally, we can conclude that $\xi_{\mathrm{d}t} \triangleq \xi_{\mathrm{d}t'}$ for non-overlapping time intervals $\mathrm{d}t$ and $\mathrm{d}t'$ which means *independent* increments. Therefore the driving function is fixed to Brownian motion of some variance κ without bias due to the additional reflection symmetry $x \leftrightarrow -x$. □

Note that h_t satisfies the Langevin equation

$$\mathrm{d}h_t = \frac{2\mathrm{d}t}{h_t} + \mathrm{d}\xi_t\,, \tag{4.14}$$

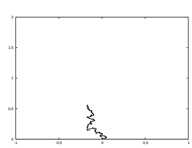
which is also known as the Bessel equation, governing the radial distance R from the origin of a Brownian particle in d dimensions:

$$\mathrm{d}R_t = \frac{\kappa(d-1)\mathrm{d}t}{2R_t} + \mathrm{d}\xi_t\,. \tag{4.15}$$

Therefore we can infer from the known (non-)recurrence properties of the Bessel process in d dimensions that the SLE curve $\gamma_{(0,t]}$ exhibits three *phases*:

① $0 < \kappa \leq 4$: the curve is a. s. a simple path.

② $4 < \kappa < 8$: the curve is self-touching.



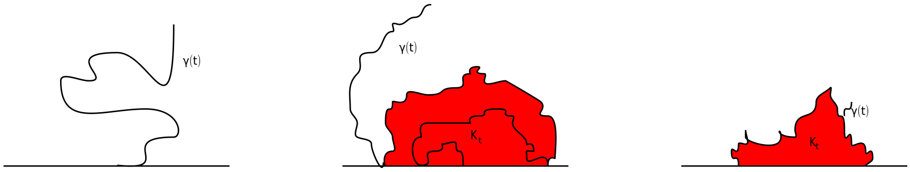


Figure 4.4: The three phases of SLE traces $\gamma(t)$ and their hulls K_t (colored subdomains).

③ $\kappa \geq 8$: the curve is space-filling.

Note that from these points follows immediately that $\gamma_{(0,t]}$ will a. s. (not) touch the boundary for $\kappa(\leq) > 4$.

4.3 SLE Variants

There have been quite a few versions of the SLE defined up to now, referring to the different standard domains $\mathbb{H}$, or $\mathbb{D}$ or the infinite strip, growing from boundary to other boundary or interior points, i. e. chordal, radial or bilateral SLE. Additionally, SLE $(\kappa, \vec{\rho}$ referring to different boundary conditions or multiple SLE describing several interfaces will briefly reviewed in the following.

4.3.1 Chordal SLE on the Upper Half-Plane

Chordal Stochastic Löwner *Evolution* is the study of the Löwner equation with a Gaussian random variable, i. e. one-dimensional Brownian motion of variance κ, with $\xi_t = \sqrt{\kappa} B_t$ as driving process in the upper half-plane.

It is easy to see from (4.11) that the differential equation for chordal SLE reads as follows:

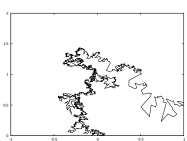
$$\mathrm{d}g_t = \frac{2\mathrm{d}t}{g_t - \xi_t}\,, \quad g_0(z) = z\,, \quad g_t(z) = z + \frac{2t}{z} + \mathcal{O}(z^{-2}) \text{ for } z \to \infty\,. \tag{4.16}$$

The (hydrodynamical) normalization at infinity and the initial condition fix the ambiguities analogously to the three parameters of global conformal invariance.

The Hausdorff dimensions of the SLE curve and the boundary of the SLE hull, which locally looks like the dual SLE with speed $16/\kappa$ ([100, 101, 102, 103, 104, 105, 106, 107]) are given by

$$\dim_\gamma(\kappa) = \max\{1 + \frac{\kappa}{8}, 2\}\,, \tag{4.17}$$

$$\dim_K(\kappa) = \begin{cases} \dim_\gamma(\kappa) & \text{for} \quad 0 < \kappa \leq 4 \\ \dim_\gamma(16/\kappa) & \text{for} \quad \kappa \geq 4 \end{cases}\,. \tag{4.18}$$



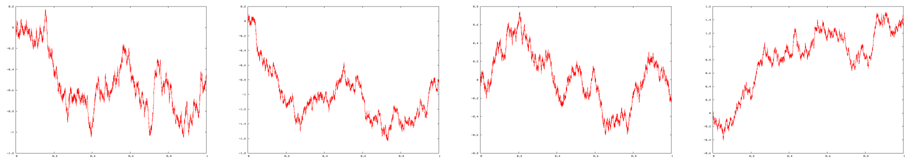


Figure 4.5: Four BROWNIAN motion samples [109]. From these samples, the SLE curves (cf. figures 4.6a-4.8d) with speeds $\kappa = 1, 1.5, 2, \dots, 8$ have been generated.

Note that the curve (or trace) can be obtained via the LÖWNER mapping via $\gamma_t = \lim_{\epsilon \downarrow 0} g_t^{-1}(\xi_t + \mathrm{i}\epsilon)$.

Additionally, the scaling property of BROWNIAN motion implies the scaling of the SLE processes and hulls:

$$g_t(z) = \frac{1}{a} g_{a^2 t}(az) \quad \text{in law}\,, \tag{4.19}$$

$$K_t = \frac{1}{a} K_{a^2 t} \quad \text{in law}\,. \tag{4.20}$$

Locality can be proven for $\kappa = 6$ which means that for these SLEs, perturbations of the domain away from the hull do not affect the SLE as is the case for percolation [43, 40, 41, 42]. For other values of κ though, this leads to a modification of the measure, i. e. so-called restriction measures [108], that account for the changes due to the perturbation. This will become important when talking about multiple SLEs [60, 67] and / or Conformal Loop Ensembles CLEs [36, 37].

4.3.2 SLE($\kappa, \vec{\rho}$)

SLE($\kappa, \vec{\rho}$) [108] is ordinary SLE plus n special boundary points ("force-points") $Y_j \in \mathbb{R}$, providing the possibility of multiple boundary conditions without extra interfaces. The force-points contribute to the driving process with strength ρ_j. At these points and the starting and end point of the SLE curve, X_0 and $X_\infty = \infty$, the boundary conditions change. There also exists an extension to bulk force-points appearing in pairs which we will discuss in greater detail in chapter 5.

Starting from chordal SLE, we choose the SLE starting point $X_0 \in \mathbb{R}$ and n other boundary points $Y_0^1, \dots, Y_0^n \in \mathbb{R}$, as well as some parameters $\rho_1, \dots, \rho_n \in \mathbb{R}$. Introducing $Y_t^i = g_t(Y_0^i)$, we get new driving processes that are martingales under the SLE($\kappa, \vec{\rho}$) measure:

$$\mathrm{d}\xi_t = \sqrt{\kappa}\mathrm{d}B_t \quad \rightarrow \quad \mathrm{d}X_t = \sqrt{\kappa}\mathrm{d}B_t + \sum_{i=1}^{n} \frac{\rho_i}{X_t - Y_t^i}\mathrm{d}t \tag{4.21}$$

$$\mathrm{d}Y_t^i = \frac{2}{Y_t^i - X_t}\mathrm{d}t\,. \tag{4.22}$$

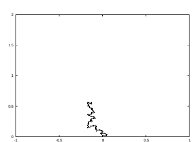
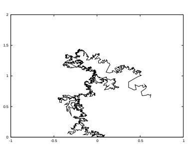


Figure 4.6: Phase 1 ($0 \leq \kappa \leq 4$): the SLE trace is a simple path. The paths above have $\kappa = 1, 1.5, \ldots, 4$ [109].



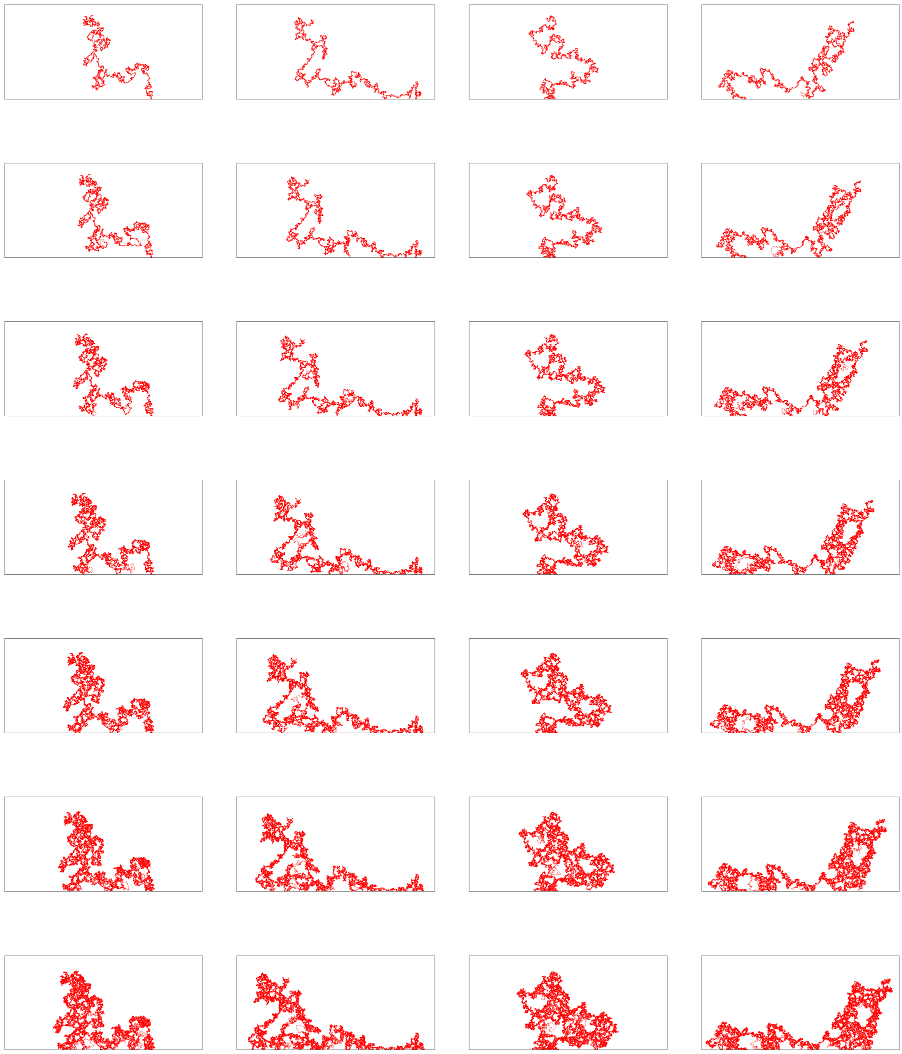
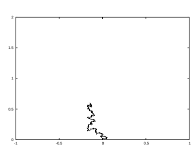


Figure 4.7: Phase 2 ($4 < \kappa < 8$): the SLE trace is self-touching. The paths above have $\kappa = 4.5, 5, \ldots, 7.5$ [109].



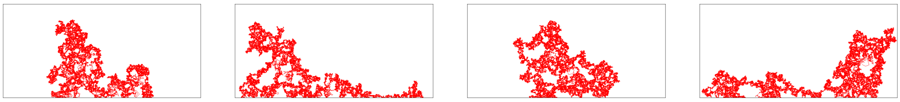


Figure 4.8: Phase 3 ($\kappa = 8$): the SLE trace is space-filling [109].

SLE($\kappa, \kappa-6$) can be viewed as standard SLE from one boundary point to another (up to normalization). In addition, there also exist extensions for force-points in the bulk [68] within the unification of radial, chordal and dipolar SLE. In this extension, radial SLE, i. e. SLE from a boundary to an interior point, becomes chordal SLE($\kappa, \kappa - 6$) where $\kappa - 6$ is the strength of the force of the interior force-point.

4.3.3 Multiple SLE

Multiple SLE [60, 67] is a more general framework for including more than two *curve-creating* boundary conditions. Here, the movement of the boundary points is not just passively driven by the Brownian motion of one SLE but every point moves under the influence of an *independent* martingale.

Multiple SLE can be obtained from m single chordal *interacting* SLEs in the same domain, requiring conformal invariance, reparameterisation invariance and absolute continuity. Allowing local growth at m tips in the upper half-plane results in a modified Löwner mapping G_t that describes m single SLEs in a single equation:

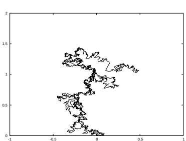
$$\mathrm{d}g_t^i(z) = \frac{2c_t^i}{g_t^i - \xi_t^i}\mathrm{d}t \quad \text{for } i = 1, \ldots, m \quad \rightarrow \quad \mathrm{d}G_t(z) = \sum_{i=1}^{m} \frac{2a_t^i \mathrm{d}t}{G_t(z) - X_t^i}\,. \tag{4.23}$$

Defining $G_t =: H_t^i \circ g_t^i$, we can specify the relationship between the old and the new driving parameters $X_t^i = H_t^i(\xi_t^i)$ and time parameterizations $a_t^i = H_t^{i\prime}(\xi_t^i)^2 c_t^i$. Loosely speaking, H_t^i is the mapping that removes the remaining $m-1$ SLE traces from the setting after the action of g_t^i.

The driving processes under the multiple SLE measure become

$$\mathrm{d}\xi_t^i = \sqrt{\kappa_i}\mathrm{d}B_t^i \quad \rightarrow \quad \mathrm{d}X_t^i = \sqrt{a_t^i \kappa_i}\mathrm{d}B_t^i + \kappa_i a_t^i \partial_{x_t^i} \log Z[x_t]\mathrm{d}t + \sum_{k \neq i} \frac{2a_t^k}{X_t^i - X_t^k}\mathrm{d}t\,. \tag{4.24}$$

From the requirement that the local growth of the single SLEs is commutative, we get $\kappa_i = \kappa_j$ or $\kappa_i = 16/\kappa_j$.



4.3.4 Other SLE variants

Radial SLE [35] is SLE on the unit disc $\mathbb{D}$, describing growing curves that start from the boundary point $1 \in \partial\mathbb{D}$ to the *interior* point $0 \in \mathbb{D}$. Choosing the standard time parameterization, $a_t^i = 1$ for all i, ensures that no curve can be disconnected from infinity.

Dipolar SLE [110] is another SLE on $\mathbb{D}$ that starts from a boundary point X_0 and ends on a point in a specified boundary interval $[X_-, X_+]$.

There have been several attempts to extend the SLE formalism to other domains, e. g. to *fractions* of the upper half-plane $\mathbb{H}_n = \{z = r \exp i\varphi \,:\, r \in (0,\infty), \varphi \in (0,\pi/n)\}$[74, 75].

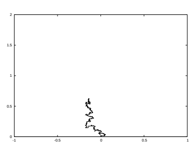
Additionally, SLE on *multiply connected domains* has been investigated [111, 112]. Here, the moduli space enters the picture and the interactions with the moduli $\mathbf{M}$, describing the conformal equivalence class, has to be taken into account. The growing curve induces a motion on the boundary of the domain but in order to recover the curve itself it inside the domain, we have to know which moduli are present. If the connectivity of the domain is greater than one, only the boundary motion together with the motion of the moduli is a Markov process that satisfies Brownian scaling. This becomes important when talking about n-point functions in bCFT and their relation to multiple SLE because local field insertions can be viewed as marked points and therefore require the inclusion of moduli. For example, the force-points of SLE$(\kappa, \vec{\rho})$ which are marked points on the boundary or the bulk of the domain serve as moduli and therefore its driving process can also be obtained from the more general setting of SLE on multiply connected domains.

Other groups have also been studying other driving processes such as Lévy processes [72, 73], including discontinuous driving processes leading to branching SLE traces.

4.3.5 Unification of the Description of SLE Variants

In the context of SLEs on multiply connected domains, there exists [68] a unified description for chordal SLE describing curves that start and end on the same boundary and radial SLE, i. e. a limiting case of an SLE starting and ending on different boundaries (bilateral SLE). Within this framework SLE with multiple boundary conditions can also be described.

In chapter 5, we will provide a unification for other features of SLE, not concentrating on the domain and the nature of start- and endpoints but rather on the different types of boundary conditions, i. e. force-points and start- and endpoints of curves. Reviewing how SLE$(\kappa, \vec{\rho})$ and multiple SLE can be obtained from a change of measure via the corresponding bCFT partition function, we will include both in a multiple SLE$(\kappa, \vec{\rho})$ framework. This unification is important when investigating the relationship between SLE and bCFT, e. g. to find the most general SLE from an underlying bCFT or Statistical Physics model, i. e. the most general physically



relevant SLE.

4.4 Connection to Mathematical Physics

The connection between mathematics and physics…
Is it only that physicists talk in the language of mathematics?
It is more.

YU I. MANIN [113]

4.4.1 The Connection to Conformal Field Theory

As motivated above, SLEs are random growth processes of hulls in the upper half-plane whose traces exhibit the properties of physical cluster interfaces in the scaling limit of two-dimensional Statistical Physics models. In [55], a relation between this description and BCFT, which has also been conjectured to describe these models, has been derived. This first attempt is based on a group theoretical formulation of SLE processes. It identifies the proper boundary states creating the interfaces by imposing appropriate boundary conditions, i. e. those with a vanishing descendant on level two. This way, an infinite set of SLE martingales, i. e. zero modes, is defined via the existence of null vectors in the appropriate VIRASORO module. From this follows a two-to-one relationship between the variance of the BROWNian motion of the SLE driving process $X_t = \sqrt{\kappa}B_t$ to the central charge c of the corresponding BCFT.

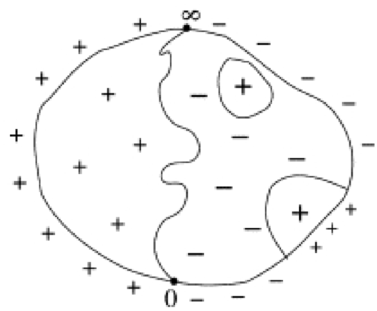


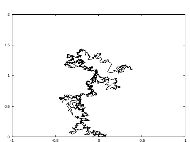
Figure 4.9: A typical SLE setting illustrated with the ISING model. A domain containing an interface created by changing boundary conditions between spins + and − at the origin and infinity [64].

4.4.2 SLE as a Random Walk on the Virasoro Group

To illustrate the connection to BCFT as first done in [55] we start with the usual LÖWNER equation and define $h_t(z) := g_t(z) - \xi_t$, satisfying the stochastic differential equation (4.14)

$$\mathrm{d}h_t = \frac{2\mathrm{d}t}{h_t} - \mathrm{d}\xi_t \,. \tag{4.25}$$

For any time t an element $\mathfrak{g}_{h_t}$ of the germs of holomorphic functions at infinity, N_-, of the form $z + \sum_{m\leq -1} h_m z^{m-1}$, is associated to $h_t(z)$. According to ITÔs formula,



this satisfies:

$$\mathfrak{g}_{h_t}^{-1} \cdot \mathrm{d}\mathfrak{g}_{h_t} = \mathrm{d}t\left(-2l_{-2} + \frac{\kappa}{2}l_{-1}^2\right) + \mathrm{d}\xi_t l_{-1}\,, \tag{4.26}$$

with $l_n = -z^{n+1}\partial_z$. In CFT, the l_n correspond to the generators L_n of the Virasoro algebra $\mathfrak{vir}$ [55]:

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,, \tag{4.27}$$

and has been shown [55] that there exists a homomorphism $\mathfrak{g}_h \to G_h$ such that G_h is an operator acting on appropriate representations of $\mathfrak{vir}$, satisfying an equation analogous to (4.26). Now it can be checked that $G_{h_t}|h_{(1,2)}\rangle$ and $G_{h_t}|h_{(2,1)}\rangle$ are local martingales, i. e. the expectation value of (4.26) vanishes, if the boundary fields $\psi_h(\xi_t)$ are primary fields of weights $h_{(1,2)}(\kappa) = \frac{6-\kappa}{2\kappa}$ or $h_{(2,1)} = h_{(1,2)}(16/\kappa)$. This can be seen by realizing that these fields have a degenerate descendent on level two‡:

$$\left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2\right)|h\rangle = 0\,, \tag{4.28}$$

if the following relation between the central charge of the CFT c and the speed of the Brownian motion κ holds [55]:

$$c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa} \leq 1\,. \tag{4.29}$$

If we choose $h = h_{(1,2)}$ or $h_{(2,1)}$ from the Kač table, we can see that (4.28) is precisely the null vector equation of level two in the minimal CFT model with central charge given in (4.29). Hence we can deduce that a correlation function involving $\psi_h(\xi_t)$ in conformal field theory is at the same time a martingale in SLE.

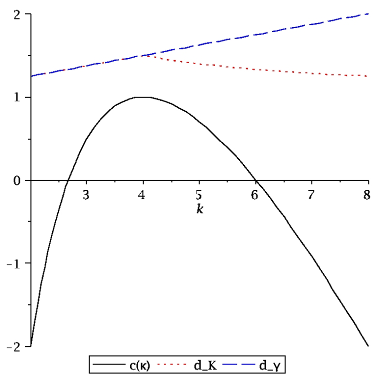
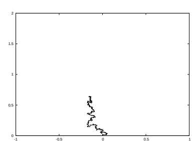


Figure 4.10: Graphical representation of (4.29).

Remark 5. Furthermore, from naive considerations it is natural to expect that characteristic quantities of the fractal curves like the Hausdorff dimension, describing the properties of coverings of the object, should be related to the measure, largely determined by the partition function of the physical model. Furthermore, it is straightforward to look for the conformal weights of bCFT fields, reflecting their behavior under rescalings when investigating e. g. hitting probabilities of one-dimensional objects like the boundary of the SLE domain.

‡Note that in [88] it has been proposed that this identification is only true modulo a phase that accounts for the branch points of CFT correlation functions. However, to keep things simple, we will not concentrate on this subtlety here.



4.5 SLE as the Scaling Limit of Lattice Models

Apart from the fact that SLE fulfils the mathematical conditions that are equivalent to the physical properties of critical curves, there are many theorems that relate the SLE curve, its hull or the surrounded or spanning clusters as well as other geometrical objects to those found in the scaling limit of critical phenomena in Statistical Physics on the lattice (cf. section 3.3).

4.5.1 Random Walk and Its Variants

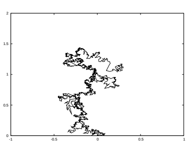
Although the scaling limit of the ordinary random walk, i. e. Brownian motion, cannot be described by the SLE formalism since it is self-crossing, rendering the Riemann mapping theorem inapplicable, variations of the random walk can be accessed. However, its external perimeter can be shown to correspond to the self-avoiding walk [115], i. e. an SLE curve of speed $\kappa = 8/3$. Therefore, its fractal dimension is $\mathrm{d}_H = 4/3$ which had already been conjectured by Mandelbrot [96].

By construction, the Loop-Erased Random Walk (LERW) is self-avoiding, since all loops are removed along its way which is the historical reason why it was introduced as the simplest model of a self-avoiding random walk (SAW). It has the Markov property and been proven to exhibit a conformally invariant scaling limit [46]: SLE of speed $\kappa = 2$, hence it is non-self-touching. The LERW is closely related to the Uniform Spanning Tree (UST). A spanning tree is a collection of vertices and edges, forming a tree without loops or cycles while uniform means that it is randomly picked with equal probability among all possible spanning trees. It can be shown that the distribution of a path connecting two points on the tree is the same as that of the LERW. The hull of the UST is a random plane-filling Peano curve of fractal dimension $\mathrm{d}_H = 2$ and therefore described by SLE with $\kappa = 8$ [46].

Self-Avoiding Random Walk (SAW) is a random walk which is a priori conditioned not to cross itself and serves as a model for polymers in a dilute solution. It satisfies the restriction property, i. e. its distribution is not dependent on conditioning to hit a bulge or not, and is therefore expected to correspond to SLE with $\kappa = 8/3$.

4.5.2 Percolation, the Ising Model and other O(n) models

> *Ising solved the one-dimensional model, [...], and on the basis of the fact that the one-dimensional model had no phase transition, he asserted that there was no phase transition in any dimension. [...] It is ironic that on the basis of an elementary calculation and erroneous conclusion, Ising's name has become among the most commonly mentioned in the theoretical physics literature.*
>
> Barry Simon



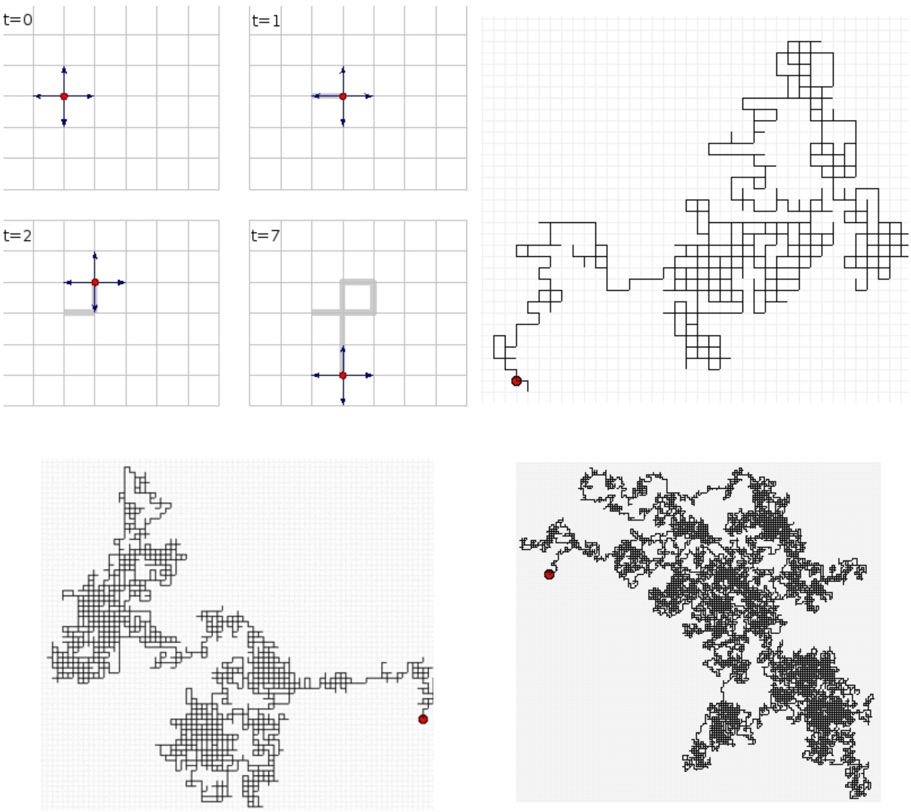
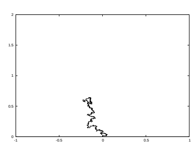


Figure 4.11: (Symmetric) Random Walk and its scaling limit: Brownian motion [114]. On the square lattice, a particle moves randomly, with equal probability in one of the four possible direction. Its trajectory is a random walk which, in the scaling limit, converges to Brownian motion.



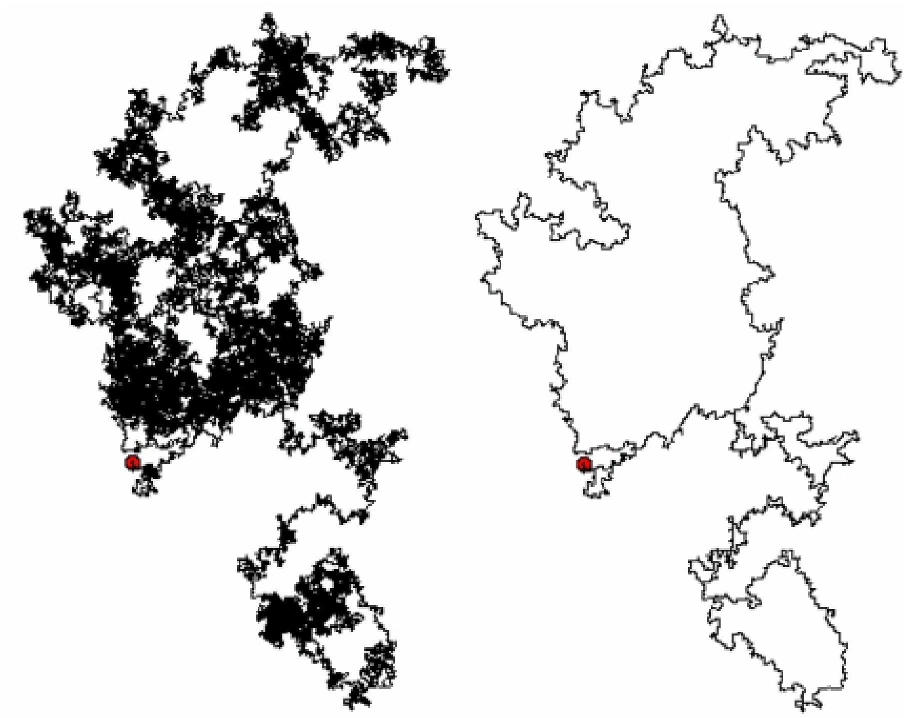
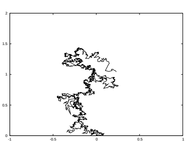


Figure 4.12: Brownian motion and its external perimeter (self-avoiding random walk) [114].



The scaling limit of site percolation on the triangular lattice has been proven to be SLE with $\kappa = 6$ by Smirnov [43]. Its growth rule is entirely local, e. g. on the hexagonal lattice: toss a coin to decide whether to go right or left (if there is still a choice), which results in a strongly meandering path. It has the only value of κ, for which SLE satisfies the *locality* property, i. e. its cluster boundaries are not deformed upon changes of the boundary – it simply does not feel the boundary unless it encounters a boundary point.

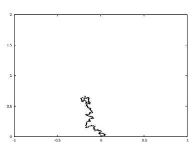
Percolation is the $n \to 1$ limit of the $O(n)$ model whose configurations can be described by clusters and graphs on the dual lattice, allowing for an SLE interpretation of the crossing domain wall. This leads to the identification $n = -2\cos\frac{4\pi}{\kappa}$ for $8/3 < \kappa < 4$. Therefore, the Ising model with $n = 2$ corresponds to $\kappa = 3$ [116] which means that its crossing domain wall is non-self-intersecting. Due to the interactions of the spins, the interfaces in this model are "stiffer" than e. g. those of the percolation model which yields a lower value of κ.

Remark 6. The pictures in our flip-books [117] are growing SLE curves, describing Ising cluster boundaries for $\kappa = 3$ on odd pages and percolation cluster boundaries for $\kappa = 6$ on even pages (flip backwards). The code used for generating the pictures is taken from Tom Kennedy's homepage [109].

Lattice Model	Model Class	κ	$c(\kappa)$	d_γ	d_K
Loop-erased random walk [35, 46]		2	-2	$\frac{5}{4}$	-
Self-avoiding random walk [118]	$n = 0$	$\frac{8}{3}$	0	$\frac{4}{3}$	-
Ising cluster boundaries [119]	$n = 2$	3	$\frac{1}{2}$	$\frac{11}{8}$	-
Dimer tilings [119, 120]	$Q = 4$	4	1	$\frac{3}{2}$	-
Harmonic explorer [121]		4	1	$\frac{3}{2}$	-
Level lines of Gaussian field [45]		4	1	$\frac{3}{2}$	-
FK cluster boundaries [119]	$Q = 2$	$\frac{16}{3}$	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{11}{8}$
Percolation cluster boundaries [35]	$Q = 1$(bond $\square$)	6	0	$\frac{7}{4}$	$\frac{4}{3}$
[43, 122]	$n = 1$ (site, $\triangle$)				
Uniform spanning trees [46]	$Q = 0$	8	-2	2	$\frac{5}{4}$

Table 4.1: Lattice Models

These are examples of lattice models for which a correspondence with SLE and hence CFT has been conjectured or proven. For $\kappa \leq 4$, the dimension d_K of the outer boundary of the hull equals d_γ, i. e. that of the trace, which is indicated by -. The values Q and n stand for the respective Q-state Potts (section 3.3.3) and $O(n)$-models (section 3.3.2) with $n = -2\cos(4\pi/\kappa) = \sqrt{Q}$.



4.6 Mathematical Interest

The physicist cannot understand the mathematician's care in solving an idealized physical problem. The physicist knows the real problem is much more complicated. It has already been simplified by intuition which discards the unimportant and often approximates the remainder.

RICHARD FEYNMAN

For mathematicians, other questions than the relation to the scaling limit of Statistical Physics models is of interest. Famous examples are crossing probabilities such as CARDY's formula for percolation [43], probabilities of passing to the right or left of points or hitting probabilities of intervals on the real [60] or imaginary line, as well as discs in the upper half-plane [123], the relation to harmonic measures [124] or the GAUSSian Free Field [45].

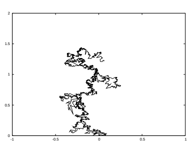
Remark 7. In two of our papers [69, 70], which we quote in chapter 6, we took advantage of the probability of the SLE trace hitting a two-dimensional ball $\mathcal{B}_\epsilon(z_0)$ of radius ϵ, located at $z_0 \in \mathbb{H}$ as derived in [123].

Theorem 4 (Disc-Hitting Probability). Let $\alpha(z_0) \in (0, \pi)$ be the argument of z_0. Then, if $\kappa \in (0, 8)$, we have the following estimate:

$$\mathbf{P}(\mathcal{B}_\epsilon(z_0) \cap \gamma_{[0,\infty)}) \asymp \left(\frac{\epsilon}{\mathfrak{Im}(z_0)}\right)^{2-d_\gamma} (\sin\alpha(z_0))^{8/\kappa-1} \,. \tag{4.30}$$

Proof. The proof of this theorem can be found in appendix A.1. □

For a deeper understanding of our argumentation, the details of the proof of this probability are essential. The key point lies in the distinction between the contributions of the local properties of the SLE trace, e. g. its HAUSDORFF dimension $d_\gamma = \min\{2, 1 + \kappa/8\}$, and its global properties, e. g. its initial behavior depending on $\sin\alpha(z_0)$.



5 SLE Variants from CFT Partition Functions

Originally, SLE has been introduced as a theory of conformally invariant probability measures on continuous curves in two dimensions. However, there is a variety of possible measures being absolutely continuous with respect to the single SLE measure, providing other SLE variants such as SLE$(\kappa, \vec{\rho})$ [108] or multiple SLE [67]. With respect to these measures, there are BCFT observables whose BCFT expectation values fulfil the SLE martingale property [60]: the time variation of their expectation value vanishes. Among these we mention in particular the correlation functions of the boundary fields, i. e. the partition function, also playing a crucial role in the derivation and classification of SLE variants.

From the BCFT point of view, it is a fact that the BCFT expectation values depend on the boundary conditions and thereby on the interfaces they create in the domain: The BCFT expectation values are given by fractions of the correlation functions of the fields contained in an observable and the boundary condition changing fields, divided by the correlation function of only the latter fields. Therefore, it is straightforward to assume that all physically relevant SLE measures also exhibiting a BCFT description can be obtained in a unified way from the boundary conditions of the model.

For some SLE variants, this relationship has already been established. However, the most general case including bulk- and boundary force-points as well as multiple interfaces in the same domain has not been covered yet. Therefore, the scope of this section is first to review how the different SLE variants [108, 67] have been derived from BCFT. Second, we will discuss the relation between bulk force-points in SLE$(\kappa, \vec{\rho})$ [68] and BCFT observables [60] to clarify which kinds of fields contribute to the partition function. Having provided the underlying framework, we motivate how the multiple SLE$(\kappa, \vec{\rho})$ setting should look like and provide a prove for the general form of the driving processes. In this context, we also comment on the relationship between the SLE$(\kappa, \vec{\rho})$ and multiple SLE, which has, in our opinion, been misunderstood in preceding work [125, 101]. We conclude these considerations with an interpretation of the martingale connecting the single SLE measure to those of other SLE variants.

5.1 SLE Variants via New Measures

Currently, there exist two versions of (chordal) SLE variants, obtained by a change of measure from single SLE: SLE $(\kappa, \vec{\rho})$ [108] and multiple SLE [67]. In the following, we briefly review the connection of their respective measures to that of single SLE and how this is related to the boundary field part of the BCFT partition function.

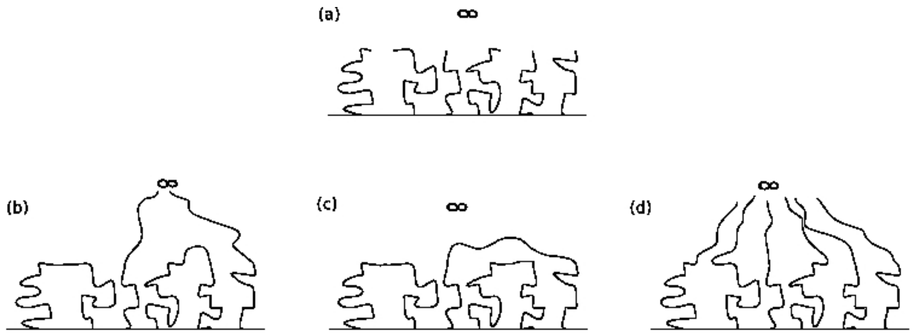


Figure 5.1: Arc configurations in multiple SLE. Starting with $m = 6$ curves in (a), we can end up in e. g. one of the configurations (b), (c) or (d), depending on the number of curves $2k = 4, 6, 0$ joining while $m - 2k = 2, 0, 6$ are growing up to infinity.

Afterwards, we will put a special emphasis on the extension of $\mathrm{SLE}(\kappa, \vec{\rho})$ to bulk force-points in the context of its relation of the Coulomb Gas approach in order to discuss the treatment of bulk fields in bCFT in the context of SLE.

5.1.1 Multiple SLE

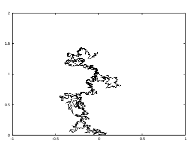
Multiple (chordal) SLE describes $m \geq 1$ interfaces with interactions in a bounded simply connected 2D-domain D which is conformally equivalent to $\mathbb{H}$. Starting at m boundary points, the interfaces either pair up or grow to infinity, forming so-called arc configurations α_{m-2n} that depend on the number of curves pairing up, $2m \leq k$, and the number of curves joining at infinity, $m - 2k$. In the corresponding bCFT the set of topologically inequivalent arc configuration is fixed by the weight of the field placed at infinity, h_{m-2k}, defined by

$$2\kappa h_n(\kappa) = n(2(n+2) - \kappa)\,. \tag{5.1}$$

The Löwner equation for multiple SLE can be obtained from the m single Löwner equations:

$$\mathrm{d}g_t^i(z) = \frac{2c_t^i}{g_t^i - \xi_t^i}\mathrm{d}t \quad \text{for } i = 1, \ldots, m \quad \rightarrow \quad \mathrm{d}G_t(z) = \sum_{i=1}^{m} \frac{2a_t^i \mathrm{d}t}{G_t(z) - X_t^i}\,. \tag{5.2}$$

Defining $G_t =: H_t^i \circ g_t^i$, the relationship between the old and the new driving parameters $X_t^i = H_t^i(\xi_t^i)$ and time parameterizations $a_t^i = H_t^{i\prime}(\xi_t^i)^2 c_t^i$ can be specified. Loosely speaking, H_t^i is the mapping that removes the remaining $m-1$ SLE traces from the setting after the action of g_t^i.



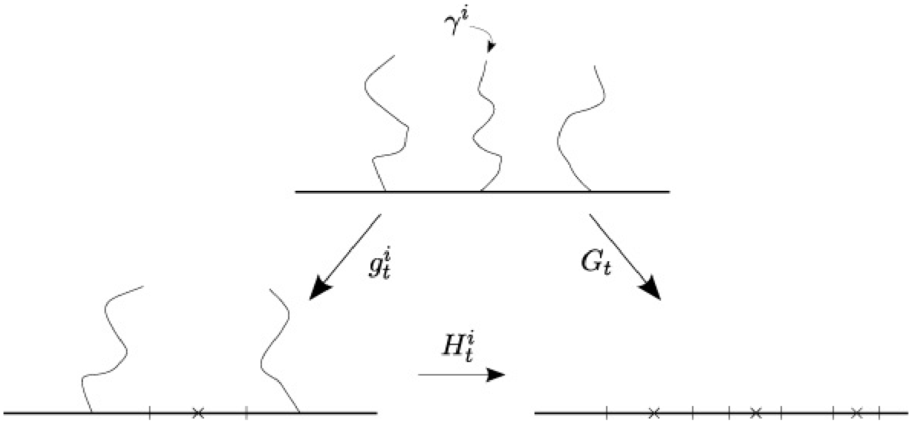


Figure 5.2: Schematic drawing of the relationship between the single Löwner mapping g_t^i, the multiple Löwner mapping G_t and H_t^i [67].

The driving processes under the multiple SLE measure become

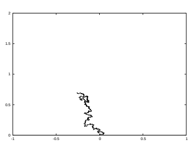
$$\mathrm{d}\xi_t^i = \sqrt{c_t^i \kappa_i}\,\mathrm{d}B_t^i \quad \rightarrow \quad \mathrm{d}X_t^i = \sqrt{\kappa_i a_t^i}\mathrm{d}B_t^i + \kappa_i a_t^i \partial_{x_t^i} \log Z[x_t]\mathrm{d}t + \sum_{k \neq i} \frac{2a_t^k}{X_t^i - X_t^k}\mathrm{d}t\,. \tag{5.3}$$

In going from the single to the multiple SLE probability measure, conformal and reparameterisation invariance has to be enforces [67]. This results in a change of the single SLE measure due to the interactions between the formerly independent m single SLE traces which are governed by the *conditions* imposed on the measure. The new measure for multiple SLE, $\mathbf{P}_{\text{new}}$, is therefore the old measure of m single SLEs, $\mathbf{P}_{\text{old}}$, *conditioned* on non-intersecting hulls, commutative growth etc. [65]. Such pairs of measures as $\mathbf{P}_{\text{new}}$ and $\mathbf{P}_{\text{old}}$ are known to be absolutely continuous w.r.t. each other and therefore the change from $\mathbf{P}_{\text{old}}$ to $\mathbf{P}_{\text{new}}$ is given by a martingale M_t: the Radon-Nikodým derivative (theorem 8). According to Girsanov's theorem (theorem 9), it can be expressed by the exponential of some other local martingale L_t:

$$M_t = \frac{\mathrm{d}\mathbf{P}_{\text{new}}}{\mathrm{d}\mathbf{P}_{\text{old}}} = \exp(L_t - \frac{1}{2}\langle L, L\rangle_t)\,, \tag{5.4}$$

The martingale L_t has been computed [67] imposing the constraints mentioned above, i. e. conformal reparameterisation invariance, directly to the driving function (5.3) after applying the corresponding transformations, i. e. conformal transformations and time changes, as well as commutative growth conditions.

This way, Graham [67] showed that multiple (chordal) SLE with m interfaces starting from the boundary can be obtained from m ordinary single SLEs with in-



teractions, weighted by a local bounded martingale given by

$$M_t \quad := \quad Z_{\text{b.c.}}[x_t] \prod_{i=1}^{m} \left(H_t^{i\prime}(w_t^i) \right)^{h_i} \exp\left(\frac{c_i}{6} \int_0^t \mathcal{S} H_s^i(w_s^i) c_s^i \mathrm{d}s \right)$$
$$\exp\left(- \int_0^t \frac{1}{Z[x_s]} \mathcal{D}_{-2}^m(x_s^i, \{x_s^k\}_{k \neq i}) Z_{\text{b.c.}}[x_s] a_s^i \mathrm{d}s \right) . \tag{5.5}$$

The notation is taken from (4.23) while $\mathcal{S}$ denotes the SCHWARZian derivative and the $\mathcal{D}_{-2}^m(x_s^i, \{x_s^k\}_{k \neq i})$ are given by

$$\mathcal{D}_{-2}^m(x_s^i, \{x_s^k\}_{k \neq i}) = \frac{\kappa_i}{2} \partial_{x_s^i}^2 - 2 \sum_{k \neq i} \left(\frac{h_k}{(x_s^k - x_s^i)^2} - \frac{1}{(x_s^k - x_s^i)} \partial_{x_s^i} \right) , \tag{5.6}$$

i. e. the differential operators annihilating $Z_{\text{b.c.}}[x_s]$, the correlation function of the curve-creating boundary fields, due to the existence of their vanishing level-two descendants. Note that the partition function of the corresponding bCFT system is given by $Z[x_t] = Z_{\text{b.c.}}[x_t] Z_{\text{free}}$.

