Diplomarbeit in Physik

Conformal Field Theory and Percolation

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Bonn, im November 2005

Abstract

In this thesis, important features of two dimensional bond percolation on an infinite square lattice at its critical point within a conformal field theory (CFT) approach are presented. This includes a level three null vector interpretation for Watts' differential equation [78] describing the horizontal vertical crossing probability within this setup. A unique solution among the minimal models, $c_{(6,1)} = -24$ seems to be a good candidate, satisfying the level two differential equation for the horizontal crossing probability derived by Cardy [7] as well.

Commonly assumed to be a truly scale invariant problem, percolation nevertheless is usually investigated as a $c = 0$ CFT. Moreover this class of CFTs is important for the study of percolation or quenched disorder models in general. Since $c_{(3,2)} = 0$ as a minimal model only consists of the identity field, following Cardy [9] different approaches to get a non trivial CFT whose partition functions differ from one as suggested by the work of Pearce and Rittenberg [69] are presented. Concentrating on a similar ansatz for logarithmic behavior as for the triplet series ($c_{(p,1)}$), we examine the properties of such a CFT based on the extended Kac-table for $c_{(9,6)} = 0$ using a general ansatz for the stress energy tensor residing in a Jordan cell of rank two. We will derive the interesting OPEs in this setup (i.e. of the stress energy tensor and its logarithmic partner) and illustrate it by a bosonic field realization. We will give a motivation why the augmented minimal model seems to be more promising than the previous approaches and present an example of a tensor construction as a fourth ansatz to solve the $c \to 0$ problem as well.

Contents

Introduction

In 1984, conformal field theory (CFT) has been invented by Belavin, Polyakov and Zamoldichikov [4] as a suitable description for scale invariant one-dimensional quantum or two-dimensional classical problems, occurring mostly in the fields of statistical and condensed matter physics (i.e. the critical Ising model) or string theory. They are characterized by one parameter, the so-called central charge, which is related to the trace of the stress energy tensor and thus proportional to the Casimir energy (the change in vacuum energy density due to a change of the macroscopic scale). Hence the case of vanishing central charge arises when the macroscopic scale approaches infinity, meaning that CFTs with $c = 0$ are truly scale invariant. Usually, a special subclass of rational CFTs (i.e. with $c \in \mathbb{Q}$) is investigated and we will concentrate on it in this thesis, too.

Famous examples of CFTs are the Ising model with central charge $c = 1/2$ which can be generalized to the Q-state Potts model [70] exhibiting a central charge of $c = 1 - \frac{6}{m(m-1)}$ which is related to the number of different possible spin states, Q, via $\sqrt{Q} = -2\cos^2(\pi/m)$, the abelian Sandpile model with $c = -2$ or the (fractional) Quantum Hall effect. Most known models belong to the so called minimal models (the special subclass of rational CFTs as we have called them before) where the central charge can be characterized by two coprime parameters, $p > q$, via $c_{(p,q)} = 1 - 6\frac{(p-q)^2}{pq}$. They are called minimal models since they only consist of a usually small and always finite set of $\frac{1}{2}(p-1)(q-1)$ primary fields and their descendants.

Nine years later, Gurarie [39] observed the appearance of logarithmic singularities in correlation functions of certain CFTs. The first and best studied example for theories with this behavior is $c_{(2,1)} = -2$ which belongs to a special subclass of the minimal models, i.e. the $c_{(p,1)}$ CFTs, whose Kac-table is empty in the standard formalism for minimal models. Thus they are technically described by $c_{(3p,3)}$ and are all known to contain indecomposable representations and thus exhibit logarithmic behavior. Due to this feature, these theories are called logarithmic CFTs (LCFTs). It is conjectured that all rational CFTs can be extended to LCFTs by taking the augmented Kac table for $p, q \to (2n+1)p, (2n+1)q$ ($n \in \mathbb{N}$).

(L)CFTs describe the critical behavior at so-called second order phase transitions or continuous transitions, where (in contrast to first order transitions) we do not have a finite jump in some macroscopic observable (e.g. the temperature) itself but observe singular behavior in its derivative. In the Ising model, for example, this phase transition occurs at the self dual point of the model which is defined through the relation between the high temperature disordered and the low temperature ordered phase of the spins.

Conformal field theories are especially interesting to study in these two dimensional systems. For any higher dimension, conformal invariance is nothing more than the invariance under translations, dilations, rotations and special conformal transformations, but, in two dimensions, the conformal (Virasoro-)algebra turns out to be infinite dimensional. The high symmetry in two dimensions imposes severe constraints on the available models which is the reason for the fact that in the case of minimal models their properties are solely fixed by their central charge.

This thesis will concentrate on a very special case of conformal field theories whose properties have been under constant quarrel for the last years - the case of vanishing central charge. It is widely believed to describe quenched disorder problems, self-avoiding polymers and, above all, percolation. Percolation is a very suitable showcase since its properties are very well known due to various numerical calculations. In most cases, bond percolation on an infinitely large square lattice, whose bonds are open (or closed) with a probability p (or $(1-p)$), is investigated. Studying percolation means asking the question how probable it is that there exists a crossing from one to another side at the critical probability for bonds to be open or closed: $p_c = 1 - p_c = 1/2$. The critical probability p_c is the point at which the crossing probability as a function of p jumps from zero to one. A less scientific examples would be the preparation of espresso, which, in english, as a matter of fact can be referred to as "percolating coffee".

Since the early nineties, percolation has been of interest to mathematical physicists, e.g. John Cardy [7], who found that the two solutions of a second order differential equation describing the horizontal crossing probability in two dimensional bond percolation can formally be obtained from a $c = 0$ minimal model and thus rational CFT with a differential operator acting on a correlation function of four $h = 0$ fields. Starting from this point, G. M. T. Watts [78] motivated a fifth order differential equation which should describe the horizontal vertical crossing probability. This equation can be simplified to a third order differential equation which, however, lacks a consistent interpretation as a level three null vector condition within the $c = 0$ conformal field theory.