In the ranges $h_i = \frac{6-\kappa_i}{2\kappa_i} = h_{(1,2)} \geq \frac{5}{8}$, i. e. $\kappa_i \leq \frac{8}{3}$, M_t is a bounded martingale $0 \leq M_t \leq \left(H_t^{i\prime}(w_t^i) \right)^{h_i} \leq H_t^{i\prime}(w_t^i) \leq 1$ [67]. M_t is defined up to the intersection time t where the j^{th} trace intersects the hull A_i of the i^{th} trace [67].

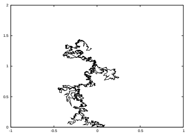
Choosing the time parameterization $a_t^i = 1$ for all $i = 1, \dots, m$ ensures that none of the traces can be disconnected from infinity which otherwise might lead to intersection of traces. This guarantees physically meaningful configurations where the hull created by one curve can not enclose the tip of another. In addition, it can be shown that in this parameterization, the tips collide with probability one only at their endpoints [67]. After this intersection time, the SLE is again well-defined, only without these two traces.

In addition to the range of definition shown in [67], we can follow [126], arguing that M_t still does not blow up if we extend it to $\frac{8}{3} \leq \kappa \leq 4$. Therefore we may assume that M_t satisfies $0 \leq M_t \leq \left(H_t^{i\prime}(w_t^i) \right)^{h_i} \leq 1$ for $\kappa_i \leq 4$. Knowing that c_i is the same for κ_i and $16/\kappa_i$, we can even extend the range to any values of κ_i and h_i within the infinitesimal approach to multiple SLE [60, 67].

The $\mathcal{D}_{-2}^m(x_s^i, \{x_s^k\}_{k \neq i})$ annihilate $Z[x_s] = Z_{\text{b.c.}}[x_s] Z_{\text{free}}$, the partition function of the corresponding bCFT system at time s. This is why the last exponential factor in (5.5) equals one. Introducing coordinates on each curve, $2t^i(s) = \text{hcap}(K_s^i)$, the individual parameterizations $c_s^i = \frac{\mathrm{d}}{\mathrm{d}s} t^i(s)$ and the arguments of the exponentials become an integral over a one-form

$$\mathrm{d}R = - \sum_i \frac{c_i}{6} \mathcal{S} H_s^i(w_s^i) \mathrm{d}t^i . \tag{5.7}$$

It is easy to see that this integral vanishes, because $c_i = c_j$ and the integral over the one-form must not depend on the integration path in "time space" [67]. Therefore, the first exponential factor in (5.5) reduces to one, too.



This shows that the weighting martingale is given by

$$M_t = Z[x_t] \prod_{i=1}^{m} \left(H_t^{i\prime}(\xi_t^i) \right)^{h_i} = Z_{\text{b.c.}}[\xi_t] \,, \tag{5.8}$$

since $(H_t^i)^{-1}(x_t^i) = \xi_t^i$. This means that the change of measure is determined by the correlation function of the boundary fields, i. e. the fields creating and interacting with the SLE interfaces.

5.1.2 SLE($\kappa, \vec{\rho}$) and the Coulomb Gas Approach

In [61, 66], SLE($\kappa, \vec{\rho}$) has been investigated in the context of the Coulomb gas approach to minimal CFT models. In the CFT picture (cf. section 3.3.1), the fields are described by vertex operators (3.63) which are primary fields of weight $h_\alpha = \alpha^2 - 2\alpha\alpha_0$, wherein α_0 is the background charge, encoding the curvature of the theory concentrated at infinity which is connected to the central charge of the bCFT.

In the following, we consider a minimal CFT model of central charge

$$c = 1 - 6\frac{(\kappa-4)^2}{4\kappa} = 1 - 24\alpha_0^2 \,, \tag{5.9}$$

i. e. $\alpha_0 = \frac{\kappa-4}{2\sqrt{\kappa}}$ on a domain D conformally equivalent to $\mathbb{H}$. We assume that there exists an interface starting from 0, enforced by a boundary condition changing operator of weight $h_{(1,2)}$. Moreover, we assume additional boundary condition changing operators located at $n+1$ points $y_t^1, \ldots, y_t^n, \infty$ on the boundary ∂D (the force-points). They are given by vertex operators V_{α_j} of weight (cf. (3.64))

$$h_j = \frac{\rho_j(\rho_j + 4 - \kappa)}{4\kappa} \,, \quad \alpha_j = \frac{\rho_j}{2\sqrt{\kappa}} \,. \tag{5.10}$$

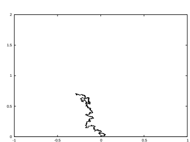
Note that the SLE curve-creating boundary fields, $\psi h_{(1,2)}$, correspond to vertex operators, too, with $\rho_\xi = 2$, i. e. $\alpha_\xi = \frac{1}{\sqrt{\kappa}}$.

The neutrality condition (3.66) requires the following relationship for the strengths of the forces:

$$\sum_{j=1}^{n} \rho_j + \rho_\infty = \kappa - 6 \,. \tag{5.11}$$

Within boundary CFT, this results in a partition function of the type $Z = Z_{\text{b.c.}} Z_{\text{free}}$ with (cf. (3.68))

$$\begin{aligned} Z_{\text{b.c.}} &= \prod_{j=1}^{n} (y_j - \xi_t)^{\frac{\rho_j}{\kappa}} \prod_{1 \le i < k \le n} (y_k - y_i)^{\frac{\rho_i \rho_k}{2\kappa}} \\ &= \langle V_{\alpha_\infty}(\infty), V_{\alpha_1}(y_1) \ldots V_{\alpha_n}(y_n) \psi_{h_{(1,2)}}(\xi_t) \rangle \,. \end{aligned} \tag{5.12}$$



In [65] it has been shown, that a chordal $\mathrm{SLE}(\kappa,\vec{\rho})$ defined via the Löwner equation

$$\mathrm{d}g_t(z) = \frac{2\mathrm{d}t}{g_t(z) - X_t}\,, \tag{5.13}$$

with driving process

$$\mathrm{d}\xi_t = \sqrt{\kappa}\mathrm{d}B_t \quad \longrightarrow \quad \mathrm{d}X_t = \sqrt{\kappa}\mathrm{d}B_t + \sum_{i=1}^{n} \frac{\rho_i}{X_t - Y_t^i}\mathrm{d}t \tag{5.14}$$

and force-points y_t^i on the boundary, obeying

$$\mathrm{d}Y_t^i = \frac{2}{Y_t^i - X_t}\mathrm{d}t\,, \tag{5.15}$$

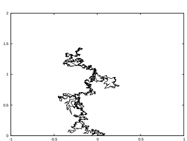
is equivalent to an ordinary SLE process weighted by a martingale M_t. In complete analogy to the last section, the martingale M_t can be obtained via comparison with (5.14). This modifies the ordinary SLE driving function from Brownian motion of speed κ to the one with the additional drift terms (5.14). This way it has been proven that, again, the martingale is given by the boundary part of the partition function:

$$\begin{aligned} M_t \quad &:= \prod_{j=1}^{n} \left(|g_t'(y_0^j)|^{\frac{(4-\kappa+\rho_j)\rho_j}{4\kappa}} |\xi_t - y_t^j|^{\frac{\rho_j}{\kappa}} \right) \prod_{1\leq j<j'\leq n} |y_t^j - y_t^{j'}|^{\frac{\rho_j\rho_{j'}}{2\kappa}} \\ &= \prod_{j=1}^{n} |g_t'(y_0^j)|^{h_j} \langle V_{\alpha_\infty}(\infty), V_{\alpha_1}(y_t^1)\ldots V_{\alpha_n}(y_t^n)\psi_{h_{(1,2)}}(\xi_t)\rangle \\ &= \prod_{j=1}^{n} |g_t'(y_0^j)|^{h_j} Z_{\mathrm{b.c.}}[\xi_t, y_t] = Z_{\mathrm{b.c.}}[\xi_t, y_0]\,. \end{aligned} \tag{5.16}$$

Remark 8. Note that the boundary force-point ensemble of $\mathrm{SLE}(\kappa,\vec{\rho})$ can be extended to bulk force-points [68]. While the boundary force-points are represented by vertex operators located on the boundary, the bulk force-points are described by local vertex operators, i. e. pairs of chiral vertex operators that interact with their mirror-images. This is due to the fact that if the SLE curve is to feel the force of the bulk force-points, the respective fields have to *interact*. However, in the mirror-image approach, interaction of fields with the boundary means that the interact with their mirror-images first. This is done by performing the OPE of the chiral vertex operator with its mirror-image:

$$V_{\alpha_k}(z_t^k)V_{\bar{\alpha}_k}(z_t^{k*}) = \sum_{h_{\alpha_k+\bar{\alpha}_k}} \sum_{|Y|=0}^{\infty} \mathfrak{Im}(z_t^k)^{|Y|+h_{\alpha_k+\bar{\alpha}_k}-h_{\alpha_k}-h_{\bar{\alpha}_k}} L_{-Y} V_{\alpha_k+\bar{\alpha}_k}(\mathfrak{Re}(z_t^k))\,. \tag{5.17}$$

This shows that, effectively, a bulk force-field pair represented by a chiral vertex operator and its mirror-image, interacting with the boundary, is represented by a



vertex operator $V_{\alpha_k+\bar{\alpha}_k}(y_t^k)$, located at $y_t^k := \mathfrak{Re}(z_t^k)$ on the boundary. The charge of the vertex operator is of course determined by the fusion rules. The fusion channels always contain the identity with $\alpha = 2\alpha_0$ and that of another representation of twice the charge $2\alpha_k$. The first case would correspond to no interaction with the boundary, hence the bulk force-field would, by definition, be no force-field and not be present in the boundary partition function as explained in detail for boundary force-fields in SLE$(\kappa, \vec{\rho})$. Therefore we argue that in the case of bulk *force*-fields, only the other fusion channel is selected, since, as we will see in the following, a force-point of twice the strength appears in the driving process when the bulk force-point pair approaches the boundary.

This is completely consistent with the weighting martingale for the generalization of ordinary SLE$(\kappa, \vec{\rho})$ to bulk force-points [68]:

$$\begin{aligned} M_t \quad := \quad & \prod_{j=1}^{n} \left(|g_t'(z_t^j)|^{\frac{(8-2\kappa+\rho_j)\rho_j}{8\kappa}} (\mathfrak{Im}(z_t^j))^{\frac{\rho_j^2}{8\kappa}} |\xi_t - z_t^j|^{\frac{\rho_j}{\kappa}} \right) \\ & \prod_{1\leq j<j'\leq n} \left(|z_t^j - z_t^{j'}||\overline{z_t^j} - z_t^{j'}| \right)^{\frac{\rho_j\rho_{j'}}{4\kappa}} . \end{aligned} \tag{5.18}$$

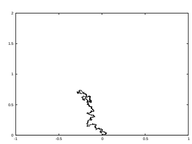
If all of the points are on the real axis as in the COULOMB gas approach, this reduces to (5.16). As said above, taking the mirror-image approach seriously, (5.18) is the same as (5.16): in the COULOMB gas approach, the charges of fused vertex operators (corresponding to the bulk field and its mirror-image) just add up. Moreover, the charge of the field and its mirror-image are the same and therefore $\alpha_k + \alpha_{n+k} = 2\alpha_k$ which shows that the terms appear twice in (5.19) since $\alpha_k \propto \rho_k$.

With the help of GIRSANOV's theorem, we can derive, the extra drift term of SLE$(\kappa, \vec{\rho})$:

$$\frac{\mathrm{d}\langle \xi, M\rangle_t}{M_t} = \sum_{j=1}^{2n} \rho_j \mathfrak{Re}\left(\frac{1}{\xi_t - y_t^j}\right) , \tag{5.19}$$

which is exactly what we expect from the mirror-image approach in the limit $z_t^{n+k} \to z_t^{k*}$, since

$$\begin{aligned} \lim_{z_t^{n+j}\to z_t^{j*}} \sum_{j=1}^{2n} \mathfrak{Re}\left(\frac{\rho_j}{\xi_t - z_t^j}\right) \quad &= \quad \lim_{z_t^{n+j}\to z_t^{j*}} \left[\sum_{j=1}^{n} \mathfrak{Re}\left(\frac{\rho_j}{\xi_t - z_t^j}\right) + \sum_{j=1}^{n} \mathfrak{Re}\left(\frac{\rho_{n+j}}{\xi_t - z_t^{n+j}}\right)\right] \\ &= \quad \lim_{z_t^{n+j}\to z_t^{j*}} \sum_{j=1}^{n} \left(\frac{\rho_j(\xi_t - \mathfrak{Re}(z_t^j)}{(\xi_t - \mathfrak{Re}(z_t^j))^2 - \mathfrak{Im}(z_t^j)^2} \right. \\ & \qquad \left. + \frac{\rho_{n+j}(\xi_t - \mathfrak{Re}(z_t^{n+j})}{(\xi_t - \mathfrak{Re}(z_t^{n+j}))^2 - \mathfrak{Im}(z_t^{n+j})^2} \right) \\ &= \quad \sum_{j=1}^{n} \frac{2\rho_j}{\xi_t - y_t^j} . \end{aligned} \tag{5.20}$$



5.1.3 CFT Observables and Bulk Force Points in SLE($\kappa, \vec{\rho}$)

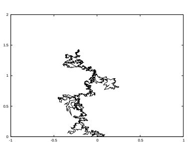
In section 5.2, we want to derive the most general SLE measure that can be obtained from bCFT. This means that we have to include boundary fields that create interfaces, i. e. boundary fields with vanishing descendant on level two depending on stochastic coordinates, and boundary fields that serve as force-points without own motion, as well as bulk fields. However, in the context of bulk force-points in the respective extension of SLE($\kappa, \vec{\rho}$) [68], the question how to take care of bulk fields arises.

Obviously, there are two types of bulk fields: those which are present in the boundary part of the partition function due to their interactions with the SLE curve and their mirror-images in SLE($\kappa, \vec{\rho}$) settings [68] and those which are not. Fields of the first type are, by definition, bulk force-fields while we will call those of the second type observables throughout this thesis. Their presence or absence in $Z_{\mathrm{b.c.}}$ determines the allowed channels in the OPE with their mirror-images and therefore the possibility of interactions with the curve-creating boundary fields.

From the definition of the SLE driving function, it follows that only *force*-fields in the bulk or on the boundary can have an effect on the movement of the trace since other objects do not contribute to $Z_{\mathrm{b.c.}}$. Therefore, the bulk and boundary force-fields should be the only fields with non-trivial OPE with the curve-creating boundary field: Physically, the existence of forces is always due to interactions which means that, locally, the presence of another field should have an effect which can be computed by the OPE of the interacting fields. Since the locations of observable bulk fields, by definition, are not allowed to exert forces on the curve (which is the boundary), the corresponding fields do *not interact*. Therefore, in the OPE of the observable field with its mirror-image which models the interaction with the boundary, only the identity may propagate such that, as a result, their interaction with the boundary is trivial – being the identity on the boundary means that the OPE of the curve-creating boundary field with such a field has no effect.

In contrast to this, the locations of bulk force-fields are bulk force-points in SLE($\kappa, \vec{\rho}$) and present in the driving process. For bulk force-points this means that the force-point corresponding to the effective force-field on the boundary which is obtained from the interaction of the bulk force-field with its mirror-image gives the effective contribution to the driving process as discussed in remark 8

Therefore, we stress that bCFT bulk fields contained in observables, should be considered as SLE($\kappa, \vec{\rho}$) bulk force-points or observable fields, depending on their interaction with their mirror-image. Those with non-trivial interactions with an SLE curve-creating boundary field, have to exhibit a non-trivial boundary field in the OPE with their mirror-image. In contrast to this, the observable fields whose locations are not present in the SLE driving process may only have trivial interactions with the curve-creating boundary field and therefore only the identity channel of the OPE with their mirror-image contributes. Unfortunately, this distinction has not been included in the literature on SLE and the OPE concept in bCFT, e. e. [110, 60] so



far since the concept of bulk force-points had not been known at that time.

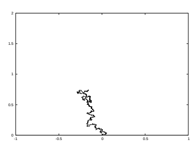
Remark 9. If there are both types of bulk fields present in a BCFT, we can assume that due to vanishing interactions, any correlation function of boundary and bulk fields factorises into two parts: the boundary part, consisting of boundary and bulk force-point fields, and the bulk part, consisting of the bulk fields only. This illustrates the problem of the SLE-BCFT correspondence in general. The SLE formalism is not suitable to yield information on the local properties of the theory which are represented by local bulk fields in the corresponding BCFT. Therefore, extensions such as Conformal Loop Ensembles (CLEs) may be better candidates for a one-to-one correspondence with BCFT. Moreover, this hints at the question to which extend the whole information on BCFT can be inferred from the boundary conditions, i. e. to which extend BCFT is effectively a one-dimensional problem. This is closely related to the DIRICHLET problem, stating that harmonic functions on bounded simply connected domains are completely determined by their boundary values. However, in BCFT we are not dealing with harmonic, i. e. holomorphic, functions but rather with meromorphic functions for which we need additional information on the order and the locations of their poles: meromorphic functions are entirely determined by the degrees of their poles and their zeros and their modulus on the boundary.

Again, the global point of view of SLE can be illustrated by the fact that if we do not include bulk observables, the whole theory is a priori fixed by the (global) boundary conditions and there are no singularities in the bulk – the theory is quasi one-dimensional. Therefore, in connection to SLE, the only interesting observable has to be connected to the correlation function of the boundary fields, i. e. the partition function. Its inverse describes the change of the measure needed to go from single SLE to other SLE variants:

$$\mathrm{d}\mathbf{P}_{\text{single}} = Z[x_t, y_t]^{-1}\mathrm{d}\mathbf{P}_{\text{variant}} \,. \tag{5.21}$$

This means that probabilities in other SLE variants are weighted by the inverse partition function which makes it a natural martingale to study in the context of SLE variants ant their connection to BCFTs. More precisely, especially when investigating so-called topological features of the SLE, e. g. approaching traces in multiple SLE or interactions with force-points that can serve as end-points of an SLE trace, we should concentrate on this object. This is the reason why we will consider the martingale $Z[x_t, y_t]^{-1}$ in chapter 6b.

Remark 10. In the context of bulk force-points, an interpretation of the vortex operators of the general COULOMB gas picture as SLE curve-creating fields due to the interaction with their respective mirror-images has been proposed [88]. These *vortex* operators arise when instead of characterizing the vertex operators by their charges α and $\bar{\alpha}$, we introduce electric and magnetic charges $e = (\alpha + \bar{\alpha})/2$ and $m = (\alpha - \bar{\alpha})/2$. In this notation, the spinless operators are characterizes either by $\alpha = \bar{\alpha}$, i. e. purely magnetic vertex operators with $m = 0$, or $\alpha = 2\alpha_0 - \bar{\alpha}$, i. e. vortex operators of electric charge $e = \alpha_0$ and arbitrary magnetic charge $m \in \mathbb{Z}/2$. The latter are the n



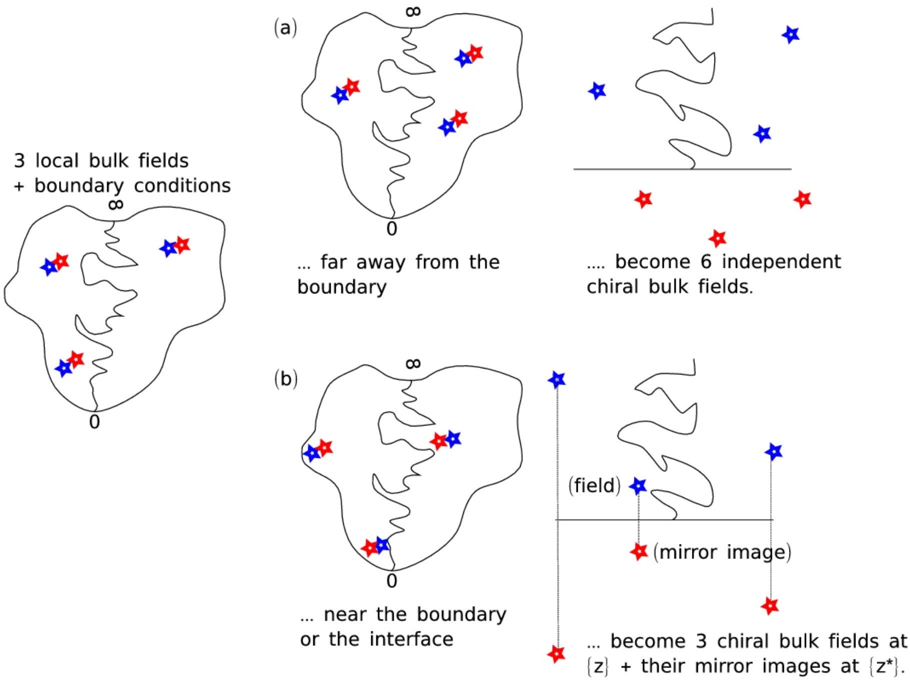
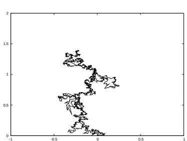


Figure 5.3: Observable SLE. A typical observable consisting of $2n$ chiral bulk fields can be in either of two situations: (a) represents the case where the bulk fields are never close to the SLE trace or its image, the boundary, while in (b) they are. In case (a), the correlation function factorises according to the cluster decomposition theorem and there is no interaction between the bulk fields and the SLE trace, hence the bulk fields should be considered as an observable and not as SLE force-points. However, in case (b), they are close to the boundary and therefore interact with their mirror-images. According to BCFT rules, their interaction is described by the outcome of the OPE of the bulk field $\phi_k(y_k)$ with its mirror-image $\phi_{n+k}(y_k^*)$ – an new boundary field $V_{2\alpha_k}(\mathfrak{Re}(y_k))$. Since the locations of these boundary fields then serve as usual force-points, the locations of the corresponding bulk fields are considered as bulk force-points for the SLE trace.



curves creating bulk operators of weight

$$h_{(0,n/2)} = \frac{4n^2 - (\kappa - 4)^2}{16\kappa} \,. \tag{5.22}$$

This is why it is usually argued that a single critical curve going through a point z is created by the bulk field $V_{\alpha_{(0,1)}}(z)$, since it creates $n = 2$ curves at its location. On the boundary, the non-trivial part of the OPE of this vortex operator with its mirror-image contains the boundary field $\psi_{h_{(1,2)}}(\mathfrak{Re}(z))$. However, this is *not* a curve-creating field since the coordinate it depends on is not stochastic in nature, which is an important subtlety that has been overlooked in [88] and the publications build on that paper. It may be interesting to consider vortex operators at stochastically moving locations in the bulk, i. e. at coordinates $z_t = \sqrt{\kappa}B_t + \mathrm{i}\mathfrak{Im}(y_t)$. These would, in principle, be able to yield curve-creating boundary fields but the question how the growth of the curve from the boundary point can be described by a Löwner-like evolution arises. There may be some applications in the context on SLE on multiply connected domains, but the work on this and related questions is still ongoing [127].

5.1.4 A Motivation for Multiple SLE($\kappa, \vec{\rho}$)

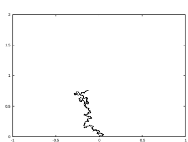
On the background of the preceding work, it is straightforward to combine both SLE variants to a multiple SLE($\kappa, \vec{\rho}$). This framework has been motivated by Dubédat [125, 101] while investigating a commutation relation for two SLE($\kappa, \vec{\rho}$) processes in the same domain. Consequently, we propose that the driving functions for the m curve-creating fields ψ at boundary points x_t^i and the n force-fields at boundary points y_t^q have to be modified accordingly:

$$\mathrm{d}X_t^i = \sqrt{\kappa_i}\mathrm{d}B_t^i + \kappa_i \partial_{x_t^i} \log Z[x_t, y_t]\mathrm{d}t + \sum_{l \neq i}^{m} \frac{2}{X_t^l - X_t^i}\mathrm{d}t \,, \tag{5.23}$$

$$\mathrm{d}Y_t^j = \sum_{l=1}^{m} \frac{2}{Y_t^i - X_t^l}\mathrm{d}t \,, \tag{5.24}$$

which we will show in the following section.

Note that, of course, the boundary partition function $Z[x_t, y_t]$ now is proportional to the correlation function of the m curve-creating and the n force-fields, paired with a boundary field of suitable weight at infinity. However, the classification of arc configurations by these weights at infinity as introduced in section 5.1.1 does not work any more. The curves are also allowed to stop at one of the force-points, similar to the interpretation of single SLE($\kappa, \kappa - 6$) as single SLE from the origin to the force-point. Furthermore, it is easy to see that for $m = 1$, (5.23) and (5.24) are equivalent to the SLE($\kappa, \vec{\rho}$) driving processes while for $n = 0$, they give back those of multiple SLE.



5.2 Unified Description from BCFT Partition Functions

From the previous sections, it has become quite obvious that the SLE measure in presence of a single interface only, i. e. no force-points, connecting two boundary points, is related to the most simple non-trivial BCFT partition function. It consists of the minimum of two boundary condition changing fields of which one is at infinity. Such a correlation function is known to be constant. Reminding ourselves that all basic information of a boundary CFT is encoded in its partition function,

$$Z_{\text{BCFT}} = Z_{\text{b.c.}} Z_{\text{free}} \,, \tag{5.25}$$

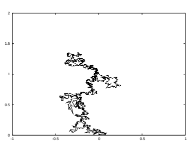
and the fact that any minimal model of CFTs with $c \leq 1$ can be described via the Coulomb Gas approach [90, 91], it is straightforward to assume that any (multiple) SLE$(\kappa, \vec{\rho})$ can be obtained via the change of measure method as illustrated in the previous section for the cases of multiple SLE and SLE$(\kappa, \vec{\rho})$.

As we will show in this section, this can be achieved in the following way: Every field which is to give rise to an SLE interface receives its own driving process as in multiple SLE while all other boundary or bulk fields become part of the force-point ensemble as in SLE$(\kappa, \vec{\rho})$. Interpreting the martingale $Z_{\text{b.c.}}$ as a change of measure, we can therefore find the appropriate SLE measure for any minimal BCFT. Note that our extension of the previous papers provides a way to connect all BCFTs exhibiting at least one interface created by a $\psi_{h_{(1,2)}}$ or $\psi_{h_{(2,1)}}$ boundary condition changing fields to SLE variants in a unified way. Of course, via the unification of SLE descriptions (see section 4.3.5), it is easy to extend this to other types of SLE, e. g. dipolar SLEs etc., too, in the same manner as it has been done for the simpler variants.

5.2.1 Derivation of Multiple SLE$(\kappa, \vec{\rho})$

The basic question we will answer is what kinds of SLEs can arise when considering m interfaces created by $\psi_{h_{(1,2)}}$ or $\psi_{h_{(2,1)}}$ boundary condition changing fields at x_t^i in the presence of a number of boundary changes implemented by other boundary condition changing fields, represented by vertex operators $V_{\alpha_j}(y_t^j)$ (which could also be the OPE outcome of bulk vertex operator pairs). From the considerations above, we know that these should, in the most general case, be of the multiple SLE(κ, ρ)-type. The key to the answer is to investigate BCFT expectation values of observables [60], i. e. products of local primary bulk fields at $(z_k, z_k^*) = (z_k, z_{k+o})$,

$$\mathcal{O}^{\text{CFT}} = \frac{\prod_{k=1}^{2o} G_t'(z_k)^{h_k} \langle \psi_{h_\infty}(\infty), \prod_{i=1}^{m} \psi_{h_2}(x_t^i) \prod_{j=1}^{m} V_\alpha(y_t^j) \prod_{k=1}^{2o} \phi_{h_k}(z_t^k) \rangle}{\langle \psi_{h_\infty}(\infty), \prod_{i=1}^{m} \psi_{h_2}(x_t^i) \prod_{j=1}^{m} V_\alpha(y_t^j) \rangle} \,. \tag{5.26}$$



These fields are expected to be local multiple SLE$(\kappa, \vec{\rho})$ martingales, hence the drift of their time derivative has to vanish. Taking the ansatz

$$\mathrm{d}X_t^i = \sqrt{\kappa_i}\mathrm{d}B_t^i + f_i \mathrm{d}t \quad \text{for } j = 1, \ldots, m\,, \tag{5.27}$$

$$\mathrm{d}Y_t^j = \sum_{l=1}^{m} \frac{2}{Y_t^i - X_t^l}\mathrm{d}t \quad \text{for } j = 1, \ldots, n\,. \tag{5.28}$$

we will motivate how to show that $f_i = \kappa_i \partial_{x_t^i} \log Z[x_t, y_t] + \sum_{l \neq i}^{m} \frac{2}{X_t^l - X_t^i}$ which proves then our proposal for multiple SLE$(\kappa, \vec{\rho})$ as the most general SLE from BCFT partition function possible. Let the numerator be N_t while the denominator is D_t. Remembering that the multiple Löwner equation is given by

$$\mathrm{d}G_t(z) = \sum_{i=1}^{m} \frac{2}{G_t(z) - X_t^i}\mathrm{d}t\,, \tag{5.29}$$

we can easily compute its derivative with respect to z:

$$\mathrm{d}G_t'(z) = -\sum_{i=1}^{m} \frac{2}{(G_t(z) - X_t^i)^2}\mathrm{d}t\,. \tag{5.30}$$

In addition, we know how the variation of primary fields as functions of stochastic (X_t^i) or deterministic (Y_t^j, Z_t^k) variables (corresponding to the processes) looks like:

$$\mathrm{d}\psi_{h_i}(X_t^i) = \left(\frac{\kappa_i}{2}\partial_{x_t^i}^2 \mathrm{d}t + \partial_{x_t^i}\mathrm{d}X_t^i\right)\psi_{h_i}(X_t^i)\,, \tag{5.31}$$

$$\mathrm{d}V_{\alpha_j}(Y_t^j) = \left(\sum_{i=1}^{m} \frac{2}{Y_t^j - X_t^i}\partial_{y_t^j}\right)V_{\alpha_j}(Y_t^j)\,, \tag{5.32}$$

$$\mathrm{d}\phi_{h_k}(Z_t^k) = \left(\sum_{i=1}^{m} \frac{2}{Z_t^k - X_t^i}\partial_{z_t^k}\right)\phi_{h_k}(Z_t^k)\,, \tag{5.33}$$

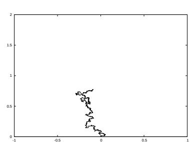
while the variation of the Jacobian factor is given by

$$\mathrm{d}G_t'(z_k)^{h_k} = -\sum_{i=1}^{m} \frac{2h_k}{Z_t^k - X_t^i}G_t'(z_k)^{h_k}\mathrm{d}t\,. \tag{5.34}$$

Starting with this and inserting that due to the null-descendants on level-two of the curve-creating boundary fields some terms vanish, it is quite lengthy but easy to compute (e. g. following [60]) that

$$\mathrm{d}N_t = \mathcal{D}N_t \qquad \mathrm{d}D_t = \mathcal{D}D_t$$

$$\mathcal{D} = \sum_{i=1}^{m}\left(\sum_{k=1}^{2o} \frac{2h_k}{(X_t^i - Z_t^k)^2}\right)\mathrm{d}t + \mathrm{d}X_t^i \partial_{x_t^i}\,. \tag{5.35}$$



Requiring that the Itô derivative of the fraction N_t/D_t has no drift, we get

$$\mathrm{d}\frac{N_t}{D_t} = \sum_{i=1}^{m} \left(f_i - \kappa_i \frac{\partial_{x_t^i} D_t}{D_t} - \sum_{l\neq i}^{m} \frac{2}{X_t^l - X_t^i} \right) \partial_{x_t^i} \frac{N_t}{D_t} \mathrm{d}t + \sqrt{\kappa_i}\partial_{x_t^i} \frac{N_t}{D_t} \mathrm{d}B_t^i \,, \qquad (5.36)$$

which leads to the definition of the f_i above. Obviously,

$$\frac{\partial_{x_t^i} D_t}{D_t} = \partial_{x_t^i} \log Z_{\text{b.c.}}[x_t, y_t] \,, \qquad (5.37)$$

which proves the relation between bCFT and multiple SLE$(\kappa, \vec{\rho})$.

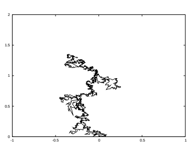
Note that, of course, this can straightforwardly be extended to the case of bulk force-points, i. e. the case where observables come close to the boundary analogous to the way presented in [68].

5.2.2 Analogies between SLE$(\kappa, \vec{\rho})$ and multiple SLE

Another question that arises is why multiple SLE is so similar to SLE$(\kappa, \vec{\rho})$. From a physical point of view, it is quite natural that there is a strong relationship between multiple SLE and SLE$(\kappa, \vec{\rho})$: both describe Statistical Physics systems at their critical point on domains, only with different boundary conditions and interfaces. From the mathematical point of view, the associated martingale that connect their measures to that of ordinary SLE can be obtained via the corresponding bCFT correlation function of boundary operators, i. e. the factor by which the partition function of the system differs from the free case. However, they are not the same, not even subsets of each other as we will argue in the following.

It is straightforward to see that SLE$(\kappa, \vec{\rho})$ with $\vec{\rho} = (2, 2, \ldots, 2)$ constitutes multiple SLE with a special choice of time parameterization. In terms of the formalisms typically used for the description of SLE$(\kappa, \vec{\rho})$, the motion of the coordinates of the force-points is determined by the Löwner equation which is driven by the Brownian motion of the SLE coordinate. The passive nature of this motion is captured by the fact that is has no motion of its own, i. e. $\kappa_j = 0$ and vanishing time parameterization. For $\rho_j = 2$, the j^{th} field has the correct weight $h_{(1,2)}$, but it does not create an interface. Applying the inverse of the Löwner mapping, the field only moves along the boundary, never entering the bulk. But if we set $a_t^j = 0$ in (4.24) for all j except the i^{th} which is set to one, we obtain the same system of SDEs for the SLE$(\kappa, \vec{\rho})$ force-points and the driving process of SLE itself:

$$\begin{aligned} \mathrm{d}X_t^i &= \sqrt{\kappa_i a_t^i}\mathrm{d}B_t^i + \kappa_i a_t^i \partial_{X_t^i} \log Z[x_t]\mathrm{d}t + \sum_{k\neq i} \frac{2a_t^k}{X_t^i - X_t^k}\mathrm{d}t \,, \\ &\to \sqrt{\kappa}\mathrm{d}B_t^i + \sum_{i(\neq k)=1}^{n} \frac{\rho_k}{X_t^k - X_t^i}\mathrm{d}t \quad \text{for the } i^{\text{th}} \qquad (5.38) \\ &\to \frac{2}{X_t^j - X_t^i}\mathrm{d}t \quad \text{for all } j \neq i \,, \qquad (5.39) \end{aligned}$$



for $Z[x_t]$ being the partition function (3.68) of the COULOMB gas approach to SLE$(\kappa, \vec{\rho})$.

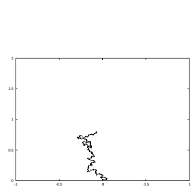
This stands in contrast to the considerations made in [125, 101] which state that multiple SLE should be a special case of SLE$(\kappa, \vec{\rho})$. However, this can not be true since it is not possible to obtain additional independent BROWNian motions needed for the driving functions of the multiple SLE interfaces from a special choice of SLE$(\kappa, \vec{\rho})$ driving functions.

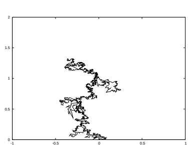
5.2.3 Interpretation for the Change of Measure M_t

Now let us briefly comment on the meaning of the "weighting by a local martingale". Mathematically speaking, the change of measure by a local martingale changes the way probabilities have to be measured in a particularly easy way. If we want to measure with our old and known measure, we have to consider the following:

$$\mathbf{P}^{\text{old}}(A) = \mathbb{E}(1_A) = \mathbb{E}^{\text{new}}(1_A/M_t)M_0 = \mathbf{P}^{\text{new}}(A/M_t)M_0 \,. \tag{5.40}$$

In the case of the SLE change of measure martingales, e. g. (5.5) and (5.18), M_t is given by the boundary part of the BCFT partition function $Z_{\text{b.c.}}[x_t, y_t]$, evaluated at the LÖWNER-time dependent coordinates. This way, (5.40) is exactly what we expect from considerations in Statistical Physics. $Z^{-1}_{\text{b.c.}}[x_t, y_t]$ adjusts the measure of the free case to that of specific boundary conditions present. Therefore, it is the most natural martingale to study within the BCFT-SLE correspondence.





6 SLE Martingales from Fusion in CFT

In the past two decades, boundary Conformal Field Theory (bCFT) [25, 31] became a very important and nowadays standard method to approach problems in the continuum limit of critical systems on simply connected compact two-dimensional domains with boundaries in Statistical Physics. More recently, it appeared that Stochastic Löwner Evolution (SLE) [35] could similarly be applied to such problems. From the point of view of physics, the former approach concentrates on local objects, i. e. the fields, and their correlation functions, while the latter describes the properties of non-local interfaces, starting and ending on the boundary.

Since both theories are successfully used to model the same physics, it is generally assumed that they can be related to each other. This idea has first been addressed by Bauer and Bernard [55], who found a one-to-two correspondence of the value c of the central charge of minimal bCFTs with the speed κ of the driving function of SLE. More precisely, considering both methods on the same domain, they have shown that the insertion of certain boundary condition changing fields, $\psi(\xi_t)$ and $\psi(\infty)$ ensures the existence of an interface in the bCFT picture. The two boundary fields are located at the image ξ_t of the endpoint of the interface growing with time t, and infinity, respectively. It can be calculated explicitly that the bCFT expectation value of some observable is conserved in mean during the SLE process, i. e. it is an SLE martingale, if the curve-creating field $\psi(\xi_t)$ exhibits a null descendant on level two. This condition also fixes the dimension $h_{(1,2)}(\kappa) = (6-\kappa)/2\kappa$ or $h_{(2,1)} = h_{(1,2)}(16/\kappa)$ of $\psi(\xi_t)$ and the central charge $c(\kappa) = (3\kappa - 8)(6-\kappa)/2\kappa$ in terms of κ.

However, a natural question is what the meaning of the other fields with dimensions $h_{(r,s)}$ in the spectrum of the bCFT could be. Do they give rise to interfaces as well? Or can they just be included as force-points as shown in the SLE$(\kappa, \vec{\rho})$ case? One should expect that they are related to SLE quantities as well, as these other fields can be obtained in a straightforward way by taking successively operator product expansions (OPEs) with the fundamental fields of dimension $h_{(1,2)}$ and $h_{(2,1)}$. This results in an infinite series of descendants of a finite set of well-known other primary fields. Applying the OPE to fields in the already identified martingale, we expect the outcome to contain new SLE martingale candidates from bCFT expectation values.

6.1 Statistical Physics Expectation Values and SLE Martingales

Due to the formulation of Statistical Physics, there exists a very basic connection between it and Probability Theory, and therefore discrete SLE. This is mostly due to the fact that the formalism of Statistical Physics is based on partition functions which, from a mathematical point of view, are unnormalized probability distributions. Therefore, essential operations such as conditional expectation values can be naturally taken. This provides the basis for a relation between interfaces in conformally invariant Statistical Physics and discrete SLE: conditioning expectation values on the existence of these interfaces is the same as considering the Statistical Physics model on the corresponding slit domain.

6.1.1 Probability Description of Statistical Physics

Mathematically speaking, a Statistical Physics model is a finite but usually large set of possible states $S = \{s\}$ on a finite grid. Each state $s \in S$ receives a BOLTZMANN weight $w(s) = \exp(-\beta H(s))$ such that the partition function,

$$Z = \sum_{s \in S} w(s)\,, \tag{6.1}$$

normalizes the weights to probabilities of the states s:

$$\mathbf{P}(s) = \frac{w(s)}{Z}\,, \tag{6.2}$$

i. e. the probability distribution is the GIBBS distribution at temperature T.

Due to the finiteness of the set of states S, the power set $\mathcal{P}(S)$ can be taken as the natural sigma algebra. From (6.1) and (6.2) it follows that the expectation value of a random variable $O : S \to \mathbb{C}$ to be

$$\mathbb{E}(O) = \langle O \rangle = \frac{1}{Z} \sum_{s \in S} O(s) w(s)\,. \tag{6.3}$$

Of course, it collections of subsets of states, e. g. $(S_\alpha)_{\alpha \in I}$, such that $\bigcup_{\alpha \in I} S_\alpha = S$ are taken, then the collection of all unions $\mathcal{F} = \{\bigcup_{\alpha \in I'} S_\alpha : I' \subset I\rangle$, too, forms a sigma algebra over S. More precisely, since S is finite, any allowed sigma algebra over S is of that type.

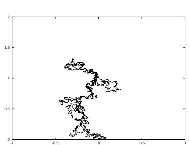
6.1.2 Martingales in Statistical Physics

Now consider a filtration $(\mathcal{F}_t)_{t \geq 0}$ on a collection of disjoint sets $(S_\alpha^{(t)})_{\alpha \in I_t}$. Then the partial partition function is given by

$$Z_\alpha^{(t)} = \sum_{s \in S_\alpha^{(t)}} w(s)\,, \tag{6.4}$$

and the conditional expectation values are given by

$$\begin{aligned} \langle O \rangle_t &= \mathbb{E}(O|\mathcal{F}_t) \\ &= \sum_{\alpha \in I_t} \frac{1}{Z_\alpha^{(t)}} \sum_{s \in S_\alpha^{(t)}} O(s) w(s) 1_{S_\alpha^{(t)}}\,. \end{aligned} \tag{6.5}$$



These conditional expectation values are, by definition, martingales, since

$$\mathbb{E}\left(\mathbb{E}(O|\mathcal{F}_t)|\mathcal{F}_s\right) = \mathbb{E}(O|\mathcal{F}_s) \quad \text{for all} \quad 0 \leq s < t\,. \tag{6.6}$$

Of course, the probability of an event $S_\alpha^{(t)}$, i. e. a specific collection of states, is $\mathbf{P}(S_\alpha^{(t)}) = Z_\alpha^{(t)}/Z^{(t)}$.

6.1.3 The Connection Between Statistical Physics and SLE

To draw a connection to discrete SLE, the disjoint subsets $(S_\alpha)_{\alpha\in I}$ are taken as configurations conditioned to exhibit certain interfaces connected to the boundary of the domain. This fixes the states along some path on the lattice to be in one state on the left and in another on the right hand side of the path.

Considering a model on a simply connected domain $D \subset \mathbb{C}$ with boundary conditions on ∂D such that there exist m mutually non-touching interfaces growing from the boundary into the domain in time. The time parameterization of the i^{th} interface γ^i is given by $t \mapsto \gamma_t^{(i)}(s)$ for $i = 1, \ldots, m$. This gives a natural filtration of the interface: $\mathcal{F}_t = \sigma(\gamma_{t'}^{(i)} : 0 \leq t' \leq t, i = 1, \ldots, m)$ up to time t.

By conformal invariance in the continuum limit, conditioning on the existence of the interfaces up to time t, $\gamma_{(0,t]}^{(i)}$, is the same as considering the model on a smaller domain D_t where these interfaces are cut out. For times $s > t$, the interfaces in D_t start growing at the tips of the cut-out $\gamma_{(0,t]}^{(i)}$.