Cardy's results have been checked by numerical simulations (e.g. see Langlands et. al. [56]) and turned out to be extremely successful. In addition, they have been compared to the results of Stochastic or Schramm Loewner Evolution (SLE (κ)) describing the random walk of speed κ of some particle in the upper half complex plane hitting the real axis several times. Depending on which part of the real axis it hits first, this situation corresponds to having a horizontal crossing after a conformal transformation or not. SLE(κ) can be extended to SLE(κ, ρ) (i.e. multiple random walks) which has been used by Dubedat in 2004 to proof an SLE interpretation of Watts' third order differential equation.

In this thesis, this third order differential equation will be given an LCFT interpretation, being the only possible Kac table based CFT including Watts' differential equation as a level three null vector condition. Surprisingly, the resulting theory is not the from former considerations expected $c = 0$ CFT but one of the $c_{(p,1)}$ set, namely with $p = 6$ and thus $c = -24$. After further research on the arguments for $c = 0$ it comes to notice that the conditions derived up to now only fix the central charge up to 24 which is due to the modular properties of CFTs. Thus $c = -24$ seems to be an interesting alternative to the $c = 0$ assumption for percolation.

Although our results suggest otherwise, we do not wish to rule out the $c = 0$ ansatz for percolation completely. Thus, it is interesting and necessary to shed some light on this struggling CFT. There has already been extensive research about the $c = 0$ case and its proper treatment within a CFT setup, e.g. whether it should be described by a non-rational CFT or via a Kac-table based ansatz. From the work of Pearce and Rittenberg [69] it is known, that there are applications where its field content should extend the Kac table of the usual minimal model $c_{(3,2)} = 0$ since they found a deviation of the partition function from one and, particularly, a weight $h = 1/3$ field, which is present in the extended Kac-table. Due to divergences in the OPE of two primary fields in a CFT with vanishing central charge, the usual approach is the assumption of a Jordan cell connection of the stress energy tensor $T(z)$ with a new field, its so-called logarithmic partner $t(z)$. It is assumed to be generated by L_0, the zeroth mode of the Laurent expansion of $T(z)$ and to be of rank two. The indecomposable connection of these two fields is a crucial property, otherwise we could separate them again by a change of basis which would not solve the divergence problem. Within this ansatz,several approaches can be chosen, e.g. strictly based on Kac table fields (eventually extending it similar to the $c_{(p,1)}$ LCFTs) or including fields outside the conformal grid and thus being non-rational.

The first approach leads to a $c_{(9,6)} = 0$ LCFT with a rank two Jordan structure for the fields of the boundary on the Kac table and a rank three structure for those inside, leaving those on the corners in usual irreducible representations. Research on the precise structure of these models is currently going on [24]. The non-rational LCFT ansatz has already been discussed in the literature, e.g. by Kogan and Nichols [53] or Gurarie and Ludwig [41]. In contrast to the one based on the extended Kac-table which will be discussed in this thesis, it excludes a Jordan cell structure for the identity and thus $t(z)$ has to lie outside the Kac table. Both are looking for states that are non orthogonal to $L_{-2}|0\rangle = T(0)|0\rangle$ to ensure that this state does not vanish. This turns out to be rather difficult since there are no predictions for the exact form from symmetry algebras of $c = 0$ so far and the structure of the representations contained in $c = 0$ is still unknown. An additional difficulty arises for the second ansatz since a priori there are no constraints on fields outside the Kac table.

Outline

In this section, we give a short summary of the five parts of this thesis.

In the first chapter some excerpts from introductions to (L)CFT will be given requiring basic knowledge of field theories and algebra, particularly of Lie algebras and central extensions. Basically following the approach of Di Francesco [14] (with some aspects taken from Gaberdiel [31], Ginsparg [36] and Cardy [8] as well), we will go through some general statements on the conformal group and the consequences of conformal invariance in general and especially in two dimensions, introducing objects like the conformal Ward identities, the concept of the conformal generators L_n, primary fields and correlation functions as well as operator product expansions (OPEs) and a brief excursus on the central charge and its physical meaning. Additionally some features of the stress energy tensor will be stated, particularly its form in a free Boson construction as well as minimal models, including Verma modules, the Kac table and null vectors. At the end of the first part we will give an overview on selected aspects that change when considering logarithmic conformal field theory. If not denoted otherwise, the LCFT material can be found in Flohr's "Bits and pieces in logarithmic conformal field theory" [23].

The second chapter will concentrate on a brief discussion of $c = 0$ candidates. We will present the various possible models of percolation, e.g. bond or site percolation on square or triangular lattices and their various applications such as forest fires or resistance networks. In the second half of this chapter we will discuss how percolation can be formulated in theory, including statistical physics, CFT and SLE approaches. The literature for the phenomenological part of this chapter can be found in Grimmett [37] whereas the link to conformal field theory and Stochastic (or Schramm) Loewner Evolution (SLE) is taken from Cardy [8, 10].

In the third chapter, the implications of numerical simulations of percolation will be treated following the content of our first paper [28]. Particularly, we will apply our knowledge gained in the first chapter, above all the null vector conditions of the second and third level (with derivations in the appendix),to the research of Cardy [7] and Watts [78] to derive answers to the open questions. Additionally, $c_{(6,1)} = -24$ will be tested on all statements formerly used within $c = 0$ as percolation, including its modular properties and field content. This will be followed by a brief covering of SLE implications for percolation, pointing out the limits of constraints from this side as well as confirming conclusions.