Therefore, the conditional expectation values (6.5) are martingales under the discrete SLE measure with a natural extension to the continuum limit. This is why, in the continuum limit, conditional expectation values of Statistical Physics are believed to be SLE martingales, too.

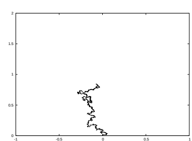
6.1.4 The Connection between Statistical Physics and CFT

As motivated above, Statistical Physics exhibits a natural description via discrete SLE. Therefore, its continuum limit should be described by ordinary SLE. However, this continuum limit of Statistical Physics has also been conjectured to correspond to CFTs [25]. If this is true, CFT expectation values as being continuum versions of (6.5) should exhibit the SLE martingale property, too, which we will review in the following [60].

Assuming its existence, the continuum limit of the Statistical Physics model at its critical point, i. e. the corresponding CFT can be taken. In this picture, observables are products of primary fields their expectation values can be expressed by normalized CFT correlation functions:

$$\langle O \rangle = \frac{\sum_{s\in S} O(s) w(s) S}{Z} \to \langle O \rangle_D^{\text{BCFT}}\,, \tag{6.7}$$

where the BCFT expectation value on the domain D is given by the correlation function of the observable and the boundary fields on ∂D divided by the correlation



function of boundary fields alone, i. e. the boundary part of the BCFT partition function $Z = Z_{\text{b.c.}} Z_{\text{free}}$:

$$Z_{\text{b.c.}} = \langle \psi_\infty, \psi_1 \dots \psi_m V_{\alpha_1} \dots V_{\alpha_p} \rangle Z_{\text{free}} \,, \tag{6.8}$$

where the ψ denote SLE curve-creating fields while the V_{α_q} are ordinary boundary fields, represented by vertex operators (3.60). This notation is motivated by the vertex operator formulation of CFT, the Coulomb Gas (see section 3.3.1) and its relation to SLE (see section 5.1.2).

In the CFT picture, the martingale (6.5) can be expressed by

$$\langle O \rangle_t = \sum_{\alpha \in I_t} \frac{1}{Z_\alpha^{(t)}} \sum_{s \in S_\alpha^{(t)}} O(s) w(s) 1_{S_\alpha^{(t)}} \to \langle O \rangle_{D_t}^{\text{BCFT}} \,, \tag{6.9}$$

i. e. the BCFT expectation value of the continuum limit of the Statistical Physics observable considered.

6.2 BCFT Observables as SLE Martingales

In the following we will consider SLE with a multiple Löwner mapping $G_t : D_t \to D$. Applying G_t to the right hand side of (6.9), takes the BCFT expectation value from D_t to D. This yields an equation on the full domain D:

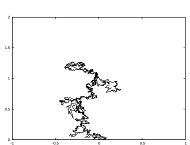
$$\langle O \rangle_t \to \frac{\langle \psi_\infty, {}^{G_t} O \psi_1 \dots \psi_m V_{\alpha_1} \dots V_{\alpha_p} \rangle_D^{\text{CFT}}}{\langle \psi_\infty, \psi_1 \dots \psi_m V_{\alpha_1} \dots V_{\alpha_p} \rangle_D^{\text{CFT}}} \,, \tag{6.10}$$

which is easy to compute using CFT methods. The superscript G_t denotes the action of the Löwner mapping on the observable O.

As a martingale with respect to the SLE measure, the SLE expectation value of (6.10) should have a vanishing (Löwner-) time-derivative which we will show in the following. The curve-creating boundary fields $\psi_1, \dots, \psi_m$ are located at points $x_t^1, \dots, x_t^m, \infty$ on the boundary, which correspond to the value of the stochastic driving process X_t^i at time t. Their variation in Löwner time according to the multiple Löwner mapping G_t is given by

$$\mathrm{d}G_t = \sum_{i=1}^{m} \frac{2\mathrm{d}t}{G_t(z) - X_t^i} \,, \quad G_0(z) = z \,. \tag{6.11}$$

In addition to these fields, there are other fields, located at bulk or boundary points which are only passively moved in Löwner time according to $G_t(z)$. By passively moved we mean that the processes Z_t formed by the locations z_t of the corresponding fields are deterministic. The n bulk fields $\phi_{h_j}(z_j, \bar{z}_j)$ are of local nature and factorise into two chiral bulk fields $\phi_{h_j}(z_j)$ and $\phi_{h_j}(\bar{z}_j) = \phi_{h_j}(z_{j+n})$ in the mirror-field approach to BCFT as reviewed in section 3.2.5. The boundary conditions and the



interactions of the bulk with the boundary fields are obtained by taking the limit $z_{j+n} \to z_j^*$. As discussed in the previous section 5.2, the observable O is given by the product of bulk fields that are far away from the boundary, i. e. not interacting with it:

$$O = \prod_{j=1}^{n} \phi_{h_j,\bar{h}_j}(z_j, \bar{z}_j) = \prod_{j=1}^{2n} \phi_{h_j}(z_j) \,. \tag{6.12}$$

The other class of passively driven bulk and boundary fields, interacting with the boundary, correspond to force-points in multiple SLE(κ, ρ) (see section 5.1.3). The bulk force-points are represented by local vertex operators $V_{\alpha_q,\bar{\alpha}_q}(z_q, \bar{z}_q)$. Each of them factorises into two chiral vertex operators in the mirror-image approach to bCFT. By definition, they can be replaced by their OPE

$$V_{\alpha_q}(z_q) V_{\bar{\alpha}_q}(z_q^*) = \sum_{h_{\alpha_q+\bar{\alpha}_q}} \sum_{|Y|=0}^{\infty} \mathfrak{Im}(z_q)^{|Y|+h_{\alpha_q+\bar{\alpha}_q}-h_{\alpha_q}-h_{\bar{\alpha}_q}} L_{-Y} V_{2\alpha_q}(y_q) \,, \tag{6.13}$$

with $\mathfrak{Re}(z_q) =: y_q$ and $h_{\alpha_q+\bar{\alpha}_q} = h_{2\alpha_q}$ or $h_{\alpha_q+\bar{\alpha}_q} = 0$ for true bulk force-fields as argued in chapter 5.

Therefore, the OPE only contains the $h_{2\alpha_q}$ channel and the bulk force-field pair $V_{\alpha_q}(z_q)V_{\bar{\alpha}_q}(z_q^*)$ is effectively equivalent to the presence of the boundary force-field $V_{2\alpha_q}(\mathfrak{Re}(z_q))$.

Despite their differences, both types of passive fields exhibit a similar variation under the mapping G_t. It is given by:

$$\mathrm{d}G_t'(z) = \sum_{i=1}^{m} \frac{2G_t'(z)}{(G_t(z) - X_t^i)^2} \mathrm{d}t \,. \tag{6.14}$$

Then, defining $z_t^j = G_t(z_j)$ and applying the chain rule, it is easy to compute that

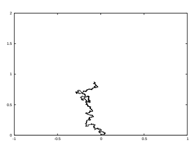
$$\mathrm{d}\phi(z_t^j) = \sum_{i=1}^{m} \frac{2}{G_t(z_j) - X_t^i} \partial_{z_t^j} \phi(z_t^j) \mathrm{d}t \,. \tag{6.15}$$

Allying the product rule and inserting the Löwner equation for $\mathrm{d}G_t$, it can be shown that

$$\frac{\mathrm{d}\left(\phi(z_t^j) G_t'(z_j)^{h_j}\right)}{G_t'(z_j)^{h_j}} = -\sum_{j=1}^{n} 2 \left(\frac{h_j}{(G_t(z_j) - x_t^i)^2} - \frac{1}{G_t(z_j) - x_t^i} \partial_{z_t^j} \right) \phi_{h_j}(z_t^j) \mathrm{d}t \,. \tag{6.16}$$

For the boundary fields that depend on a stochastic variable, we have to take the Itô-calculus, i. e. the variation of the field itself gives us:

$$\mathrm{d}\psi(x_t^i) = \partial_{x_t^i} \psi(x_t^i) \mathrm{d}X_t^i + \frac{\kappa_i}{2} \partial_{x_t^i}^2 (x_t^i) \mathrm{d}t \,, \tag{6.17}$$



Extending the computations done in [60] to our general approach to multiple SLE$(\kappa, \vec{\rho})$, we arrive at:

$$\begin{aligned}
\mathrm{d}\langle O\rangle &= \sum_{i=1}^{m} \frac{\mathcal{D}_{-2}^{m,p,2n}(x_t^i; \{x_t^l\}_{l\neq i}, \{y_t^q\}, \{z_t^k\})\langle\psi(\infty), ({}^{G_t}\prod_{k=1}^{2n}\phi(z_t^k)) \times \text{b.f.}\rangle_D^{\text{CFT}}}{\langle\psi(\infty), \prod_{i=1}^m \psi_i(x_t^i) \prod_{q=1}^{p} V_{\alpha_q}(y_t^q)\rangle_D^{\text{CFT}}}\mathrm{d}t \\
&+ \sum_{i=1}^{m} \frac{\mathcal{D}_{-2}^{m,p}(x_t^i; \{x_t^l\}_{l\neq i}, \{y_t^q\})\langle\psi(\infty), \psi_1 \ldots \psi_m V_{\alpha_1} \ldots V_{\alpha_p}\rangle_D^{\text{CFT}}}{\langle\psi(\infty), \psi_1 \ldots \psi_m V_{\alpha_1} \ldots V_{\alpha_p}\rangle_D^{\text{CFT}}}\langle O\rangle\mathrm{d}t \\
&+ \sum_{i=1}^{m} \partial_{x_t^i}\langle O\rangle\sqrt{\kappa}\mathrm{d}X_t^i\,, \qquad (6.18)
\end{aligned}$$

wherein b.f. stands for the product of boundary fields $\prod_{i=1}^{m}\psi_i(x_t^i)\prod_{q=1}^{p}V_{\alpha_q}(y_t^q)$. If the curve-creating boundary fields exhibit a vanishing descendant on level two, we know that the $\mathcal{D}$'s annihilate the correlation functions and $\mathrm{d}\langle X_t^i\rangle = 0$ under the multiple SLE measure. Therefore, the expectation value of the right hand side of (6.18) vanishes and the BCFT expectation value with additional bulk- and boundary force-fields, $\langle O\rangle$ can be considered an SLE martingale.

Remark 11. SLE is a random process in time, moving in a domain with a moving boundary. However, it models processes which are *not generated in time* by the formation of the SLE path. The configurations corresponding to SLE are rather weighted by an equilibrium measure (the inverse partition function), reflecting the mean growth rule for the SLE path.

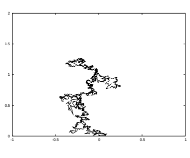
This observation is quite similar to the quasi-ergodic hypothesis in thermodynamics: time-averages can be related to ensemble averages, or more precisely:

Theorem 5 (Quasi-Ergodic Hypothesis). Over large enough time periods, the time spent by a particle in some region of phase space of microstates of the same energy is proportional to the volume of this region. This means that all accessible microstates are equally probable over a long period of time. Therefore, the time-average of each quantity equals the ensemble average at an arbitrary point in time.

This behavior implies that such systems do not depend on their initial states anymore which explains why the driving process needed to describe physically relevant interfaces has to be MARKOV.

6.3 Motivation for New SLE Martingales

SLE martingales as constructed above can only be obtained from the BCFT objects containing at least one of two types of BCFT boundary fields: $\psi_{h_{(1,2)}}$ or $\psi_{h_{(2,1)}}$. However, it is generally hoped for to find a relationship between *all* types of BCFT



boundary conditions that lead to interfaces and SLE objects since the two theories describe the same scaling limit of Statistical Physics models. Therefore, our primary goal is to find BCFT objects containing other boundary fields than the two mentioned above that are at the same time SLE martingales, *independent* of the presence of $\psi_{h_{(1,2)}}$ or $\psi_{h_{(2,1)}}$ boundary fields. A straightforward way to obtain other types of boundary fields in BCFT starting from the presence of $\psi_{h_{(1,2)}}$ or $\psi_{h_{(2,1)}}$, is to investigate the short-distance behavior of the boundary fields $\psi_{h_{(1,2)}}$ or $\psi_{h_{(2,1)}}$, using their operator product expansion (OPE). The OPE contains fields corresponding to the fusion product of two copies of the representations of highest weights $h_{(1,2)}$ or $h_{(2,1)}$, i. e. the identity, $(h_{(1,1)}, c_{p,q})$, and $(h_{(1,3)}, c_{p,q})$ or $(h_{(3,1)}, c_{p,q})$, respectively.

Another motivation for analyzing the OPE of these boundary fields is the existence of so-called topological observables [60] (cf. chapter 5). In this context, curves starting at the boundary in multiple SLE either pair up or grow all the way up to infinity, corresponding to fusion to the identity or an appropriate field of the same weight as that at infinity. In SLE$(\kappa, \vec{\rho})$ this can also mean that an SLE curve ends on another boundary point than ∞.

In order to succeed in finding new SLE martingales starting from the ones already known, e. g. $\langle O_t \rangle$ (cf. (6.10)), we have to ask what the joint short-distance behavior of the null-vector operator (3.39) acting on a correlation function wherein we replace two boundary fields with their OPE looks like. This is what we will derive in the following.

From the physical picture, it follows that the OPE of a curve-creating boundary field $\psi(x_i)$ should only be taken with fields that are allowed to *interact* with it, i. e.

① a neighboring curve-creating boundary field $\psi(x_{i+1})$,,

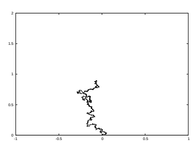
② ❶ or a neighboring boundary force-field $V_{\alpha_r}(y_r)$,

❷ or bulk force-field pairs $V_{\alpha_r,\bar{\alpha}_r}(z_r, z_r^*)$, effectively describing a boundary force-field $V_{2\alpha_r}(y_r)$ with $y_r = \mathfrak{Re}(z_r)$.

i. e. those fields that differ from the identity on the boundary.

As put up to discussion in section 5.1.3, we motivated that bulk *observables* do not interact with the curve. Therefore, their expectation values may not be the most natural quantities to consider when investigating the short-distance behavior of curve-creating boundary with other fields. By definition, when coming close to the boundary, the interactions of observable fields with their mirror-images are not allowed to result in boundary fields different from the identity. If otherwise, they would have to contribute to the boundary partition functions and therefore they would be boundary *force*-fields, whose locations correspond to force-points, contributing to the driving process. From this we conclude that the most natural martingale to consider in this context is the inverse partition function, $Z[x_t, y_t]^{-1}$, emerging in the study of probabilities of events (cf. section 5.2.3).

This is a very subtle point, unnoticed in the literature so far, of which we, too, became aware only very recently. Up to now, only expectation values of observables



have been considered as martingales corresponding to SLE quantities, e. g. by Bauer, Bernard and Kyölä in [60], on whose considerations we based our first paper [69]. Therefore we decided to put this new insight up for discussion and present our original idea by including the original version of our paper [69] considering single SLE in chapter 6a. Starting from this well-formulated basis presenting the main idea and interpretation, we then apply the same methods used in 6a to SLE($\kappa, \vec{\rho}$) with bulk force-points and multiple SLE (partly following [70], where the non-trivial boundary part of the partition function provides a natural basis for the insights presented above.

6.3.1 Short Distance Limits for Null Vector Conditions

For the derivation and interpretation of new martingales in the following two parts (a and b) of this chapter, we need to start with the investigation of the short-distance expansions in the context of null-vector conditions. This is due to the fact that the martingale property of bCFT quantities means that we have to check the Löwner variation of the bCFT object. This typically leads to a null-vector condition on a correlation function of primary fields. However investigating the OPE of a null descendant of a primary field with another primary, we found that there are two things that have to be considered:

① the distance dependence of the correlation function arising from the OPE,

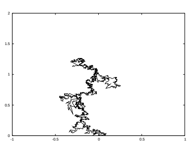
② the distance dependence of the differential operator due to the change of coordinates.

Only the first dependence has been considered in preceding work, e. g. [110, 60]. However, when speaking about SLE-bCFT correspondence, we always have to check the martingale properties of the bCFT objects possibly leading to additional contributions of type ②. Hence, we start by examining what happens, if we take both dependencies into account.

The differential equations in question are bCFT null-vector conditions, they are of the form (cf. (3.39 and (3.40))

$$\sum_{|\{k\}|=m} \mathcal{L}_{-\{k\}} \langle 0|\phi_{h_{r_1,s_1}}(z_1,\bar{z}_1)\cdots\phi_{h_{r_l,s_l}}(z_l,\bar{z}_l)\cdots\phi_{h_{r_n,s_n}}(z_n,\bar{z}_n)|0\rangle = 0\,. \qquad (6.19)$$

Now we use the OPE to determine the contribution of ① and apply a Taylor expansion to get a hand on the effect of ② e. g. for $z_1 \to z_2$. This will be done in two steps: first we will reformulate the differential operators $(z_1, z_2) \to (z, \epsilon)$, then we will provide instructions how to obtain the correct coefficients $\beta^{|Y|}(r', s')$ in the OPE.



6.3.2 Step 1: Coordinate Transformation

To obtain the transitions from $(z_1, z_2) \to (z, \epsilon)$, we define:

$$\begin{aligned} \epsilon &:= z_1 - z_2\,, &(6.20)\\ z &:= z_1\,. &(6.21)\end{aligned}$$

With little effort, we can transform (3.40) for $z_l = z_1$ to

$$\begin{aligned}\tilde{\mathcal{L}}_{-k}(z,\epsilon;\{z_i\}_{i\neq 1,2}) &= \frac{h(k-1)}{\epsilon^k} - \frac{1}{\epsilon^{k-1}}\partial_\epsilon + \sum_{i=2}^{n} \frac{(k-1)h_i}{(z_i - z)^k} + \frac{1}{(z_i-z)^k}\partial_{z_i}\,,\\ &=: \epsilon^{-k}(h_1(k-1) - \epsilon\partial_\epsilon) + \mathcal{L}^{n-1}_{-k}(z)\,. \qquad (6.22)\end{aligned}$$

6.3.3 Step 2: OPE Coefficients

The basic tool to get the correct values for the OPE coefficients $\beta^{|Y|}_{(r',s')}$ is global conformal invariance [23]. Starting with

$$\phi_h(z')(\mathrm{d}z')^h = \phi_h(z)(\mathrm{d}z)^h\,, \qquad (6.23)$$

where $z' = z + f(z)$ for any conformal transformation $f(z)$, we get for infinitesimal $f(z)$(:

$$(\phi_h(z) + f(z)\partial_z\phi_h(z))(\mathrm{d}z)^h(1 + hf'(z)) = \phi_h(z)(\mathrm{d}z)^h\,. \qquad (6.24)$$

Laurent expanding $f(z) = \sum_k f_k z^{k+1}$ and $\delta_f L_{-j}\phi_h(z) = f_k L_k L_{-j}\phi_h(z)$, the action of the L_k on the OPE of $\phi_{h_1}(z_1)$ and $\phi_{h_2}(z_2)$ can be computed:

$$\begin{aligned}&\sum_Y \epsilon^{|Y|}\beta^{|Y|}_{(r',s')} L_k L_{-j}\phi_{h_{(r',s')}}(z) \qquad (6.25)\\ &\quad = (h_1(k+1)\epsilon^k + \epsilon^{k+1}\partial_\epsilon)\sum_Y \epsilon^{|Y|}\beta^{|Y|}_{(r',s')} L_{-j}\phi_{h_{(r',s')}}(z)\,.\end{aligned}$$

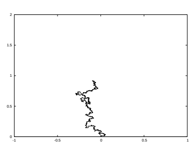
This implies the following rule of thumb for the computation of the $\beta^{|Y|}_{(r',s')}$:

① For any level $|Y|$ we take the $|Y|$ equations

$$L_k\phi^{(|Y|)}_{h_{(r',s')}}(z) = \big(h_1(k+1) + |Y| - k + h_{(r',s')} - h_1 - h_2\big)\,\phi^{(|Y|-k)}_{h_{(r',s')}}(z)\,, \quad (6.26)$$

with $k = 1, \ldots, |Y|$ and

$$\phi^{(|Y|)}_{h_{(r',s')}}(z) = \sum \beta^{|Y|}_{(r',s')} L_{-Y}\beta_{(r',s')}\phi_{h_{(r',s')}}(z)\,. \qquad (6.27)$$



② The expression

$$\phi^{(|Y|)}_{h_{(r',s')}}(z) = L_k \sum \beta^{|Y|}_{(r',s')} L_{-Y} \beta_{(r',s')} \phi_{h_{(r',s')}}(z)\,, \qquad (6.28)$$

$$= \sum \beta^{|Y|}_{(r',s')} [L_k, L_{-Y}] \beta_{(r',s')} \phi_{h_{(r',s')}}(z)\,, \qquad (6.29)$$

has to be checked algebraically.

From the comparison of the coefficients of $\phi^{(|Y|-k)}_{h_{(r',s')}}(z)$ in ① and ②, we can extract the coefficients $\beta^{|Y|}_{(r',s')}$. Note that while the exact values of the $\beta^{|Y|}_{(r',s')}$ depend on the choice of z and ϵ, the final result does not.

The resulting differential equation is on level $m = r' \cdot s'$ since the other contributions vanish. Therefore, we directly consider the $|Y| = r' \cdot s' - r_1 \cdot s_1$ channel when starting with the differential equation due to the null vector on the state corresponding to $\phi_{h_1}(z_1)$.

6.3.4 Result

In the case of SLE-bCFT correspondence, we have type $h_{(1,2)}$ and $h_{(2,1)}$ fields due to which we know that a differential equation arises. Therefore we will apply the above to two $(r,s) = (1,2)$ fields, i. e. $\phi_{h_1} = \phi_{h_2} = \phi_{h_{(1,2)}}$ with $h_{(1,2)} = \frac{6-\kappa}{2\kappa}$. The corresponding representations can fuse into the identity or the $(1,3)$ representation:

$$(h_{(1,2)}, c_{p,q}) \times (h_{(1,2)}, c_{p,q}) = (h_{(1,1)}, c_{p,q}) + (h_{(1,3)}, c_{p,q})\,. \qquad (6.30)$$

Since global conformal invariance is equivalent to the only possible null vector condition on level $m = 1$, the behavior of the identity channel is trivial and we will not discuss it in greater detail here.

To derive the differential equation of the second channel in lowest order, we have to take $|Y| = 1 \cdot 3 - 1 \cdot 1 = 1$ and start with step 1: the computation of the transformed differential operators. More precisely, we have to compute (6.22) for $k = 1, 2$ since the null state is given by $(\frac{\kappa}{2} L_{-1}^2 - 2L_{-2})|h_1\rangle = |0\rangle$.

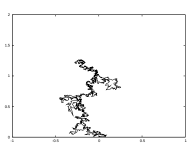
$$\tilde{\mathcal{L}}_{-1} = \mathcal{L}_{-1}(z) - \partial_\epsilon\,, \qquad (6.31)$$

$$\tilde{\mathcal{L}}_{-2} = \mathcal{L}_{-2}(z) - \frac{1}{\epsilon}\partial_\epsilon + \frac{h_1}{\epsilon^2}\,. \qquad (6.32)$$

These transformed differential operators now act on a correlation function containing the fusion product of $\phi_{h_1}(z_1)$ and $\phi_{h_2}(z_2)$ of which we will choose only the $h_{(1,3)} = h$ channel:

$$\Big[\frac{\kappa}{2}\left(\partial_\epsilon^2 - 2\partial_\epsilon \mathcal{L}_{-1}(z) + \mathcal{L}_{-1}^2(z)\right) \qquad (6.33)$$

$$-2\left(\mathcal{L}_{-2}(z) - \frac{1}{\epsilon}\partial_\epsilon + \frac{h_1}{\epsilon^2}\right)\Big] \sum_Y \epsilon^{|Y|+h-2h_1} \langle \phi_h^{|Y|}(z) \phi_{h_3}(z_3) \cdots \phi_{h_n}(z_n)\rangle\,.$$



After executing the derivatives with respect to ϵ and re-ordering the terms of the various powers of ϵ, we get:

$$\begin{aligned} 0 &= \left[\frac{\kappa}{2}(3+h-2h_{(1,2)})(2+h-2h_{(1,2)})-2(3+h-3h_{(1,2)})\right]\phi_h^{(3)} \\ &\quad -\kappa(3+h-2h_{(1,2)})L_{-1}\phi_h^{(2)}+\left(\frac{\kappa}{2}L_{-1}^2-2L_{-2}\right)\phi_h^{(1)} . \end{aligned} \tag{6.34}$$

With a lengthy calculation (see appendix A.2.1), it can be checked that the coefficients $\beta^{(1)}_{h_{(1,3)}}$ lead in fact to the known level three null vector

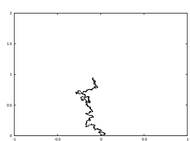
$$\left(\frac{\kappa}{2}L_{-1}^3+\left(\frac{\kappa}{2}-4\right)L_{-2}L_{-2}-4\left(\frac{8}{\kappa}-1\right)L_{-3}\right)|h_{(1,3)}\rangle . \tag{6.35}$$

6a Towards an Interpretation of Fusion in Stochastic Löwner Evolution

In the first part of this chapter, we will consider a single chordal SLE setting and its corresponding BCFT on the upper half-plane with two boundary fields $\psi_{h_{(1,2)}}$ located at ξ_t and ∞. In the bulk, we will add a couple of marked points that are represented by primary fields ϕ whose product serves as an observable. Here, and in the following, we always will denote boundary fields with ψ and bulk fields with ϕ. We will show explicitly that upon performing the OPE between one of the bulk fields and the boundary field $\psi_{h_{(1,2)}}$ located at the tip of the SLE interface does not spoil the martingale property of the BCFT expectation value of the observable if the expansion is valid. In addition, if the bulk field ϕ is of weight $h_{(1,2)}$, too, we will propose an interpretation of this martingale in terms of the SLE event of interfaces intersecting a small-sized disc around the point marked by $\phi_{h_{(1,2)}}$ [123]. In doing so, the short-distance scaling behavior of the expansion becomes meaningful in terms of the probability of the SLE event.

We should mention that our work fits nicely into the context of preceding work by Bauer and Bernard identifying BCFT objects that are also SLE martingales. In [60], the OPE of boundary fields only, determined by the fusion product of their representations $(h_{(1,2)}, c_{p,q}) \times (h_{(1,2)}, c_{p,q}) = (h_{(1,1)}, c_{p,q}) + (h_{(1,3)}, c_{p,q})$, has been related to possible interface configurations in multiple SLE. Therein, the identity OPE channel is interpreted in terms of interfaces pairing up while the other channel corresponds to interfaces growing to infinity. In addition, in [110] the same SLE probability has been investigated in terms of single SLE related to bulk fields of weight $h_{(0,1)}(\kappa) = (8-\kappa)/16$.

We will motivate that in the verification of the martingale property of the expanded BCFT expectation value of the observable, a peculiarity arises. Its scaling behavior



is governed by the OPE of the level two *null-descendant* of $\psi_{h_{(1,2)}}$ with $\phi_{h_{(1,2)}}$. This differs from that of the OPE of the *primary* field $\psi_{h_{(1,2)}}$ itself with $\phi_{h_{(1,2)}}$. From this, we receive additional contributions to the short-distance scaling behavior of the martingale containing the fused boundary field $\psi_{h_{(1,3)}}$. This allows for a more intuitive interpretation of the probability of the SLE trace hitting the disc than the one given in [110]. Furthermore, our result is more general since it naturally yields the correct exponent of the angular behavior of the probability while staying within the field content of the minimal bCFT models.

We assume that our audience is familiar with SLE and bCFT, and refer to standard texts on SLE [128, 62, 129, 130] and CFT [31, 80, 131, 132] for further reading.

6.4 Review of Single SLE and CFT

We start by collecting some standard results in SLE and CFT which will be needed in the following.

Within its application to the scaling limit of two-dimensional Statistical Physics, the Löwner equation describes the growth of a physical domain interface $\gamma_{(0,t]}$ on the upper half-plane $\mathbb{H}$, starting at the origin and aiming at infinity. The evolution of γ_t is given by the pre-images of the singularities of the Löwner differential equation

$$\mathrm{d}g_t = \frac{2\mathrm{d}t}{g_t - \xi_t}\,, \tag{6.36}$$

with initial condition $g_0 = z$ and hydrodynamical normalization $g_t(z) = z + \frac{2t}{z} + \mathcal{O}(z^{-2})$ for $z \to \infty$. The Löwner mapping $g_t(z)$ is well-defined for all times $t < T(z)$ with

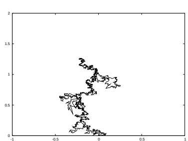
$$T(z) := \sup\{t \geq 0 \,:\, \min_{s\in[0,t]} |g_s(z) - \xi_s| > 0\}\,. \tag{6.37}$$

The hull of the SLE is defined as

$$K_t = \{z \in \overline{\mathbb{H}} \,:\, T(z) \leq t\}\,, \tag{6.38}$$

such that $g_t : \mathbb{H}/K_t \to \mathbb{H}$. The complement of the hull, $\mathbb{H}_t = \mathbb{H}/K_t$, is the unbounded connected component of $\mathbb{H}/\gamma_{(0,t]}$ where the tip of the curve is given by $\gamma_t := \lim_{\delta\to 0} g_t^{-1}(\xi_t + \mathrm{i}\delta)$. Oded Schramm [35] proved that choosing the driving function to be $\xi_t = \sqrt{\kappa}B_t$, i.e. one-dimensional Brownian motion of speed $\kappa \in \mathbb{R}^+$, results in conformally invariant interfaces $\gamma_{(0,\infty)}$ exhibiting the Markov property. Such interfaces have the properties of physically relevant interfaces of the statistical models.

In the following we will review how a relationship between the two mathematical models, SLE and bCFT, that are conjectured to describe the scaling limit of models in two-dimensional Statistical Physics, has been established [60].



6.4.1 BCFT Observables as SLE Martingales

From a physicist's point of view, martingales in SLE are expectation values of observables in the scaling limit of two-dimensional Statistical Physics models at criticality [60]. This is motivated by their rather tautological correspondence in the discrete case. However, this scaling limit is conjectured to be described by BCFTs as well. Therefore, SLE martingales are expected to be expressible in terms of BCFT expectation values $\mathcal{O}^{\mathrm{CFT}}$. These are suitably normalized correlation functions of BCFT observables, $O^{\mathrm{CFT}}(\{z_l, \bar{z}_l\}) := \prod_{l=1}^{n} \phi(z_l, \bar{z}_l)$, i. e. of products of local primary fields. Throughout this paper, we will consider such a BCFT expectation values

$$\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t} := \frac{\left\langle \psi(\infty), O^{\mathrm{CFT}}(\{z_l\})\psi(\xi_t)\right\rangle}{\langle \psi(\infty), \psi(\xi_t)\rangle}\,, \tag{6.39}$$

in the presence of two boundary fields $\psi(\infty)$ and $\psi(\xi_t)$.

In the standard mirror-image approach to BCFT, these n *local* primary fields on $\mathbb{H}$ are regarded as $2n$ *independent chiral* fields, $\phi(z_l, \bar{z}_l) = \phi(z_l)\phi(z_{l+n})$, on $\mathbb{C}$ with $z_{n+l} = \bar{z}_l$ in abuse of notation. Taking the limit $z_{n+l} \to z_l^*$, where $*$ denotes the complex conjugate, imposes the boundary conditions on the real line.

Considering the BCFT expectation value of the observable in an SLE domain $\mathbb{H}_t$ at time t, it has been shown [110] that it fulfils the SLE martingale condition, i. e. that its probabilistic expectation value $\mathbb{E}(\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t})$ has vanishing time derivative. Using this fact, BCFT expectation values can be identified as SLE martingales:

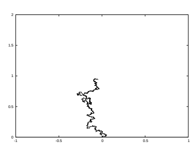
$$\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t}(\xi_t; \{z_t^l\}) := \frac{\left\langle \psi(\infty), {}^{g_t}O^{\mathrm{CFT}}(\{z_l\})\psi(\xi_t)\right\rangle_{\mathbb{H}}}{\langle \psi(\infty), \psi(\xi_t)\rangle_{\mathbb{H}}} \equiv O_t^{\mathrm{SLE}}\,, \tag{6.40}$$

where the superscript g_t denotes the action of the Löwner mapping. On the left hand side, we suppress the dependence on the coordinate fixed under the mapping g_t, which is ∞. In order to show that the probabilistic expectation value of $\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t}$ has vanishing time derivative and can therefore be identified as an SLE martingale, the variation of the observable with respect to g_t has to be computed. Introducing $z_t^l := g_t(z_l)$ with $z_0^l = z_l$ for $l = 1, 2, \ldots, 2n$, this is done with the help of the Löwner equation and the transformation properties of primary fields, i. e. covariant tensors of rank h, in CFT:

$$\mathrm{d}\left({}^{g_t}O^{\mathrm{CFT}}(\{z_t^l\})\right) = -2\sum_{l=1}^{2n}\left(\frac{h_l}{\left(z_t^l - \xi_t\right)^2} - \frac{1}{z_t^l - \xi_t}\partial_{z_t^l}\right) {}^{g_t}O^{\mathrm{CFT}}(\{z_l\})\mathrm{d}t\,. \tag{6.41}$$

As a function of a stochastic variable ξ_t, the variation of the boundary field has to be computed according to the Itô calculus

$$\mathrm{d}(\psi(\xi_t)) = \partial_{\xi_t}\psi(\xi_t)\mathrm{d}\xi_t + \frac{\kappa}{2}\partial^2_{\xi_t}\psi(\xi_t)\mathrm{d}t\,. \tag{6.42}$$



Together, the variation with respect to the LÖWNER time becomes a differential equation in the coordinates of the SLE domain

$$\mathrm{d}\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t}(\xi_t;\{z^l_t\}) = \mathcal{D}^{2n}_{-2}(\xi_t;\{z^l_t\})\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t}(\xi_t;\{z^l_t\})\mathrm{d}t + \partial_{\xi_t}\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t}(\xi_t;\{z^l_t\})\mathrm{d}\xi_t\,, \quad (6.43)$$

where the first differential operator is defined as

$$\mathcal{D}^{2n}_{-2}(\xi_t;\{z^l_t\}) := \frac{\kappa}{2}\partial^2_{\xi_t} - 2\sum_{l=1}^{2n}\left(\frac{h_l}{\left(z^l_t-\xi_t\right)^2} - \frac{1}{z^l_t-\xi_t}\partial_{z^l_t}\right). \quad (6.44)$$

Taking the probabilistic expectation value in (6.43), it is obvious that due to the vanishing probabilistic expectation value of BROWNian motion, $\mathbb{E}(B_t)=0$, the term proportional to $\mathrm{d}\xi_t$ disappears. Moreover, the term proportional to $\mathrm{d}t$ can be identified as the level two null-vector condition emerging due to the existence of a level two null descendant of the boundary field $\psi_{h_{(1,2)}}$ in CFT. Therefore, if the boundary condition changing field at ξ_t is of type $\psi_{h_{(1,2)}}$, the probabilistic expectation value of (6.43) is time independent. This is why the BCFT expectation value $\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t}(\xi_t;\{z^l_t\})$ can be interpreted as an SLE martingale.

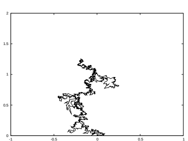
6.4.2 Fusion in CFT and the OPE

In CFTs with degenerate representations, primary fields $\phi_{h_{(r,s)}}$ correspond to highest weight representations $(h_{(r,s)}, c_{p,q})$ of the Kac-table. The fusion rules of these representations tell us which primaries and descendants appear in the operator product expansion of two given fields. In principle, if the two fundamental fields $\phi_{h_{(2,1)}}$ and $\phi_{h_{(1,2)}}$ are present in a theory, all other fields may be generated by the OPE of a suitable number of copies of these two fundamental fields, i. e. consecutive fusion of the respective representations. Therefore, if we want to identify other fields from the Kac-table with SLE objects, it is natural to study the OPE of the fields contained in the SLE martingale (6.40).

The OPE of two primary fields is given by (e. g. cf. [23])

$$\phi_{h_{(r_0,s_0)}}(z)\phi_{h_{(r_1,s_1)}}(w) = \sum_{h_{(r',s')},Y} g_{(r',s')}(z-w)^{|Y|-\mu}\beta_{Y,(r',s')}L_{-Y}\phi_{h_{(r',s')}}(z)\,. \quad (6.45)$$

Here, $Y=\{k_1,k_2,\ldots,k_n\}$, $k_1\geq\ldots\geq k_n$, denotes a multi-index such that $|Y| = \sum_{i=1}^n(-k_i)$ is the level of the linear combination of products of descendant operators $L_{-Y}=L_{-k_1}\ldots L_{-k_n}$ with coefficients $\beta_{Y,(r',s')}$. The exponent μ is defined through the dimensions of the Kac-table fields involved, $\mu = h_{(r',s')} - h_{(r_0,s_0)} - h_{(r_1,s_1)}$. The coefficient of their three-point function, $g_{(r',s')}$, is only non-zero for weights $h_{(r',s')}$ that appear in the fusion product $(h_{(r_0,s_0)}, c_{p,q})\times(h_{(r_1,s_1)}, c_{p,q})$ of the representations belonging to the primary fields $\phi_{h_{(r_0,s_0)}}$ and $\phi_{h_{(r_1,s_1)}}$. This implicitly defines the range of the sum.



For our purposes, it will be sufficient to know that

$$(h_{(1,2)}, c_{p,q}) \times (h_{(1,2)}, c_{p,q}) = (h_{(1,1)}, c_{p,q}) + (h_{(1,3)}, c_{p,q}) \,. \tag{6.46}$$

This means that in the OPE of two fields of weight $h_{(1,2)}$ the first summation is over terms with $h_{(1,1)} = 0$ and $h_{(1,3)}$ only:

$$\begin{aligned} \phi_{h_{(1,2)}}(z)\phi_{h_{(1,2)}}(w) &= \sum_Y g_{(1,1)}(z-w)^{|Y|-\mu}\beta_{Y,(1,1)}L_{-Y}\mathbb{I}(z) \\ &+ \sum_Y g_{(1,3)}(z-w)^{|Y|-\mu}\beta_{Y,(1,3)}L_{-Y}\phi_{h_{(1,3)}}(z) \,, \end{aligned} \tag{6.47}$$

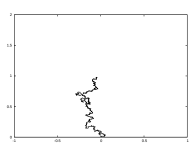
The OPE in CFT is defined in correlation functions which are analytical objects. As such it is valid on a suitable coordinate patch and may exhibit singular behavior on its boundary. In particular this means that applying the OPE to two fields at x_i, x_j in a correlation function containing other fields located at y_k, the OPE is convergent if

$$\epsilon := |x_i - x_j| < \min_k |x_i - y_k| \,. \tag{6.48}$$

In words, this means that in radial order only adjacent fields can be replaced by their OPE. In the following, we will refer to this condition as the distance ϵ being small. However, in adjacent coordinate patches where the OPE in the distance ϵ is not valid, there exist always other expansions. Expanding the operator product in parameters δ that can be considered as small in these other patches, e. g. $(1-\epsilon)$ or $1/\epsilon$, then results in an equally valid expression. This way we can always find similar expansions to (6.47), only in terms of some suitable δ e. g. chosen from $\{\epsilon, (1-\epsilon), 1/\epsilon\}$. Therefore, taking the OPE on the *whole* domain can be done by partitioning the domain into suitable coordinate patches, on which we expand the operator product in a suitable choice of δ. In the following we will denote the coordinate patch in which the expansion in the small distance ϵ is valid by D_ϵ and the collection of coordinate patches by $D := \bigcup_\delta D_\delta$.

6.5 SLE Interpretation of the Fusion Product

As motivated earlier, we would like to check the martingale property of the bCFT expectation value (6.40) after applying the OPE to the product of the boundary field $\psi_{h_{(1,2)}}(\xi_t)$ with a bulk field $\phi(z_t^j)$ contained in the observable. If after doing so, its time derivative still vanishes when taking the probabilistic expectation value, we can identify the outcome with a new SLE martingale containing fields of other dimension than $h_{(1,2)}$ or $h_{(2,1)}$ in the spirit of [60]. Therefore we start with the simplest scenario which is the situation where the bulk field $\phi(z_t^j)$ is of the same dimension $h_{(1,2)}$ as the boundary field $\psi_{h_{(1,2)}}(\xi_t)$. Investigating the variation of the bCFT expectation value of the observable (6.43) in the short-distance limit, we have to replace the two fields $\psi_{h_{(1,2)}}(\xi_t)$ and $\phi_{h_{(1,2)}}(z_t^j)$ by their OPE (6.47). In addition, we have to re-express



the differential operator $\mathcal{D}_{-2}^{2n}(\xi_t;\{z_t^l\})$ in the new coordinates, i. e. replace (ξ_t, z_t^j) by $(\xi_t, \xi_t - z_t^j)$, to analyze its action on the expanded BCFT expectation value.

Before we proceed, we would like to remark that the considerations up to here and in the following sections stated in terms of κ and $h_{(r,s)}$ are equally valid for $r \leftrightarrow s$, i. e. $\kappa \leftrightarrow 16/\kappa$. This connection between the two values of κ corresponding to the same central charge is called duality and will become important later on, when we discuss the physical interpretation of our results.

6.5.1 Martingales from Fusion in Correlation Functions

Taking a look at the OPE of the boundary field $\psi_{h_{(1,2)}}(\xi_t)$ with the bulk field $\phi_{h_{(1,2)}}(z_t^j)$, we see that from the fusion rules (6.46) for the representations corresponding to to the two fields of weight $h_{(1,2)}$ we receive two contributions:

$$\begin{aligned}\psi_{h_{(1,2)}}(\xi_t)\phi_{h_{(1,2)}}(z_t^j) &= \sum_Y g_{(1,1)}(z-w)^{|Y|-\mu}\beta_{Y,(1,1)}L_{-Y}\mathbb{I}(\xi_t) \\ &\quad + \sum_Y g_{(1,3)}(z-w)^{|Y|-\mu}\beta_{Y,(1,3)}L_{-Y}\psi_{h_{(1,3)}}(\xi_t)\,,\end{aligned} \tag{6.49}$$

For the identity and its descendants, a peculiarity arises. Taking the identity channel corresponds to the situation where the two fields annihilate, i. e. the point z_t^j gets actually hit by the image of the tip ξ_t of the trace [60]. However, we do not consider SLE$(\kappa, \vec{\rho})$ here, therefore the SLE curve is prohibited to stop at other points than ∞ by the boundary conditions. This goes well along with the SLE result [123] that for vanishing distance and $0 < \kappa < 8$, the probability of the SLE curve to hit a point is zero. Therefore, we will restrict our investigations to the fusion channel containing the $h_{(1,3)}$ representation in the following. On the BCFT side, the selection of the $(h_{(1,3)}, c_{p,q})$ fusion channel can be done by choosing an appropriate weight for the out-state corresponding to $\psi(\infty)$.

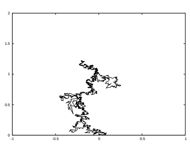
Introducing the notation $\epsilon := \xi_t - z_t^j$, we can express $\mathcal{D}_{-2}^{2n}(\xi_t;\{z_t^l\})$ in terms of the new coordinates, $(\xi_t, z_t^j) \rightarrow (\xi_t, \epsilon)$. In addition, we replace the product $\psi_{h_{(1,2)}}(\xi_t)\phi_{h_{(1,2)}}(z_t^j)$ in the BCFT expectation value of the observable $\mathcal{O}_{\mathbb{H}_t}^{\mathrm{CFT}}(\xi_t;\{z_t^l\})$ by the OPE of $\psi_{h_{(1,2)}}(\xi_t)$ with $\phi_{h_{(1,2)}}(z_t^j)$. This results in an ϵ-dependent sum of descendant operators acting on the fused boundary field $\psi_{h_{(1,3)}}(\xi_t)$ contained in the new BCFT expectation value

$$\tilde{O}_{\mathbb{H}_t}^{\mathrm{CFT}}(\xi_t;\{z_t^l\}_{l\neq j}) := \left\langle \psi_{h_\infty}(\infty), \tilde{O}^{\mathrm{CFT}}(\{z_t^l\}_{l\neq j})\psi_{h_{(1,3)}}(\xi_t)\right\rangle\,, \tag{6.50}$$

where

$$\tilde{O}^{\mathrm{CFT}}(\{z_t^l\}_{l\neq j}) := \prod_{l\neq j}^{2n} \phi_l(z_t^l)\,. \tag{6.51}$$

Note that we expanded the OPE of the boundary field $\psi_{h_{(1,2)}}(\xi_t)$ with the bulk field $\phi_{h_{(1,2)}}(z_t^j)$ such that the resulting field $\psi_{h_{(1,3)}}(\xi_t)$ is located on the boundary again.



Furthermore, we inserted the fact that the two-point function of the two fields at ξ and ∞ is constant and can therefore be set to one.

With the notation in (6.51), if ϵ is small in the sense defined in (6.48), we can rewrite (6.43), suppressing the coordinate dependence of the BCFT expectation values $\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t}$ in the following. If ϵ is not small, i.e. if we are outside the coordinate patch D_ϵ, we abstain from taking any expansion at all. We will denote this by restricting ourselves in the first case to the coordinate patch D_ϵ by writing $|_{D_\epsilon}$. Analogously, in the second case, we restrict ourselves to all other coordinate patches $D\backslash D_\epsilon$ which we will denote by $|_{D\backslash D_\epsilon}$.

$$\begin{aligned}\mathrm{d}\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t} \asymp & \left[\frac{\kappa}{2}\left(\partial_\epsilon^2 - 2\partial_\epsilon\partial_{\xi_t} + \partial_{\xi_t}^2\right) - 2\left(\frac{h_{(r,s)}}{\epsilon^2} - \frac{1}{\epsilon}\partial_\epsilon + \sum_{\substack{l=1\\l\neq j}}^{2n}\frac{h_l}{(z_t^l-\xi_t)^2} - \frac{1}{z_t^l-\xi_t}\partial_{z_t^l}\right)\right] \\ & \sum_Y \epsilon^{|Y|-\mu}\beta_{Y,(1,3)}L_{-Y}\tilde{\mathcal{O}}^{\mathrm{CFT}}_{\mathbb{H}_t}(\xi_t;\{z_t^l\}_{l\neq j})\Big|_{D_\epsilon}\mathrm{d}t \\ & +\mathcal{D}^{2n}_{-2}(\xi_t;\{z_t^l\})\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t}(\xi_t;\{z_t^l\})\Big|_{D\backslash D_\epsilon}\mathrm{d}t + \partial_{\xi_t}\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t}\mathrm{d}\xi_t\,. \end{aligned} \tag{6.52}$$

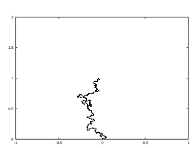
The rather lengthy procedure to obtain the ϵ-dependences is illustrated in [133].

The first non-zero contribution to the sum in (6.52) arises from the $|Y|=1$ term which yields $\epsilon^{1-\mu}$ as a common pre-factor for all terms:

$$\begin{aligned}\mathrm{d}\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t} \asymp\ & \epsilon^{1-\mu}\left[\sum_{|Y|=1}\epsilon^{|Y|}\tilde{\beta}_{Y,(1,3)}L_{-Y}\mathcal{D}^{2n-1}_{-3}(\xi_t;\{z_t^l\}_{l\neq j})\tilde{\mathcal{O}}^{\mathrm{CFT}}_{\mathbb{H}}(\xi_t;\{z_t^l\}_{l\neq j})\mathrm{d}t\right]\Big|_{D_\epsilon} \\ & +\mathcal{D}^{2n}_{-2}(\xi_t;\{z_t^l\})\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t}(\xi_t;\{z_t^l\})\Big|_{D\backslash D_\epsilon}\mathrm{d}t + \partial_{\xi_t}\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t}\mathrm{d}\xi_t \\ =:\ & \epsilon^{1-\mu}\mathcal{D}^{2n-1}_{-3}(\xi_t;\{z_t^l\}_{l\neq j})\left[\sum_{k=1}\epsilon^k\tilde{\mathcal{O}}^{\mathrm{CFT}}_k(\xi_t;\{z_t^l\}_{l\neq j})\mathrm{d}t\right]\Big|_{D_\epsilon} \\ & +\mathcal{D}^{2n}_{-2}(\xi_t;\{z_t^l\})\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t}(\xi_t;\{z_t^l\})\Big|_{D\backslash D_\epsilon}\mathrm{d}t + \partial_{\xi_t}\mathcal{O}^{\mathrm{CFT}}_{\mathbb{H}_t}\mathrm{d}\xi_t \end{aligned} \tag{6.53}$$

where $\tilde{\mathcal{O}}^{\mathrm{CFT}}_{\mathbb{H}}(\xi_t;\{z_t^l\}_{l\neq j})$ is the BCFT expectation value now containing the boundary field $\psi_{h_{(1,3)}}(\xi_t)$ instead of the boundary field $\psi_{h_{(1,2)}}(\xi_t)$ and the bulk field $\phi_{h_{(1,2)}}(z_t^j)$. The index k denotes the presence of a descendant on level k of the boundary field $\psi_{h_{(1,3)}}(\xi_t)$. The level-three differential operator is given by

$$\begin{aligned}\mathcal{D}^{2n-1}_{-3}(\xi_t;\{z_t^l\}_{l\neq j}) \quad := \quad & \frac{\kappa}{2}\partial^3_{\xi_t} - 2\left(\sum_{\substack{l=1\\l\neq j}}^{2n}\frac{h_j}{(z_t^l-\xi_t)^2} - \frac{1}{z_t^l-\xi_t}\partial_{z_t^l}\right)\partial_{\xi_t} \\ & + h_{(1,3)}\left(\sum_{\substack{l=1\\l\neq j}}^{2n}\frac{2h_l}{(z_t^l-\xi_t)^3} - \frac{1}{(z_t^l-\xi_t)^2}\partial_{z_t^l}\right). \end{aligned} \tag{6.54}$$



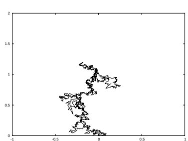
Of course, switching to other coordinates must not change the result if the expansion is valid. This is a simple argument why every order in ϵ has to vanish under the action of $\mathcal{D}_{-3}^{2n-1}(\xi_t; \{z_t^l\}_{l\neq j})$.

Again, the term proportional to $\mathrm{d}\xi_t$ in (6.53) vanishes when we take the probabilistic expectation value. The crucial point is that the terms proportional to $\mathrm{d}t$ are zero as well.

This follows since the BCFT expectation value $\tilde{\mathcal{O}}_{\mathbb{H}}^{\mathrm{CFT}}(\xi_t; \{z_t^l\}_{l\neq j})$ now contains the boundary field $\psi_{h_{(1,3)}}(\xi_t)$, and (6.54) is precisely the differential operator arising due to its null descendant on level three.

This directly leads to the conclusion that, for small ϵ in the sense of (6.48) on the coordinate patch D_ϵ, the martingale identified in (6.43) remains a martingale in this expansion since the time derivative of its probabilistic expectation value vanishes. If ϵ is small, the martingale contains a field with null descendant on level three while exhibiting the scaling behavior $\epsilon^{1-\mu}$. If ϵ not small, we are in the coordinate patches $D\backslash D_\epsilon$ where we abstain from any expansions and therefore do not change the martingale property. Investigating different expansions in detail however, as we will argue, is not necessary. Martingales resulting thereof cannot be identified as *new* SLE martingales corresponding to BCFT expectation values yet, since the respective SLE probabilities have not been discussed in the literature so far. They are only defined via the complement and therefore provide no *additional* information on whether our interpretation for the complementary event is valid.

In the following, we want to interpret the new SLE martingale with the help of the probability of the SLE trace coming closer to a point z_j in the upper half-plane than some given threshold ϵ. For this setting, the corresponding BCFT observable consists only of the minimum number of bulk fields needed to mark the point, i. e. it is given by $n = 1$ local bulk fields. In this case, the distinction between ϵ small and large as stated in (6.48) is not really necessary. The whole BCFT ensemble corresponding to the complete time evolution of an SLE can be considered simultaneously. This is due to global conformal invariance of the BCFT expectation values which are, in the mirror-image approach, built from chiral fields on the full complex plane. One parameter of global conformal invariance is fixed by the requirement that ∞ is mapped to ∞. The remaining two free parameters can be used to fix the relative positions of the boundary field $\psi_{h_{(1,2)}}(\xi_t)$ and the bulk field $\phi_{h_{(1,2)}}(z_t^j)$. Therefore their distance ϵ, can be made time-independent and hence be set to its probabilistic expectation value. We can illustrate this explicitly by considering the setting we want to interpret in the following. It consists of the SLE curve-creating boundary field $\psi_{h_{(1,2)}}(\xi_t)$, its partner at infinity $\psi_{h_{(1,2)}}(\infty)$ and the observable marking the point which is equivalent to $2n = 2$ chiral fields at z_t^1 and $z_t^{n+1} = z_t^2$. It is known that such a four-point function effectively depends only on one parameter, since due to global conformal invariance we have the freedom to fix three coordinates. This can be done by fixing infinity and the distances between the curve-creating boundary field and both bulk fields, respectively, i. e. choosing a transformation that keeps $\epsilon = \xi_t - z_t^1$ and $\delta = \xi_t - z_t^2$



constant, leaving just one effective LÖWNER time-dependent coordinate describing the growth of the interface. Depending on the relative size of ϵ and δ, the distinction in (6.48) can be done for all times.

What remains to be done is to give an SLE-interpretation to the ϵ-dependent pre-factor of the martingale and the event corresponding to the BCFT expectation value of the observable, $\tilde{\mathcal{O}}(\xi_t, \{z_l\}_{l\neq j})$ and its descendants.

6.5.2 Interpretation of the Short-Distance Pre-Factor

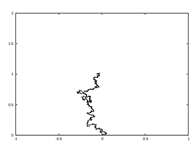
We are in the convenient situation that the probability of the SLE event corresponding to the BCFT situation considered above, i. e. of the SLE curve-creating boundary field $\psi_{h_{(1,2)}}(\xi_t)$ coming close to a point z_t^j marked by the bulk field $\phi_{h_{(1,2)}}(z_t^j)$, has already been studied in [123]. There, the probability of the SLE$_\kappa$ trace, $\gamma_{(0,\infty)}$, intersecting a ball $\mathcal{B}_\epsilon(z_j)$ of radius ϵ centred at a point z_j in the upper half-plane has been derived for $0 < \kappa < 8$. It is given by

$$\begin{aligned} \mathbf{P}_{\epsilon,z_j} &:= \mathbf{P}\left(\gamma_{(0,\infty)} \cap \mathcal{B}_\epsilon(z_j) \neq \emptyset\right) \\ &\asymp \left(\frac{\epsilon}{\mathfrak{Im}(z_j)}\right)^{2-d_\gamma} (\sin\alpha(z_j))^{8/\kappa-1} , \end{aligned} \tag{6.55}$$

where $d_\gamma = \min\{2, 1+\kappa/8\}$ is the HAUSDORFF dimension of the SLE trace and $\alpha(z_j) = \log(z_j/\bar{z}_j)$. For z_j far away from the boundary, the angle dependent prefactor is of order one and slowly varying with the distance from the boundary. It can therefore be neglected since the derivation of (6.55) has been accomplished modulo constant factors. This is the case in the first SLE phase, i. e. $0 < \kappa \leq 4$, where the SLE curve is a non self-touching simple path. For z_j near the boundary, the angle dependent factor stays important, since the variation w. r. t. the angle diverges for $\mathfrak{Im}(z_j) \to 0$: $\partial_x \sin(x)^{8/\kappa-1} \propto \cos(x)\sin(x)^{8/\kappa-2}$ where the exponent $8/\kappa - 2 < 0$ for $4 < \kappa < 8$. This is the self-touching SLE phase, where the curve is allowed to come close to the boundary. Therefore, we will only give an interpretation to this factor for $4 < \kappa < 8$ in the next section.

Now let us motivate, how we will use the probability in (6.55) for our interpretation in the following. Remember that in the BCFT picture, the marked point z_t^j is associated with a bulk field $\phi_{h_{(1,2)}}$ located at z_t^j. Therefore, we do not want its pre-image z_j to be disconnected from infinity since, in this case, the field would be removed from $\mathbb{H}_t$ by applying the LÖWNER mapping g_t. Hence, we are rather interested in the behavior of the *boundary* of the SLE *hull*, ∂K_t, than in the behavior of the trace itself. More precisely, instead of the HAUSDORFF dimension d_γ of the trace $\gamma_{(0,t]}$ we will employ the HAUSDORFF dimension d_K of the boundary of the hull ∂K_t in (6.55).

In the parameter range $0 < \kappa \leq 4$, where the SLE curve $\gamma_{(0,t]}$ is a simple path, ∂K_t is given by $\gamma_{(0,t]}$ itself and its HAUSDORFF dimension is $d_K = 1+\kappa/8$. For $4 < \kappa < 8$, we are in the self-touching phase, where ∂K_t is conjectured [100, 101, 102, 103, 104,



105, 106, 107] to be locally described by the dual SLE trace of speed $16/\kappa$. This is why it is generally assumed that $d_K = 1 + 2/\kappa$ in this range of κ.

Now let us define the event $A := \{d = \mathrm{dist}(z_j, \gamma) : d \leq \epsilon\}$ such that $\mathbf{P}(A) = \mathbf{P}_{\epsilon, z_j}$. It is evident that

$$\begin{aligned} \mathbb{E}(O_t^{\mathrm{SLE}}) &= \mathbf{P}(A)\mathbb{E}(O_t^{\mathrm{SLE}} \Big| A) + \mathbf{P}(\bar{A})\mathbb{E}(O_t^{\mathrm{SLE}} \Big| \bar{A}) \\ &\asymp \left(\frac{\epsilon}{\mathfrak{Im}(z_j)}\right)^{2-d_K} (\sin\alpha(z_j))^{8/\kappa - 1}\, \mathbb{E}(O_t^{\mathrm{SLE}} \Big| A) + \mathbf{P}(\bar{A})\mathbb{E}(O_t^{\mathrm{SLE}} \Big| \bar{A}) . \end{aligned} \tag{6.56}$$

Let us take ϵ to be the maximal distance of two fields for which we are allowed to perform an OPE between them in BCFT, i. e. the maximal ϵ fulfilling the condition (6.48)

$$\epsilon := |\gamma_t - z_j| < \min_k |\gamma_t - z_k| , \tag{6.57}$$

where the z_k are the locations of the other bulk fields in the BCFT picture.

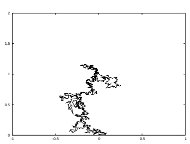
Then A is the event that the boundary field $\psi_{h_{(1,2)}}(z_t^j)$ is at least ϵ-close to the bulk field $\phi_{h_{(1,2)}}(\xi_t)$, i. e. A is the event of being on the coordinate patch D_ϵ. This means that on the BCFT side it follows from (6.53) that

$$\begin{aligned} \mathbb{E}(\mathcal{O}_{\mathbb{H}_t}^{\mathrm{CFT}}) &= \mathbb{E}\Big(\mathcal{O}_{\mathbb{H}_t}^{\mathrm{CFT}}\Big|_{D_\epsilon}\Big) + \mathbb{E}\Big(\mathcal{O}_{\mathbb{H}_t}^{\mathrm{CFT}}\Big|_{D\setminus D_\epsilon}\Big) \\ &\asymp \epsilon^{1-\mu}\mathbb{E}\Big(\sum_{k=1}\epsilon^k \tilde{\mathcal{O}}_k^{\mathrm{CFT}}\Big|_{D_\epsilon}\Big) + \mathbb{E}\Big(\mathcal{O}_{\mathbb{H}_t}^{\mathrm{CFT}}\Big|_{D\setminus D_\epsilon}\Big) . \end{aligned} \tag{6.58}$$

Comparing (6.56) to (6.58), we know from our heuristic picture that the first and second terms of the equations should be equal, respectively. This is supported by the scaling exponent of the ϵ behavior, because the OPE exponent $1-\mu$ in (6.54) equals the SLE probability exponent $1 - d_K$ of the outer boundary of an SLE hull intersecting $\mathcal{B}_\epsilon(z_j)$. This can be shown if we assign the two mutually dual boundary fields with null descendant on level two of weight $h_{(2,1)}(\kappa)$ and $h_{(1,2)}(\kappa) = h_{(2,1)}(16/\kappa)$, to the simple and the self-touching SLE phase in the following way:

- $0 < \kappa \leq 4$: As mentioned above, for this range of parameters, the boundary of the SLE hull is equivalent to the trace itself. Therefore its dimension is given by $d_K = 1 + \frac{\kappa}{8}$. In the BCFT picture, the boundary field $\psi_{h_{(2,1)}}(\xi_t)$ is located at the image of the SLE trace. The SLE event of this trace intersecting $\mathcal{B}_\epsilon(z_j)$ then corresponds to the BCFT event of $\psi_{h_{(2,1)}}(\xi_t)$ approaching a chiral bulk field $\phi_{h_{(2,1)}}(z_t^j)$ of the same weight located at z_t^j. This way, from $h_{(2,1)} = \frac{3\kappa-8}{16}$ and $h_{(3,1)} = \frac{\kappa-2}{2}$, it follows that $\mu = h_{(3,1)} - 2h_{(2,1)} = \frac{\kappa}{8}$. The scaling exponents in (6.56) and (6.58) are identical, given by

$$1 - \mu = 1 - \frac{\kappa}{8} = 2 - d_K . \tag{6.59}$$



- $4 < \kappa \leq 8$: In this range of parameters, the boundary of the SLE hull is conjecturally given by the trace of the dual SLE of speed $16/\kappa$. Therefore its dimension is given by $d_K = 1 + \frac{2}{\kappa}$. Consequently, in the BCFT picture, we have to place the dual boundary field $\psi_{h_{(1,2)}}(\xi_t)$ at its tip. Then, the SLE event of the dual SLE trace intersecting $\mathcal{B}_\epsilon(z_j)$ corresponds to the BCFT event of $\psi_{h_{(1,2)}}(\xi_t)$ approaching a chiral bulk field $\phi_{h_{(1,2)}}(z_t^j)$ of the same weight located at z_t^j. In this case, $h_{(1,2)} = \frac{6-\kappa}{2\kappa}$ and $h_{(1,3)} = \frac{8-\kappa}{\kappa}$ imply that $\mu = h_{(1,3)} - 2h_{(1,2)}$, and we have
$$1 - \mu = 1 - \frac{2}{\kappa} = 2 - d_K \,. \tag{6.60}$$

Unfortunately, up to now, no explicit probability for the complementary event has been derived yet, i. e. it is only given by $1 - \mathbf{P}(A)$. Therefore this leaves an open question whether the interpretation works for the second terms in (6.56) and (6.58). On the BCFT side, we would have to employ other expansions than in ϵ, e. g. $1/\epsilon$ or $(1-\epsilon)$, if ϵ can not be considered as small. Physically, this is the case in which other fields than the one located at z_j are closer to the curve-creating field in radial ordering.

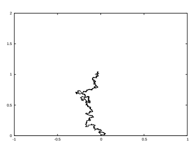
We want to stress again that we do not have to care about the dependence on the Löwner time on the CFT side. The situations at different times are connected by conformal transformations under which the theory is invariant. Hence even though the distance of the SLE trace tip to the marked point varies in time, this does not affect our interpretation. There always exists a suitable mapping on the CFT side which transforms the coordinates in such a way that infinity is preserved and z_t^j and ξ_t are at a short distance where the interpretation is valid if this is the case at one Löwner time instant.

6.5.3 Interpretation of the Angular Pre-Factor

To be honest, we have not provided a full interpretation of the formula (6.55) yet. It contains additional dependencies on the angle, $\alpha(z_j) := \log(z_j/z_j^*)$, as well as on $\mathfrak{Im}(z_j)$. The latter remains an open question in our ansatz. We will illustrate our idea for the interpretation of the angular term with our toy model again. Therefore let us consider a setting with the curve-creating boundary field $\psi_{h_{(1,2)}}(\xi_t)$, its partner at ∞ and $n = 1$ local bulk fields, i. e. two chiral bulk fields $\phi_{h_{(1,2)}}(z_1)$ and $\phi_{h_{(1,2)}}(z_2)$. First we perform the OPE on the boundary field $\psi_{h_{(1,2)}}(\xi_t)$ and the bulk field $\phi_{h_{(1,2)}}(z_1)$, neglecting higher order terms. Then we use global conformal invariance to map $z_2 \to 0$. This yields a three-point correlation function

$$\begin{aligned}\langle \psi_{h_{(1,2)}}(\infty), \psi_{h_{(1,3)}}(z_1)\phi_{h_{(1,2)}}(0)\rangle \quad &\propto \quad |z_1 - 0|^{h_{(1,2)} - h_{(1,2)} - h_{(1,3)}} \\ &= \quad \sin\alpha(z_1)^{h_{(1,3)}} \mathfrak{Im}(z_1)^{-h_{(1,3)}} \propto \mathbf{P}_{\epsilon, z_1} \,,\end{aligned} \tag{6.61}$$

since $h_{(1,3)} = 8/\kappa - 1$.



This interpretation agrees with the results of the previous section: For $4 < \kappa < 8$, the results of section 6.5.2 provide us with the $h_{(1,3)}$ field needed in (6.61), which is precisely the phase where the self-touching SLE trace is allowed to come near the boundary again and the angular dependence can become dominant if $\sin\alpha(z_j)$ is small.

In [110], Bauer and Bernard gave a CFT interpretation for (6.55), too, however in terms of the local bulk field $\phi_{h_{(0,1)}}(z,\bar{z})$. While, at first sight, this provided them with the correct scaling in terms of ϵ, there are three points where we come to alternative conclusions than their proposal. First of all, the weight of the local bulk field, $h_{(0,1)}$, gives rise to a different exponent for the angular dependence, i. e. $-2h_{(0,1)} = \kappa/8 - 1$ instead of $8/\kappa - 1$. Second, fields of that weight are formally part of the *extended* Kac-table which usually leads to extensions of the models, i. e. logarithmic CFTs [134, 135, 136, 137]. Third, as pointed out earlier in the proof of the martingale property, we have to take the contributions of the descendant operator of the boundary field into account. Apparently, this has been neglected in [110] spoiling the correct ϵ scaling behavior of the probability of the martingale.

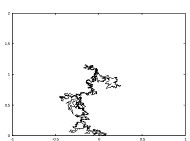
6b CFT Interpretation of Merging Multiple SLE Traces

In part b of this chapter, we will consider the inverse partition function of a BCFT model on the upper half-plane as an SLE martingale. To keep things simple, we will consider two illustrative toy models where a curve-creating boundary field comes close to a bulk point.

The first toy model contains two SLE curve-creating boundary fields $\psi_{h_{(1,2)}}$ located at x_t^1 and x_t^2 on the real line and their counterpart, $\psi_{h_{(1,3)}}$ located at infinity. In addition to these fields, we will assume the existence of p boundary force-fields, represented by vertex operators V_{α_q} of strength $\rho_q = 2\alpha_q\sqrt{\kappa}$, located at $y_t^q \in \mathbb{R}$. In the most simple case, $p = 0$.

The second toy model contains just one SLE curve-creating boundary field $\psi_{h_{(1,2)}}$ located at $\xi_t \in \mathbb{R}$ and its counterpart at infinity, $\psi_{h_{(1,3)}}$. In addition to these fields, we will again assume the existence of p boundary force-fields, again represented by vertex operators $V_{\alpha_q}(y_t^q)$. However, in the second model, one of the boundary force-fields, $V_{2\alpha_r}(y_t^r)$ say, is of weight $h_r = 2\alpha_r(2\alpha_r - 2\alpha_0) = h_{(1,2)}$ as a result of the presence of a bulk force-field pair of weight $h_{(0,1)}$, located at z_t^r and z_t^{r*} with $y_t^r = \mathfrak{Re}(z_t^r)$. In the most simple case, $p = 2$ here.

We will show explicitly that after performing the OPE between the two curve-creating boundary fields in the first and the curve-creating boundary field and the r^{th} force-field in the second toy model, the $Z[x_t, y_t]^{-1}$ stays a martingale. We will interpret the short-distance scaling behavior of this martingale in terms of the prob-



ability of an SLE curve coming close to a point in the upper half-plane [123], namely the pre-image of x_t^2 in the first and z_t^r in the second toy model.

This approach is based on a different point of view than preceding work [110, 60, 69] considering the martingale property of bCFT expectation values of observables. According to the discussion in chapter 5, bulk fields close to the boundary are interpreted in terms of bulk force-fields of an SLE($\kappa, \vec{\rho}$), yielding non-trivial interactions with the curve-creating boundary field in the mirror-image approach. However, other aspects are quite similar to [69]: we again use the fact that the short-distance scaling behavior is governed by the OPE of a *null-descendant* of the curve-creating boundary field $\psi_{h_{(1,2)}}$ with another boundary field of the same weight. This allows for a more intuitive interpretation of the probability of the SLE trace hitting the disc than the one given in [110].

6.6 Review of Multiple SLE and CFT

As an extension of single SLE, multiple SLE as introduced in [60, 67] (cf. section 4.3.3 or 5.1.1), is described by the multiple Löwner equation (4.23) for m traces

$$\mathrm{d}G_t(z) = \sum_{i=1}^{m} \frac{2\mathrm{d}t}{G_t(z) - X_t^i} . \tag{6.62}$$

with driving processes (4.24)

$$\mathrm{d}X_t^i = \sqrt{\kappa}_i \mathrm{d}B_t^i + \kappa_i \partial_{x_t^i} \log Z[x_t, y_t]\mathrm{d}t + \sum_{k \neq i} \frac{2}{X_t^i - X_t^k}\mathrm{d}t \tag{6.63}$$

with $\kappa_i = \kappa_j$ or $\kappa_i = 16/\kappa_j$ for $\kappa_i \in \mathbb{R}^+$

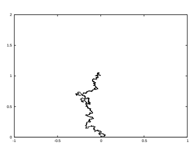
In the following we will proceed analogously to the approach presented before, since the basic idea and the interpretation in SLE are quite similar. To keep things simple, we will illustrate our ideas with the two representative examples mentionned. The generalization to other situations is straightforward.

6.6.1 The OPE for Boundary Fields

Instead of considering the OPE of a boundary SLE curve-creating field $\psi_{h_i}(x_t^i)$ with a the bulk fields $\phi_{h_l}(z_l)$ contained in an observable as done in chapter 6a, we consider the measure martingale $Z^{-1}[x_t, y_t]$, with

$$Z[x_t, y_t] = Z_{\mathrm{b.c.}}[x_t, y_t] Z_{\mathrm{free}} = \langle \psi(\infty), \psi_{h_1}(x_t^1) \ldots \psi_{h_m}(x_t^m) V_{\alpha_1}(y_t^1) \ldots V_{\alpha_p}(y_t^p) \rangle . \tag{6.64}$$

We will investigate the short-distance limit of the fields contained in the boundary part $Z_{\mathrm{b.c.}}[x_t, y_t]$, i. e. a curve-creating boundary field $\psi_{h_i}(x_t^i)$ with vanishing descendant on level two coming close to a neighboring field of the same weight. This neighboring field can either be another curve-creating boundary field $\psi_{h_{i+1}}(x_t^{i+1})$ or



the boundary force-field $V_{2\alpha_r}(y_t^r)$, resulting from the interaction of a bulk force-field $V_{\alpha_r}(z_t^r)$ with its mirror-image $V_{\alpha_r}(z_t^r*)$.

Remark 12. The boundary force-field $V_{2\alpha_r}(y_t^r)$ is assumed to be created by the interaction of a bulk force-field $V_{\alpha_r}(z_t^r)$ with its mirror-image $V_{\bar{\alpha}_r}(z_t^{r*})$. If near the boundary, i. e. the SLE trace, a bulk field interacts with the boundary. In the mirror-image approach to BCFT this implies two things:

① First, the boundary conditions become important and the bulk force-field $V_{\alpha_r}(z_t^r)$ interacts with its mirror-image $V_{\bar{\alpha}_r}(z_t^{r*})$. This interaction is modeled by taking their OPE:

$$V_{\alpha_r}(z_q)V_{\bar{\alpha}_r}(z_t^{r*}) = \sum_{k=0}^{\infty} \mathfrak{Im}(z_t^r)^{k+h_{2\alpha_r}-2h_{\alpha_r}} L_{-\{k\}} V_{2\alpha_r}(y_t^r)\,, \tag{6.65}$$

where the identity channel of fusion is prohibited by definition due to non-vanishing interactions with the boundary and $y_t^r := \mathfrak{Re}(z_t^r)$.

② The second effect is that these boundary force-fields $V_{2\alpha_r}(y_t^r)$ interact with the SLE curve-creating boundary field which we will discuss in the following. Since we want to consider the OPE of the curve-creating boundary field with one of the same weight, the bulk force-field and its mirror-image should effectively correspond to a boundary field $V_{2\alpha_r}(y_t^r)$ of the same weight as the curve-creating boundary field $\psi(\xi_t)$. This is the case if $V_{\alpha_r}(z_t^r)$ is the vertex operator of charge (cf. 3.65)

$$\alpha_{(r,s)} = \alpha_{(0,1)} = \alpha_0 - \frac{1}{2}\left(\frac{1}{2}\alpha_-\right)\,, \tag{6.66}$$

or $\alpha_{(1,0)}$, depending on the weight of the curve-creating boundary field, $h_{(2,1)}$ or $h(1,2)$, respectively. This can be shown by explicitly computing the OPE of $V_{\alpha_{(0,1)}}(z_t^r)$ with $V_{\alpha_{(0,1)}}(z_t^{*r})$, which results in a vertex operator of twice the charge:

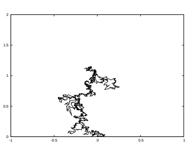
$$2\alpha_{(0,1)} = 2\left[\alpha_0 - \frac{1}{2}\left(\frac{1}{2}\alpha_-\right)\right] = \alpha_0 - \frac{1}{2}\left(1\alpha_+ + 2\alpha_-\right) = \alpha_{(2,1)}\,, \tag{6.67}$$

which is the vertex operator $V_{\alpha_{(2,1)}}(y_t^r)$. Analogously, taking the OPE of $V_{\alpha_{(1,0)}}(z_t^r)$ with $V_{\alpha_{(1,0)}}(z_t^r*)$ results in $V_{\alpha_{(1,2)}}(y_t^r)$.

For the next step, i. e. the investigation of the OPE of fields contained in (6.64) and checking the martingale property, we have to distinguish between the two cases mentioned in section 6.3:

① curve-creating boundary fields ψ,

② and bulk force-fields V_{α_q}.



In case ①, we take the OPE of $\psi_{h_i}(x_t^i)$ with $\psi_{h_{i+1}}(x_t^{i+1})$:

$$\begin{aligned}\psi_{h_{(1,2)}}(x_t^i)\psi_{h_{(1,2)}}(x_t^{i+1}) &= \sum_Y g_{(1,1)}(x_t^{i+1}-x_t^i)^{|Y|+\mu_{(1,1)}}\beta_{Y,(1,1)}L_{-Y}\mathbb{1}(x_t^i) \\ &\quad +\sum_Y g_{(1,3)}(x_t^{i+1}-x_t^i)^{|Y|+\mu_{(1,3)}}\beta_{Y,(1,3)}L_{-Y}\psi_{h_{(1,3)}}(x_t^i)\,,\end{aligned} \tag{6.68}$$

while in case ②, we consider the OPE of $\psi_{h_i}(x_t^i)$ with $V_{\alpha_q}(y_t^q)$:

$$\begin{aligned}\psi_{h_{(1,2)}}(x_t^i)V_{\alpha_q}(y_t^q) &= \sum_Y g_{(1,1)}(x_t^i-y_t^r)^{|Y|+\mu_{(1,1)}}\beta_{Y,(1,1)}L_{-Y}\mathbb{1}(x_t^i) \\ &\quad +\sum_Y g_{(1,3)}(x_t^i-y_t^r)^{|Y|+\mu_{(1,3)}}\beta_{Y,(1,3)}L_{-Y}\psi_{h_{(1,3)}}(x_t^i)\,,\end{aligned} \tag{6.69}$$

which is always valid since with a suitable rescaling, we can ensure the short-distance conditions:

$$\delta \;:=\; |x_t^{i+1}-x_t^i| < \min_{k,p}\{|x_t^i-x_t^k|,|y_t^p-x_t^i|\}\,, \tag{6.70}$$

$$\epsilon \;:=\; |y_t^r-x_t^i| < \min_{k,p}\{|x_t^i-x_t^k|,|y_t^p-x_t^i|\}\,, \tag{6.71}$$

which we will again refer to as δ and ϵ being small.

Now, the question arises, whether we have to take into account the OPE channel of the two fields which contains the identity. For this toy model, this is especially easy to answer, since the topology of the arc configuration is fixed by the weight at infinity. Choosing $h_\infty = h_{(1,3)}$ in the first toy model with $m=2, p=0$ and $h_\infty = h_{(1,3)}$ in the second toy model with $m=1, p=2$, we can be sure that this channel will not be selected, and we can concentrate on the non-trivial part of the fusion product.

6.6.2 New Martingales from Fusion

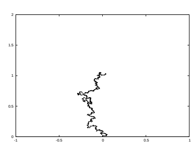
For the next step, we have to distinguish between the two cases mentioned in section 6.3 again: the boundary curve-creating field $\psi_{h_i}(x_t^i)$ coming close to

① another curve-creating boundary fields $\psi_{h_{i+1}}(x_t^{i+1})$,

② or a boundary force-field $V_{2\alpha_r}(y_t^r)$, resulting from the interaction of a bulk force-field pair with the boundary

We start with the time-variation of the martingale (6.64) whose short-distance behavior we want to investigate in this section:

$$\begin{aligned}-\mathrm{d}Z[x_t,y_t]^{-1} &= \sum_{i=1}^m \frac{\mathcal{D}_{-2}^{m,p}(x_t^i;\{x_t^l\}_{l\neq i},\{y_t^q\})Z[x_t,y_t]}{Z^2[x_t,y_t]}\mathrm{d}t \\ &\quad +\sum_{i=1}^m \partial_{x_t^i}Z[x_t,y_t]^{-1}\sqrt{\kappa}\mathrm{d}B_t^i\,.\end{aligned} \tag{6.72}$$



In order to perform the OPE in the first case, we have to replace the boundary fields $\psi_{h_{(1,2)}}(x_t^i)\psi_{h_{(1,2)}}(x_t^{i+1})$ by their OPE (6.68). Similarly, in the second case, we have to replace the boundary fields $\psi_{h_{(1,2)}}(x_t^i)V_{\alpha_q}(y_t^q)$ by their OPE (6.69). Hence, we again have to expand the differential operators, according to the method presented at the beginning of this chapter. In the first case, the m differential operators $\mathcal{D}_{-2}^{m,p}(x_t^i;\{x_t^l\}_{l\neq i},\{y_t^q\})$ become $\tilde{\mathcal{D}}_{-2}^{m-1,p}(x_t^i,\delta)$ and $\bar{\mathcal{D}}_{-2}^{m-1,p}(x_t^j,\delta)$ for $j\neq i,i+1$, which are given by

$$\begin{aligned}\bar{\mathcal{D}}_{-2}^{m-1,p}(x_t^j,\delta) &= \mathcal{D}_{-2}^{m-1,p}(x_t^j;\{x_t^k\}_{k\neq j,i+1},\{y_t^q\}) \\ &+\sum_{n=1}^{\infty}(x_i-x_j)^{-2n-2}\left(\frac{\delta}{2}\right)^{2n}\left((h_i+h_{i+1})(n+1)+(x_i-x_j)\frac{\partial}{\partial x_t^i}+\delta\frac{\partial}{\partial\delta}\right),\end{aligned} \tag{6.73}$$

$$\begin{aligned}\tilde{\mathcal{D}}_{-2}^{m-1,p}(x_t^i,\delta) &= \mathcal{D}_{-2}^{m-1,p}(x_t^i;\{x_t^k\}_{k\neq i,i+1},\{y_t^q\}) \\ &+\frac{\kappa}{2}\left(\frac{\partial^2}{\partial\epsilon^2}-2\frac{\partial}{\partial\epsilon}\frac{\partial}{\partial x_t^i}\right)-2\left(\frac{h_i}{\delta^2}-\frac{1}{\delta}\frac{\partial}{\partial\delta}\right).\end{aligned} \tag{6.74}$$

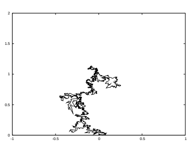
As shown before, $\tilde{\mathcal{D}}_{-2}^{m-1,p}(x_t^i,\delta)$ and $\bar{\mathcal{D}}_{-2}^{m-1,p}(x_t^j,\delta)$ acting on the expanded partition function, i. e. the correlator with the OPE of $\psi_{h_i}(x_t^i)$ with $\psi_{h_{i+1}}(x_t^{i+1})$ inserted, become $\mathcal{D}_{-3}^{m-1,p}(x_t^i;\{x_t^k\}_{k\neq i,i+1},\{y_t^q\})$ and $\mathcal{D}_{-2}^{m-1,p}(x_t^j;\{x_t^k\}_{k\neq j,i+1},\{y_t^q\})$, respectively, with the definitions

$$\begin{aligned}\mathcal{D}_{-2}^{m-1,p}(x_t^j) &= \frac{\kappa}{2}\partial_{x_t^j}^2-2\sum_{k\neq j,i,i+1}^{m}\left(\frac{h_k}{(x_t^k-x_t^j)^2}-\frac{1}{x_t^k-x_t^j}\partial_{x_t^j}\right) \\ &-2\sum_{q=1}^{p}\left(\frac{h_q}{(y_t^q-x_t^j)^2}-\frac{1}{y_t^q-x_t^j}\partial_{x_t^j}\right)-2\left(\frac{h_{(1,3)}}{(x_t^i-x_t^j)^2}-\frac{1}{x_t^i-x_t^j}\partial_{y_t^q}\right),\end{aligned} \tag{6.75}$$

$$\begin{aligned}\mathcal{D}_{-3}^{m-1,p}(x_t^i) &= \frac{\kappa}{2}\partial_{x_t^i}^3-2\sum_{k\neq i,i+1}^{m}\left(\frac{h_k}{(x_t^k-x_t^i)^2}-\frac{1}{x_t^k-x_t^i}\partial_{x_t^k}\right)\partial_{x_t^i} \\ &-2\sum_{q=1}^{p}\left(\frac{h_q}{(y_t^q-x_t^i)^2}-\frac{1}{y_t^q-x_t^i}\partial_{y_t^q}\right)\partial_{x_t^i} \\ &+h_{(1,3)}\sum_{k\neq i,i+1}^{m}\left(\frac{2h_k}{(x_t^k-x_t^i)^3}-\frac{1}{(x_t^k-x_t^j)^2}\partial_{x_t^k}\right) \\ &+h_{(1,3)}\sum_{q=1}^{p}\left(\frac{2h_q}{(y_t^q-x_t^i)^3}-\frac{1}{(y_t^q-x_t^j)^2}\partial_{y_t^q}\right).\end{aligned} \tag{6.76}$$

In the second case, i. e. the OPE of $\psi_{h_i}(x_t^i)$ with $V_{\alpha_q}(y_t^q)$, we get analogous formulas for $\mathcal{D}_{-2}^{m,p-1}(x_t^j)$ and $\mathcal{D}_{-3}^{m,p-1}(x_t^i)$.

This results in the following for our toy models:



① For case ① in section 6.3, we take a model with only two curve-creating boundary fields $\psi(x_t^1)$ and $\psi(x_t^2)$ of the same weight. Both shall have vanishing descendants on level two, ensuring the existence of two SLE curves starting on the real line, going to infinity. At infinity, another boundary field $\psi(\infty)$ with vanishing descendant on level three is located. In addition, we allow for p other boundary force-fields $V_{\alpha_q}(y_t^q)$, e. g. resulting from bulk force-fields. Hence, the boundary part of the partition function (modulo Z_{free}) is given by

$$Z[x_t, y_t] = \langle \psi_{h_{(1,3)}}(\infty), \psi_{h_{(1,2)}}(x_t^1)\psi_{h_{(1,2)}}(x_t^2) \prod_{q=1}^{p} V_{\alpha_q}(y_t^q) \rangle \,, \tag{6.77}$$

which, after inserting the OPE (6.68), becomes

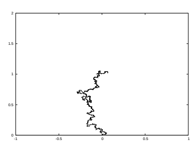
$$\begin{aligned} Z^{\text{exp}}[x_t, y_t] &= \sum_Y g_{(1,3)}(x_t^{i+1} - x_t^i)^{|Y| + h_{(1,3)} - 2h_{(1,2)}} \beta_{Y,(1,3)} \\ &\qquad \mathcal{L}_{-Y} \langle \psi_{h_{(1,3)}}(\infty), \psi_{h_{(1,3)}}(x_t^1) \prod_{q=1}^{p} V_{\alpha_q}(y_t^q) \rangle \\ &= \sum_Y g_{(1,3)}(x_t^{i+1} - x_t^i)^{|Y| + h_{(1,3)} - 2h_{(1,2)}} \beta_{Y,(1,3)} \mathcal{L}_{-Y} \tilde{Z}[x_t, y_t] \,. \end{aligned} \tag{6.78}$$

② For case ② in section 6.3, we take a model with only one curve-creating boundary field $\psi_{h_\psi}(\xi_t)$ with vanishing descendant on level two and its partner at infinity. In addition, we will assume the presence of a bulk force-field $V_{\alpha_r}(z_t^r)$ and its mirror-image, effectively resulting in the boundary force-field $V_{2\alpha_r}(y_t^r)$ with $y_t^r = \mathfrak{Re}(z_t^r)$ and $\alpha_r = \alpha_\psi = \alpha_{(1,2)}$. To keep things simple, in the partition functions that follow, the interaction between the bulk force-field and its mirror image will be performed already, omitting the pre-factors depending on $\mathfrak{Im}(z_t^r)$.

$$Z[x_t, y_t] = \langle \psi_{h_{(1,3)}}(\infty), \psi_{h_{(1,2)}}(\xi_t) V_{\alpha_{(1,2)}}(y_t^r) \prod_{q \neq r}^{p} V_{\alpha_q}(y_t^q) \rangle \,. \tag{6.79}$$

which, after inserting the OPE (6.69), becomes

$$\begin{aligned} Z^{\text{exp}}[x_t, y_t] &= \sum_Y g_{(1,3)}(x_t^i - y_t^r)^{|Y| + h_{(1,3)} - 2h_{(1,2)}} \beta_{Y,(1,3)} \\ &\qquad \mathcal{L}_{-Y} \langle \psi_{h_{(1,3)}}(\infty), \psi_{h_{(1,3)}}(\xi_t) \prod_{q \neq r}^{p} V_{\alpha_q}(y_t^q) \rangle \\ &= \sum_Y g_{(1,3)}(x_t^i - y_t^r)^{|Y| + h_{(1,3)} - 2h_{(1,2)}} \beta_{Y,(1,3)} \mathcal{L}_{-Y} \bar{Z}[x_t, y_t] \,. \end{aligned} \tag{6.80}$$



The same thing could also be done for the second possible choice of boundary fields with vanishing descendant on level two by switching from κ to $16/\kappa$, i. e. $(r,s) \to (s,r)$.