The fourth chapter will deal with the contents of our second paper [27]. Discussing divergence problems at $c \to 0$, following Cardy [9] we will present four solutions, of which one is trivial ($c = 0$ as a minimal model), only consisting of the identity. Two solutions use different LCFT approaches, of which the first is based on the $c_{(9,6)} = 0$ augmented minimal model and thus a rational LCFT. Since it has not been discussed in papers before, our presentation of this approach will dominate the main part of the fourth chapter as our solution to the $c \to 0$ catastrophe, constructing the operator product expansions for the two interesting operators – the stress energy tensor and its logarithmic partner field – within this ansatz. Additionally we present a fourth loophole (a tensor ansatz) at the end of the discussion and give examples for this approach and the rational LCFT. The second of the two LCFT solutions (the non-rational ansatz chosen by Kogan and Nichols [53] or Gurarie and Ludwig [41]) includes fields outside the Kac table without a logarithmic partner for the identity. After a brief introduction to this approach, we will present several arguments why in our opinion the such an LCFT approach to the $c \to 0$ catastrophe is less suitable than the Kac-table based ansatz introduced in our paper.

This will conclude the research part of this thesis, leaving the fifth chapter for remarks, conclusions and an outlook where further research on this subject needs to be done.

1. Conformal Field Theory

1.1. The conformal group

In d dimensions, the conformal transformations consist of

- translations: $x^\mu \to x'^\mu = x^\mu + a^\mu$,
- dilations: $x^\mu \to x'^\mu = \alpha x^\mu$,
- rigid rotations: $x^\mu \to x'^\mu = M^\mu_\nu x^\nu$ and
- special conformal transformations: $x^\mu \to x'^\mu = \frac{x^\mu - b^\mu x^2}{1-2bx+b^2x^2}$.

In two dimensions, things can be simplified. Considering the complex coordinates $z = z^0 + \mathrm{i}z^1$ and $\bar{z} = z^0 - \mathrm{i}z^1$ and enforcing that any conformal transformation, $(z, \bar{z}) \to (w(z, \bar{z}), \bar{w}(z, \bar{z}))$, should leave the metric tensor invariant up to a scale, we see that the conditions arising are just the Cauchy-Riemann equations

$$\partial_{\bar{z}} w(z, \bar{z}) = 0 \quad \text{and} \quad \partial_z \bar{w}(z, \bar{z}) = 0\,. \tag{1}$$

The solutions for the former (latter) are (anti)holomorphic mappings of the complex plane onto itself. Thus the conformal group in two dimension is nothing else than the set of all analytic maps. It is infinite dimensional since the Laurent series $f(z) = \sum_n f_n z^n$ has an infinite number of parameters, determining the behavior of any analytic function in a small disc around any point by definition. The group multiplication, of course, is the composition of such mappings.

Thus it is obvious, that in two dimensions, due to the infinite symmetry (Virasoro-) algebra we have a very special and restrictive situation and thus there is hope for concrete predictions of our two dimensional conformal field theory.

There are six special generators among the infinite set contained in the symmetry algebra, which can be integrated to the global conformal group generating the so-called special conformal transformations (or as well Moebius transformations) and given by

$$f(z) = \frac{az+b}{cz+d} \quad \text{with} \quad ad - cb \neq 0\,. \tag{2}$$

These are the only possible globally defined holomorphic mappings. Analogously, this can be written down for the antiholomorphic case, too, to get the other three parameters. It can easily be verified that the global conformal group is isomorphic to $SL(2, \mathbb{C}) \cong SO(3, 1)$.

1.2. N-point functions and correlators

1.2.1. The transformation of primary fields

It can be shown, that under a conformal map $z \to w(z) = z + \epsilon(z)$ any primary field $\phi(z)$ with conformal weight h transforms as

$$\phi(z) \to \phi'(w) = \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^{-h} \phi(z)\,, \tag{3}$$

introducing the scale factor $\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|$ which is the same scale factor allowed as a transformation of the metric under conformal mappings. Thus, for an infinitesimal transformation, expanding the mapping into its Laurent series, yields

$$\delta_\epsilon \phi(z) \equiv \phi'(w) - \phi(z) = -h\phi\partial_z\epsilon(z).$$

Fields that transform this way under any local conformal transformation are called primary fields. Any field which is not primary is generally called secondary. A fact which might be bemusing is that there may be fields which are primary and secondary at the same time, e.g. the stress energy tensor at $c = 0$.

1.2.2. Correlation functions

From the property (3) follows immediately that the two-point function of two spinless fields is proportional to their distance, meaning that

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle = \begin{cases} \frac{C_{12}}{|z_1-z_2|^{2h_1}} & \text{for } h_1 = h_2 \\ 0 & \text{for } h_1 \neq h_2 \end{cases}, \tag{4}$$

and, analogously,

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}}{|z_1-z_2|^{h_1+h_2-h_3}|z_2-z_3|^{h_2+h_3-h_1}|z_1-z_3|^{h_3+h_1-h_2}} . \tag{5}$$

The first non trivial n-point function is the four-point function, depending on the so called cross (or anharmonic) ratio of the coordinates $\eta_1 = \frac{z_{01}z_{23}}{z_{02}z_{13}}$ (or $\eta_2 = \frac{z_{01}z_{23}}{z_{12}z_{03}}$) with $z_{ij} = |z_i - z_j|$. Hence the four-point function is given by

$$\langle\phi_0(z_0)\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = F(\eta_1,\eta_2)\prod_{i<j}^{3} z_{ij}^{H/3-h_i-h_j} \tag{6}$$

with $H = \sum_{i=0}^{3} h_i$. We can transform any four-point function to a form in which three of the fields are taken at fixed points, most times chosen to be $z_1 \to 0$, $z_2 \to 1$ and $z_3 \to \infty$. Here $z_0 \to z$ is the only independent variable on which the correlator depends contrary to the case of the two and three point functions which are completely fixed up to a normalization constant.

In general, four-point functions can be reduced to three-point functions with the help of the operator algebra,

$$\phi_1(z)\phi_2(0) = \sum_p \sum_{\{k\}} C_{12}^{p\{k\}} z^{h_p-h_1-h_2+K} \phi_p^{\{k\}}(0) , \tag{7}$$

with $K = \sum_{i=1}^{k} k_i$ and $\{k\} = \{k_1,\ldots,k_l\}$. The coefficients $C_{12}^{p\{k\}}$ are the coefficients of the possible three point functions including $\phi_2(0), \phi_1(z)$ and any $\phi_p(\infty)$ such that the three point function does not vanish. Thus any OPE of primary fields or their descendants can be deduced from the form of their respective three-point functions or by a differential equation $F(z)$ has to solve.

The discussion of symmetries of n-point functions will be omitted here and can be found e.g. in Di Francesco [14].

1.2.3. The conformal Ward Identities

Although it is not the original form of the Ward identities, the most convenient way to denote them is the following formula:

$$\delta_\epsilon\langle X\rangle = -\frac{1}{2\pi i}\oint_C \mathrm{d}z\, \epsilon(z)\langle T(z)X\rangle . \tag{8}$$

with $z \to z + \epsilon(z)$ being an infinitesimal conformal transformation as introduced in (3). Thus the conformal Ward identities relate the variation of some n-point function $X = \phi_0(z_0)\ldots\phi_{n-1}(z_{n-1})$

under a local conformal transformation to the stress energy tensor $T(z)$ (which is assumed to be regular). The contour C has to include all fields contained in X (i.e. the contour runs around all points z_i). For primary fields we know that $\delta_\epsilon \phi(z) = 0$ for global conformal transformations and thus we can deduce three conditions for n-point functions from the Ward-identities (8)

$$\sum_i \partial_{z_i} \langle X(z_0 1, \ldots, z_{n-1}) \rangle = 0\,, \tag{9}$$

$$\sum_i (z_i \partial_{z_i} + h_i) \langle X(z_0, \ldots, z_{n-1}) \rangle = 0\,, \tag{10}$$

$$\sum_i (z_i^2 \partial_{z_i} + 2 z_i h_i) \langle X(z_0, \ldots, z_{n-1}) \rangle = 0\,. \tag{11}$$

1.2.4. Free fields and the operator product expansion for chiral local operators

As commonly known, singularities arise in correlation functions when the coordinates on which the involved fields depend approach each other. To state the behavior of these divergences, the operator product expansion (OPE) is introduced. It consists of a sum of well defined operators, multiplied by a possibly divergent function of the two coordinates as they approach each other. Thus, in general, we have

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n}\,. \tag{12}$$

In practice, the contributions of the OPE may be calculated via the Wick theorem or known two and three-point functions (see (7)). For non-chiral or non-local operators, the condition $n \in \mathbb{Z}$ does not hold any more.

1.3. The central charge

The OPE of the stress energy tensor with itself,

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,, \tag{13}$$

where $\sim$ means equality modulo the non divergent parts as $z \to w$ which are omitted in the OPE, is obviously not the same as expected for a primary field (if $c \neq 0$). In fact, $T(z)$ is only quasi primary and thus exhibits a term proportional to $(z-w)^{-2h}$ in the OPE with itself which comes with a constant c, the so called central charge depending on the CFT in which it is defined. There is always a way to find a free Boson realization for the stress energy tensor for any central charge $c = 1 - 24\alpha_0^2$:

$$T(z) = \frac{1}{2} {:}\partial\varphi\partial\varphi{:}(z) + \mathrm{i}\sqrt{2}\alpha_0 {:}\partial^2\varphi{:}(z)\,. \tag{14}$$

The central charge or conformal anomaly shows up when a macroscopic scale is introduced into the system. It corresponds to a soft breaking of conformal symmetry since for $c \neq 0$ the theory is not truly scale invariant any more. It can be shown to be proportional to the Casimir energy or the free energy per unit length of a periodic statistical system. The central charge may also differ from zero while investigating CFTs on a curved space.

1.4. Conformal generators and the Hilbert space

1.4.1. Radial quantization

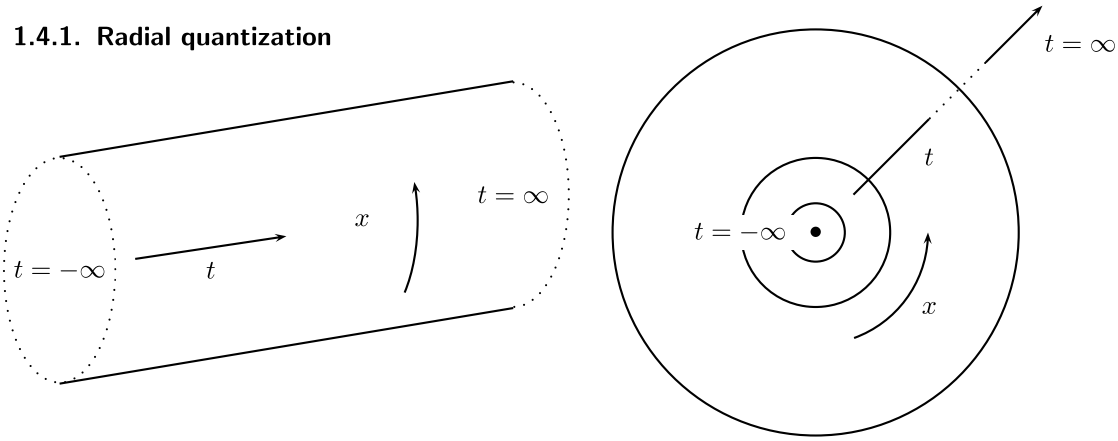

Figure 1: Deformed cylinder mapped onto the complex plane.

In radial quantization, the space direction is chosen to be along concentric circles centered at the origin and time the distance of any point to the origin. This can easily be obtained by compactifying space on a cylinder with circumfence L while the time is along the flat direction. Deforming this cylinder with a mapping $z = \exp(2\pi(t + \mathrm{i}x)/L)$, we find ourselves on the Riemann sphere with $t = -\infty$ representing the origin $z = 0$ and $t = +\infty$ lying at $z = +\infty$.