With these definitions, we can compute the short-distance limit in the variation of the martingale considered in the first toy model ①. Inserting the OPE of $\psi_{h_{(1,2)}}(x_t^1)$ with $\psi_{h_{(1,2)}}(x_t^2)$ and the differential operator $\tilde{\mathcal{D}}_{-2}^{1,p}(x_t^1,\delta)$ into (6.72) in the special case of the first toy model, we see that the variation of the martingale (6.64) is given by

$$\begin{aligned} -\mathrm{d}Z[x_t,y_t]^{-1} \quad &\asymp \quad \delta^{1-\mu}\frac{\mathcal{D}_{-3}^{1,p}(x_t^1;\{y_t^q\})\left[\sum_{k=0}\delta^k\mathcal{L}_{-\{k\}}\tilde{Z}[x_t,y_t]\right]}{\tilde{Z}_e[x_t,y_t]^2}\mathrm{d}t \\ &\quad + \sum_{l=1}^{2}\partial_{x_t^l}Z[x_t,y_t]^{-1}\mathrm{d}X_t^l \qquad (6.81) \end{aligned}$$

The terms proportional to $\mathrm{d}X_t^l$ vanish after taking the expectation value. Furthermore, the terms proportional to $\mathrm{d}t$ vanish as well, since, $\tilde{Z}[x_t,y_t]$ contains the boundary field $\psi_{h_{(1,3)}}(x_t^1)$ for which $\mathcal{D}_{-3}^{1,p}(x_t^1;\{y_t^q\})$ is the null-vector operator arising due to the level-three null-descendant of $\psi_{h_{(1,3)}}(x_t^i)$ as argued above. Therefore, the expanded correlation function, $\tilde{Z}_e[x_t,y_t] = \sum_{k=0}\delta^k\mathcal{L}_{-\{k\}}\tilde{Z}[x_t,y_t] = \tilde{Z}[x_t,y_t] + \mathcal{O}(\delta^2)$, is a martingale, too, exhibiting the overall short-distance scaling behavior $\delta^{1-\mu}$.

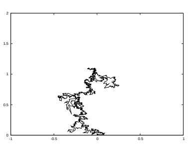
The variation of the martingale of the second toy model ② can be computed analogously. Inserting the OPE of $\psi_{h_{(1,2)}}(\xi_t)$ with $V_{\alpha_{(1,2)}}(y_t^r)$ and the differential operator $\tilde{\mathcal{D}}_{-2}^{1,p-1}(x_t^1,\epsilon)$ into (6.72) in the special case of the first toy model, we see that the martingale (6.64) is given by

$$\begin{aligned} -\mathrm{d}Z[x_t,y_t]^{-1} \quad &= \quad \mathrm{d}\langle\psi(\infty),\psi_{h_{(1,2)}}(\xi_t)V_{\alpha_1}(y_t^1)\dots V_{\alpha_p}(y_t^p)\rangle^{-1} \qquad (6.82) \\ &\asymp \quad \epsilon^{1-\mu}\frac{\mathcal{D}_{-3}^{1,p-1}(\xi_t;\{y_t^q\}_{q\neq r})\left[\sum_{k=0}\epsilon^k\mathcal{L}_{-\{k\}}\bar{Z}[x_t,y_t]\right]}{\bar{Z}_e[x_t,y_t]^2}\mathrm{d}t \\ &\quad + \partial_{\xi_t}Z[x_t,y_t]^{-1}\mathrm{d}X_t \qquad (6.83) \end{aligned}$$

The term proportional to $\mathrm{d}X_t$ vanishes after taking the expectation value in a similar way as in the first toy model. The terms proportional to $\mathrm{d}t$ vanish, since $\bar{Z}[x_t,y_t]$ contains the boundary field $\psi_{h_{(1,3)}}(\xi_t)$ for which $\mathcal{D}_{-3}^{1,p-1}(\xi_t;\{y_t^q\}_{q\neq r})$ is the null-vector operators arising due to the level-three null-descendant of $\psi_{h_{(1,3)}}(\xi_t)$. Therefore, the expanded correlation function, $\bar{Z}_e[x_t,y_t]\sum_{k=0}\epsilon^k\mathcal{L}_{-\{k\}}\bar{Z}[x_t,y_t] = \bar{Z}[x_t,y_t] + \mathcal{O}(\epsilon^2)$, is a martingale, too, exhibiting the overall short-distance scaling behavior $\epsilon^{1-\mu}$.

6.6.3 Interpretation

As shown above, the short-distance scaling behavior of $Z[x_t,y_t]^{-1}$ exhibits the exponent $1-\mu = 2-d_K$ again, so that we can draw analogous conclusions as in the case considered in chapter 6a or [69, 70]. We suggest to interpret this pre-factor again



as the probability of the SLE trace created by $\psi_{h_{(1,2)}}(x_t^i)$ or $\psi_{h_{(1,2)}}(\xi_t)$ intersecting discs of size δ around $z := G_t^{-1}(x_t^{i+1})$ and of size ϵ around $z := g_t^{-1}(y_t^r)$, respectively:

$$\begin{aligned} \mathbf{P}_{\epsilon,z} &:= \mathbf{P}\left(\gamma_{(0,\infty)} \cap \mathcal{B}_\epsilon(z) \neq \emptyset\right) \\ &\asymp \left(\frac{\epsilon}{\mathfrak{Im}(z)}\right)^{2-d_K} , \end{aligned} \tag{6.84}$$

or analogously for $\mathbf{P}_{\delta,z}$. Recall that $d_K = 1 + \frac{\kappa}{8}$ for $0 < \kappa \leq 4$, i. e. $2 - d_K = 1 - \frac{\kappa}{8}$, and $d_K = 1 + \frac{2}{\kappa}$ for $4 < \kappa < 8$, i. e. $2 - d_K = 1 - \frac{2}{\kappa}$. For $\kappa \geq 8$, the trace is space filling, therefore the probability of hitting a point in the upper-half-plane is one. Also remember that from (6.47), it follows that the OPE exponent of an $h_{(r,s)}$ with an $h_{(k,l)}$ field in the $h_{(m,n)}$-branch is given by

$$\mu = h_{(m,n)} - h_{(r,s)} - h_{(k,l)} \,. \tag{6.85}$$

From this and

$$h_{(1,2)} = \frac{6-\kappa}{2\kappa} \quad \text{and} \quad h_{(1,3)} = \frac{8-\kappa}{\kappa}\,, \tag{6.86}$$

$$h_{(2,1)} = \frac{3\kappa-8}{16} \quad \text{and} \quad h_{(3,1)} = \frac{\kappa-2}{2}\,, \tag{6.87}$$

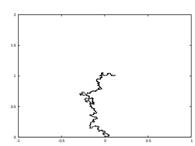
it follows that $\mu \in \{2/\kappa, \kappa/8\}$ and therefore the short-distance scaling exponent due to taking the OPE in $\mathrm{d}Z[x_t, y_t]^{-1}$ is given by $1 - 2/\kappa$ in the $(1, s)$-case and $1 - 8/\kappa$ in the $(r, 1)$-case.

The interpretation of the ϵ and δ scaling behaviors can be done analogously to the bCFT expectation value considered in chapter 6a. We can again distinguish between the two SLE phases in both toy models:

① In the first toy model, the $0 < \kappa \leq 4$ phase corresponds to two neighboring SLE curve-creating fields of weight $h_{(2,1)}$ coming δ-close to each other. In the SLE picture, this means that an SLE trace intersects a ball of radius δ around a neighboring trace. The $4 < \kappa < 8$ phase corresponds to two neighboring SLE curve-creating fields of weight $h_{(1,2)}$, describing the boundaries of two neighboring SLE hulls coming δ-close to each other.

② In the second toy model, the $0 < \kappa \leq 4$ phase corresponds to the boundary SLE curve-creating fields of weight $h_{(2,1)}$ coming ϵ-close to the force-field of the same weight, i. e. the corresponding SLE curve intersects an ϵ-sized ball around the force-point. In the $4 < \kappa < 8$ phase, we observe the boundary of an SLE hull approaching the force-point. This is modeled by an SLE curve-creating field of weight $h_{(1,2)}$ interacting with a force-field of the same weight.

The difference to the case considered in chapter 6a is that the point that is approached by the trace now lies on the *boundary*, which can also be a neighboring trace, or is at least described by a boundary field. This conveniently ensures that the OPE can



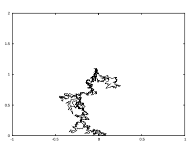
always be taken since, by definition, no other field can move in between if it has not been in between before. This feature is ensured by the LÖWNER mapping of multiple SLE$(\kappa, \vec{\rho})$ itself: SLE curves cannot disconnect each other from infinity and force-points can not move across each other. The angular dependence on $\sin(\alpha_0/2)$ where $\alpha_0 = z_r/z_r^*$ and the dependence on the imaginary part $\mathfrak{Im}(z_r)$ can be checked by considering the complete change of measure $M_t/M_0 = Z_0/Z_t$, i. e. taking into account the contributions of Z_0 (cf. chapter 5). This can be illustrated by considering only the four-point function $\langle \psi_{h_{(1,3)}}(\infty), \psi_{h_{(1,2)}}(\xi_t) V_{\alpha_r}(z_t^r) V_{\alpha_r}(z_t^{r*}) \rangle$ at time $t = 0$, i. e. $\langle \psi_{h_{(1,3)}}(\infty), \psi_{h_{(1,2)}}(0) V_{\alpha_r}(z_r) V_{\alpha_r}(z_r^*) \rangle$. This four-point function can be exactly solved using the null-vector constraints on the primary fields, similarly to the correlation function considered in [110] only with a different field at infinity. Note that this is not meaningful in the case of two curve-creating boundary fields approaching each other since at time zero, the fields are on the boundary.

Remark 13. Note that the question if, when imposing the boundary conditions in the model considered in chapter 6a, we have to compute the interaction of the bulk fields $\phi_{h_{(1,2)}}$ or $\phi_{h_{(2,1)}}$ with their mirror-images first and then perform the OPE with the curve-creating boundary field $\psi_{h_{(1,2)}}$ or $\psi_{h_{(2,1)}}$ or if it is allowed to do it the other way around as done in 6a, is subject to an ongoing discussion.

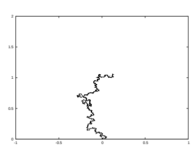
It is answered when adopting the point of view we suggested in section 5.1.3 and used in chapter 6b. For bulk force-fields, the order of expansions is set by definition to compute the interaction with their mirror-images first to yield the correct interaction with the boundary, i. e. to obey the boundary conditions. This way, directly imposing the boundary conditions by setting the coordinate of the mirror-image to the complex conjugate of the field, the procedure is consistent with most of the BCFT literature. This provides another argument why the martingales considered in part b may be more reasonable than the one chosen in [69].

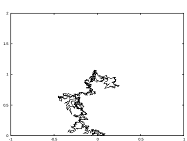
Remark 14. Within the bulk force-point approach, we have to include fields of weight $h_{(0,1)}$ or $h_{(1,0)}$ from the border of the KAČ table in our model. In principle, this may lead to a logarithmic CFTs [138, 134, 135, 136, 137], whose connection to SLE has been discussed in very few articles so far [139, 140, 141] and can therefore not be considered as well-understood. Nevertheless, the existence of e. g. non-trivial $h = 0$ boundary fields in connection to SLE has been observed in other contexts [142], too. Therefore it seems reasonable to expect those fields to play a role in the context of SLE. This is furthermore supported by recent numerical results [141].

Remark 15. The considerations of this chapter can not be generalized easily to the case of *true* boundary force-points. In this case, we would have to consider different probabilities since the SLE trace can only approach such points from "half of" the directions. Therefore, we would have to consider probabilities of the SLE trace to hit *boundary intervals* or *half-discs* on the boundary. The exponents appearing in such proabilities fit nicely into our observations: The presence of a boundary field of weight $h_{(1,3)}$ in the self-touching phase also appears in the study of covariant measures of SLE on the boundary of the domain. In recent papers [143, 144, 145] it has been



shown, that the HAUSDORFF dimension of $\gamma \cap \mathbb{R}$ is a. s. $d_{\gamma\cap\mathbb{R}} = 2 - 8/\kappa = 1 - h_{(1,3)}$. Therefore, the probability that an infinitesimal interval $[x, x+\epsilon]$ on the boundary is hit by the trace scales as $(\epsilon/x)^{1-d_{\gamma\cap\mathbb{R}}} = (\epsilon/x)^{h_{(1,3)}}$. This has also been addressed in [110] and related to the fusion of two weight $h_{(1,2)}$ fields. However, in the case of single SLE this should rather be related to some kind of self-interaction or as a case of SLE$(\kappa, \kappa - 6)$ where the trace is absorbed at the boundary force-point of strength $\kappa - 6$ since the boundary of the domain is the past of the SLE itself. This can then be interpreted within the fusion of two $(h_{(1,2)}, c_{p,q})$ representations since such a boundary force-field is a vertex operator of charge $\alpha_{(1,2)}$.





7 From the κ-Relation to Infinitesimal SLE Variants

Chordal Stochastic Löwner evolution describes the scaling limit of critical curves connected to the boundary in Statistical Physics models on domains conformally equivalent to the upper half-plane. In the case of more than one curve, the original approach by Schramm [35] for one curve is not sufficient since in their joint description, each SLE perturbs the time scales of the other SLEs. However, a natural condition is that the collection of SLEs should be invariant under global time reparameterizations [125]. From this follows that the order of the infinitesimal growth of the curve tips should not result in a different outcome. This means that the infinitesimal generators of interacting curves, whose action on SLE martingales describe the time evolution due to the driving processes, should satisfy a commutation relation.

In the connection to rational boundary Conformal Field Theory, which is conjectured to describe the same models, the partition function can be shown to be an SLE martingale. As such, its infinitesimal time evolution is determined by the action of the driving process generators on it. Explicitly calculating the effect of the commutation condition, an algebraic relation between the respective speeds κ of the driving processes simultaneously present in the same multiple SLE setting can be found [67].

In the language of bCFT, the generators of the driving processes are differential operators that impose level-two null vector equations, arising due to the presence of null-descendants on level two of the boundary fields contained in the partition function. Those boundary fields precisely corespond to the representations that, after consecutive fusion of sufficiently many of them, produce any representation of the bCFT. On the level of fields, the fusion of representations is done by taking the short-distance expansion or operator product expansion (OPE) of the corresponding fields. Since the bCFT partition function is proportional to the correlation function of all boundary fields, it is straightforward to study the OPE between two boundary fields to see how their joint growth, i. e. the infinitesimal generator of their joint driving process, can be expressed in SLE terms. As an outcome, we expect a modified commutation relation since from the bCFT point of view, fusion of representations does not lead to a different central charge. From its properties, we can deduce if joint growth descriptions of SLE traces are consistent with the OPE concept in bCFT, e. g. by checking if the κ relation holds. If it does, the infinitesimal Löwner equation for the joint process can be traced back.

In this paper, we will investigate the joint growth two curves enforced by the existence of two boundary fields of the same type. This will be done by considering their

OPE, containing descendants of the identity and a field with vanishing descendant on level three. Requiring the commutation relation for the infinitesimal generator of their joint driving process, we will show that the same conditions of the κ values have to be satisfied. Afterwards we will extract the corresponding infinitesimal LÖWNER map arising due to the joint driving process, comparing our result to previous work on driving processes related to boundary fields with descendants on level three.

7.1 The κ Condition for Commuting SLEs

Multiple chordal SLE as introduced by [60] describes the simultaneous growth of several non-crossing interfaces in the closure of the upper half-plane. The start- and endpoints of the interfaces are located on the infinite line. In general, it is easiest to think of m curves, starting on the real axis at points w_0^i of which $m-2k$ go to infinity while the remaining $2k$ pair up to form a total of $m-k$ curves.

In more mathematical terms, multiple SLE can be regarded as m single chordal SLEs with interactions in the same domain. On short time scales, each curve evolves under the influence of an independent martingale, represented by an infinitesimal differential operator which we call $\mathcal{D}_{-2}^m$, i. e. they should be absolutely continuous with respect to each other. Their interactions are governed by additional drift terms of the driving function. Additionally, they are conformally invariant and their infinitesimal local growth is commutative to yield reparameterisation invariance for the curves [60, 125, 67]. Allowing local growth at m tips in the upper half-plane results in a modified Löwner mapping G_t that describes m single SLEs of type (4.16) in just one equation:

$$\begin{aligned} \mathrm{d}g_t^i(z) &= \frac{2c_t^i}{g_t^i-\xi_t^i}\mathrm{d}t \quad \text{for } i=1,\ldots,m \\ &\rightarrow \quad \mathrm{d}G_t(z)=\sum_{i=1}^{m}\frac{2a_t^i\mathrm{d}t}{G_t(z)-X_t^i}\,. \end{aligned} \tag{7.1}$$

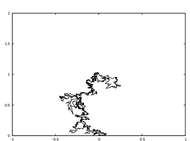
The multiple SLE equation has the usual initial condition $g_0^i(z)=G_0(z)=z$ and is hydrodynamically normalized at infinity. Setting the half-plane capacity of the joint hull $K_{t^1,\ldots,t^m}=\bigcup_{m=1}^{m}K_{t^i}$ to $\mathrm{hcap}(K_{t^1,\ldots,t^m})=2t$ in

$$G_t(z)=z+\frac{\mathrm{hcap}(K_{t^1,\ldots,t^m})}{z}+\mathcal{O}(z^{-2}) \quad \text{for } z\to\infty\,, \tag{7.2}$$

fixes the individual parameterization of the curves since they given through the dependence on their own time $t^i(t)$:

$$a_t^i=\frac{\mathrm{d}t^i(t)}{\mathrm{d}t}\partial_{t^i}\mathrm{hcap}(K_{t^1,\ldots,t^m})\,. \tag{7.3}$$

Defining $G_t=:H_t^i\circ g_t^i$, we can specify the relationship between the driving parameters from the single SLE picture, ξ_t^i, and the multiple SLE picture, $X_t^i=H_t^i(\xi_t^i)$, as well



as that for the time parameterizations $a_t^i = {H_t^i}'(\xi_t^i)^2 c_t^i$. Loosely speaking, H_t^i is the mapping that removes the remaining $m-1$ SLE traces from the setting after the action of g_t^i.

The implication of the above is a change of the drift term in the driving functions from purely Brownian motion to:

$$\begin{aligned} \mathrm{d}\xi_t^i &= \sqrt{c_t^i \kappa_i}\mathrm{d}B_t^i \\ \to & \qquad \mathrm{d}X_t^i = \sqrt{\kappa_i a_t^i}\mathrm{d}B_t^i + \kappa_i a_t^i \partial_{x_t^i} \log Z[x_t]\mathrm{d}t + \sum_{k \neq i} \frac{2a_t^k}{X_t^i - X_t^k}\mathrm{d}t\,, \end{aligned} \tag{7.4}$$

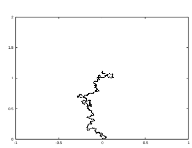
where in connection to rational BCFT, $Z[x_t]$ is identified with the BCFT partition function $Z[x_t] = Z_{\mathrm{b.c.}}[x_t]Z_{\mathrm{free}}$ [60, 67] when investigating physically relevant SLEs.

This infinitesimal approach to multiple SLE is based on three requirements for going from m interacting single SLEs to a unified description: conformal invariance, reparameterisation invariance of the curves and absolute continuity of the multiple SLE with respect to the single SLE measure. The first states that the form of the stochastic differential equation which is satisfied by the driving function should not change under conformal transformations. Being "absolutely continuous w. r. t. a single SLE" means that multiple SLEs should locally "look like a single SLE". The most important for this section is reparameterisation invariance. It says that if we choose a time parameterization such that first the i^{th} of the multiple SLE curves grows until hcap $(K_t^i) = \epsilon_i$ and then the j^{th} until hcap $(K_t^j) = \epsilon_j$, the equally valid reversed order time parameterization should yield the same results. Explicitly calculating the commutator of the expectation values results in a requirement for the respective speeds of the single SLEs, i. e. the κ-relation, which we will review in the following.

7.1.1 Commutation Relations ins SLE

The ansatz to demand that the single SLE generators should commute has first been brought up by J. Dubédat [125, 146]. The starting point of his idea were natural examples of commutation relations from SLE properties such as reversibility, duality, locality and restriction. Defining this global commutation relation of geometric nature results in algebraic conditions in terms of the infinitesimal generators of the SLE processes as has been shown in [125, 67]. Appropriate domain Markov conditions for the joint law of m single SLEs as one multiple SLE imposes commutation conditions on the generators of the driving processes (7.4), and therefore on the drift terms.

As a Markov process, the driving processes in the multiple Löwner equation



result in a generator, given by the operator

$$\begin{aligned}\mathcal{L}_t \quad &:= \quad \sum_{i=1}^{m} a_t^i \left[\frac{\kappa_i}{2} \frac{\partial}{\partial {x_t^i}^2} - 2 \sum_{k \neq i}^{m} \left(\frac{h_k}{(x_t^k - x_t^i)^2} - \frac{1}{x_t^k - x_t^i} \frac{\partial}{\partial x_t^k} \right) \right] \\ &=: \quad \sum_{i=1}^{m} a_t^i \mathcal{D}_{-2}^m (x_t^i) \,, \end{aligned} \tag{7.5}$$

where we suppress the dependence on the other coordinates in the definition of the $\mathcal{D}_{-2}^m(x_t^i)$. They are precisely the operators that impose the null descendant constraints from BCFT on a correlation function containing the appropriate boundary fields†.

The next step is to consider two infinitesimal time steps of size δ in which two of the m SLE processes, say i and j, shall grow independently of the others until their respective capacities have reached the values $\mathrm{hcap}[K_t^i] = \epsilon_i \ll 1$ and $\mathrm{hcap}[K_t^j] = \epsilon_j \ll 1$. Let the LÖWNER equation of the single SLE processes be given by

$$\mathrm{d}g_t^k(z) = \frac{2 c_t^k \mathrm{d}t}{g_t^k - \xi_t^k} \,, \tag{7.6}$$

and $G_t = H_t^k \circ g_t^k$ with $X_t^k = H_t^k(\xi_t^k)$ and $a_t^k = H_t^{k\prime}(\xi_t^k)^2 c_t^k$. Hence we have

$$\begin{aligned} a_0^i \delta \quad &= \quad c_0^i \delta = \epsilon_i \,, & (7.7) \\ a_\delta^j \delta \quad &= \quad H_\delta^j(\xi_\delta^j)^2 c_\delta^j \delta = H_\delta^{j\prime}(\xi_t^j)^2 \epsilon_j \,, & (7.8) \end{aligned}$$

while $a_0^k = 0, k \neq i$ and $a_\delta^k = 0, k \neq j$.

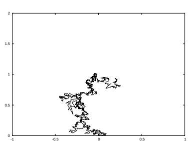
In this case, the LÖWNER map of the curve i at time δ is given by H_δ^j since the full multiple SLE map $G_{2\delta} = H_\delta^j \circ g_t^i$. This gives us

$$H_\delta^j(z) = z + \frac{2\epsilon_i}{z - x_0^i} + \mathcal{O}(\epsilon_i) \quad \rightarrow \quad H_\delta^{j\prime}(\xi_\delta^j)^2 = 1 - \frac{4\epsilon_i}{(x_0^j - x_0^i)^2} + \mathcal{O}(\epsilon_i^2) \tag{7.9}$$

since the coordinates do not change up to order ϵ, i. e. $x_0^j := H_0^j(w_0^j) = x_\delta^j + \mathcal{O}(\epsilon) = \xi_\delta^j + \mathcal{O}(\epsilon)$.

The expectation value of some functional f_t at time $t = 2\delta$ can therefore be expressed via the action of $\exp \mathcal{L}_\delta$ with the parameters specified in equations (7.7) and

†As motivated in e. g. [147], the infinitesimal generator of a random process acting on a function of it is given by the time derivative of its expectation value. This explains the observation of the previous chapter, where the time derivative of the expectation value of a BCFT expectation value resulted in a spatial differential equation on it.



(7.8) up to order ϵ^2:

$$\begin{aligned}
\mathbb{E}\left[f_{2\delta}|x_0\right] &= \left(1+\delta\mathcal{L}_0+\frac{\delta^2}{2}\mathcal{L}_0^2\right)\left(1+\delta\mathcal{L}_\delta+\frac{\delta^2}{2}\mathcal{L}_\delta^2\right)\mathbb{E}\left[f_0|x_0\right] \\
&= \left(1+\delta a_0^i\mathcal{D}_{-2}(x_0^i)+\frac{\delta^2}{2}{a_0^i}^2\mathcal{D}_{-2}(x_0^i)^2\right) \\
&\qquad \left(1+\delta a_\delta^j\mathcal{D}_{-2}(x_0^j)+\frac{\delta^2}{2}{a_\delta^j}^2\mathcal{D}_{-2}(x_0^j)^2\right)\mathbb{E}\left[f_0|x_0\right] \\
&= \Bigg[1+\epsilon_i\mathcal{D}_{-2}(x_0^i)+\left(1-\frac{4\epsilon_i}{(x_i-x_j)^2}\right)\epsilon_j\mathcal{D}_{-2}(x_0^j)+\frac{\epsilon_i^2}{2}\mathcal{D}_{-2}(x_0^i)^2 \\
&\qquad +\frac{\epsilon_j^2}{2}\mathcal{D}_{-2}(x_0^j)^2+\epsilon_i\epsilon_j\mathcal{D}_{-2}(x_0^i)\mathcal{D}_{-2}(x_0^j)\Bigg]\mathbb{E}\left[f_0|x_0\right]
\end{aligned}$$

This result has to be equal to that of the situation with the roles of i and j interchanged. Subtracting the two possible orderings and requiring them to equal zero, only the terms proportional to $\epsilon_i\epsilon_j$ survive. Division by this factor leads to DUBÉDAT's commutation requirement [125]:

$$\left\{\left[\mathcal{D}_{-2}(x_0^i),\mathcal{D}_{-2}(x_0^j)\right]-\frac{4}{(x_i-x_j)^2}\left(\mathcal{D}_{-2}(x_0^j)-\mathcal{D}_{-2}(x_0^i)\right)\right\}\mathbb{E}\left[f_0|x_0\right]=0\,. \tag{7.10}$$

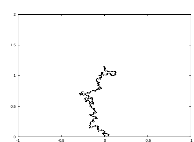
In the following we will use the short-hand notation $x_i \equiv x_0^i$.

Remark 16. It is essential to realize, that reverse engineering of equations (7.10) and (7.10) provides us with the information on the infinitesimal LÖWNER equation for the i^{th} and j^{th} trace. This is due to the fact that the *coefficients* of the infinitesimal generators $\mathcal{D}_{-2}(x_0^j)$ and $\mathcal{D}_{-2}(x_0^i)$ in (7.10) are the deviations of $H_\delta^{j\prime}$ and $H_\delta^{i\prime}$ (see (7.9)) from the identity of the respective LÖWNER mappings as can be inferred from (7.10).

7.1.2 The κ Condition in Multiple SLE

Within the connection to BCFT, it has been shown that BCFT expectation values of observables are at the same time SLE martingales if they contain boundary fields with vanishing descendant at level two. In the case of multiple SLE, the most simple of such martingales are correlation functions of boundary fields of weight $h_{(1,2)}$ or $h_{(2,1)}$ at the start- and end-points of the interfaces, i.e. $\psi_1(x_1), \ldots, \psi_m(x_m)$, and $\psi_{h_\infty}(\infty)$, imposing the boundary conditions. This correlation function is also known as the pre-factor of the partition function of the system: $Z_{\text{BCFT}} = Z_{\text{b.c.}} Z_{\text{free}}$ where Z_{free} is the partition function of the same system with free boundary conditions. Therefore, one of these special martingale expectation value for $\mathbb{E}\left[f_0|x_0\right]$ is chosen [67], although the consideration would work for other martingales as well:

$$Z_{\text{b.c.}} = \langle\psi(\infty),\psi_1(x_1)\ldots\psi_m(x_m)\rangle\,. \tag{7.11}$$



This way, a relation for κ_i and κ_j [67] from explicit computation of (7.10) emerges as follows: On the one hand side, it is known that the outcome of the commutation relation must yield zero since the operators are the null-descendant operators on the boundary fields contained in the correlation function. On the other hand, their action on the correlation function can be computed explicitly, resulting in the r.h.s. of the following algebraic relation:

$$K(\kappa_i, \kappa_j) := -3\frac{(\kappa_i - \kappa_j)(16 - \kappa_i\kappa_j)}{\kappa_i\kappa_j(x_i - x_j)^4} = 0\,. \tag{7.12}$$

This is a the consistency requirement for the single components of multiple SLE: $\kappa_i = \kappa_j$ or $\kappa_i = 16/\kappa_j$, also known from general considerations in CFT: all fields of one theory should belong to representations of the same central charge c. Indeed, it is the same for κ and $16/\kappa$:

$$c = \frac{(3\kappa - 8)(\kappa - 6)}{2\kappa}\,. \tag{7.13}$$

As motivated above, the natural next step is covered by our research presented in the following. We check if the κ-relations derived in [67, 125] still holds in the case of curves approaching each other, i. e. fusion in CFT. This is what we expect from CFT where it is known that fusion does not to produce fields of representations not belonging to the collection of representations of the same central charge.

7.2 The κ Condition for Joint Processes

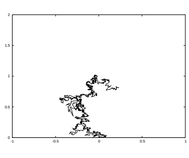
A more explicit statement for the κ relation is that the boundary fields that are allowed to create curves in the same SLE domain are the two fields with vanishing descendant on level two: $\psi_{h_{(1,2)}}$ and $\psi_{h_{(2,1)}}$. These two are the so-called fundamental fields since they correspond to the two representations out of which by consecutive fusion one can compose any representation of the theory contained the Kač-table,

$$h_{(r,s)} = \frac{(r\kappa - 4s)^2 - (\kappa - 4)^2}{16\kappa} \quad \text{with} \quad 1 \le r < q\,,\ 1 \le s < p\,, \tag{7.14}$$

$$c_{p,q} = 1 - 6\frac{(p-q)^2}{pq} \quad \text{for} \quad p, q \in \mathbb{N} \text{ coprime, i. e. } \kappa = q/p\,. \tag{7.15}$$

For example the representation $(h_{(r,s)}, c_{p,q})$ corresponding to the field $\psi_{h_{(r,s)}}$, is contained in the fusion product of at least $(r-1)$ copies of $(h_{(2,1)}, c_{p,q})$ and $(s-1)$ copies of $(h_{(1,2)}, c_{p,q})$ representations. On the level of fields, this is done by taking the operator product expansion (OPE) (3.46), i. e. the short-distance product of two fields.

Hence, from a bCFT point of view it is important to investigate if upon performing such an OPE between two curve-creating boundary fields in an expectation value spoils the consistency requirement of commuting generators. If not, we can use the result to see how the infinitesimal Löwner equation for the joint growth looks like.



7.2.1 The Generators of the Joint Processes

To check if the requirement of commuting growth of a joint generator of two processes and another generator is consistent with the κ relation, we first have to compute the joint generator. Therefore, we have to expand the $\mathcal{D}_{-2}(x_i)$ in new coordinates and letting it act on the martingale (the BCFT partition function), wherein we perform the OPE between the two boundary fields whose curves join. Then we compute the commutation relation again and see if we arrive at the same algebraic relation for the κ_i. Afterwards, we can deduce how the infinitesimal LÖWNER equation belonging to the new interfaces looks like via comparison with (7.10).

We consider the OPE between two neighboring boundary fields‡ $\psi_{h_{(1,2)}}(x_t^i)$ and $\psi_{h_{(1,2)}}(x_t^{i+1})$, denoting their distance by $\epsilon = x_t^i - x_t^{i+1}$:

$$\begin{aligned}\psi_{h_{(1,2)}}(x_i)\psi_{h_{(1,2)}}(x_{i+1}) &= \sum_Y g_{(1,1)}\epsilon^{|Y|-\mu}\beta_{Y,(1,1)}L_{-Y}\mathbb{I}(x_i) \\ &\quad + \sum_Y g_{(1,3)}\epsilon^{|Y|-\mu}\beta_{Y,(1,3)}L_{-Y}\psi_{h_{(1,3)}}(x_i)\,, \quad (7.16)\end{aligned}$$

Obviously, we get two contributions: one from the identity and one from descendants of $\psi_{h_{(1,3)}}(x_i)$. We will leave out the part of the fusion product of their representations yielding the identity since this situation results in joining tips where the growth stops [60], so that no new LÖWNER-like evolution equations can be motivated.

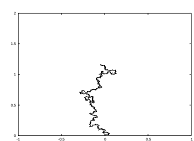
For the limit $x_{i+1} \to x_i$ $(i, i+1 \neq j)$ we introduce the notation $\tilde{\mathcal{D}}_{-2}(x_i)$ and $\bar{\mathcal{D}}_{-2}(x_j)$ for the infinitesimal generators $\mathcal{D}_{-2}(x_k)$ expressed in the new coordinates $(x_i, x_{i+1}) \to (x_i, \epsilon)$, given by

$$\begin{aligned}\bar{\mathcal{D}}_{-2}(x_j) &= \mathcal{D}_{-2}^{m-1}(x_j; \{x_k\}_{k\neq j,i+1}) \quad (7.17)\\ &\quad + \sum_{n=1}^{\infty}(x_i - x_j)^{-2n-2}\left(\frac{\epsilon}{2}\right)^{2n}\left((h_i + h_{i+1})(n+1) + (x_i - x_j)\frac{\partial}{\partial x_i} + \epsilon\frac{\partial}{\partial \epsilon}\right)\\ \tilde{\mathcal{D}}_{-2}(x_i) &= \mathcal{D}_{-2}^{m-1}(x_i; \{x_k\}_{k\neq i,i+1}) + \frac{\kappa}{2}\left(\frac{\partial^2}{\partial\epsilon^2} - 2\frac{\partial}{\partial\epsilon}\frac{\partial}{\partial x_i}\right) - 2\left(\frac{h_i}{\epsilon^2} - \frac{1}{\epsilon}\frac{\partial}{\partial\epsilon}\right)\,. \quad (7.18)\end{aligned}$$

The operator product expanded partition function can be written as follows:

$$\begin{aligned}Z_{\text{BCFT}}^{\text{exp}} &= \sum_k \epsilon^{k-\mu}\langle\psi_\infty(\infty), \psi_1(x_1)\ldots\psi_{i-1}(x_{i-1})\psi_{h_{(1,3)}}^{(k)}(x_i)\psi_{i+2}(x_{i+2})\ldots\\ &\quad \psi_{j-1}(x_{j-1})\psi_j(x_j)\psi_{j+1}(x_{j+1})\ldots\psi_m(x_m)\rangle Z_{\text{free}}\,, \quad (7.19)\end{aligned}$$

‡Of course, the considerations can be done also for $\psi_{h_{(2,1)}}$ boundary fields. Just change $\kappa \to 16/\kappa$.



wherein each $\psi^{(k)}_{h_{(1,3)}}$ is a combination of descendant fields of level $k = |Y|$, with $y = \{k_1, \ldots, k_n\}$, $\sum_i k_i = k$, i.e. $\psi^{(k)}_{h_{(1,3)}} = \sum_{|Y|=k} \beta_{\{Y\}} L_{-\{Y\}} \psi_{h_{(1,3)}}$, normalized such that $\beta_{\{0\}} = 1$.

Acting on (7.19), the terms in the two operators $\tilde{\mathcal{D}}_{-2}(x_i)$ and $\bar{\mathcal{D}}_{-2}(x_j)$ as defined in (7.17) and (7.18) pick up different k modes of the descendants of ψ_{x_i} in the correlation function. This results in the necessity of shifting the summation indices to group the same orders of ϵ again.

7.2.2 Commutation Part I: One Joint Process

Now we calculate the commutation relation for the fused process:

$$\begin{aligned} \left\{ \left[\tilde{\mathcal{D}}_{-2}(x_i), \bar{\mathcal{D}}_{-2}(x_j)\right] \right. &- \left. \left(\frac{4}{(x_i - x_j)^2} \bar{\mathcal{D}}_{-2}(x_j) - \frac{4}{(x_i - x_j)^2} \tilde{\mathcal{D}}_{-2}(x_i) \right) \right\} Z^{\text{exp}}_{\text{BCFT}} \\ &= K(\kappa_i, \kappa_j) Z^{\text{exp}}_{\text{BCFT}} = 0\,. \end{aligned} \tag{7.20}$$

A lengthy computation (see appendix A.3.1) shows that, indeed, the terms cancel in such a way that we get (modulo factors in ϵ), we get:

$$\begin{aligned} \left\{ [\mathcal{D}_{-3}(x_i), \mathcal{D}_{-2}(x_j)] \right. &- \left. \left(\frac{4F(x_i; x_j)}{(x_i - x_j)^2} \mathcal{D}_{-2}(x_j) - \frac{6}{(x_i - x_j)^2} \mathcal{D}_{-3}(x_i) \right) \right\} \tilde{Z}_{\text{b.c.}} \\ &= K(\kappa_i, \kappa_j) \cdot G(x_i; x_j) \tilde{Z}_{\text{b.c.}} = 0\,. \end{aligned} \tag{7.21}$$

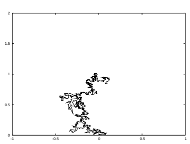
in which the factors are given by:

$$F(x_i; x_j) := -\frac{4}{(x_i - x_j)} \left(h_i + (x_i - x_j) \frac{\partial}{\partial x_i} \right), \tag{7.22}$$

$$G(x_i; x_j) := 4 \left(\frac{2h_i}{x_i - x_j} + \frac{\partial}{\partial x_i} \right) = 8 \left(\frac{\beta_{\{0\}} L_0}{x_i - x_j} + \beta_{\{1\}} L_{-1} \right). \tag{7.23}$$

The distortion arises due to the mixing of contributions of different orders in ϵ when performing the short-distance limit. However, since we only want to know if the κ relation holds and if it does what the pre-factor of the new generator hinting at the corresponding infinitesimal LÖWNER mapping looks like, we leave aside the details here, concentrating on the major implications: $K(\kappa_i, \kappa_j)$ and the pre-factor of $\mathcal{D}_{-3}(x_i)$.

Now we will double-check this result by computing the commutation relation for two pairs of joint processes.



7.2.3 Commutation Part II: Two Joint Processes

Furthermore, we have to calculate if the commutation relation works for two joint processes, too. Therefore, in addition to the limit $x_i \to x_{i+1}$, we consider the limit $x_j \to x_{j+1}$, for $x_i, x_{i+1} \neq x_j, x_{j+1}$. This means that we are considering the action of

$$\begin{aligned}\tilde{\tilde{\mathcal{D}}}_{-2}(x_i) &= \mathcal{D}_{-2}^{m-2}(x_i; \{x_k\}_{k\neq i,i+1,j,j+1} + \frac{\kappa}{2}\left(\frac{\partial^2}{\partial\epsilon^2} - 2\frac{\partial}{\partial\epsilon}\frac{\partial}{\partial x_i}\right) - 2\left(\frac{h_i}{\epsilon^2} - \frac{1}{\epsilon}\frac{\partial}{\partial\epsilon}\right) \\ &+ \sum_{n=1}^{\infty}(x_j - x_i)^{-2n-2}\left(\frac{\delta}{2}\right)^{2n}\left((h_j + h_{j+1})(n+1) + (x_j - x_i)\frac{\partial}{\partial x_j} + \delta\frac{\partial}{\partial\delta}\right),\end{aligned}$$

and the analogously defined $\tilde{\tilde{\mathcal{D}}}_{-2}(x_j)$ on the twice-expanded partition function

$$\begin{aligned}Z_{\text{bCFT}}^{2\times\ \text{exp}} &= \sum_{k,l}\epsilon^{k-\mu_k}\delta^{l-\mu_l}\langle\psi_\infty(\infty), \psi_1(x_1)\ldots\psi_{i-1}(x_{i-1})\psi_{h_{(1,3)}}^{(k)}(x_i)\psi_{i+2}(x_{i+2}) \\ &\ldots\psi_{j-1}(x_{j-1})\psi_{h_{(1,3)}}^{(l)}(x_j)\psi_{j+2}(x_{j+2})\ldots\psi_m(x_m)\rangle Z_{\text{free}}\,. \qquad (7.24)\end{aligned}$$

in Dubédats commutation relation:

$$\begin{aligned}\Big\{\Big[\tilde{\tilde{\mathcal{D}}}_{-2}(x_i), \tilde{\tilde{\mathcal{D}}}_{-2}(x_j)\Big] &- \frac{4}{(x_i - x_j)^2}\Big[\tilde{\tilde{\mathcal{D}}}_{-2}(x_j) - \tilde{\tilde{\mathcal{D}}}_{-2}(x_i)\Big]\Big\} Z_{\text{bCFT}}^{2\times\ \text{exp}} \\ &= K(\kappa_i, \kappa_j) Z_{\text{bCFT}}^{2\times\ \text{exp}} = 0 \qquad (7.25)\end{aligned}$$

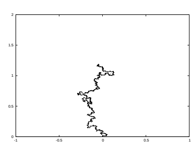
After an even lengthier computation, we we arrive at the following commutation relation:

$$\begin{aligned}\Big\{[\mathcal{D}_{-3}(x_i), \mathcal{D}_{-3}(x_j)] &- \frac{6}{(x_i - x_j)^2}\Big[F(x_j; x_i)\tilde{\tilde{\mathcal{D}}}_{-3}(x_j) - F(x_i; x_j)\tilde{\tilde{\mathcal{D}}}_{-2}(x_i)\Big]\Big\}\tilde{\tilde{Z}}_{\text{b.c.}} \\ &= K(\kappa_i, \kappa_j) H(x_i, x_j)\tilde{\tilde{Z}}_{\text{b.c.}} = 0 \qquad (7.26)\end{aligned}$$

where the pre-factor is given by

$$H(x_i, x_j) = 4^2\left(-\frac{5h_ih_j}{(x_i - x_j)^2} + \frac{2h_i}{(x_i - x_j)}\frac{\partial}{\partial x_j} + \frac{2h_j}{(x_j - x_i)}\frac{\partial}{\partial x_i} - \frac{\partial^2}{\partial x_i \partial x_j}\right).$$

Hence we can see that the concept of fusion makes sense with respect to the κ-relation as long as other SLEs are sufficiently far away. Therefore we will go on and extract the properties of a Löwner equation for the fused process for small t in the next section.



7.3 Interpretation

7.3.1 Extraction of the Löwner-like Equation

In the last section we have shown that the commutation requirement of joint growth of two neighboring traces in a multiple SLE setting with that of another trace or another joint growth of two neighboring traces stays valid. Now the question how the Löwner equation describing the infinitesimal joint growth processes looks like.

From comparison with (7.10) we see that the coefficient of the infinitesimal generator gives us the square of the derivative of the infinitesimal Löwner mapping $H_t{}'(z)$ belonging to the process. Taking the coefficient of $\mathcal{D}_{-3}$, we see that the $\tilde{H}_t{}'(z)$ for infinitesimal t is up to order $\mathcal{O}(\epsilon^2)$ given by

$$\tilde{H}_t^{i\prime}(z)^2 = 1 - \frac{6\delta c_0^j}{(z-x_0^i)^2} \quad \Rightarrow \quad \tilde{H}_t^i(z) = z - \frac{3t}{z} + \mathcal{O}(t^2)\,, \tag{7.27}$$

Comparing (7.27) with the ordinary single SLE mapping for infinitesimal t, i. e.

$$g_t(z) = z - \frac{2t}{z} + \mathcal{O}(t^2)\,, \tag{7.28}$$

suggests to draw the following conclusions:

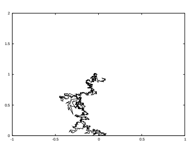
① The κ-relation, i. e. $K(\kappa_i, \kappa_j) = 0 \quad \rightarrow \quad \kappa_i \in \{\kappa_j, 16/\kappa_j\}$ stays exactly the same. This hints at the fact that on the SLE side, too, fusion of two representations does not lead to representations of other central charges. This supports us in hoping that at least to some extent, the concept of the OPE can be carried over to the SLE picture.