Within our radial quantization, the time ordering corresponds to radial ordering , i.e.

$$\mathcal{R}\Phi_1(z)\Phi_2(w) = \begin{cases} \Phi_1(z)\Phi_2(w) & \text{if } |z| > |w| \\ \pm\Phi_2(w)\Phi_1(z) & \text{if } |w| > |z| \end{cases}, \tag{15}$$

where $\pm$ stands for Bosons or Fermions, respectively. Within correlation functions, all fields have to be time ordered (which means radially ordered) for the left hand side to be a well defined operator.

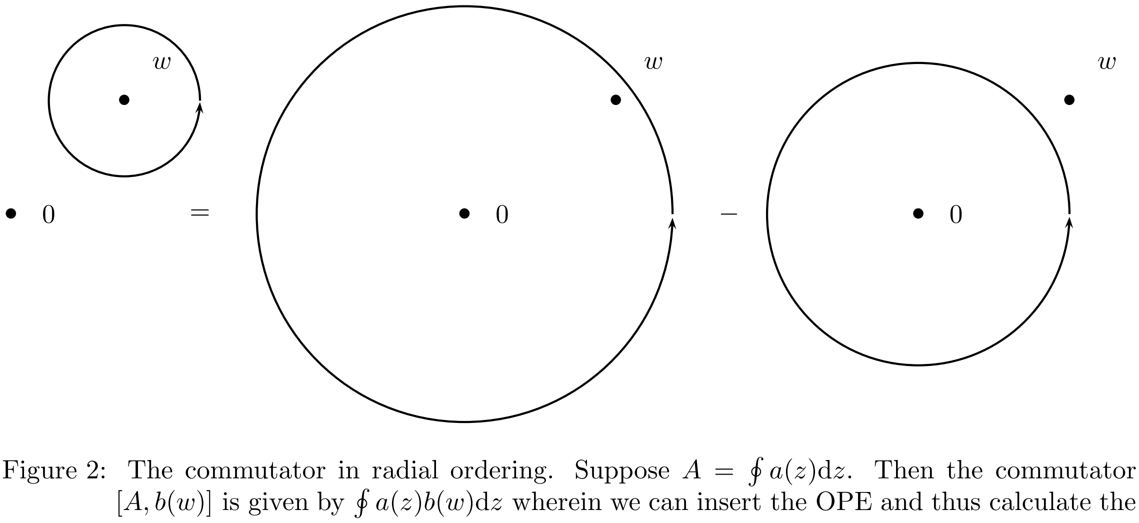

Figure 2: The commutator in radial ordering. Suppose $A = \oint a(z)\mathrm{d}z$. Then the commutator $[A, b(w)]$ is given by $\oint a(z)b(w)\mathrm{d}z$ wherein we can insert the OPE and thus calculate the value of the commutator.

From contour integration it is obvious that the equal time commutator of two variables $A = \oint a(z)\mathrm{d}z$ and $B = \oint b(w)\mathrm{d}w$ can be written as

$$[A,B] = [\oint a(z)\mathrm{d}z, \oint b(w)\mathrm{d}w] = \oint_0 \mathrm{d}w \oint_w \mathrm{d}z\, a(z)b(w). \tag{16}$$

An illustrative example can be found in figure 2.

1.4.2. Asymptotic states

Since the concept of Hilbert spaces is extremely useful in theoretical physics, we assume the existence of a vacuum state $|0\rangle$ from which we can build up the whole space by applying creation operators L_{-n}. In complete analogy to ordinary field theories, we define asymptotic fields $\phi_{in} \propto \lim_{z\to 0}\phi(z)$ and the corresponding operator $|\phi_{in}\rangle = \lim_{z\to 0}\phi(z)|0\rangle$. Keeping the custom that positive frequency states annihilate this vacuum, we have to introduce the mode expansion of conformal fields as follows

$$\phi(z) = \sum_{n\in\mathbb{Z}} z^{-n-h}\phi_n \tag{17}$$

$$\phi_n = \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}z\, z^{n+h-1}\phi(z)\,. \tag{18}$$

Thus in order to get a regular expression for $z \to 0$, we have to require $\phi_n|0\rangle = 0$ for $n > -h$. The Hermitian conjugate now is obviously denoted by $\phi_n^\dagger = \phi_{-n}$.

A concrete example is the expansion of a free boson which reads as

$$\varphi(z) = \varphi_0 - \mathrm{i}\pi_0 \log(z) + \mathrm{i}\sum_{n\neq 0}\frac{1}{n}a_n z^{-n}\,. \tag{19}$$

Obviously, $\varphi(z)$ is not primary, contrary to its derivative $\partial\varphi(z) = -i\sum_n a_n z^{-n}$ with $a_n = \pi_0$.

1.4.3. The conformal generators

Applying what we have just learned to the conformal Ward identities (8), we observe that they translate into

$$\begin{aligned}\delta_\epsilon\phi(z) &= -\frac{1}{2\pi i}\oint_C \mathrm{d}z\,\epsilon(w)T(w)\phi(z)\\ &= -[Q_\epsilon,\phi(z)]\,,\end{aligned} \tag{20}$$

where Q_ϵ denotes the conformal charge given by $Q_\epsilon = \frac{1}{2\pi\mathrm{i}}\oint dz\,\epsilon(z)T(z)$ with $\epsilon(z) = \sum_{n\in\mathbb{Z}}\epsilon_n z^n$. Writing down the mode expansion of the stress energy tensor as a weight two field

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n\,, \tag{21}$$

$$L_n = \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}z\, z^{n+1}T(z)\,, \tag{22}$$

we see that after an expansion of the conformal transformation, the conformal charge is in fact given by

$$Q_\epsilon = \sum_{n\in\mathbb{Z}}\epsilon_n L_n\,. \tag{23}$$

The L_n are the so-called the conformal generators for the infinitely many local conformal transformations of the Hilbert space. $L_{\pm 1}$ and L_0 form a closed subalgebra generating the global conformal (or $SL(2,\mathbb{C})$) transformations. Particularly, $L_0 + \bar{L}_0$ generate the dilations or time translations in radial ordering and therefore are proportional to the Hamiltonian of the system.