② The Löwner-like equation for the joint processes differs from the ordinary one. More precisely, the half-plane capacity seems to receive an effective change from $2t$ to $3t$.

Taking advantage of the property of $\text{hcap}(t)$:

$$\text{hcap}(rt) = r^2 \text{hcap}(t)\,, \tag{7.29}$$

we can assume for (7.27), i. e. $r^2 = 3/2$, we have a change of the time parameterization from t to $\sqrt{2/3}t$. Naively, this would mean that the effect of merging SLEs traces infinitesimally yields a factor of $\sqrt{3/2}$ "faster" SLE. However, since the κ-relation still holds, this can not be the answer to our question. The speed κ has to stay fixed when interpreting this pre-factor. This constraint is supported by the fact that this effect is independent of κ, i. e. we have the same effect for the second channels of fusion of $(h_{(1,2)}, c_{p,q}) \times (h_{(1,2)}, c_{p,q}) \rightarrow (h_{(1,1)}, c_{p,q}) + (h_{(1,3)}, c_{p,q})$ as for $(h_{(2,1)}, c_{p,q}) \times (h_{(2,1)}, c_{p,q}) \rightarrow (h_{(1,1)}, c_{p,q}) + (h_{(3,1)}, c_{p,q})$. One would expect that we would also be able to switch from the SLE-like process that belongs to $\phi_{(1,3)}(x_t^i)$ to the one belonging to $\phi_{(3,1)}(x_t^i)$ by $\kappa \rightarrow 16/\kappa$. With a constant factor, this does not



work anymore since a faster SLE process for one of the duality pair would have to result in a slower process for the other and vice versa.

However, this is only a local effect while the κ-relation hints at the global properties since the growth processes see each other only from far away. To give a final interpretation of this result, more information on local processes corresponding to $\psi_{h_{(1,3)}}$ or $\psi_{h_{(3,1)}}$ boundary fields has to be collected. There are important things left to be done: we did not prove that we are allowed to perform a short-distance expansion of the infinitesimal generators of the SLE processes. Historically, multiple SLE has been defined assuming that the other growth points are *sufficiently far away*. Furthermore, our result only provides a proposal how the *infinitesimal* LÖWNER equation for an object corresponding to a growing interface existing due to a boundary condition changing operator with vanishing descendant on level three may look like. This is not easy to generalize to the finite case. Moreover, taking th eOPE approach to yield boundary fields with vanishing descendants on higher levels than two, automatically leads to an *infinite sum of descendands* of the other boundary condition changing field which may also not necessarily be a desired feature. Therefore, much work remains to be done and we propose to take our results as a motivation.

7.3.2 Relation to Preceding Work

Unfortunately, up to now, there has been little research aiming at stochastic LÖWNER-like processes that are suitable to describe other fields of the BCFT KAČ-table. Basically, there has been only one proposal by Rasmussen and Lesage [74] that provides a LÖWNER equation for processes similar to those we investigated. Therefore, we will briefly review their results in the following, comparing them to our observations.

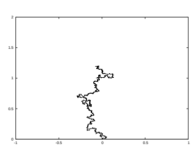
Rasmussen and Lesage [74] started with the idea to restrict the LÖWNER evolution to integer fractions of $\mathbb{H}$, which later on has been extended by Moghimi-Araghi et. al. [75] to integer fractions of the full complex plane $\mathbb{C}$, i. e. half-integer fractions of $\mathbb{H}$. The case studied here corresponding to a boundary field with vanishing descendant on level three, should correspond to an $n = \frac{3}{2}$ fraction and hence SLE on two thirds of the upper half-plane. The LÖWNER-like stochastic differential equation is given by

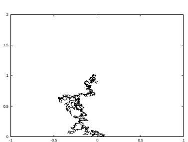
$$\mathrm{d}g_t(z) = \frac{2\mathrm{d}t}{g_t^{n-1}(z)\,(g_t^n(z) - \xi_t)}\,, \tag{7.30}$$

and has the following expansion for small t (or large z):

$$g_t(z) = z + \frac{2t}{z^{2n-1}} + \mathcal{O}(t^2) = z + \frac{2t}{z^2} + \mathcal{O}(t^2)\,. \tag{7.31}$$

In comparison with our result, this means that the first term in the expansion of the LÖWNER equation that differs from the identity is not of order z^{-1} but rather z^{-2}. This rules out any connection to our approach, although, with the help of additional jump processes, Moghimi-Araghi et. al. [75] arrive at a stochastic process which then leads to the level-three null vector equation of BCFT as well.





8 Related Work and Promising Directions

Our original work in the previous sections points at the question if it is possible, in general, to describe curves that arise due to the presence of other boundary condition changing fields than of type $(1, 2)$ or $(2, 1)$ in a LÖWNER-like manner. In the following, we motivate why we can not directly obtain them from processes driven by ordinary BROWNian motion. This leads to the conclusion that in order to describe other curves *directly* and not e. g. via a limiting procedure, we have to take other processes into account. We will present two possibilities. The first is based on the concept of fractional BROWNian motion whose variance scales with smaller fractions of the exponent of time than one which is the case for ordinary BROWNian motion. Second, we introduce on the LÉVY processes as done by [72, 73] with a short comment on their application in SLE on fractions of the upper half-plane [74, 75].

8.1 Generators from Variants of Brownian Motion

8.1.1 Ordinary Brownian Motion

There exists a simple argument why it is impossible to get higher order differential equations from ordinary BROWNIAN terms alone. It is due to the fact that the infinitesimal generators for processes driven by BROWNian motion are functions of BROWNian motion and hence their dynamics are computed via their ITÔ derivative. Since the variance of BROWNian motion is proportional to the time

$$\mathrm{d}B_t^2 = \mathrm{d}t\,, \tag{8.1}$$

only terms in the TAYLOR expansion in the heuristic derivation of the ITÔ derivative (section A.4.6) up to order two in the derivative with respect to the spatial coordinate are possible. This gives an easy and straightforward argument why it is impossible to yield higher order equations in a naive manner, i. e. without adding extra terms such as jump processes to the driving process.

8.1.2 Fractional Brownian Motion

As mentioned above, the key point why ordinary BROWNian motion does not work for higher-order differential equations is due to the scaling behavior of its variance. Therefore it is straightforward to look for processes of other variance time-scaling. One of them is

Definition 7 (Fractional Brownian Motion)**.** A d-dimensional fractional BROWNian motion with HURST parameter $H \in (0,1)$ is a GAUSSian process

$$B_t = (B_t^1, \dots, B_t^d), \quad t \geq 0\,, \tag{8.2}$$

where the B_t^i are d independent centred GAUSSian processes with covariance function

$$\mathbb{E}(\langle B_t, B_s \rangle) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)\,. \tag{8.3}$$

It can be shown that FBM has a continuum limit with p finite variation for $1/p < H$ For $H = 1/2$ we get ordinary BROWNian motion. It is a self-similar process, i. e. for all $\alpha > 0$, we have

$$\alpha^{-H} B_{\alpha t} = B_t \tag{8.4}$$

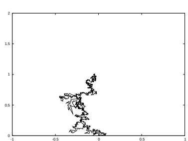
in distribution. Its increments are stationary but only independent for $H = 1/2$; for greater values of H we have long range dependences while for smaller values, the increments are negatively correlated. The scaling property in (8.4) may be crucial to yield higher order derivative terms in the differential of some functional of the fractional BROWNian motion.

Therefore, the main difference between *fractional* and ordinary BROWNian motion is that FBM is neither MARKOV nor a semi-martingale which are basic requirements for the SLE driving processes up to now. However, there are also physically relevant ergodic processes which are not MARKOV, e. g. in the study of stochastic forces describing the interaction between a small system and its large environment. Additionally, self-similar processes such as FBM appear quite naturally in hydrodynamics and in models for long-time correlations in stock markets.

For FBM there exists also infinitesimal generators from which we can deduce that the invariant measure for the stochastic differential equation for FBM has to satisfy an infinite dimensional system of partial differential equations [148]. In addition, stochastic integration with respect to FBM can be defined [149]. Therefore, we think that considering SLE type growth generated by FBM could lead to interesting new results, eventually containing a relation to curve-creating boundary fields in BCFT with vanishing descendant on other level than two.

8.2 SLE with Lévy-type Driving Processes

There have been a a couple of other approaches to SLE that do not fulfil all of the original conditions such as continuity of the driving process or the martingale property. One of them addresses SLE on fractions of the upper half-plane. It is built on discontinuous driving processes with jumps that allow for higher order derivatives and hence can provide correspondences to fields in the KAČ-table on places (r,s) with $r \cdot s > 2$. This is why we review SLE with discontinuous driving processes, or, more precisely, LÉVY processes, here.



Ordinary SLE variants driven by continuous BROWNian motion produce *continuous* fractal traces. Adding jump processes, e. g. of LÉVY-type [150],

$$\mathrm{d}x_t = \sqrt{\kappa}\mathrm{d}B_t + c^{1/\alpha} L_\alpha(t)\,, \tag{8.5}$$

results in branching traces [72, 73].

LÉVY processes are continuous-time stochastic processes, starting at the origin with stationary independent increments. Furthermore, they are everywhere right-continuous and have left limits everywhere. The most famous examples are WIENER and POISSON processes. However, both of these examples are continuous-time MARKOV processes and therefore produce no jumps. They differ by the probability distribution of their increments: while the increments of the former are normally distributed, the latter follow a POISSON distribution.

Any LÉVY process can be split up into three components: a drift, a diffusion component and a jump component. These three components (the LÉVY-KHINCHIN representation of the process) are fully determined by the (α, σ^2, B_t) where α is the jump parameter, $\sigma^2 = \kappa$ the variance and B_t a BROWNian motion as done in (8.5).

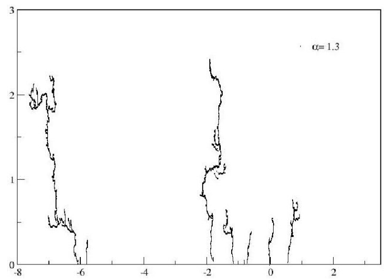


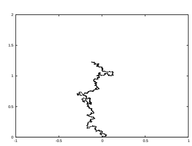
Figure 8.1: Branching SLE trace for LÉVY distributed forcing $\alpha = 1,3$, $\kappa = 4,3$ [72].

Dependent on the parameter α of the LÉVY distribution, different phases exist with a phase transition at $\alpha = 1$. More precisely, the growth process changes qualitatively and singularly at this value. It continues indefinitely in the vertical direction for $\alpha > 1$, goes as $\log(t)$ for $\alpha = 1$ and saturates for smaller values. This results in different scales of the proability density to directions along and perpendicular to the boundary. For the former, the scale is $t^{1/\alpha}$ while for the latter we have a constant plus $t^{1-1/\alpha}$ while at the critical point, it scales as $\ln t$.

As ordinary SLE is reversible, assuming the same feature for jump-SLE, going backwards in time should naturally lead to events that look like merging SLE traces, i. e. events that could correspond to fusion of boundary condition changing fields. This may be a feature worth investigating in the conetext of the SLE-bCFT correspondence.

8.3 SLE on Fractions of the Upper Half Plane

Initially, SLE variants on integer fractions of the upper half-plane have been investigated [74]. This has been extended to integer fractions of the whole complex plane



in [75], in principle yielding differential equations of all (and not only even) orders. It has been done as follows: on $\mathbb{H}/n$, regard the LÖWNER equation

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) = \frac{2}{g_t^{n-1}(z)(g_t^n(z) - \xi_t)}\,, \tag{8.6}$$

where $g_t : \mathbb{H}_t/n \to \mathbb{H}/n$. Analogously define $f_t^n = g_t^n - \xi_t$.

In order to get terms proportional to a third order derivative, additional terms have to be added to the driving function. In [75], this has been done by adding discontinuous processes as we review in the following.

8.3.1 Driving Functions with Jump Processes

In addition to the ordinary WIENER noise $\mathrm{d}B_t$ of variance $b(f_t)$ with drift $a(f_t)\mathrm{d}t$, Moghimi-Araghi et. al. [75] added POISSON-distributed jumps to the differential of the driving process. These jumps are described by the stochastic differential equation

$$\mathrm{d}x = \sum_i J_i \delta(t - t_i)\mathrm{d}t = J\mathrm{d}N\,, \tag{8.7}$$

which means that the process x is constant with jumps of height J_i at times t_i which shall be distributed according to some density function $\rho(J)$:

$$\rho(J) = \delta(J) + \frac{1}{(2n)!}\delta^{2n}(J)\,. \tag{8.8}$$

The first term ensures the normalizability of $\rho(J)$ while the second provides the necessary terms for higher order differentail operators.

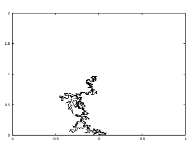
8.3.2 The cases $n = 3/2$ and $n = 2$

For $n = 3/2$, the jump density is given by $\rho(J) = \delta(J) + \partial_J^3\delta(J)$ and resulting in the following differential

$$\mathrm{d}f_t(z) = \frac{2}{f_t(z)^2}\mathrm{d}t - \frac{2\sqrt{\kappa}}{3\sqrt{f_t(z)}}\mathrm{d}B_t + c(f_t)J\mathrm{d}N\,. \tag{8.9}$$

For the special choices $\kappa = \frac{8}{h+1}$ and $c(f_t) = \frac{-2}{(h+2)(h-1)}$, the random walk on the VIRASORO group is given by

$$\begin{aligned} G_t^{-1}\mathrm{d}G_t &= \frac{2(h+2)}{h-1}\left(L_{-3} - \frac{2}{h+1}L_{-2}L_{-1} + \frac{1}{(h+1)(h+2)}L_{-1}^3\right)\mathrm{d}t \\ &\quad + \frac{2\sqrt{\kappa}}{3\sqrt{f_t(z)}}L_{-1}\mathrm{d}B_t\,, \end{aligned} \tag{8.10}$$



which, after taking the expectation value is equivalent to the level-three null vector operator acting on a state of weight $h_{(1,3)} = 8/\kappa - 1$:

$$\left(L_{-1}^3 - 2(h+2)L_{-2}L_{-1} + (h+1)(h+2)L_{-3}\right)|h_{(1,3)}\rangle = 0\,. \tag{8.11}$$

However, in the case $n = 2$, i.e. the Yang-Lee Edge Singularity, investigated in [74] a particularity arises: by chance, we do not need any jump processes to describe the fourth-order differential equation in the Yang-Lee Edge Singularity model. The driving process is given by

$$\mathrm{d}f_t = \frac{2}{f_t^{2n-1}(z)} - \frac{1}{nf_t^{n-1}(z)}\sqrt{\kappa}\mathrm{d}B_t\,, \tag{8.12}$$

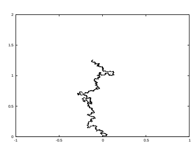
giving rise to a random walk on the Virasoro group:

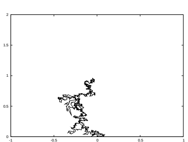
$$G_t^{-1}\mathrm{d}G_t = \left[\left(2 - \frac{\kappa(n-1)}{2n^2}\right)L_{-2n} + \frac{\kappa}{2n^2}L_{-n}^2\right]\mathrm{d}t + \frac{\sqrt{\kappa}}{n}L_{-n}\mathrm{d}B_t\,. \tag{8.13}$$

For $n = 2$, i.e. on the half of the upper half-plane, its expectation value gives rise to the level-four null operator of the Yang-Lee Edge Singularity:

$$\left(3L_{-4} - 5L_{-2}^2\right)|h_{(1,4)}\rangle = 0\,. \tag{8.14}$$

It is easy to check that this is the only case where the procedure works without additional terms [151]. Therefore, this procedure is not a candidate for an ordinary SLE variant since it is not possible to consider SLE on fractions of $\mathbb{H}$, yielding $n > 2$ null vector conditions without discontinuous driving processes, inevitably leading to branching curves. However, discussions about possible corresponding bCFT objects to branching SLE curves may nevertheless be interesting.





9 Closing Remarks

9.1 Conclusion

> *No one can be said to understand a paper unless he is able to generalize the paper.* From this definition the following corollary follows: *No author can be said to understand his most recent paper.*
>
> BARRY MCCOY [152]

In this thesis, we extended the relationship between BCFT partition functions and SLE variants [108, 67], based on the change of the single SLE measure, to multiple SLE $(\kappa, \vec{\rho})$. This is the most general SLE variant, as long the dynamic treatment of curves is limited to those created by boundary fields with vanishing descendant on level two. Our framework describes the presence of multiple interfaces, corresponding to boundary condition changing fields with vanishing descendant on level two, and force-points on the boundary and in the bulk as well as observables, corresponding to boundary and bulk vertex operators in the COULOMB gas formalism. In this context, we provided an interpretation for the weighting martingale as naturally emerging from the normalization of expectation values in Statistical Physics. In addition, we proved the relationship between multiple SLE and SLE$(\kappa, \vec{\rho})$, as SLE$(\kappa, 2, \ldots, 2)$ being a special case of multiple SLE where all but one curve simply do not grow into the interior of the domain, ruling out previous ideas [125, 101]. Furthermore, we clarified the role of bulk observables in BCFT, motivating that a clear distinction between bulk field and mirror-image pairs becoming the identity near the boundary [60], i. e. observables, on the one hand and those whose OPEs yield non-trivial boundary fields, i. e. SLE$(\kappa, \vec{\rho})$ force-fields [68], on the other hand has to be made.

In part a of chapter 6, we have shown that certain BCFT expectation values provide us with new SLE martingales containing curve-creating fields of other dimension than $h_{(1,2)}$ or $h_{(2,1)}$. This has been accomplished by considering the OPE of the curve-creating boundary field $\psi_{h_{(1,2)}}(\xi_t)$ with a bulk field $\phi_{h_{(1,2)}}(z_t^j)$ in a BCFT expectation value. The emerging scaling law for short distances receives an interpretation via the probability of the corresponding SLE event, i. e. the boundary of an SLE hull approaching the marked point $z_j = g_t^{-1}(z_t^j)$ [123]. We found that the different SLE phases, $0 < \kappa \leq 4$ and $4 < \kappa < 8$, correspond to the BCFT boundary fields $\psi_{h_{(2,1)}}$ and $\psi_{h_{(1,2)}}$, respectively, in agreement with previous results [153]. This is further supported by the identification of the exponent of the angular part of the SLE probability with $h_{(1,3)}$, i. e. the weight of the relevant branch of the fusion product of the corresponding representations $(h_{(1,2)}, c_{p,q}) \times (h_{(1,2)}, c_{p,q}) = (h_{(1,1)}, c_{p,q}) + (h_{(1,3)}, c_{p,q})$.

In part b of chapter 6, we provided amendments to our preceding work, including the insights about bulk fields as bulk force-points as argued in chapter 5. On this basis, we considered the inverse partition function as a martingale, performing analogous investigations to those done in chapter 6a for bulk force-fields from SLE$(\kappa, \vec{\rho})$. Furthermore, we extended the analysis to the fusion of two neighboring curve-creating boundary fields occurring in multiple SLE. Choosing illustrative toy models, we first consider the OPE of a curve-creating boundary field with a bulk force-field and second with another curve-creating boundary field. This procedure allows an analogous interpretation to 6a: the emerging scaling law for short-distances can again be interpreted as the probability of the corresponding SLE event, i. e. the boundary of an SLE hull approaching the force-point or a neighboring SLE curve.

In chapter 7 of this thesis, we investigated the constraints of global time parameterizations on the joint growth of two SLE traces, which are generated by two boundary fields with null descendant on level two of the same type. Based on bCFT considerations, we motivated that the joint growth should be due to a field with null descendant on level three emerging from the short-distance product of the original fields. We proved that the algebraic κ-relation [125, 67] holds for the joint description, too. From this result we motivate a proposal for the infinitesimal Löwner mapping for the joint process, i. e. the infinitesimal slit mapping for the curve created by the boundary field with null descendant on level three, comparing it to preceding work [74, 75].

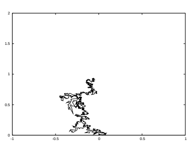
9.2 Comments, Outlook and Open Questions

Problems worthy of attack prove their worth by fighting back.

Paul Erdos

From a mathematical point of view, SLE is a major step towards the understanding of fractal shapes in the 2D continuum limit. It is not only a framework for describing one-parameter families of fractal curves but also provides tools which led to many new results. Canonical further derivations such as the proof of scaling limits, e. g. of the FK representation of the Potts model or the SAW, are being developed at the moment. Of course one is also interested in natural extensions of SLE, e. g. by considering other stochastic driving processes such as Lévy processes [72, 73].

From a physical point of view, there has been some progress with respect to the connection to bCFT, e. g. in a series of papers, Bauer and Bernard [54, 55, 56, 57, 58, 59, 60] presented several ways how SLE results can be obtained from bCFT and in [154], Cardy provides a way how to connect multiple SLE to Dyson's Brownian motion. Apart from this, little progress has been achieved with respect to the relation of curves emerging from boundary condition changing operators that exhibit vanishing descendants on other levels than two, although corresponding scaling exponents also appear in SLE computations. However fixing the central charge in terms of κ and



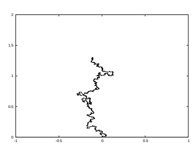
identifying the SLE curves with physical interfaces created by boundary operators with vanishing descendant on level two, is far from providing a complete proof of the SLE-bCFT correspondence. Furthermore, almost all of the results acheived by physicists that continued along this path, lack mathematical rigor. Therefore they may at most be considered as *motivations* for SLE objects as resulting from the presence of other bCFT fields, or as *predictions*, not as proofs. Our results, too, do not make an exeption here.

The sluggish progress at this point may be due to the most obvious limitations of SLE: it is only able to address the critical domain walls and not the full configurations of clusters and loops as is done by the lattice descriptions of the models. In addition, it leaves aside the local observables and powerful algebraic structure known from CFT, although there are some publications trying to find a suitable representation to identify the Virasoro algebra in SLE [56, 155]. However, there is a good chance that this will be resolved by a method extending SLE: Conformal Loop Ensembles (CLEs) [36, 37, 45] introduced in 2005.

The Conformal Loop Ensembles CLE(κ), defined for $8/3 \leq \kappa \leq 8$, are random collections of countably many disjoint loops in an SLE domain. The set of outermost loops in a CLE(κ) has the same law as the set of loop soup cluster boundaries for a loop soup of intensity $c = (3\kappa - 8)(6 - \kappa)/2\kappa$. These loop soup measures [130] are the restriction measures of SLE curves, conditioned not to intersect some hull. At any time instant, CLE(8/3) is a. s. empty, i. e. without macroscopic loops. For $8/3 \leq \kappa \leq 4$, the loops are a. s. simple and disjoint. For $4 < \kappa \leq 8$, the loops may be self- and mutually intersecting. CLE(8) a. s. consists of a single, space-filling loop. Within CLE, the main local observable, i. e. the stress-energy tensor, has been defined, reproducing results from CFT such as Ward identities, its transformation property with the Schwarzian derivative (leading to the Virasoro algebra) and its relation to small variations of the boundary [156, 153]. Furthermore, the *full* scaling limit of two-dimensional critical percolation has been proven to correspond to CLE(6) [44].

The question remains if it is possible to find a one-to-one relation between quantities of a theory which is based on *local* objects like bCFT and those of another which concentrates completely on *non-local* objects like SLE. From the considerations above, it seems by all means that CLE provides a more promising basis. However, even the CLEs require the presence of boundary fields with vanishing descendant on level two to fix the correspondence to bCFT which is a constraint not fulfilled by a *general* bCFT model. Therefore, even in this extension to SLE, it is questionable on a general basis if a full correspondence can be acheived.

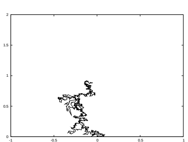
Some more concrete questions of mostly mathematical interest concerning SLE as the scaling limit of Statistical Physics models instead of their correspondence to bCFT, may be answered more easily. One is related to the insight into spin correlations in the Potts or $O(n)$ models, i. e. are the full scaling limits of these models given by some CLE(κ)? Do the height functions in theses models have scaling limits given by multiples of Gaussian Free Fields subject to suitable boundary



conditions? Related to that, is the scaling limit of loops of critical FK clusters in the Q-state Potts model on a planar lattice given by some CLE(κ)? Another open problem of physical interest is the investigation of finite size effects which has not yet been addressed at all, although the results are important for the significance of numerical simulations. Additionally, no interpretation of the free energy of the domain walls and of the interaction energy associated with several domain walls has been provided yet.

One of the major limitations of the theory of conformally invariant systems and critical phenomena in general is that it works only in *two* dimensions. Understanding similar problems in three or more dimensions is significantly harder and therefore most are still unsolved. Unfortunately, the methods used in SLE can not be carried over to other dimesions since theorems like the Riemann theorem and the richness of the conformal group is unique to two dimensions. Furthermore, numerics suggests that the answers may not be as nice, e. g. scaling exponents may not be rational numbers any more.

> *"Would you tell me, please, which way I ought to go from here?" "That depends a good deal on where you want to get to," said the Cat. "I don't much care where –" said Alice. "Then it doesn't matter which way you go," said the Cat. "– so long as I get somewhere," Alice added as an explanation. "Oh, you're sure to do that," said the Cat, "if you only walk long enough."*
>
> Lewis Carroll [157]



A Appendix

A.1 Proof of the Probability of Hitting a Disc

For the proof of theorem 4, we need the following lemma:

Theorem 6. Let (X_t) be a diffusion on the interval $[0,1]$, obeying

$$\mathrm{d}x_t = \sigma \mathrm{d}B_t + f(X_t)\mathrm{d}t\,, \tag{A.1}$$

where (B_t) is standard BROWNian motion, $\sigma \in \mathbb{R}^+$, f smooth on $(0,1)$ satisfying suitable boundary conditions. Then L defined by

$$L\phi = \frac{\sigma^2}{2}\phi'' + f\phi'\,, \tag{A.2}$$

is the generator of the diffusion process. Let λ be its leading eigenvalue. Then for $t \to \infty$, the probability $\mathbb{P}_t$ that the diffusion is defined up to time t tends to zero as

$$\mathbf{P}_t \asymp \exp(-\lambda t)\,. \tag{A.3}$$

With these result, we can proceed to do the

Proof of Theorem 4. Let δ_t be the EUCLIDEAN distance between z and the tip of the hull of the SLE trace K_t. (δ_t) is then a non-increasing process and $\lim_{t\to\infty}\delta_t = \mathrm{dist}(z_0, \gamma_{[0,\infty)})$. Applying the KÖBE $\frac{1}{4}$-theorem to the LÖWNER map g_t, we get

$$\delta_t \asymp \frac{\mathfrak{Im}(g_t(z_0))}{|g_t'(z_0)|}\,. \tag{A.4}$$

Mapping the problem onto the unit disc where z_0 is mapped to zero and ξ_t is mapped to a random process on the unit circle, we introduce

$$\tilde{g}_t\,:\, z \mapsto \frac{g_t(z) - g_t(z_0)}{g_t(z) - g_t(\bar{z}_0)} \quad \text{and} \quad w \mapsto \tilde{g}_t\left(\frac{w(\overline{g_t(z_0)} - g_t(z_0))}{w-1}\right)\,. \tag{A.5}$$

In this geometry, we can simplify (A.4) to

$$\delta_t \asymp |\tilde{g}_t'(z_0)|^{-1}\,. \tag{A.6}$$

Taking the time derivative of $\tilde{g}_t(z)$,

$$\partial_t \tilde{g}_t(z) = \frac{2(\tilde{\beta}_t - 1)^3}{\left(g_t(z_0) - \overline{g_t(z_0)}\right)^2 \tilde{\beta}_t^2} \cdot \frac{\tilde{\beta}_t \tilde{g}_t(z)\,(\tilde{g}_t(z) - 1)}{\tilde{g}_t(z) - \tilde{\beta}_t}\,, \tag{A.7}$$

where the random process β_t is mapped onto the unit circle,

$$\tilde{\beta}_t = \frac{\beta_t - g_t(z_0)}{\beta_t - \overline{g_t(z_0)}}\,, \tag{A.8}$$

and reparameterizing the curve,

$$\mathrm{d}s = \frac{(t-1)^4}{\left|g_t(z_0) - \overline{g_t(z_0)}\right|^2 \tilde{\beta}_t^2}\mathrm{d}t\,, \tag{A.9}$$

we can introduce the random process $h_s = \tilde{g}_{t(s)}$ which satisfies

$$\partial_s h_s(z) = \tilde{X}(\tilde{\beta}_{t(s)}, h_s(z))\,, \quad \text{with} \quad \tilde{X}(\xi, w) = \frac{2\xi w(w-1)}{(1-\xi)(w-\xi)}\,. \tag{A.10}$$

Rewriting $\tilde{\beta}_{t(s)} = \exp(i\alpha_s)$, this can be described by the diffusion process α_s on the interval $(0, 2\pi)$ with

$$\mathrm{d}\alpha_s = \sqrt{\kappa}\mathrm{d}B_s + \frac{\kappa - 4}{2}\cot\frac{\alpha_s}{2}\mathrm{d}s\,, \quad \alpha_0 = 2\alpha(z_0)\,. \tag{A.11}$$

Differentiating (A.10) with respect to z at $z = z_0$ and taking the real part on both sides, it can be shown that

$$\partial_s \log|h_s'(z_0| = 1\,, \tag{A.12}$$

which means that a.s. for all $s > 0$, we have $|h_s'(z_0)| = |h_0'(z_0)|\exp s$. Together with (A.6), we can deduce

$$\delta_{(t(s)} \asymp \delta_0 \exp(-s) \asymp \mathfrak{Im}(z_0)\exp(-s)\,. \tag{A.13}$$

At the stopping time, $\tau_{z_0} = \inf\{t : z_0 \in K_t\}$, we can have one of to cases

① z_0 is on the trace, i. e. $\delta_t \to 0$ and $s \to \infty$ and (α_s) does not touch $\{0, 2\pi\}$,

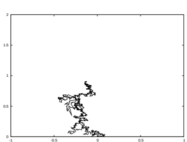
② z_0 is not on the trace, i. e. $\delta_t \to \mathrm{dist}(z_0, \gamma_{[0,\infty)}) > 0$ and (α_s) reaches the boundary of the interval $(0, 2\pi)$ at time

$$s_0 := \log\delta_0 - \log(\mathrm{dist}(z_0, \gamma_{[0,\infty)})) + \mathcal{O}(1)\,. \tag{A.14}$$

For the surviving time S of (α_s), from (A.13) follows that

$$\mathrm{dist}(z_0, \gamma_{[0,\infty)}) \asymp \delta_0 \exp(-S)\,, \tag{A.15}$$

and estimating the probability that z_0 is ϵ-close to the trace becomes equivalent to estimating the probability that (α_s) survives up to time $\log(\delta_0/\epsilon)$.



For $\kappa \leq 4$, the drift of α_s is towards the boundary and therefore it dies a.s. in finite time. Employing that the principal eigenfunction (EF) with eigenvalue (EV) of the generator L_k of the diffusion process α_s is

$$EF = \left(\sin \frac{x}{2}\right)^{8/\kappa-1}, \quad EV = 1 - \frac{\kappa}{8}, \tag{A.16}$$

we can deduce from theorem 6 that for α_0 far from the boundary

$$\begin{aligned} \mathbf{P}(S > s) &\asymp \exp(-EV \cdot s) \quad \text{from which follows} \\ \mathbf{P}(\mathrm{dist}(z_0, \gamma_{[0,\infty)}) &\asymp \exp\left((1-\kappa/8)\log\frac{\epsilon}{\delta_0}\right) \asymp \left(\frac{\epsilon}{\delta_0}\right)^{1-\kappa/8}. \end{aligned} \tag{A.17}$$

Now we have to take the initial value α_0 into account. Introducing the bounded local martingale

$$X_s := \sin\left(\frac{\alpha_s}{2}\right)^{8/\kappa-1} \exp\left((1-\kappa/8)s\right), \tag{A.18}$$

for $s < S$ and $X_s = 0$ for $s \geq S$, we can equate its expected value at time zero and s, $\mathbb{E}(X_0) = \mathbb{E}(X_s)$,

$$\sin\left(\frac{\alpha_0}{2}\right)^{8/\kappa-1} = \exp\left((1-\kappa/8)s\right)\mathbf{P}(S \geq s)\mathbb{E}\left[\sin\left(\frac{\alpha_s}{2}\right)^{8/\kappa-1}\middle| S \geq s\right], \tag{A.19}$$

to obtain

$$\mathbf{P}(S \geq s) \asymp \exp\left((1-\kappa/8)s\right)\sin\left(\frac{\alpha_0}{2}\right)^{8/\kappa-1}. \tag{A.20}$$

This is the surviving probability of (α_s) up to to time $\log(\delta_0/\epsilon)$ and hence the probability that z_0 is ϵ-close to the trace as mentioned above.

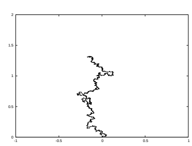
□

A.2 Auxiliary Calculation for Chapter 6 and 7

A.2.1 Preliminaries

Setting $|Y| = 1$, we get a level-three differential equation. For simplicity, we will introduce new coefficients:

$$\begin{aligned} 0 &= \left[\frac{\kappa}{2}(3-2h_1+h)(2-2h_1+h) - 2(3h_1-3-h)\right]\phi_h^{(3)} \qquad \text{(A.21)} \\ &\quad -\kappa(2-2h_1+h)L_{-1}\phi_h^{(2)} + \frac{\kappa}{2}L_{-1}^2\phi_h^{(1)} - 2L_{-2}\phi_h^{(1)} \\ &=: \left[V_1^1\left(\beta_{111}^1 L_{-1}^3 + \beta_{12}^1 L_{-1}L_{-2} + \beta_{21}^1 L_{-2}L_{-1} + \beta_3^1 L_{-3}\right)\right. \qquad \text{(A.22)} \\ &\quad \left. + V_2^1 L_{-1}\left(\beta_{11}^1 L_{-1^2} + \beta_2^1 L_{-2}\right) + L_{-1}^2\beta_1^1 L_{-1} + V_3^1 L_{-2}\beta_1^1 L_{-1}\right]\phi_h(x) \\ &=: K_1^3 L_{-1}^3 + K_{12}^3 L_{-1}L_{-2} + K_{21}^3 L_{-2}L_{-1} + K_3^3 L_{-3}, \qquad \text{(A.23)} \end{aligned}$$



with

$$\begin{aligned} K_1^1 &:= V_1^3\beta_{111}^1 + V_2^3\beta_{11}^1 + \beta_1^1\,, && \text{(A.24)}\\ K_{12}^1 &:= V_1^3\beta_{12}^1 + V_2^3\beta_2^1\,, && \text{(A.25)}\\ K_{21}^1 &:= V_1^3\beta_{21}^1 + V_3^3\beta_1^1\,, && \text{(A.26)}\\ K_3^1 &:= V_1^3\beta_3^1\,, && \text{(A.27)} \end{aligned}$$

using the commutator $L_{-1}L_{-2} = L_{-2}L_{-1} + L_{-3}$.

Knowing the form of the algebraic level-three null operator:

$$L_{-1}^3 - (h+1)(L_{-2}L_{-1} + L_{-1}L_{-2}) + (h+1)^2 L_{-3}\,, \qquad \text{(A.28)}$$

we only have to compare the coefficients:

$$\begin{aligned} 0 &= K_1^3 L_{-1}^3 + K_{12}^3 L_{-1}L_{-2} + K_{21}^3 L_{-2}L_{-1} + K_3^3 L_{-3}\\ &= L_{-1}^3 - (h+1)L_{-2}L_{-1} - (h+1)L_{-1}L_{-2} + (h+2)(h+1)L_{-3}\,. && \text{(A.29)} \end{aligned}$$

Hence all that is left to be shown is:

$$\begin{aligned} -2(h+1) &= \frac{K_{12}^3 + K_{21}^3}{K_1^3} =: H_{12}^3 + H_{21}^3 =: I_2^3\,, && \text{(A.30)}\\ h(h+1) &= \frac{K_3^3 + K_{12}^3}{K_1^3} =: H_{12}^3 + H_3^3 =: I_3^3\,, && \text{(A.31)} \end{aligned}$$

for which we have to compute the exact values of the $\beta_{\{\cdots\}}^1$. Therefore we reexpress the variables in terms of the new weight h:

$$\begin{aligned} c &= -\frac{(3h-1)(h-2)}{h+1}\,, && \text{(A.32)}\\ h_2 &= h_1 = \frac{1}{8}(3h-1)\,. && \text{(A.33)} \end{aligned}$$

A.2.2 Computation of the Coefficients

Now we will use our rule of thumb stated above:

① $|Y| - k = 1$

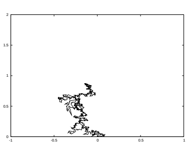
① from covariance:

$$\begin{aligned} L_1\phi_h^{(1)}(x) &= [h_1(1+1) + 1 - 1 - (2h_1 - h)]\,\phi_h^{(1-1)}(x)\\ &= h\phi_h(x)\,, && \text{(A.34)} \end{aligned}$$

② algebraically:

$$\begin{aligned} L_1\phi_h^{(1)}(x) &= L_1L_{-1}\beta_1^1\phi_h(x)\\ &= 2h\beta_1^1\phi_h(x)\,. && \text{(A.35)} \end{aligned}$$

Hence it follows that $\beta_1^1 = 1/2$.



② $|Y| - k = 2$

① from covariance:

$$\begin{aligned} L_1\phi_h^{(2)}(x) &= [h_1(1+1)+2-1-(2h_1-h)]\,\phi_h^{(2-1)}(x) \\ &= (h+1)\phi_h^{(1)}(x) \\ &= \frac{h+1}{2}L_{-1}\phi_h(x)\,, \end{aligned} \tag{A.36}$$

$$\begin{aligned} L_2\phi_h^{(2)}(x) &= [h_1(2+1)+2-2-(2h_1-h)]\,\phi_h^{(2-2)}(x) \\ &= (h+h_1)\phi_h(x)\,, \end{aligned} \tag{A.37}$$

② algebraically:

$$\begin{aligned} L_1\phi_h^{(2)}(x) &= L_1\left(\beta_{11}^1 L_{-1}^2 + \beta_2^1 L_{-2}\right)\phi_h(x) \\ &= \left(2(2h+1)\beta_{11}^1 + 3\beta_2^1\right)L_{-1}\phi_h(x)\,, \end{aligned} \tag{A.38}$$

$$\begin{aligned} L_2\phi_h^{(2)}(x) &= L_2\left(\beta_{11}^1 L_{-1}^2 + \beta_2^1 L_{-2}\right)\phi_h(x) \\ &= \left(6h\beta_{11}^1 + (4h+c/2)\beta_2^1\right)\phi_h(x)\,, \end{aligned} \tag{A.39}$$

comparing the coefficients, it follows:

$$\beta_{11}^1 = \frac{1}{8}\frac{h+1}{h+2} \tag{A.40}$$

$$\beta_2^1 = \frac{1}{4}\frac{h+1}{h+2} \tag{A.41}$$

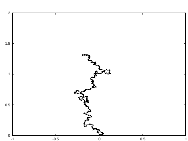
③ $|Y| - k = 3$

① from covariance:

$$\begin{aligned} L_1\phi_h^{(3)}(x) &= [h_1(1+1)+3-1-(2h_1-h)]\,\phi_h^{(3-1)}(x) \\ &= (h+2)\phi_h^{(2)}(x) \\ &= (h+2)\left(\beta_{11}^1 L_{-1}^2 + \beta_2^1 L_{-2}\right)\phi_h(x)(x)\,, \end{aligned} \tag{A.42}$$

$$\begin{aligned} L_2\phi_h^{(3)}(x) &= [h_1(2+1)+3-2-(2h_1-h)]\,\phi_h^{(3-2)}(x) \\ &= (h+1+h_1)\phi_h^{(1)} \\ &= \frac{h+1+h_1}{2}L_{-1}\phi_h(x)\,, \end{aligned} \tag{A.43}$$

$$\begin{aligned} L_3\phi_h^{(3)}(x) &= [h_1(3+1)+3-3-(2h_1-h)]\,\phi_h^{(3-3)}(x) \\ &= (h+2h_1)\phi_h(x)\,, \end{aligned} \tag{A.44}$$



② algebraically:

$$\begin{aligned}
L_1\phi_h^{(3)}(x) &= L_1\left(L_{-1}^3\beta^1_{111} + L_{-1}L_{-2}\beta^1_{12} + L_{-2}L_{-1}\beta^1_{21} + L_{-3}\beta^1_3\right)\phi_h(x) \\
&= \left(\left[6(h+1)L_{-1}^2\right]\beta^1_{111} + \left[3L_{-1}^2 + 2(h+2)L_{-2}\right]\beta^1_{12}\right. \\
&\quad \left. + \left[2hL_{-2} + 3L_{-1}^2\right]\beta^1_{21} + \left[4L_{-2}\right]\beta^1_3\right)\phi_h(x) \\
&= \left(\left[6(h+1)\beta^1_{111} + 3\beta^1_{12} + 3\beta^1_{21}\right]L_{-1}^2\right. \\
&\quad \left. + \left[2(h+2)\beta^1_{12} + 2h\beta^1_{21} + 4\beta^1_3\right]L_{-2}\right)\phi_h(x)\,, \qquad \text{(A.45)} \\
L_2\phi_h^{(3)}(x) &= L_2\left(L_{-1}^3\beta^1_{111} + L_{-1}L_{-2}\beta^1_{12} + L_{-2}L_{-1}\beta^1_{21} + L_{-3}\beta^1_3\right)\phi_h(x) \\
&= \left(\left[6(3h+1)\right]\beta^1_{111} + \left[5 + 4(h+1) + \frac{c}{2}\right]\beta^1_{12}\right. \\
&\quad \left. + \left[4(h+1) + c/2\right]\beta^1_{21} + [5]\,\beta^1_3\right)L_{-1}\phi_h(x)\,, \qquad \text{(A.46)} \\
L_3\phi_h^{(3)}(x) &= L_3\left(L_{-1}^3\beta^1_{111} + L_{-1}L_{-2}\beta^1_{12} + L_{-2}L_{-1}\beta^3_{21} + L_{-3}\beta^1_3\right)\phi_h(x) \\
&= \left(24h\beta^1_{111} + (16h+2c)\beta^1_{12} + 10h\beta^1_{21} + (6h+2c)\beta^1_3\right)\phi_h(x)\,, \qquad \text{(A.47)}
\end{aligned}$$

comparing the coefficients, it follows:

$$\beta^1_{111} = \text{free}\,, \tag{A.48}$$

$$\beta^1_{12} = h(h+1)\beta^1_{111} - \beta^1_3 - \frac{(h-3)(h-1)}{48}\,, \tag{A.49}$$

$$\beta^1_{21} = -(h+2)(h+1)\beta^1_{111} + \beta^1_3 + \frac{(h-3)(h-1)}{48}\,, \tag{A.50}$$

$$\beta^1_3 = \text{free}\,. \tag{A.51}$$

This directly leads us to the non-normalized coefficients

$$K^3_1 = \frac{1}{16(h+2)}\left[48\beta^1_{111}(h+2)(h+3) - (h^2+2h-7)\right] \tag{A.52}$$

$$K^3_{12} = h(h+1)K^3_1 - 3\beta^1_3(h+3) \tag{A.53}$$

$$K^3_{21} = -(h+2)(h+1)K^3_1 + 3(h+3)\beta^1_3 \tag{A.54}$$

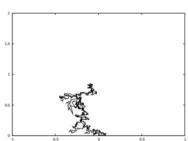
$$K^3_3 = 3(h+3)\beta^1_3 \tag{A.55}$$

and the normalized

$$H^3_{12} = h(h+1) - \frac{48\beta^1_3(h+2)(h+3)}{48\beta^1_{111}(h+2)(h+3) - (h^2+2h-7)}\,, \tag{A.56}$$

$$H^3_{21} = \frac{48\beta^1_3(h+2)(h+3)}{48\beta^1_{111}(h+2)(h+3) - (h^2+2h-7)} - (h+2)(h+1)\,, \tag{A.57}$$

$$H^3_3 = \frac{48\beta^1_3(h+2)(h+3)}{48\beta^1_{111}(h+2)(h+3) - (h^2+2h-7)}\,. \tag{A.58}$$



Hence we get the desired result

$$
\begin{aligned}
I_2^3 = H_{12}^3 + H_{12}^3 &= -2(h+1)\,, && \text{(A.59)}\\
I_3^3 = H_{12}^3 + H_3^3 &= h(h+1)\,, && \text{(A.60)}
\end{aligned}
$$

i. e. the level-two null vector condition on a primary $\phi_{(1,2)}$ field translates to a level-three null vector equation on a primary $\phi_{(1,3)}$ field after fusion with another $\phi_{(1,2)}$ field.