1.4.4. The Virasoro algebra

The algebra of the conformal generators for the holomorphic and the antiholomorphic part is a direct sum of two Virasoro algebras, i.e.

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}\,, \tag{24}$$

which can be computed by inserting (22) into (16).

Some facts about the Hilbert space can easily be seen at this point. Since we claim that the energy momentum tensor shall be well defined for $z \to 0$ which means that $T(z)|0\rangle$ is non divergent in this limit, taking equation (21) into account, we know that for $n \geq -1$ we have $L_n|0\rangle = 0$ should hold when.

Computing the commutator of the L_n with some primary field of weight h,

$$[L_n, \phi_h(z)] = h(n+1)z^n\phi_h(z) + z^{n+1}\partial_z\phi_h(z)\,, \tag{25}$$

for all $n \geq -1$, we see that for the $z \to 0$ limit and $n = 0$, primary field states are eigenstates of the Hamiltonian, since $L_0\phi_h(0) = h\phi_h(0)$. We will denote it by $|h, \bar{h}\rangle = \phi_{h,\bar{h}}(0)|0\rangle$ as an eigenstate of $\mathcal{H} \propto L_0 + \bar{L}_0$ with eigenvalue $h + \bar{h}$.

1.4.5. Descendant fields

To any primary field of dimension h belongs a tower of so called descendant fields, $p(n)$ (partitions of the number n) on each level n with conformal weight $h + n$. They are built by application of a set of $L_{-\{n\}}$ on the asymptotic state $|h\rangle$, i.e. a descendant of level n is given by

$$\phi^{-\{n\}}(w) \equiv L_{-k_1}L_{-k_2}\cdots L_{-k_l}\phi_h(w) =: L_{-\{n\}}\phi_h(w)\,, \tag{26}$$

with $\sum_i k_i = n$ and $k_l \geq \ldots \geq k_1$. Within a correlation function, the action of the L_n on the asymptotic state can be expressed by a differential operator using (25)

$$\langle\phi^{-n}(w)X\rangle \equiv \sum_{|\{n\}|=n} \beta^{\{n\}}\mathcal{L}_{-\{n\}}\langle\phi_h(w)X\rangle\,, \tag{27}$$

with

$$\mathcal{L}_{-n} = \sum_i \left(\frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i}\right)\,. \tag{28}$$

The set of a primary field and all its descendants is called a conformal family. The action of the stress energy tensor on one of its members will (by definition) only yield members of this family again.

The states obtained by applying our primary fields to the vacuum, $\phi_h|0\rangle = |h\rangle$, are highest weight states with respect to the Virasoro algebra, obeying

$$L_0|h\rangle = h|h\rangle\,, \tag{29}$$

$$L_n|h\rangle = 0 \quad \text{for } n > 0\,. \tag{30}$$

The so called Verma module $V(c, h)$ consists of all descendant states of a certain highest weight state $|h\rangle$ and thus our Hilbert space is a direct sum of all available Verma modules of the system, i.e. $\mathcal{H} = \bigoplus_{h,\bar{h}} M_{h,\bar{h}}V(c,h) \otimes V(c,\bar{h})$, with $M_{h,\bar{h}}$ most commonly being a diagonal matrix. Terms of the same conformal weight are not excluded in the sum and it is not required to be finite.

To each Verma module, a generating function can be assigned which is called the character of the module, defined by

$$\begin{aligned}\chi_{(c,h)}(\tau) &= \mathrm{Tr}\, q^{L_0-\frac{c}{24}} \\ &= \sum_{n=0}^{\infty} \dim(h+n) q^{n+h-\frac{c}{24}} \,, \end{aligned} \tag{31}$$

with $q \equiv \exp(2\pi i\tau)$ and $\dim(h+n)$ denoting the number of linearly independent states at level n of the module. The exponent $c/24$ is a remnant of the modular properties of the CFT.

1.5. Minimal models and the Kac-table

From commutativity and associativity of the OPE, constraints on the operator algebra can be deduced. These translate into constraints on the OPE which means that certain fields do not arise for a given value of the central charge. For the special subclass of the rational CFTs, the minimal models, i.e. with central charges parameterized by two coprime integers $p > q$, $c = 1-6(p-q)^2/(pq)$, it can be shown that from the otherwise infinite number of conformal fields that can be present in the theory, only a finite set of primary fields (with infinitely many descendants) is allowed to show up. Following Belavin, Polyakov and Zamolodchikov [4] we see that the allowed values of $h(c)$ are given by the Kac determinant

$$\det M^l = \alpha_l \prod_{r,s\geq 1, rs\leq l} (h - h_{r,s}(c))^{p(l-rs)} = 0 \,, \tag{32}$$

with $p(l-rs)$ being the number of all partitions of the positive integer $l - rs$ and α_l a positive constant depending on h. Thus the allowed weights for the central charges may be expressed by

$$h_{(r,s)} = \frac{(pr-qs)^2 - (p-q)^2}{4pq} \,, \tag{33}$$

$$c_{(p,q)} = 1 - 6\frac{(p-q)^2}{pq} \,, \tag{34}$$

with p, q being coprime.

For $p = q + 1$ these theories are unitary, meaning that they do not contain any negative norm states. Arranging the conformal weights $h_{(r,s)}$ on a grid, we get the so called Kac-table which is symmetric in $r \to q-r$ and $s \to p-s$. Thus it is sufficient to state the Kac-table for $1 \leq r < q$ and $1 \leq s < p$ and identify those states that correspond to each other under this symmetry. We have $\frac{1}{2}(p-1)(q-1)$ independent primary fields in each Kac-table. These theories contain an infinite set of null vectors within each Verma module $V_{(h,c)}$ and are therefore referred to as minimal models $\mathcal{M}_{(h,c)}$ when those null states have been divided out. Of course, their field content is still solely made up out of the finite number of local fields that form the Kac-table and their descendants.