A.3 Commutation of Joint Generators

In order to compute the commutation relation for joint growth processes, we first perform the short-distance expansion of the generators and the partition function, before carrying out the derivatives etc. in the computation of the commutation relation for the outcomes. In the case of just one joint process, we therefore have to consider the action of $\tilde{\mathcal{D}}_{-2}(x_t^i)$ and $\bar{\mathcal{D}}_{-2}(x_t^j)$ on the expanded partition function $Z^{\exp}_{\text{BCFT}}$ (7.19). This is known to result in $\mathcal{D}_{-3}^{m-1}(x_t^i; \{x_t^k\}_{k\neq i,i+1})$ and $\mathcal{D}_{-2}^{m-1}(x_t^j; \{x_t^k\}_{k\neq j,j+1})$, respectively, defined as follows:

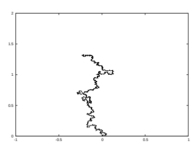
$$
\begin{aligned}
\mathcal{D}_{-2}^{m-1}(x_t^j) &= \frac{\kappa}{2}\partial^2_{x_t^j} - 2\sum_{k\neq j,i,i+1}^{m}\left(\frac{h_k}{(x_t^k - x_t^j)^2} - \frac{1}{x_t^k - x_t^j}\partial_{x_t^j}\right)\\
&\quad -2\left(\frac{h_{(1,3)}}{(x_t^i - x_t^j)^2} - \frac{1}{x_t^i - x_t^j}\partial_{x_t^k}\right)\,, && \text{(A.61)}\\
\mathcal{D}_{-3}^{m-1}(x_t^i) &= \frac{\kappa}{2}\partial^3_{x_t^i} - 2\sum_{k\neq i,i+1}^{m}\left(\frac{h_k}{(x_t^k - x_t^i)^2} - \frac{1}{x_t^k - x_t^i}\partial_{x_t^k}\right)\partial_{x_t^i}\\
&\quad + h_{(1,3)}\sum_{k\neq i,i+1}^{m}\left(\frac{2h_k}{(x_t^k - x_t^i)^3} - \frac{1}{(x_t^k - x_t^j)^2}\partial_{x_t^k}\right)\,, && \text{(A.62)}
\end{aligned}
$$

when acting on the fused partition function $\tilde{Z}[x_t] := \tilde{Z}_{\text{b.c.}} Z_{\text{free}}$ given by:

$$
\begin{aligned}
\tilde{Z}_{\text{b.c.}} &= \langle \psi_\infty(\infty), \psi_1(x_t^1)\ldots\psi_{i-1}(x_t^{i-1})\psi_{h_{(1,3)}}(x_t^i)\psi_{i+2}(x_t^{i+2})\ldots\\
&\quad \psi_{j-1}(x_t^{j-1})\psi_j(x_t^j)\psi_{j+1}(x_t^{j+1})\ldots\psi_m(x_t^m)\rangle\,. && \text{(A.63)}
\end{aligned}
$$

Note that we leave aside possible higher orders in the short-distance, assuming that the distance between the two joining curves is much smaller than the distance to the other curves. Since our computations are only for an infinitesimal time, this is a quite reasonable assumption.

In the case of a pair of joint processes, we have to consider the action of $\tilde{\bar{\mathcal{D}}}_{-2}(x_t^i)$ and $\bar{\tilde{\mathcal{D}}}_{-2}(x_t^j)$ on the doubly expanded partition function $Z^{2\times\exp}_{\text{BCFT}}$ (7.24), resulting in



$\mathcal{D}_{-3}^{m-2}(x_t^i; \{x_t^k\}_{k\neq i,i+1,j+1})$ and $\mathcal{D}_{-3}^{m-2}(x_t^j; \{x_t^k\}_{k\neq j,j+1,i+1})$, given by

$$\begin{aligned} \mathcal{D}_{-3}^{m-2}(x_t^i) &= \frac{\kappa}{2}\partial_{x_t^i}^3 - 2\sum_{k\neq i,i+1}^{m}\left(\frac{h_k}{(x_t^k - x_t^i)^2} - \frac{1}{x_t^k - x_t^i}\partial_{x_t^k}\right)\partial_{x_t^i} \\ &\quad + h_{(1,3)}\sum_{k\neq i,i+1}^{m}\left(\frac{2h_k}{(x_t^k - x_t^i)^3} - \frac{1}{(x_t^k - x_t^j)^2}\partial_{x_t^k}\right), \end{aligned} \tag{A.64}$$

wherein $h_j = h_{(1,3)}$ acting on the doubly fused partition function $\tilde{\tilde{Z}}[x_t] := \tilde{\tilde{Z}}_{\text{b.c.}} Z_{\text{free}}$ given by:

$$\begin{aligned} \tilde{\tilde{Z}}_{\text{b.c.}} &= \langle \psi_\infty(\infty), \psi_1(x_t^1)\dots\psi_{i-1}(x_t^{i-1})\psi_{h_{(1,3)}}(x_t^i)\psi_{i+2}(x_t^{i+2})\dots \\ &\quad \psi_{j-1}(x_t^{j-1})\psi_{h_{(1,3)}}(x_t^j)\psi_{j+2}(x_t^{j+2})\dots\psi_m(x_t^m)\rangle\,. \end{aligned} \tag{A.65}$$

In CFT it is more natural to use the parameter $t = p/q = 4/\kappa$ instead of κ. Therefore, we switch $r \leftrightarrow s$ and $\kappa \leftrightarrow 16/\kappa$ in the following. Furthermore, we introduce the definitions for the differential operators $\mathcal{L}_{-m}$:

$$\mathcal{L}_{-m} = \sum_{l\neq i=1}^{n} \frac{(m-1)h_l}{(x_l - x_i)^m} - \frac{1}{(x_l - x_i)^{m-1}}\frac{\partial}{\partial x_l}\,, \tag{A.66}$$

omitting the time index t from now on, such that the null-vector operators acting on correlation functions containing $\psi_{(2,1)}$ or $\psi_{(3,1)}$ fields are given by

$$\mathcal{D}_{-2} = \mathcal{L}_{-1}^2 - t\mathcal{L}_{-2}\,, \tag{A.67}$$

and

$$\mathcal{D}_{-3} = \mathcal{L}_{-1}^3 - 4t_i\mathcal{L}_{-2}\mathcal{L}_{-1} + 2t_i(2t_i - 1)\mathcal{L}_{-3}\,, \tag{A.68}$$

respectively.

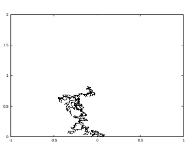
Due to the different orders in k that are picked up when performing the short-distance limit, we expect that coefficients α, β and γ arise, such that

$$\begin{aligned} [\mathcal{D}_{-3}(x_i), \mathcal{D}_{-2}(x_j)] &= \frac{\alpha}{(x_i - x_j)^2}\mathcal{D}_{-3}(x_i) \\ &\quad + \frac{1}{(x_i - x_j)^2}\left(\frac{\beta}{x_i - x_j} + \gamma\mathcal{L}_{-1}(x_i)\right)\mathcal{D}_{-2}(x_j) \end{aligned} \tag{A.69}$$

(hopefully) gives us back something proportional to the κ relation when acting on the BCFT partition function

$$Z_{\text{b.c.}} = \langle \psi_\infty(\infty), \psi_{(2,1)}(x_1)\cdots\psi_{2,1)}(x_m)\rangle\,. \tag{A.70}$$

Similar to the original case, the outcome of (7.12), we should get the κ condition back.



Analogously, we will investigate this for two short-distance limits, i. e.

$$\begin{aligned}[\mathcal{D}_{-3}(x_i), \mathcal{D}_{-2}(x_j)] &= \left(\frac{\alpha_i}{(x_i - x_j)} + \beta_i \mathcal{L}_{-1}(x_j)\right) \mathcal{D}_{-3}(x_i) \\ &\quad + \left(\frac{\alpha_j}{(x_j - x_i)} + \beta_j \mathcal{L}_{-1}(x_i)\right) \mathcal{D}_{-3}(x_j)\,, \end{aligned} \tag{A.71}$$

where the α_i, α_j and β_i, β_j have to be determined.

A.3.1 One Short-Distance Limit

The first step is to sort the single terms that can show up in the commutator. Therefore we compute

$$\begin{aligned} &[\mathcal{D}_{-3}(x_i), \mathcal{D}_{-2}(x_j)] \\ =\; & \left[\mathcal{L}_{-1}^3(x_i) - 4t_i \mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i) + 2t_i(2t_i - 1)\mathcal{L}_{-3}(x_i), \mathcal{L}_{-1}^2(x_j) - t\mathcal{L}_{-2}(x_j)\right] \\ =\; & \left[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-1}^2(x_j)\right] - t\left[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-2}(x_j)\right] \\ & +4t_i\left[\mathcal{L}_{-1}^2(x_j), \mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i)\right] - 2t_i(2t_i - 1)\left[\mathcal{L}_{-1}^2(x_j), \mathcal{L}_{-3}(x_i)\right] \\ & -2t_i t(2t_i - 1)\left[\mathcal{L}_{-3}(x_i), \mathcal{L}_{-2}(x_j)\right] + 4t_i t\left[\mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i), \mathcal{L}_{-2}(x_j)\right]\,. \end{aligned} \tag{A.72}$$

In the following, we will have a look at the following terms separately:

① coefficient proportional to $\left[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-1}^2(x_j)\right]$

② coefficient proportional to $-t\left[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-2}(x_j)\right]$

③ coefficient proportional to $4t_i\left[\mathcal{L}_{-1}^2(x_j), \mathcal{L}_{-2}\mathcal{L}_{-1}(x_i)\right]$

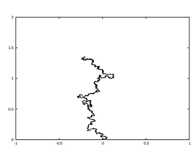
④ coefficient proportional to $-2t_i(2t_i - 1)\left[\mathcal{L}_{-1}^2(x_j), \mathcal{L}_{-3}(x_i)\right]$

⑤ coefficient proportional to $-2tt_i(2t_i - 1)\left[\mathcal{L}_{-3}(x_i), \mathcal{L}_{-2}(x_j)\right]$

⑥ coefficient proportional to $4tt_i\left[\mathcal{L}_{-2}\mathcal{L}_{-1}(x_i), \mathcal{L}_{-2}(x_j)\right]$

Term ① is easy to compute since

$$\left[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-1}^2(x_j)\right] = 0\,. \tag{A.73}$$



Term ② is already more complicated:

$$
\begin{aligned}
&\left[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-2}(x_j)\right] \\
&= \frac{\partial^3}{\partial x_i^3}\left(\sum_{l\neq j=1}^{n} \frac{h_l}{(x_l-x_j)^2} - \frac{1}{x_l-x_j}\frac{\partial}{\partial x_l}\right) \\
&\quad +3\frac{\partial^2}{\partial x_i^2}\left(\sum_{l\neq j=1}^{n} \frac{h_l}{(x_l-x_j)^2} - \frac{1}{x_l-x_j}\frac{\partial}{\partial x_l}\right)\frac{\partial}{\partial x_i} \\
&\quad +3\frac{\partial}{\partial x_i}\left(\sum_{l\neq j=1}^{n} \frac{h_l}{(x_l-x_j)^2} - \frac{1}{x_l-x_j}\frac{\partial}{\partial x_l}\right)\frac{\partial^2}{\partial x_i^2} \\
&= \frac{(-24)h_i}{(x_i-x_j)^5} + \frac{6(3h_i+1)}{(x_i-x_j)^4}\frac{\partial}{\partial x_i} + \frac{(-6)(h_i+1)}{(x_i-x_j)^3}\frac{\partial^2}{\partial x_i^2} + \frac{3}{(x_i-x_j)^2}\frac{\partial^3}{\partial x_i^3}
\end{aligned}
\tag{A.74}
$$

In the end, this leaves us with

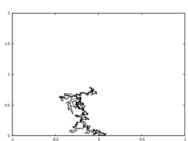
$$
\begin{aligned}
-t\left[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-2}(x_j)\right] &= \frac{24t(2t_i-1)}{(x_i-x_j)^5} + \frac{(-12)t(3t_i-1)}{(x_i-x_j)^4}\frac{\partial}{\partial x_i} \\
&\quad + \frac{12tt_i}{(x_i-x_j)^3}\frac{\partial^2}{\partial x_i^2} + \frac{(-3)t}{(x_i-x_j)^2}\frac{\partial^3}{\partial x_i^3}\,.
\end{aligned}
$$

Term ③, proportional to $4t_i$ is given by the following:

$$
\begin{aligned}
&\left[\mathcal{L}_{-1}^2(x_j), \mathcal{L}_{-2}\mathcal{L}_{-1}(x_i)\right] \\
&= \frac{\partial^2}{\partial x_j^2}\left(\sum_{k\neq\sim=1}^{n} \frac{h_k}{(x_k-x_i)^2} - \frac{1}{x_k-x_i}\frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_i} \\
&\quad +2\frac{\partial}{\partial x_j}\left(\sum_{k\neq\sim=1}^{n} \frac{h_k}{(x_k-x_i)^2} - \frac{1}{x_k-x_i}\frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} \\
&= \frac{6h_j}{(x_i-x_j)^4}\frac{\partial}{\partial x_i} + \frac{2(2h_j+1)}{(x_i-x_j)^3}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} + \frac{2}{(x_i-x_j)^2}\frac{\partial^2}{\partial x_j^2}\frac{\partial}{\partial x_i}
\end{aligned}
\tag{A.75}
$$

The final result for this term reads as follows:

$$
\begin{aligned}
4t_i\left[\mathcal{L}_{-1}^2(x_j), \mathcal{L}_{-2}\mathcal{L}_{-1}(x_i)\right] &= \frac{6t_i(3t-2)}{(x_i-x_j)^4}\frac{\partial}{\partial x_i} + \frac{12tt_i}{(x_i-x_j)^3}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} \\
&\quad + \frac{8t_i}{(x_i-x_j)^2}\frac{\partial^2}{\partial x_j^2}\frac{\partial}{\partial x_i}\,.
\end{aligned}
$$



Term ④, proportional to $-2t_i(2t_i-1)$ is equivalent to

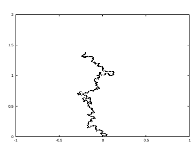
$$
\begin{aligned}
&\left[\mathcal{L}_{-1}^2(x_j),\mathcal{L}_{-3}(x_i)\right] \\
&= \frac{\partial^2}{\partial x_j^2}\left(\sum_{k\neq\sim=1}^{n}\frac{2h_k}{(x_k-x_i)^3}-\frac{1}{(x_k-x_i)^2}\frac{\partial}{\partial x_k}\right) \qquad \text{(A.76)}\\
&\quad +2\frac{\partial}{\partial x_j}\left(\sum_{k\neq\sim=1}^{n}\frac{2h_k}{(x_k-x_i)^3}-\frac{1}{(x_k-x_i)^2}\frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_j}\\
&= \frac{(-24)h_j}{(x_i-x_j)^5}+\frac{(-6)(2h_j+1)}{(x_i-x_j)^4}\frac{\partial}{\partial x_j}+\frac{(-4)}{(x_i-x_j)^3}\frac{\partial^2}{\partial x_j^2},
\end{aligned}
$$

which is not more than

$$
\begin{aligned}
-2t_i(2t_i-1)\left[\mathcal{L}_{-1}^2(x_j),\mathcal{L}_{-3}(x_i)\right] &= \frac{12t_i(2t_i-1)(3t-2)}{(x_i-x_j)^5}+\frac{18tt_i(2t_i-1)}{(x_i-x_j)^4}\frac{\partial}{\partial x_j}\\
&\quad +\frac{8t_i(2t_i-1)}{(x_i-x_j)^3}\frac{\partial^2}{\partial x_j^2}.
\end{aligned}
$$

Term ⑤, proportional to $-2tt_i(2t_i-1)$, is the most complicated term:

$$
\begin{aligned}
&[\mathcal{L}_{-3}(x_i),\mathcal{L}_{-2}(x_j)] \qquad \text{(A.77)}\\
&= \left[\sum_{k\neq\sim=1}^{n}\frac{2h_k}{(x_k-x_i)^3}-\frac{1}{(x_k-x_i)^2}\frac{\partial}{\partial x_k},\sum_{l\neq j=1}^{n}\frac{h_l}{(x_l-x_j)^2}-\frac{1}{x_l-x_j}\frac{\partial}{\partial x_l}\right]\\
&= \sum_{m\neq\sim,j=1}^{n}2h_m\frac{2x_i-x_j-x_m}{(x_m-x_j)^2(x_m-x_i)^2(x_i-x_j)^2}\\
&\qquad -\frac{2x_i-x_j-x_m}{(x_m-x_j)(x_m-x_i)^2(x_i-x_j)^2}\frac{\partial}{\partial x_m}\\
&\quad +\frac{2(3h_j-h_i)}{(x_i-x_j)^5}+\frac{1}{(x_i-x_j)^4}\left(\frac{\partial}{\partial x_i}+2\frac{\partial}{\partial x_j}\right)\\
&\quad +\sum_{m\neq\sim,j=1}^{n}\frac{6h_m}{(x_m-x_i)^3(x_i-x_j)(x_m-x_j)}\\
&\qquad -\frac{2}{(x_m-x_i)^2(x_i-x_j)(x_m-x_j)}\frac{\partial}{\partial x_m}.
\end{aligned}
$$

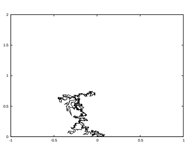


This can be reduced to

$$
\begin{aligned}
&-2tt_i(2t_i-1)\left[\mathcal{L}_{-3}(x_i),\mathcal{L}_{-2}(x_j)\right] \qquad \text{(A.78)}\\
=\quad & \frac{-2tt_i(2t_i-1)}{(x_i-x_j)^2}\sum_{m\neq\sim,j=1}^{n}2h_m\frac{2x_i-x_j-x_m}{(x_m-x_j)^2(x_m-x_i)^2}-\frac{2x_i-x_j-x_m}{(x_m-x_j)(x_m-x_i)^2}\frac{\partial}{\partial x_m}\\
&+\frac{-tt_i(2t_i-1)(9t-8t_i-2)}{(x_i-x_j)^5}+\frac{(-2)tt_i(2t_i-1)}{(x_i-x_j)^4}\left(\frac{\partial}{\partial x_i}+2\frac{\partial}{\partial x_j}\right)\\
&+\frac{-2tt_i(2t_i-1)}{(x_i-x_j)}\sum_{m\neq\sim,j=1}^{n}\frac{6h_m}{(x_m-x_i)^3(x_m-x_j)}-\frac{2}{(x_m-x_i)^2(x_m-x_j)}\frac{\partial}{\partial x_m}\,.
\end{aligned}
$$

Term ⑥, proportional to $4tt_i$ yields

$$
\begin{aligned}
&\left[\mathcal{L}_{-2}\mathcal{L}_{-1}(x_i),\mathcal{L}_{-2}(x_j)\right] \qquad \text{(A.79)}\\
=\quad & \left[\left(\sum_{k\neq\sim=1}^{n}\frac{h_k}{(x_k-x_i)^2}-\frac{1}{x_k-x_i}\frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_i},\sum_{l\neq j=1}^{n}\frac{h_l}{(x_l-x_j)^2}-\frac{1}{x_l-x_j}\frac{\partial}{\partial x_l}\right]\\
=\quad & \sum_{m\neq j,\sim=1}^{n}\frac{(2x_m-x_i-x_j)2h_m}{(x_i-x_j)(x_m-x_i)^2(x_m-x_j)^2}\frac{\partial}{\partial x_i}\\
&+\sum_{m\neq j,\sim=1}^{n}\frac{(-2)}{(x_i-x_j)(x_m-x_j)(x_m-x_i)}\frac{\partial}{\partial x_m}\frac{\partial}{\partial x_i}\\
&+\sum_{m\neq j,\sim=1}^{n}\frac{h_m}{(x_i-x_j)^2(x_m-x_i)^2}\frac{\partial}{\partial x_i}+\frac{(-1)}{(x_i-x_j)^2(x_m-x_i)}\frac{\partial}{\partial x_m}\frac{\partial}{\partial x_i}\\
&+\sum_{m\neq j,\sim=1}^{n}\frac{(-2)h_ih_m}{(x_i-x_j)^3(x_m-x_i)^2}-\frac{(-2)h_i}{(x_i-x_j)^3(x_m-x_i)}\frac{\partial}{\partial x_m}\\
&+\frac{(-2)h_i(h_j+3)}{(x_i-x_j)^5}+\frac{2h_i+2-h_j}{(x_i-x_j)^4}\frac{\partial}{\partial x_i}+\frac{(-1)}{(x_i-x_j)^3}\frac{\partial^2}{\partial x_i^2}+\frac{(-2)h_i}{(x_i-x_j)^4}\frac{\partial}{\partial x_j}
\end{aligned}
$$



This can be written as

$$
\begin{aligned}
&4tt_i\left[\mathcal{L}_{-2}\mathcal{L}_{-1}(x_i),\mathcal{L}_{-2}(x_j)\right] \qquad \text{(A.80)}\\
=\;& \frac{8tt_i}{(x_i-x_j)}\left(\sum_{m\neq j,\sim=1}^{n}\frac{(2x_m-x_i-x_j)h_m}{(x_m-x_i)^2(x_m-x_j)^2}-\frac{1}{(x_m-x_j)(x_m-x_i)}\frac{\partial}{\partial x_m}\right)\frac{\partial}{\partial x_i}\\
&+\frac{4tt_i}{(x_i-x_j)^2}\left(\sum_{m\neq j,\sim=1}^{n}\frac{h_m}{(x_m-x_i)^2}-\frac{1}{(x_m-x_i)}\frac{\partial}{\partial x_m}\right)\frac{\partial}{\partial x_i}\\
&+\frac{-8tt_i(2t_i-1)}{(x_i-x_j)^3}\left(\sum_{m\neq j,\sim=1}^{n}\frac{h_m}{(x_m-x_i)^2}-\frac{1}{(x_m-x_i)}\frac{\partial}{\partial x_m}\right)\\
&+\frac{(-2)tt_i(2t_i-1)(3t+10)}{(x_i-x_j)^5}+\frac{tt_i(16t_i-3t+2)}{(x_i-x_j)^4}\frac{\partial}{\partial x_i}\\
&+\frac{(-4)tt_i}{(x_i-x_j)^3}\frac{\partial^2}{\partial x_i^2}+\frac{(-8)tt_i(2t_i-1)}{(x_i-x_j)^4}\frac{\partial}{\partial x_j}
\end{aligned}
$$

Now let us compare the outcome of the commutation with the difference terms. It is given by

① the α-term $-\frac{\alpha}{(x_i-x_j)^2}\mathcal{D}_{-3}(x_i)$,

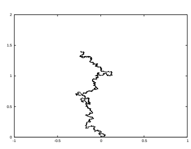
② the β- and γ-term $-\frac{1}{(x_i-x_j)^2}\left(\frac{\beta}{x_i-x_j}+\gamma\mathcal{L}_{-1}(x_i)\right)\mathcal{D}_{-2}(x_j)$.

Computing ①, gives

$$
\begin{aligned}
&-\frac{\alpha}{(x_i-x_j)^2}\mathcal{D}_{-3}(x_i)\\
=\;& -\frac{\alpha}{(x_i-x_j)^2}\left(\mathcal{L}_{-1}^3-4t_i\mathcal{L}_{-2}\mathcal{L}_{-1}+2t_i(2t_i-1)\mathcal{L}_{-3}\right)\\
=\;& -\frac{\alpha}{(x_i-x_j)^2}\frac{\partial^3}{\partial x_i^3}+4t_i\frac{\alpha}{(x_i-x_j)^2}\left(\sum_{l\neq\sim=1}^{n}\frac{h_l}{(x_l-x_i)^2}-\frac{1}{x_l-x_i}\frac{\partial}{\partial x_l}\right)\frac{\partial}{\partial x_i}\\
&-2t_i(2t_i-1)\frac{\alpha}{(x_i-x_j)^2}\left(\sum_{l\neq\sim=1}^{n}\frac{2h_l}{(x_l-x_i)^3}-\frac{1}{(x_l-x_i)^2}\frac{\partial}{\partial x_l}\right)
\end{aligned}
$$

which in comparison to the $\frac{\partial^3}{\partial x_i^3}$-terms of ② fixes α to

$$\alpha=-3t\,. \qquad \text{(A.81)}$$



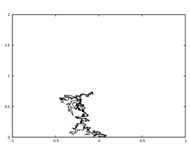
$$
\begin{aligned}
& -\frac{\alpha}{(x_i-x_j)^2}\mathcal{D}_{-3}(x_i) \\
= \quad & \frac{3t}{(x_i-x_j)^2}\frac{\partial^3}{\partial x_i^3} + \frac{(-12)tt_ih_j}{(x_i-x_j)^4}\frac{\partial}{\partial x_i} + \frac{(-12)tt_i}{(x_i-x_j)^3}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \\
& + \frac{(-12)tt_i(2t_i-1)h_j}{(x_i-x_j)^5} + \frac{(-6)tt_i(2t_i-1)}{(x_i-x_j)^4}\frac{\partial}{\partial x_j} \\
& + \frac{(-12)t_it}{(x_i-x_j)^2}\left(\sum_{m\neq\sim,j=1}^{n}\frac{h_m}{(x_m-x_i)^2} - \frac{1}{x_m-x_i}\frac{\partial}{\partial x_m}\right)\frac{\partial}{\partial x_i} \\
& + \frac{6tt_i(2t_i-1)}{(x_i-x_j)^2}\left(\sum_{m\neq\sim,j=1}^{n}\frac{2h_m}{(x_m-x_i)^3} - \frac{1}{(x_m-x_i)^2}\frac{\partial}{\partial x_m}\right)
\end{aligned}
$$

The β- and γ-term ② can be computed to be

$$
\begin{aligned}
& -\frac{1}{(x_i-x_j)^2}\left(\frac{\beta}{x_i-x_j} + \gamma\mathcal{L}_{-1}(x_i)\right)\mathcal{D}_{-2}(x_j) \\
= \quad & -\frac{1}{(x_i-x_j)^2}\left(\frac{\beta}{x_i-x_j} + \gamma\mathcal{L}_{-1}(x_i)\right)\left(\mathcal{L}_{-1}^2 - t\mathcal{L}_{-2}\right) \qquad \text{(A.82)} \\
= \quad & -\frac{\beta}{(x_i-x_j)^3}\frac{\partial^2}{\partial x_j^2} - \frac{\gamma}{(x_i-x_j)^2}\frac{\partial^2}{\partial x_j^2}\frac{\partial}{\partial x_i} \\
& + t\frac{\beta}{(x_i-x_j)^3}\left(\sum_{k\neq j=1}^{n}\frac{h_k}{(x_k-x_j)^2} - \frac{1}{x_k-x_j}\frac{\partial}{\partial x_k}\right) \\
& - t\frac{\gamma}{(x_i-x_j)^2}\left(\frac{2h_i}{(x_i-x_j)^3} - \frac{1}{(x_i-x_j)^2}\frac{\partial}{\partial x_i}\right) \\
& + t\frac{\gamma}{(x_i-x_j)^2}\left(\sum_{k\neq j=1}^{n}\frac{h_k}{(x_k-x_j)^2} - \frac{1}{x_k-x_j}\frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_i}
\end{aligned}
$$

such that comparing them to the $\frac{\partial^2}{\partial x_j^2}$-terms of ④ and the $\frac{\partial^2}{\partial x_j^2}\frac{\partial}{\partial x_i}$-terms of ③ fixes β and γ to

$$
\beta = 8t_i(2t_i-1) \quad \text{und} \quad \gamma = 8t_i\,. \qquad \text{(A.83)}
$$



$$
\begin{aligned}
&-\frac{1}{(x_i-x_j)^2}\left(\frac{\beta}{x_i-x_j}+\gamma\mathcal{L}_{-1}(x_i)\right)\mathcal{D}_{-2}(x_j)\\
=\;& \frac{(-8)t_i(2t_i-1)}{(x_i-x_j)^3}\frac{\partial^2}{\partial x_j^2}+\frac{(-8)t_i}{(x_i-x_i)^2}\frac{\partial^2}{\partial x_j^2}\frac{\partial}{\partial x_i}\\
&+\frac{8tt_i(2t_i-3)h_i}{(x_i-x_j)^5}+\frac{8tt_i}{(x_i-x_j)^4}\frac{\partial}{\partial x_i}+\frac{(-8)tt_i}{(x_i-x_j)^3}\frac{\partial^2}{\partial x_i^2}\\
&+\frac{8tt_i(2t_i-1)}{(x_i-x_j)^3}\left(\sum_{m\neq j,\sim=1}^{n}\frac{h_m}{(x_m-x_j)^2}-\frac{1}{x_m-x_j}\frac{\partial}{\partial x_m}\right)\\
&+\frac{8tt_i}{(x_i-x_j)^2}\left(\sum_{m\neq j,\sim=1}^{n}\frac{h_m}{(x_m-x_j)^2}-\frac{1}{x_m-x_j}\frac{\partial}{\partial x_m}\right)\frac{\partial}{\partial x_i}
\end{aligned}
$$

The result can be interpreted as follows: with our choices of α, β and γ, all terms cancel, yielding only non-vanishing terms proportional to ∂_{x_i} and a constant:

$$
\begin{aligned}
0 \;=\;& [\mathcal{D}_{-3}(x_i),\mathcal{D}_{-2}(x_j)]\\
&+\frac{3t}{(x_i-x_j)^2}\mathcal{D}_{-3}(x_i)-\frac{8t_i}{(x_i-x_j)^2}\left(\frac{h_i}{x_i-x_j}+\frac{\partial}{\partial x_i}\right)\mathcal{D}_{-2}(x_j)\\
=\;& \frac{12(1-tt_i)(t-t_i)}{(x_i-x_j)^4}\left(\frac{2h_i}{x_i-x_j}+\frac{\partial}{\partial x_i}\right). \qquad \text{(A.84)}
\end{aligned}
$$

This is true for $t=t_i$ or $t=\frac{1}{t_i}$, which is just another way to state the κ relation since $t=4/\kappa$. For the sake of completeness, we state the commutation relation for the appropriately normalized differential operators (modulo $\kappa/2$:

$$
\begin{aligned}
0 \;=\;& [\mathcal{D}_{-3}(x_i),\mathcal{D}_{-2}(x_j)]-\frac{2}{(x_i-x_j)^2}\left[8\left(\frac{h_i}{x_i-x_j}+\frac{\partial}{\partial x_i}\right)\mathcal{D}_{-2}(x_j)-3\mathcal{D}_{-3}(x_i)\right]\\
=\;& -\frac{12(1-\tilde{\kappa}\kappa_j)(\tilde{\kappa}-\kappa_j)}{\tilde{\kappa}\kappa_j(x_i-x_j)^4}\left(\frac{2h_i}{x_i-x_j}+\frac{\partial}{\partial x_i}\right) \qquad \text{(A.85)}
\end{aligned}
$$

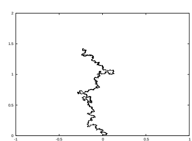
A.3.2 Result I

With the computations above, we have shown that, that the modified commutation relation acting on (A.63) is:

$$
\left[\mathcal{D}_{-3}(x_t^i),\mathcal{D}_{-2}(x_t^j)\right]-\left(\frac{4}{(x_t^i-x_t^j)^2}F(x_t^i;x_t^j)\mathcal{D}_{-2}(x_t^j)-\frac{6}{(x_t^i-x_t^j)^2}\mathcal{D}_{-3}(x_t^i)\right), \qquad \text{(A.86)}
$$

in which $F(x_t^i;x_t^j)$ is given by:

$$
F(x_t^i;x_t^j):=-\frac{4}{(x_t^i-x_t^j)}\left(h_i+(x_i-x_j)\frac{\partial}{\partial x_t^i}\right). \qquad \text{(A.87)}
$$



The terms in (A.86) acting on (A.63) give us back the κ relation with an additional factor $G(x_t^i; x_t^j)$:

$$K(\kappa_i, \kappa_j) \cdot G(x_t^i; x_t^j) \tilde{Z}_{\text{b.c.}} = 0\,, \tag{A.88}$$

which is given by

$$G(x_t^i; x_t^j) := 4\left(\frac{2h_i}{x_t^i - x_t^j} + \frac{\partial}{\partial x_t^i}\right) = 8\left(\frac{\beta_{\{0\}} L_0}{x_t^i - x_t^j} + \beta_{\{1\}} L_{-1}\right)\,. \tag{A.89}$$

A.3.3 Two Short-Distance Limits

Of course, the same computations can be done for two short-distance limits.

$$\begin{aligned}
&[\mathcal{D}_{-3}(x_i), \mathcal{D}_{-3}(x_j)] \\
= \quad & \left[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-1}^3(x_j)\right] \\
& -4t_j\left[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-2}(x_j)\mathcal{L}_{-1}(x_j)\right] + 4t_i\left[\mathcal{L}_{-1}^3(x_j), \mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i)\right] \\
& +2t_jh_j\left[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-3}(x_j)\right] - 2t_ih_i\left[\mathcal{L}_{-1}^3(x_j), \mathcal{L}_{-3}(x_i)\right] \\
& +16t_it_j\left[\mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i), \mathcal{L}_{-2}(x_j)\mathcal{L}_{-1}(x_j)\right] + 4t_it_jh_ih_j\left[\mathcal{L}_{-3}(x_i), \mathcal{L}_{-3}(x_j)\right] \\
& -8t_it_jh_j\left[\mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i), \mathcal{L}_{-3}(x_j)\right] + 8t_it_jh_i\left[\mathcal{L}_{-2}(x_j)\mathcal{L}_{-1}(x_j), \mathcal{L}_{-3}(x_i)\right]
\end{aligned} \tag{A.90}$$

In the following, we will have a look at the following terms separately:

① coefficient proportional to $[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-1}^3(x_j)]$

② coefficient proportional to $\left[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-2}(x_j)\mathcal{L}_{-1}(x_j)\right]$

③ coefficient proportional to $[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-3}(x_j)]$

④ coefficient proportional to $[\mathcal{L}_{-3}(x_i), \mathcal{L}_{-3}(x_j)]$

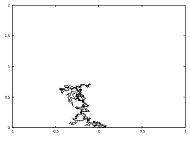
⑤ coefficient proportional to $[\mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i), \mathcal{L}_{-3}(x_j)]$

⑥ coefficient proportional to $[\mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i), \mathcal{L}_{-2}(x_j)\mathcal{L}_{-1}(x_j)]$ which will be split up into

$$\begin{aligned}
&\left(\mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i)\mathcal{L}_{-2}(x_j)\right)\mathcal{L}_{-1}(x_j) \\
&\text{and } -\left(\mathcal{L}_{-2}(x_j)\mathcal{L}_{-1}(x_j)\mathcal{L}_{-2}(x_i)\right)\mathcal{L}_{-1}(x_i)
\end{aligned}$$

The first term ① is trivial:

$$[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-1}^3(x_j)] = 0 \tag{A.91}$$



The second term ② is more complicated already, starting with

$$
\begin{aligned}
& \left[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-2}(x_j)\mathcal{L}_{-1}(x_j)\right] \\
= & \left[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-2}(x_j)\right] \mathcal{L}_{-1}(x_j) \qquad \text{(A.92)} \\
= & \frac{(-24)h_i}{(x_i-x_j)^5}\frac{\partial}{\partial x_j} + \frac{12(3t_i-1)}{(x_i-x_j)^4}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \\
& + \frac{(-12)t_i}{(x_i-x_j)^3}\frac{\partial^2}{\partial x_i^2}\frac{\partial}{\partial x_j} + \frac{3}{(x_i-x_j)^2}\frac{\partial^3}{\partial x_i^3}\frac{\partial}{\partial x_j} \qquad \text{(A.93)}
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
-4t_j\left[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-2}(x_j)\mathcal{L}_{-1}(x_j)\right] = & \frac{96t_jh_i}{(x_i-x_j)^5}\frac{\partial}{\partial x_j} + \frac{(-48)t_j(3t_i-1)}{(x_i-x_j)^4}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \\
& + \frac{48t_it_j}{(x_i-x_j)^3}\frac{\partial^2}{\partial x_i^2}\frac{\partial}{\partial x_j} + \frac{(-12)t_j}{(x_i-x_j)^2}\frac{\partial^3}{\partial x_i^3}\frac{\partial}{\partial x_j}
\end{aligned}
$$

and, as well:

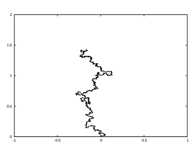
$$
\begin{aligned}
4t_i\left[\mathcal{L}_{-1}^3(x_j), \mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i)\right] = & \frac{96t_ih_j}{(x_i-x_j)^5}\frac{\partial}{\partial x_i} + \frac{48t_i(3t_j-1)}{(x_i-x_j)^4}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} \\
& + \frac{48t_it_j}{(x_i-x_j)^3}\frac{\partial^2}{\partial x_j^2}\frac{\partial}{\partial x_i} + \frac{12t_i}{(x_i-x_j)^2}\frac{\partial^3}{\partial x_j^3}\frac{\partial}{\partial x_i}
\end{aligned}
$$

The next term ③ is given by

$$
\begin{aligned}
& [\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-3}(x_j)] \\
= & \left[\frac{\partial^3}{\partial x_i^3}, \sum_{k\neq j}^{n} \frac{2h_k}{(x_k-x_j)^3} - \frac{1}{(x_k-x_j)^2}\frac{\partial}{\partial x_k}\right] \qquad \text{(A.94)} \\
= & \frac{(-120)(2t_i-1)}{(x_i-x_j)^6} + \frac{48(3t_i-1)}{(x_i-x_j)^5}\frac{\partial}{\partial x_i} \\
& + \frac{(-36)t_i}{(x_i-x_j)^4}\frac{\partial^2}{\partial x_i^2} + \frac{6}{(x_i-x_j)^3}\frac{\partial^3}{\partial x_i^3}
\end{aligned}
$$

Thus we have:

$$
\begin{aligned}
2t_j(2t_j-1)\left[\mathcal{L}_{-1}^3(x_i), \mathcal{L}_{-3}(x_j)\right] = & \frac{(-240)t_jh_jh_i}{(x_i-x_j)^6} + \frac{96t_jh_j(3t_i-1)}{(x_i-x_j)^5}\frac{\partial}{\partial x_i} \\
& + \frac{(-72)t_it_jh_j}{(x_i-x_j)^4}\frac{\partial^2}{\partial x_i^2} + \frac{12t_jh_j}{(x_i-x_j)^3}\frac{\partial^3}{\partial x_i^3}
\end{aligned}
$$



and, in the same way we get:

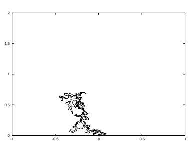
$$-2t_i(2t_i-1)\left[\mathcal{L}_{-1}^3(x_j),\mathcal{L}_{-3}(x_i)\right] = \frac{240t_ih_ih_j}{(x_i-x_j)^6}+\frac{96t_ih_i(3t_j-1)}{(x_i-x_j)^5}\frac{\partial}{\partial x_j} +\frac{72t_jt_ih_i}{(x_i-x_j)^4}\frac{\partial^2}{\partial x_j^2}+\frac{12t_ih_i}{(x_i-x_j)^3}\frac{\partial^3}{\partial x_j^3}$$

The term ④ is again a rather easy one, starting with

$$\begin{aligned}
&[\mathcal{L}_{-3}(x_i),\mathcal{L}_{-3}(x_j)] \qquad \text{(A.95)}\\
&= \left[\sum_{l\neq i}^{n}\frac{2h_l}{(x_l-x_i)^3}-\frac{1}{(x_l-x_i)^2}\frac{\partial}{\partial x_l},\sum_{k\neq j}^{n}\frac{2h_k}{(x_k-x_j)^3}-\frac{1}{(x_k-x_j)^2}\frac{\partial}{\partial x_k}\right]\\
&= \frac{12(t_j-t_i)}{(x_j-x_i)^6}+\frac{2}{(x_i-x_j)^5}\left(\frac{\partial}{\partial x_j}+\frac{\partial}{\partial x_i}\right)\\
&\quad +\frac{6}{(x_i-x_j)}\sum_{m\neq i,j}^{n}\left(\frac{(2x_m-x_j-x_i)2h_m}{(x_m-x_j)^3(x_m-x_i)^3}-\frac{1}{(x_m-x_i)^2}\frac{1}{(x_m-x_j)^2}\frac{\partial}{\partial x_m}\right)
\end{aligned}$$

Thus we have

$$\begin{aligned}
&4t_it_j(2t_i-1)(2t_j-1)\left[\mathcal{L}_{-3}(x_i),\mathcal{L}_{-3}(x_j)\right] \qquad \text{(A.96)}\\
&= \frac{48t_it_jh_ih_j(t_j-t_i)}{(x_i-x_j)^6}+\frac{8t_it_jh_ih_j}{(x_i-x_j)^5}\left(\frac{\partial}{\partial x_j}+\frac{\partial}{\partial x_i}\right)\\
&\quad +\frac{24t_it_jh_ih_j}{(x_i-x_j)}\sum_{m\neq i,j}^{n}\left(\frac{(2x_m-x_j-x_i)2h_m}{(x_m-x_j)^3(x_m-x_i)^3}-\frac{1}{(x_m-x_i)^2}\frac{1}{(x_m-x_j)^2}\frac{\partial}{\partial x_m}\right)
\end{aligned}$$