1.5.1. An example: The Ising model

A famous example of the minimal models is the Ising model. From all non trivial minimal models, it has the smallest Kac-table with a central charge of $c_{(4,3)} = 1/2$, which is given by

$\mathbb{1}_{h=0}$	$\sigma_{h=\frac{1}{16}}$	$\epsilon_{h=\frac{1}{2}}$
$\epsilon_{h=\frac{1}{2}}$	$\sigma_{h=\frac{1}{16}}$	$\mathbb{1}_{h=0}$

, (35)

with σ representing the lattice spin and ϵ the interaction energy between two nearest neighboring spins.

In statistical physics it is built by N spins on a lattice which can occupy one of two possible states usually denoted by ± 1. The partition function of the system depends on the energy between two nearest neighbor spins (σ_i, σ_j) in the same state minus the energy of those in different states. Thus the partition function is given by

$$\mathcal{Z} = \sum_{\{\sigma\}} \exp\left(-\beta \sum_{\langle ij \rangle} E_{\langle ij \rangle}\right) , \tag{36}$$

where $E_{\langle ij \rangle}$ denotes the energy per link $-J\sigma_i\sigma_j$.

While in one dimension there is no phase transition, in two dimensions we have one of second order, namely between an ordered phase at low temperature and a disordered phase at high temperature. Between those two phases the Kramers-Wannier duality is established and defines the fixed point of the temperature. As a very simple model for ferromagnetism, its critical temperature is well known to be the Curie temperature:

$$\beta J_c = -\frac{1}{2}\ln(\sqrt{2}-1) . \tag{37}$$

From the duality relation, a new operator (dual to the spin operator) arises which we call disorder operator μ. Interchanging those two operators switches from low to high temperature in the model.

In CFT, two critical exponents arise in the two-point correlation functions of

$$\langle \sigma_i \sigma_{i+n} \rangle = \frac{1}{|n|^{\eta}} , \tag{38}$$

$$\langle \varepsilon_i \varepsilon_{i+n} \rangle = \frac{1}{|n|^{4-2/\nu}} , \tag{39}$$

with η and ν being the critical exponents of the Ising model.

The Ising model is equivalent to the free Majorana fermion where the energy density corresponds to the fermion mass $\bar{\psi}\psi$. The Majorana fermions are given by $\psi(z) \propto \phi_{(2,1)}(z) \otimes \phi_{(1,1)}(z)$ and $\bar{\psi}(\bar{z}) \propto \phi_{(1,1)}(\bar{z}) \otimes \phi_{(2,1)}(\bar{z})$ and the spin is given by $\sigma(z,\bar{z}) \propto \phi_{(1,2)}(z) \otimes \phi_{(1,2)}(\bar{z})$ whereas the interacting energy is $\varepsilon(z,\bar{z}) = \mathrm{i} : \psi\bar{\psi} : (z,\bar{z}) \propto \phi_{(2,1)}(z) \otimes \phi_{(2,1)}(\bar{z})$.

The underlying $\mathbb{Z}_2$ symmetry reflects the temperature duality relation. Away from the critical point, the Ising model is no longer scale invariant and a mass term arises in the Ising action

$$S = \frac{1}{2\pi} \int \mathrm{d}^2 z \, (\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}) . \tag{40}$$

Vertex operators Since the free boson φ has vanishing scaling dimensions, it is possible to construct primary fields by exponentiating without the need to introduce a scale. Defining a vertex operator of weight $h(a) = a^2 - 2a\alpha_0$ as

$$V_a(z) = \exp(\mathrm{i}\alpha\sqrt{2}\varphi) , \tag{41}$$

it can easily be shown that V_a is primary. In minimal models, vertex operators whose charges differ by $2\alpha_0$ can be identified. This is allowed due to the insertion of screening charges into correlation function, $Q_{\pm}^{n}$, which do not affect the conformal properties of the correlators and thus do not change its physical properties.

Taking two (non interacting) Ising models with $c = \frac{1}{2} + \frac{1}{2} = 1$ yields a model which has a one to one correspondence to the free bosonic field at $c = 1$. This can easily be seen when regarding the free complex Dirac fermion

$$\mathcal{D}(z,\bar{z}) = \begin{pmatrix} D(z) \\ \bar{D}(\bar{z}) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1 + \mathrm{i}\psi_2 \\ \bar{\psi}_1 + \mathrm{i}\bar{\psi}_2 \end{pmatrix} (z,\bar{z}) \,, \tag{42}$$

which is equivalent to $D(z) = \exp(\mathrm{i}\varphi(z))$ and analogously $\bar{D}(\bar{z}) = \exp(\mathrm{i}\bar{\varphi}(\bar{z}))$. This is one of many possible examples where vertex operators may arise and in general is called "Bosonization". Within the Coulomb gas formalism this is a very helpful technique to find bosonic realizations for given models.

1.5.2. Null vectors and differential equations for correlation functions

Any state $|\chi^{(n)}_{(h,c)}\rangle$ which vanishes under all action of the conformal generators L_k generates a Verma module V_χ within the original module. It is called null state and corresponds to a secondary field which has all properties of a primary field. Quotienting all Verma modules generated by null states out of the original module, we get an irreducible representation, $M_{(c,h)}$.

The explicit form of null vectors on level n in a minimal model can be found by consequently acting on a generic state $|\chi^{(n)}_{(h,c)}\rangle = \sum_{|\{n\}|=n} \beta^{\{n\}} L_{-\{n\}}|0\rangle$ with the conformal generators L_k with $k \leq |\{n\}|$, requiring the results to vanish.