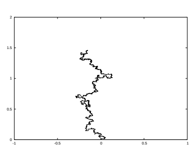


The following terms ⑤ are the second most complicated. Therefore, they are split up. We compute

$$
\begin{aligned}
&[\mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i), \mathcal{L}_{-3}(x_j)] \\
= \quad & \left[\sum_{l\neq i}^{n}\left(\frac{h_l}{(x_l-x_i)^2}-\frac{1}{(x_l-x_i)}\frac{\partial}{\partial x_l}\right)\frac{\partial}{\partial x_i}, \sum_{k\neq j}^{n}\frac{2h_k}{(x_k-x_j)^3}-\frac{1}{(x_k-x_j)^2}\frac{\partial}{\partial x_k}\right] \\
= \quad & \frac{(-6)h_i(h_j+4)}{(x_i-x_j)^6} \\
& +\frac{6}{(x_i-x_j)^5}\left(2t_i\frac{\partial}{\partial x_i}-h_i\frac{\partial}{\partial x_j}\right)+\frac{1}{(x_i-x_j)^4}\left(\frac{\partial}{\partial x_j}-2\frac{\partial}{\partial x_i}\right)\frac{\partial}{\partial x_i} \\
& +\frac{(-2)}{(x_i-x_j)^3}\left(\sum_{m\neq i,j}^{n}\frac{h_m}{(x_m-x_i)^2}-\frac{1}{(x_m-x_i)}\frac{\partial}{\partial x_m}\right)\left(\frac{3h_i}{(x_i-x_j)}-\frac{\partial}{\partial x_i}\right) \\
& +\frac{2}{(x_i-x_j)}\sum_{m\neq i,j}^{n}\frac{1}{(x_m-x_i)}\left(\frac{3h_m}{(x_m-x_j)^3}-\frac{1}{(x_m-x_j)^2}\frac{\partial}{\partial x_m}\right)\frac{\partial}{\partial x_i} \\
& +\frac{(-1)}{(x_i-x_j)^2}\sum_{m\neq i,j}^{n}\frac{2x_j-x_i-x_m}{(x_m-x_j)^2}\left(\frac{2h_m}{(x_m-x_i)^2}-\frac{1}{(x_m-x_i)}\frac{\partial}{\partial x_m}\right)\frac{\partial}{\partial x_i}
\end{aligned}
$$

Together with the coefficients, this yields

$$
\begin{aligned}
&-8t_it_j(2t_j-1)\,[\mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i), \mathcal{L}_{-3}(x_j)] \qquad \text{(A.97)}\\
= \quad & \frac{48t_it_jh_jh_i(h_j+4)}{(x_i-x_j)^6}+\frac{(-48)t_it_jh_j}{(x_i-x_j)^5}\left(2t_i\frac{\partial}{\partial x_i}-h_i\frac{\partial}{\partial x_j}\right) \\
& +\frac{(-8)t_it_jh_j}{(x_i-x_j)^4}\left(\frac{\partial}{\partial x_j}-2\frac{\partial}{\partial x_i}\right)\frac{\partial}{\partial x_i} \\
& +\frac{16t_it_jh_j}{(x_i-x_j)^3}\left(\sum_{m\neq i,j}^{n}\frac{h_m}{(x_m-x_i)^2}-\frac{1}{(x_m-x_i)}\frac{\partial}{\partial x_m}\right)\left(\frac{3h_i}{(x_i-x_j)}-\frac{\partial}{\partial x_i}\right) \\
& +\frac{-16t_it_jh_j}{(x_i-x_j)}\sum_{m\neq i,j}^{n}\frac{1}{(x_m-x_i)}\left(\frac{3h_m}{(x_m-x_j)^3}-\frac{1}{(x_m-x_j)^2}\frac{\partial}{\partial x_m}\right)\frac{\partial}{\partial x_i} \\
& +\frac{8t_it_jh_j}{(x_i-x_j)^2}\sum_{m\neq i,j}^{n}\frac{2x_j-x_i-x_m}{(x_m-x_j)^2}\left(\frac{2h_m}{(x_m-x_i)^2}-\frac{1}{(x_m-x_i)}\frac{\partial}{\partial x_m}\right)\frac{\partial}{\partial x_i}
\end{aligned}
$$



Due to the $i \leftrightarrow j$-symmetry, we get analogously get for the second term:

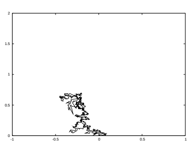
$$
\begin{aligned}
&8t_it_j(2t_i-1)\left[\mathcal{L}_{-2}(x_j)\mathcal{L}_{-1}(x_j),\mathcal{L}_{-3}(x_i)\right] \qquad \text{(A.98)}\\
=\ & \frac{-48t_it_jh_ih_j(h_i+4)}{(x_j-x_i)^6}+\frac{-48t_it_jh_i}{(x_i-x_j)^5}\left(2t_j\frac{\partial}{\partial x_j}-h_j\frac{\partial}{\partial x_i}\right)\\
&+\frac{8t_it_jh_i}{(x_i-x_j)^4}\left(\frac{\partial}{\partial x_i}-2\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_j}\\
&+\frac{(-8)t_it_jh_i}{(x_i-x_j)^3}\left(\sum_{m\neq i,j}^{n}\frac{h_m}{(x_m-x_j)^2}-\frac{1}{(x_m-x_j)}\frac{\partial}{\partial x_m}\right)\left(\frac{3h_j}{(x_i-x_j)}+\frac{\partial}{\partial x_j}\right)\\
&+\frac{(-16)t_it_jh_i}{(x_i-x_j)}\sum_{m\neq i,j}^{n}\frac{1}{(x_m-x_j)}\left(\frac{3h_m}{(x_m-x_i)^3}-\frac{1}{(x_m-x_i)^2}\frac{\partial}{\partial x_m}\right)\frac{\partial}{\partial x_j}\\
&+\frac{(-8)t_it_jh_i}{(x_i-x_j)^2}\sum_{m\neq i,j}^{n}\frac{2x_i-x_j-x_m}{(x_m-x_i)^2}\left(\frac{2h_m}{(x_m-x_j)^2}-\frac{1}{(x_m-x_j)}\frac{\partial}{\partial x_m}\right)\frac{\partial}{\partial x_j}
\end{aligned}
$$

The most complicated computation arises when considering ⑥:

$$
\begin{aligned}
&\left[\mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i),\mathcal{L}_{-2}(x_j)\mathcal{L}_{-1}(x_j)\right] \qquad \text{(A.99)}\\
=\ & \left(\mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i)\mathcal{L}_{-2}(x_j)\right)\mathcal{L}_{-1}(x_j)-\left(\mathcal{L}_{-2}(x_j)\mathcal{L}_{-1}(x_j)\mathcal{L}_{-2}(x_i)\right)\mathcal{L}_{-1}(x_i)\,.
\end{aligned}
$$

Therefore we consider and separately:, starting with

$$
\begin{aligned}
&\mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i)\mathcal{L}_{-2}(x_j) \qquad \text{(A.100)}\\
=\ & \sum_{l\neq i}^{n}\sum_{k\neq j}^{n}\left(\frac{h_l}{(x_l-x_i)^2}-\frac{1}{(x_l-x_i)}\frac{\partial}{\partial x_l}\right)\frac{\partial}{\partial x_i}\left(\frac{h_k}{(x_k-x_j)^2}-\frac{1}{(x_k-x_j)}\frac{\partial}{\partial x_k}\right)\\
=\ & \frac{(-1)}{(x_i-x_j)^2}\left(\sum_{m\neq i,j}^{n}\frac{h_m}{(x_m-x_i)^2}-\frac{1}{(x_m-x_i)}\frac{\partial}{\partial x_m}\right)\left(\frac{2h_i}{(x_i-x_j)}-\frac{\partial}{\partial x_i}\right)\\
&+\frac{(-2)h_i(h_j+3)}{(x_i-x_j)^5}+\frac{1}{(x_i-x_j)^4}\left((h_j+2+2h_i)\frac{\partial}{\partial x_i}+(-2)h_i\frac{\partial}{\partial x_j}\right)\\
&+\frac{1}{(x_i-x_j)^3}\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_i}\right)\frac{\partial}{\partial x_i}\\
&+\frac{1}{(x_i-x_j)}\sum_{m\neq i,j}^{n}\frac{1}{(x_m-x_i)}\left(\frac{2h_m}{(x_m-x_j)^2}-\frac{1}{(x_m-x_j)}\frac{\partial}{\partial x_m}\right)\frac{\partial}{\partial x_i} \qquad \text{(A.101)}
\end{aligned}
$$



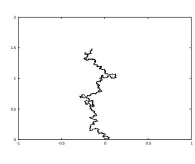
And analogously

$$
\begin{aligned}
&\mathcal{L}_{-2}(x_j)\mathcal{L}_{-1}(x_j)\mathcal{L}_{-2}(x_i) \qquad \text{(A.102)}\\
=\;& \frac{1}{(x_i-x_j)^2}\left(\sum_{m\neq i,j}^{n}\frac{h_m}{(x_m-x_j)^2}-\frac{1}{(x_m-x_j)}\frac{\partial}{\partial x_m}\right)\left(\frac{2h_j}{(x_i-x_j)}+\frac{\partial}{\partial x_j}\right)\\
&+\frac{2h_j(h_i+3)}{(x_i-x_j)^5}+\frac{1}{(x_i-x_j)^4}\left((h_i+2+2h_j)\frac{\partial}{\partial x_j}+(-2)h_j\frac{\partial}{\partial x_i}\right)\\
&+\frac{1}{(x_i-x_j)^3}\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_i}\right)\frac{\partial}{\partial x_j}\\
&+\frac{(-1)}{(x_i-x_j)}\sum_{m\neq i,j}^{n}\frac{1}{(x_m-x_j)}\left(\frac{2h_m}{(x_m-x_i)^2}-\frac{1}{(x_m-x_i)}\frac{\partial}{\partial x_m}\right)\frac{\partial}{\partial x_j}
\end{aligned}
\qquad \text{(A.103)}
$$

Putting things back together again, we get

$$
\begin{aligned}
&16t_it_j\left[\mathcal{L}_{-2}(x_i)\mathcal{L}_{-1}(x_i),\mathcal{L}_{-2}(x_j)\mathcal{L}_{-1}(x_j)\right]\\
=\;& \frac{(-16)t_it_j}{(x_i-x_j)^2}\left(\sum_{m\neq i,j}^{n}\frac{h_m}{(x_m-x_i)^2}-\frac{1}{(x_m-x_i)}\frac{\partial}{\partial x_m}\right)\left(\frac{2h_i}{(x_i-x_j)}-\frac{\partial}{\partial x_i}\right)\frac{\partial}{\partial x_j}\\
&+\frac{(-32)t_it_jh_i(h_j+3)}{(x_i-x_j)^5}\frac{\partial}{\partial x_j}+\frac{16t_it_j}{(x_i-x_j)^4}\left((h_j+2+2h_i)\frac{\partial}{\partial x_i}+(-2)h_i\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_j}\\
&+\frac{16t_it_j}{(x_i-x_j)^3}\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_i}\right)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\\
&+\frac{16t_it_j}{(x_i-x_j)}\sum_{m\neq i,j}^{n}\frac{1}{(x_m-x_i)}\left(\frac{2h_m}{(x_m-x_j)^2}-\frac{1}{(x_m-x_j)}\frac{\partial}{\partial x_m}\right)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\\
&\frac{(-16)t_it_j}{(x_i-x_j)^2}\left(\sum_{m\neq i,j}^{n}\frac{h_m}{(x_m-x_j)^2}-\frac{1}{(x_m-x_j)}\frac{\partial}{\partial x_m}\right)\left(\frac{2h_j}{(x_i-x_j)}+\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_i}\\
&+\frac{(-32)t_it_jh_j(h_i+3)}{(x_i-x_j)^5}\frac{\partial}{\partial x_i}+\frac{-16t_it_j}{(x_i-x_j)^4}\left((h_i+2+2h_j)\frac{\partial}{\partial x_j}+(-2)h_j\frac{\partial}{\partial x_i}\right)\frac{\partial}{\partial x_i}\\
&+\frac{(-16)t_it_j}{(x_i-x_j)^3}\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_i}\right)\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\\
&+\frac{16t_it_j}{(x_i-x_j)}\sum_{m\neq i,j}^{n}\frac{1}{(x_m-x_j)}\left(\frac{2h_m}{(x_m-x_i)^2}-\frac{1}{(x_m-x_i)}\frac{\partial}{\partial x_m}\right)\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}
\end{aligned}
\qquad \text{(A.104)}
$$

Now let us compare the outcome of the commutation with the difference terms. It is given by



① the $\mathcal{D}_{-3}(x_i)$ - term: $\left(\frac{\alpha_i}{(x_i-x_j)} + \beta_i \mathcal{L}_{-1}(x_j)\right) \mathcal{D}_{-3}(x_i)$

② and the $\mathcal{D}_{-3}(x_j)$ - term: $\left(\frac{\alpha_j}{(x_j-x_i)} + \beta_j \mathcal{L}_{-1}(x_i)\right) \mathcal{D}_{-3}(x_j)$.

Term ① is computed to be

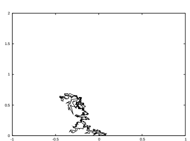
$$\left(\frac{\alpha_i}{(x_i-x_j)} + \beta_i \mathcal{L}_{-1}(x_j)\right) \mathcal{D}_{-3}(x_i) \tag{A.105}$$

$$= \frac{\alpha_i}{(x_i-x_j)} \mathcal{D}_{-3}(x_i) + \beta_i \frac{\partial}{\partial x_j} \mathcal{D}_{-3}(x_i) + \beta_i \mathcal{D}_{-3}(x_i) \frac{\partial}{\partial x_j} \tag{A.106}$$

$$\begin{aligned}
= \;& \frac{\alpha_i}{(x_i-x_j)} \left(\frac{\partial^3}{\partial x_i^3} - 4t_i \left(\frac{h_j}{(x_j-x_i)^2} - \frac{1}{(x_j-x_i)} \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_i} \right. \\
& \left. +2t_i(2t_i-1) \left(\frac{2h_j}{(x_j-x_i)^3} - \frac{1}{(x_j-x_i)^2} \frac{\partial}{\partial x_j} \right) \right) \\
& +\beta_i \left(\frac{\partial^3}{\partial x_i^3} \frac{\partial}{\partial x_j} - 4t_i \left(\frac{(-2)h_j}{(x_j-x_i)^3} - \frac{(-1)}{(x_j-x_i)^2} \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_i} \right. \\
& \left. +2t_i(2t_i-1) \left(\frac{2(-3)h_j}{(x_j-x_i)^4} - \frac{(-2)}{(x_j-x_i)^3} \frac{\partial}{\partial x_j} \right) \right) \\
& +\beta_i \left(-4t_i \left(\frac{h_j}{(x_j-x_i)^2} - \frac{1}{(x_j-x_i)} \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right. \\
& \left. +2t_i(2t_i-1) \left(\frac{2h_j}{(x_j-x_i)^3} - \frac{1}{(x_j-x_i)^2} \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_j} \right) \\
& + \frac{\alpha_i}{(x_i-x_j)} \left(-4t_i \sum_{m\neq i,j}^{n} \left(\frac{h_m}{(x_m-x_i)^2} - \frac{1}{(x_m-x_i)} \frac{\partial}{\partial x_m} \right) \frac{\partial}{\partial x_i} \right. \\
& \left. +2t_i(2t_i-1) \sum_{m\neq i,j}^{n} \left(\frac{2h_m}{(x_m-x_i)^3} - \frac{1}{(x_m-x_i)^2} \frac{\partial}{\partial x_m} \right) \right) \\
& +\beta_i \left(-4t_i \sum_{m\neq i,j}^{n} \left(\frac{h_m}{(x_m-x_i)^2} - \frac{1}{(x_m-x_i)} \frac{\partial}{\partial x_m} \right) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right. \\
& \left. +2t_i(2t_i-1) \sum_{m\neq i,j}^{n} \left(\frac{2h_m}{(x_m-x_i)^3} - \frac{1}{(x_m-x_i)^2} \frac{\partial}{\partial x_m} \right) \frac{\partial}{\partial x_j} \right)
\end{aligned} \tag{A.107}$$

Comparison with ② gives

$$\beta_i = (-12)t_j \tag{A.108}$$



and with ③ gives

$$\alpha_i = 12t_j(2t_j - 1) = 12t_j h_j \tag{A.109}$$

which yields

$$\begin{aligned}
= \quad & \frac{12t_j h_j}{(x_i - x_j)} \frac{\partial^3}{\partial x_i^3} + (-12)t_j \frac{\partial^3}{\partial x_i^3} \frac{\partial}{\partial x_j} \\
& +(-24)t_j t_i \left(2 \left(\frac{h_j(h_j-2)}{(x_i - x_j)^3} + \frac{(h_j - 1)}{(x_i - x_j)^2} \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_i} \right. \\
& \left. +h_i \left(\frac{2h_j(h_j - 3)}{(x_i - x_j)^4} + \frac{(h_j - 2)}{(x_i - x_j)^3} \frac{\partial}{\partial x_j} \right) \right) \\
& +24 t_i t_j \left(2 \left(\frac{h_j}{(x_i - x_j)^2} + \frac{1}{(x_i - x_j)} \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right. \\
& \left. +2h_i \left(\frac{2h_j}{(x_i - x_j)^3} + \frac{1}{(x_i - x_j)^2} \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_j} \right) \\
& + \frac{(-24)t_i t_j h_j}{(x_i - x_j)} \left(2 \sum_{m \neq i,j}^{n} \left(\frac{h_m}{(x_m - x_i)^2} - \frac{1}{(x_m - x_i)} \frac{\partial}{\partial x_m} \right) \frac{\partial}{\partial x_i} \right. \\
& \left. -h_i \sum_{m \neq i,j}^{n} \left(\frac{2h_m}{(x_m - x_i)^3} - \frac{1}{(x_m - x_i)^2} \frac{\partial}{\partial x_m} \right) \right) \\
& +24 t_i t_j \left(2 \sum_{m \neq i,j}^{n} \left(\frac{h_m}{(x_m - x_i)^2} - \frac{1}{(x_m - x_i)} \frac{\partial}{\partial x_m} \right) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right. \\
& \left. -h_i \sum_{m \neq i,j}^{n} \left(\frac{2h_m}{(x_m - x_i)^3} - \frac{1}{(x_m - x_i)^2} \frac{\partial}{\partial x_m} \right) \frac{\partial}{\partial x_j} \right)
\end{aligned}$$

from the requirement, that the κ-condition should be obtained.

In complete analogy, term ① is given by

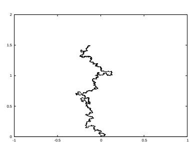
$$\left(\frac{\alpha_j}{(x_j - x_i)} + \beta_j \mathcal{L}_{-1}(x_i) \right) \mathcal{D}_{-3}(x_j) \tag{A.110}$$

such that we can exploit the $i \leftrightarrow j$-symmetry to arrive at

$$\alpha_j = -12t_i(2t_i - 1) = -12t_i h_i \tag{A.111}$$

and

$$\beta_j = 12t_i \tag{A.112}$$



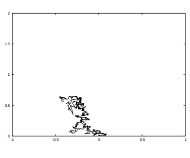
which yields

$$
\begin{aligned}
= \quad & \frac{12t_i h_i}{(x_i - x_j)} \frac{\partial^3}{\partial x_j^3} + 12t_i \frac{\partial^3}{\partial x_j^3} \frac{\partial}{\partial x_i} \\
& +(-24)t_j t_i \left(2 \left(\frac{h_i(h_i-2)}{(x_i-x_j)^3} - \frac{h_i - 1}{(x_i-x_j)^2} \frac{\partial}{\partial x_i} \right) \frac{\partial}{\partial x_j} \right. \\
& \left. -h_j \left(\frac{2h_i(h_i-3)}{(x_i-x_j)^4} - \frac{(h_i-2)}{(x_i-x_j)^3} \frac{\partial}{\partial x_i} \right) \right) \\
& +(-24)t_j t_i \left(2 \left(\frac{h_i}{(x_i-x_j)^2} - \frac{1}{(x_i-x_j)} \frac{\partial}{\partial x_i} \right) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} \right. \\
& \left. -h_j \left(\frac{2h_i}{(x_i-x_j)^3} - \frac{1}{(x_i-x_j)^2} \frac{\partial}{\partial x_i} \right) \frac{\partial}{\partial x_i} \right) \\
& + \frac{12t_i h_i}{(x_i-x_j)} \left(-4t_j \sum_{m \neq i,j}^{n} \left(\frac{h_m}{(x_m-x_j)^2} - \frac{1}{(x_m-x_j)} \frac{\partial}{\partial x_m} \right) \frac{\partial}{\partial x_j} \right. \\
& \left. +2t_j h_j \sum_{m \neq i,j}^{n} \left(\frac{2h_m}{(x_m-x_j)^3} - \frac{1}{(x_m-x_j)^2} \frac{\partial}{\partial x_m} \right) \right) \\
& +12t_i \left(-4t_j \sum_{m \neq i,j}^{n} \left(\frac{h_m}{(x_m-x_j)^2} - \frac{1}{(x_m-x_j)} \frac{\partial}{\partial x_m} \right) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} \right. \\
& \left. +2t_j h_j \sum_{m \neq i,j}^{n} \left(\frac{2h_m}{(x_m-x_j)^3} - \frac{1}{(x_m-x_j)^2} \frac{\partial}{\partial x_m} \right) \frac{\partial}{\partial x_i} \right)
\end{aligned}
$$

Putting all the terms together (which is a quite lengthy procedure!), the result can be interpreted as follows: with our choices of α_i, α_j and $\beta_i, \beta_,$, all terms cancel, yielding only non-vanishing terms proportional to $\partial_{x_i}, \partial_{x_j}$ and a constant:

$$
\begin{aligned}
0 \quad = \quad & [\mathcal{D}_{-3}(x_i), \mathcal{D}_{-3}(x_j)] \qquad \text{(A.113)} \\
& - \frac{12}{(x_i-x_j)^2} \left[t_i \left(\frac{h_i}{(x_i-x_j)} + \frac{\partial}{\partial x_i} \right) \mathcal{D}_{-3}(x_j) - t_j \left(\frac{h_j}{(x_j-x_i)} + \frac{\partial}{\partial x_j} \right) \mathcal{D}_{-3}(x_i) \right] \\
= \quad & \frac{48(t_i-t_j)(1-t_i t_j)}{(x_i-x_j)^4} \left(\frac{5h_i h_j}{(x_i-x_j)^2} - \frac{2h_i}{(x_i-x_j)} \frac{\partial}{\partial x_j} + \frac{2h_j}{(x_i-x_j)} \frac{\partial}{\partial x_i} + \frac{\partial^2}{\partial x_i \partial x_j} \right)
\end{aligned}
$$

This is true for $t = t_i$ or $t = \frac{1}{t_i}$, which is just another way to state the κ relation since $t = 4/\kappa$. For the sake of completeness, we state the commutation relation for



the appropriately normalized differential operators (modulo $\kappa/2$:

$$\begin{aligned} 0 &= [\mathcal{D}_{-3}(x_i), \mathcal{D}_{-3}(x_j)] \\ &\quad - \frac{24}{(x_i - x_j)^2} \left[\left(\frac{h_i}{(x_i - x_j)} + \frac{\partial}{\partial x_i} \right) \mathcal{D}_{-3}(x_j) - \left(\frac{h_j}{(x_j - x_i)} + \frac{\partial}{\partial x_j} \right) \mathcal{D}_{-3}(x_i) \right] \\ &= 16K(\kappa_i, \kappa_j) \left(\frac{5h_i h_j}{(x_i - x_j)^2} - \frac{2h_i}{(x_i - x_j)} \frac{\partial}{\partial x_j} + \frac{2h_j}{(x_i - x_j)} \frac{\partial}{\partial x_i} + \frac{\partial^2}{\partial x_i \partial x_j} \right) \end{aligned} \tag{A.114}$$

A.3.4 Result II

With the computations above, we see that the same prefactors (A.87) and (A.89) show up in the commutation relation which is given by

$$\left[\mathcal{D}_{-3}(x_t^i), \mathcal{D}_{-3}(x_t^j)\right] = \frac{6}{(x_t^i - x_t^j)^2} \left[F(x_t^j; x_t^i) \mathcal{D}_{-3}(x_t^j) - (i \leftrightarrow j) \right], \tag{A.115}$$

which acting on the twice-fused correlation function of the boundary fields,

$$\begin{aligned} \tilde{\tilde{Z}}_{\text{b.c.}} &= \langle \psi_\infty(\infty), \psi_1(x_t^1) \ldots \psi_{i-1}(x_t^{i-1}) \psi_{h_{(1,3)}}(x_t^i) \psi_{i+2}(x_t^{i+2}) \ldots \\ &\quad \psi_{j-1}(x_t^{j-1}) \psi_{h_{(1,3)}}(x_t^j) \psi_{j+2}(x_t^{j+2}) \ldots \psi_m(x_t^m) \rangle , \end{aligned} \tag{A.116}$$

is equivalent to

$$K(\kappa_i, \kappa_j) H(x_t^i, x_t^j) \tilde{\tilde{Z}}_{\text{b.c.}} = 0 , \tag{A.117}$$

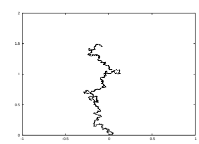
where the pre-factor is given by

$$H(x_t^i, x_t^j) = 4^2 \left(-\frac{5h_i h_j}{(x_t^i - x_t^j)^2} + \frac{2h_i}{(x_t^i - x_t^j)} \frac{\partial}{\partial x_t^j} + \frac{2h_j}{(x_t^j - x_t^i)} \frac{\partial}{\partial x_t^i} - \frac{\partial^2}{\partial x_t^i \partial x_t^j} \right) .$$

It is obviously consistent with the results of the previous computation. Due to the simplified calculation we yield the same pre-factors $F(x_t^i, x_t^j)$ and $F(x_t^j, x_t^i)$ as well as a term which can be identified with "$G(x_t^i, x_t^j)^2$" if correctly accounting for non-commuting contributions.

A.3.5 Interpretation

As we expected, there are additional factors $G(x_t^i, x_t^j)$, $F(x_t^i, x_t^j)$ and $H(x_t^i, x_t^j)$, due the fact that taking the limit and acting with the operators are two non-commutative operations. The additional factors G and F arise due to the fact that in the limit $x_t^i \to x_t^{i+1}$, we mix the $k = 0, 1$ and 2 terms of the OPE of $\phi_i(x_t^i)\phi_{i+1}(x_t^{i+1})$, when acting on it with $\tilde{\mathcal{D}}_{-2}(x_t^i)$ and $\tilde{\mathcal{D}}_{-2}(x_t^j)$. Remembering that the $(k = 1)$-term is given by $\epsilon^{1+\mu}\mathcal{L}_{-1}(x_t^i)\frac{1}{2}\tilde{Z}_{\text{b.c.}}$, we can immediately see why the additional derivative $\mathcal{L}_{-1}(x_t^i) = \partial_{x_t^i}$ arises. Due to the different orders in ϵ, the order in $(x_t^i - x_t^j)$ gets shifted, too. Therefore another $\mathcal{D}_{-2}(x_t^j)$ term shows up with an additional factor $(x_t^i - x_t^j)$ in the denominator. Physicall, $\mathcal{L}_{-1}$ can be interpreted as a translation, i. e. the joint growth stretches the real line in comparison with the ordinary growth only.



A.4 Basics in Stochastic Calculus and Probability Theory

Most of the statements in this section will be done for discrete time stochastics. However, all of them can be straightforwardly translated to the corresponding statements in continuous time. Note that this is only a short reminder where we left out some of the prerequisites in some theorems.

A.4.1 Objects in Probability Theory

Definition 8 (σ-Algebra, Filtration)**.** $\mathcal{F} \subset P(\Omega)$ (the power set of a set Ω) is called σ-algebra of Ω if $\mathcal{F} \neq \emptyset$ and

① $\Omega \subset \mathcal{F}$,

② $A \in \mathcal{F} \quad \Rightarrow \quad A^c := \Omega \backslash A \in \mathcal{F}$ (closedness under complements),

③ $A_i \in \mathcal{F} \quad \Rightarrow \quad \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ for all $i \in \mathbb{N}$ (closedness under countable unions).

A Filtration is an increasing sequence of sub-σ-algebras.

Definition 9 (Measure Space, Measurable Functions)**.** Let $\mathcal{F}$ be a σ-algebra of Ω, then the tupel $(\Omega, \mathcal{F})$ is called measure space. A function f between two measure spaces is called measurable, if its image is contained in the sigma algebra of the target space.

Definition 10 ((σ-Finite) Measure)**.** Let $(\Omega, \mathcal{F})$ be a measure space. A function $\mathbf{P} : \mathcal{F} \to \overline{\mathbb{R}}$ is called measurable, if

① $\mathbf{P}(\emptyset) = 0$,

② $\mathbf{P}(A) \geq 0$ for all $A \in \mathcal{F}$,

③ for all sequences $(A_n)_{n\in\mathbb{N}}$ of disjoint subsets of $\mathcal{F}$ the following holds:

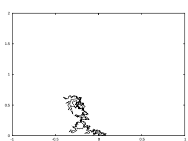
$$\mathbf{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbf{P}(A_n) \quad (\sigma - \text{additivity})\,. \tag{A.118}$$

If for one (A_n) we have $\bigcup_{n=1}^{\infty} A_n = \Omega$ and $\mathbf{P}(A_n) < \infty$ for all $n \in \mathbb{N}$, we call the measure σ-finite.

Definition 11 (Probability Space)**.** Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a measure space and $\mathbf{P}(\Omega) = 1$. Then $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space.

Definition 12 (Conformal Property for Measures)**.** Under conformal transformations g, a conformally invariant probability measure should transform as

$$\mathbf{P}_D(\gamma) = \mathbf{P}_{g(D)}(g(\gamma))\,. \tag{A.119}$$



Definition 13 (Random Variable)**.** Let $(\Omega_1, \mathcal{F}_1, \mathbf{P})$ be a probability space and $(\Omega_2, \mathcal{F}_2)$ be a measure space. Then we call the $\mathcal{F}_1$-$\mathcal{F}_2$-measurable mapping

$$X : \Omega_1 \to \Omega_2 \tag{A.120}$$

a random variable. The measure of the image $\mathbf{P}_X$ is called distribution of X.

Definition 14 (Expectation Value)**.** Let $X : \Omega \to \mathbb{R}$ be an integrable real random variable. Then its expectation value is defined via the special Lebesgue integral:

$$\mathbb{E}(X) := \int X \mathrm{d}\mathbf{P} \,. \tag{A.121}$$

Definition 15 (Indicator Function)**.** The Indicator of $A \subset \Omega$ is defined as:

$$I_A : \Omega \to \mathbb{R}\,, \quad \omega \mapsto I_A(\omega) := \begin{cases} 1 & \text{for } \omega \in A\,, \\ 0 & \text{else}\,. \end{cases} \tag{A.122}$$

Definition 16 (Stochastic Process, Sample Path)**.** Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $(S, \mathcal{S})$ a measure space and $I \subset [0, \infty)$ an index set. Then we call the family $(X_t)_{t \in I}$ of measurable mappings

$$X_t : \Omega \to S\,, \quad t \in I\,, \tag{A.123}$$

a stochastic process with state space S. For $\omega \in \Omega$, the mapping

$$X(\omega) : I \to S\,, \quad t \mapsto X_t(\omega)\,, \tag{A.124}$$

is called sample path of ω.

Definition 17 (Adapted Process)**.** The process X_t is said to be adapted to the filtration $(\mathcal{F}_t)$ if the random variable $X_t : \Omega \to S$ is a $(\mathcal{F}_t, \Sigma)$-measurable function for each t.

Heuristically, this means that the process cannot see into the future.

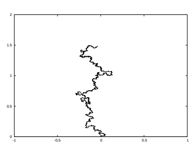
Definition 18 (Martingale)**.** A discrete martingale [158] (M_n) is a stochastic process that is adapted to a filtration $(\mathcal{F}_n)$ and for any $n \in \mathbb{N}_0$ integrable. It is called

① submartingale if $\mathbb{E}(M_n | \mathcal{F}_{n-1}) \geq M_{n-1}$,

② supermartingale if $\mathbb{E}(M_n | \mathcal{F}_{n-1}) \leq M_{n-1}$,

③ martingale if $\mathbb{E}(M_n | \mathcal{F}_{n-1}) = M_{n-1}$.

Hence, this means that the expectation value of a martingale is time independent and its time derivative has to vanish. This is also the statement of the optional stopping theorem: $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$.



Remark 17. Originally, martingale referred to a class of betting strategies popular in 18^{th} century France. The simplest of these strategies was designed for a game in which the gambler wins his stake if a coin comes up heads and loses it if the coin comes up tails. The strategy had the gambler double his bet after every loss, so that the first win would recover all previous losses plus win a profit equal to the original stake. Since a gambler with infinite wealth will with probability 1 eventually flip heads, the martingale betting strategy was seen as a sure thing by those who practised it. Unfortunately, none of these practitioners in fact possessed infinite wealth, and the exponential growth of the bets would eventually bankrupt those foolish enough to use the martingale.

Definition 19 (Stopping / Hitting Time)**.** A stopping time is a mapping

$$\tau : \Omega \to \mathbb{N}_0 \cup \{\infty\} , \tag{A.125}$$

if $\{\tau \leq n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}_0$.

A.4.2 Theorems

Theorem 7 (Optional Sampling Theorem)**.** Let (M_n) be a martingale and σ, τ some stopping times. Then we have

$$\mathbb{E}(M_\tau \big| \mathcal{F}_\sigma) = M_\sigma , \quad \sigma \leq \tau . \tag{A.126}$$

Heuristically, this means that a statement about the martingale property stays true even if we interchange the deterministic times m and n by randomly chosen stopping times σ and τ.

Theorem 8 (Radon-Nikodým Theorem)**.** Let $\mathbf{P}$ and $\tilde{\mathbf{P}}$ be two σ-finite measures on the same measurable space $(\Omega, \mathcal{F}_t)$, such that $\tilde{\mathbf{P}}$ is absolutely continuous with respect to $\mathbf{P}$. Then there exists a measurable function f, which is non-negative and finite, such that for each $A \in \mathcal{F}_t$

$$\tilde{\mathbf{P}}(A) = \int_A f \mathrm{d}\mathbf{P} , \tag{A.127}$$

with f uniquely given by the Radon-Nikodým derivative

$$f = \frac{\mathrm{d}\tilde{\mathbf{P}}}{\mathrm{d}\mathbf{P}} . \tag{A.128}$$

Theorem 9 (Girsanov's Theorem)**.** Let X_t be a stochastic process in $(\Omega, \mathcal{F}, \mathbf{P})$, satisfying the SDE

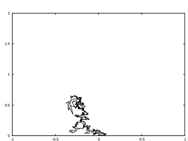
$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t . \tag{A.129}$$

If M_t is a martingale under the measure $\mathbf{P}$, satisfying

$$\mathrm{d}M_t = -\frac{\mu_t}{\sigma_t} M_t \mathrm{d}B_t , \tag{A.130}$$

then X_t is a martingale with respect to the measure $\tilde{\mathbf{P}}$ given by

$$\mathrm{d}\tilde{\mathbf{P}} = M_t \mathrm{d}\mathbf{P} . \tag{A.131}$$



A.4.3 Random Walk

Random Walk on the lattice is defined as we would naively expect with every step $\vec{\eta}_i$ random, isotropic and uncorrelated, i. e. $\langle \vec{\eta}_i \rangle = 0$ and $\langle \vec{\eta}_i^{\alpha} \vec{\eta}_j^{\beta} \rangle = \delta_{ij}\delta^{\alpha\beta}$. Hence, the end-to-end distance is given by $\vec{x} = \sum_{i=1}^{N} \vec{\eta}_i$ and the size is $R = \sqrt{\langle \vec{x}^2 \rangle} \propto \sqrt{N}$. Its fractal dimension obtained via the box procedure is $D = 2$, hence its scaling exponent is $\nu = 1/D$ due to $R \propto N^{\nu}$.

A.4.4 Brownian Motion

BROWNian Motion is the scaling limit of the unbiased random walk [159] taken in such a way that with $\eta \to 0$ and $N \to \infty$, $R \propto \sqrt{N}\eta = const.$. The resulting path of BROWNian motion is a scale-invariant, continuous but nowhere differentiable random curve with fractal dimension 2. In two dimensions it is plane filling and recurrent. Each 1D component satisfies the LANGEVIN equation

$$\frac{\mathrm{d}}{\mathrm{d}t} B_t = \eta_t , \quad \langle \eta_t \eta_s \rangle = \delta(t-s) \,, \tag{A.132}$$

where η_t is uncorrelated GAUSSian white noise. From this we can infer directly two important properties of BROWNian motion, or, mathematically a WIENER process:

① stationarity: $B_{t+\mathrm{d}t} - B_t \triangleq B_{\mathrm{d}t}$.

② independence: $B_{\mathrm{d}t}, B_{\mathrm{d}t'}$ are independent for $\mathrm{d}t \neq \mathrm{d}t'$

where $\triangleq$ denotes identically distributed. Moreover, we have $\langle |B_t - B_s|^2 \rangle = |t-s|$ and hence BROWNian motion¶ is distributed according to the normal (GAUSSian) distribution with mean zero and variance $t-s$: $\mathcal{N}(0, t-s)$.

Given a BROWNian motion, we can easily construct other examples:

① $B_{c^2 t} \triangleq cB_t$ for $c > 0$, $-B_t$,

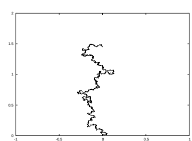
② $-B_t$,

③ $B_t := B_T - B_{T-t}$.

④ $B_t := tB_{1/t}$.

Additionally, it can be shown that it is a. s. a continuous martingale. As a WIENER process, it is recurrent in $d \leq 2$ and transient in all other dimensions.

¶Trivia: In Douglas Adam's "The Hitchhiker's Guide to the Galaxy", BROWNian motion is used to calculate the Infinite Improbability Drive that powers the spaceship "Heart of Gold", generated by a hot cup of tea.



A.4.5 The Markov Property

Consider an interface encoded by the curve γ and assume that part of it, say γ' has already been evolved. Then the conditional distribution is given by

$$\mathbf{P}_D(\gamma|\gamma') = \frac{\mathbf{P}_D(\gamma\gamma')}{\mathbf{P}_D(\gamma')} = \mathbf{P}_{D\backslash\gamma'}(\gamma)\,. \tag{A.133}$$

The last equality is the MARKOV property‡.

However, in the scaling limit, the distributions diverge and must be replaced by appropriate probability measures. Assuming that the MARKOV property continues to hold, we will interpret $\mathbf{P}$ as such a measure.

In the case of Statistical Physics models or CFT, the probability distribution is given by the ratio of the partition functions

$$\mathbf{P}(\gamma) = \frac{Z(\gamma)}{Z}\,, \tag{A.134}$$

where $Z(\gamma)$ is the partial partition function that insures the existence of the path γ and Z the full partition function. Heuristically, the MARKOV property means that the process has no memory, i. e. it does not depend on its past.

In the discrete case, a MARKOV process is a

Definition 20 (Markov Chain)**.** The sequence $(X_n)_{n\in\mathbb{N}_0}$ is called a homogeneous discrete Markov chain with state space Ω, if for every $n \in \mathbb{N}_0$ the condition

$$\mathbf{P}\left[X_{n+1} = j \middle| X_0 = i_0, \ldots, X_n = i_n\right] = \mathbf{P}\left[X_{n+1} = j \middle| X_n = i_n\right] \tag{A.135}$$

holds for all $(i_0, \ldots, i_n, j) \in \Omega^{n+2}$, for which $\mathbf{P}[X_0 = i_0, \ldots, X_n = i_n] > 0$.

Definition 21 (Recurrence)**.** A state i is recurrent iff

$$\mathbf{P}[X_n = i \text{ infinitely often } |X_0 = i] = 1\,. \tag{A.136}$$

It is called positive recurrent if $\mathbb{E}(T(X_n = i)) < \infty$ and null recurrent if $\mathbb{E}(T(X_n = i)) = \infty$.

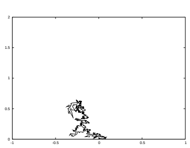
Definition 22 (Transience)**.** A state i is transient iff it is not recurrent.

A.4.6 Itô Calculus

Due to the infinite variation of BROWNIAN motion, the integral in stochastics has to be modified. Heuristically we can deduce from the TAYLOR expansion:

$$\mathrm{d}f(B_t) = f(B_{t+\mathrm{d}t}) - f(B_t) = f'(B_t)(B_{t+\mathrm{d}t}) + \dot{f}(B_t)\mathrm{d}t + \frac{1}{2}f''(B_t)\mathrm{d}t + \ldots \tag{A.137}$$

‡The theory of MARKOV processes was first discovered by Döblin [160] during World War II.



where we have inserted $(B_{t+\mathrm{d}t})^2 = \mathrm{d}t$. Hence we follow the ITÔ differential for a general stochastic process $\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}B_t$:

$$\mathrm{d}f(X_t) = f'(X_t)\mathrm{d}X_t + \left(\frac{\sigma^2}{2} f''(X_t) + \dot{f}\right) \mathrm{d}t\,. \tag{A.138}$$

The product rule of the ITÔ calculus for two stochastic processes X_t and Y_t given by $\mathrm{d}X_t = \mu_X \mathrm{d}t + \sigma_X \mathrm{d}B_t$ and $\mathrm{d}Y_t = \mu_Y \mathrm{d}t + \sigma_Y \mathrm{d}B_t$ can be stated as follows:

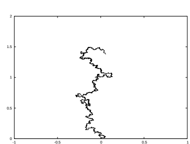
$$\begin{aligned} \mathrm{d}(X_t Y_t) &= (\mathrm{d}X_t)Y_t + X_t(\mathrm{d}Y_t) + \mathrm{d}\langle X_t, Y_t\rangle \\ &= (\mu_X Y_t + \mu_Y X_t + \sigma_X \sigma_Y)\mathrm{d}t + (\sigma_X Y_t + \sigma_Y X_t)\mathrm{d}B_t\,. \end{aligned} \tag{A.139}$$

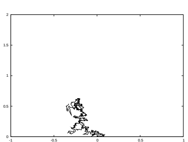
A.4.7 Bessel Process

A BESSEL process (R_t^d) of dimension d is a family of continuous MARKOV processes taking values in $\mathbb{R}_+$. For $d \in \mathbb{N}$, (R_t^d) may be represented as the EUCLIDEAN norm of BROWNIAN MOTION in $\mathbb{R}^d$. It satisfies the SDE:

$$\mathrm{d}R_t^d = \frac{d-1}{2R_t}\mathrm{d}t + \mathrm{d}\xi_t\,. \tag{A.140}$$

It is transient for $d > 2$, null recurrent for $d = 2$ and recurrent for $d < 2$.





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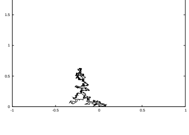

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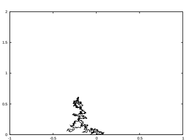
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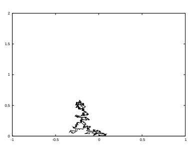

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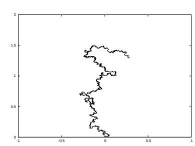

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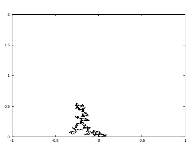

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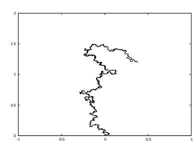

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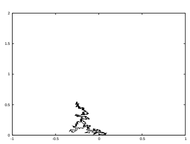

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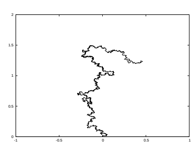

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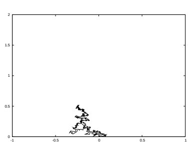

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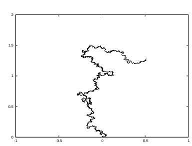

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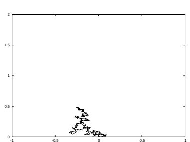

Acknowledgements

> *"But I don't want to go among mad people,"* Alice remarked. *"Oh, you can't help that,"* said the Cat: *"We're all mad here. I'm mad. You're mad."* *"How do you know I'm mad?"* said Alice. *"You must be,"* said the Cat, *"otherwise you wouldn't have come here."*
>
> LEWIS CARROLL [157]

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