This way we obtain two examples, e.g. for the second and third level:

$$|\chi^{(2)}_{(h,c)}\rangle = \left(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2 \right) |h\rangle \,, \tag{43}$$

$$h^{(2)} = \frac{1}{16} \left(5 - c \pm \sqrt{(c-1)(c-25)} \right) \,, \tag{44}$$

$$|\chi^{(3)}_{h,c}\rangle = \left(L_{-1}^3 - 2\,(h+1)\,L_{-2}L_{-1} + h\,(h+1)\,L_{-3} \right) \phi_h \,, \tag{45}$$

$$h^{(3)} = \frac{1}{6} \left(7 - c \pm \sqrt{(c-1)(c-25)} \right) \,. \tag{46}$$

As stated above, from these null vectors follow differential equations for the four-point function, leaving n possible solutions for $F(\eta)$, or, after taking the usual limit $(\{z_0, z_1, z_2, z_3\} \to \{z, 0, 1, \infty\})$, $F(z)$.

The calculations for the differential equations are presented in the appendix, therefore here we will only state the results for the second level ($h = h^{(2)}$),

$$\left(\frac{3}{2(2h+1)} \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{2z-1}{z(z-1)} \frac{\mathrm{d}}{\mathrm{d}z} - \frac{h_1}{z^2} - \frac{h_2}{(z-1)^2} + \frac{h + h_1 + h_2 - h_3}{z(z-1)} \right) F(z) = 0 \,, \tag{47}$$

and for the third level ($h = h^{(3)}$),

$$\begin{aligned} 0 \;=\; & \Bigg[\frac{\mathrm{d}^3}{\mathrm{d}z^3} + 2(h+1) \frac{2z-1}{z(z-1)} \frac{\mathrm{d}^2}{\mathrm{d}z^2} \\ + \; & (h+1) \left(\frac{h - 2h_1}{z^2} + \frac{h - 2h_2}{(z-1)^2} - 2\frac{h_3 - h - h_1 - h_2}{z(z-1)} + \frac{h}{z(z-1)} \right) \frac{\mathrm{d}}{\mathrm{d}z} \\ + \; & h(h+1) \left(-\frac{2h_1}{z^3} - \frac{2h_2}{(z-1)^3} + \frac{(2z-1)(h + h_1 + h_2 - h_3)}{z^2(z-1)^2} \right) \Bigg] F(z) \,. \end{aligned} \tag{48}$$

1.5.3. Fusion rules

In order to see which fields can arise in the OPE of primary fields, we have to check how their representations behave under fusion. For minimal models this may be motivated through null vectors and the OPE but we will only state the outcome here.

The commutative and associative fusion algebra, which is generated by the ϕ_j, $j = 1, \ldots, r$ with $\phi_1 = \mathbb{1}$ and the fusion product $\times$, is defined by its multiplication rule

$$\phi_i \times \phi_j = \sum_k \mathcal{N}_{ij}^k \phi_k. \tag{49}$$

For minimal models, we have $\mathcal{N}_{ij}^k \in \{0, 1\}$, while the coefficient vanishes if the structure constant in the corresponding OPE is zero. Thus it is given by

$$\phi_{(r,1)} \times \phi_{(k,l)} = \sum_{\substack{m=k-r+1 \\ m-k+r-1 \text{ even}}}^{k+r-1} \phi_{(m,l)} \,. \tag{50}$$

This means that in a minimal model we will not run out of the Kac-table, i.e. the fusion of two Kac-table representations always gives fields of the Kac-table again. For augmented $c_{(p,q)}$ models, this statement holds for the extended Kac table for LCFTs $c_{(3p,3q)}$.

1.6. Boundary condition changing operators

In general, we do not have to deal with infinite systems without (or rather with free) boundary conditions which are rather easy to handle. Thus we have to think about how to enforce them and which boundary conditions may be allowed in a given CFT.

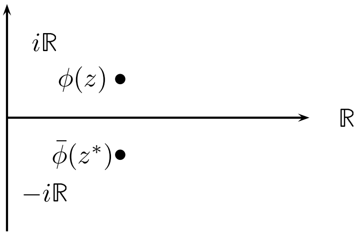

Figure 3: The bulk scaling field approaching the real axis interacts with the boundary.

The established concept is the introduction of so called boundary condition (changing) operators $\phi_B(z)$ which arise in the OPE of a bulk field with its mirror image when approaching the real axis (see figure 3). Thus it may be replaced by the OPE

$$\phi(z)\bar{\phi}(z^*) \sim \sum_i (z - z^*)^{(h_i - 2h)} \phi_B^{(i)}(x) \tag{51}$$

with $x = \mathcal{R}e(z)$. They live solely on the boundary since they only depend on the real part of the variable z. Being inserted at some point at the boundary, those operators change the conditions from one to another by changing the analyticity condition at the part of the boundary at which they are inserted. For example, in the Ising model, the $\phi_{(1,2)}$ operator as a weight $h = 1/16$-field generates a branch cut in the OPEs in which it is involved. Thus if we cross from one to the other half of the Riemann sphere, we pick up a phase and the analyticity condition changes from $\phi(z) = \bar{\phi}(z^*)$ to something else. This approach to boundary conditions is sufficient since any two-dimensional region with a piecewise differentiable boundary can be mapped onto the upper half plane and it can be shown that the Kac-table fields are in one-to-one correspondence with the possible different boundary condition changing operators. In the Ising model, the change of boundary condition a to b is described by the insertion of a boundary condition changing operator $\phi_{(a|b)}$ with weight

$$(+|+) \text{ or } (-|-) \quad \rightarrow \quad h = 0\,, \tag{52}$$

$$(f|f) \quad \rightarrow \quad h = 0, 1/2\,, \tag{53}$$

$$(+|-) \quad \rightarrow \quad h = 1/2\,, \tag{54}$$

$$(-|f) \text{ or } (+|f) \quad \rightarrow \quad h = 1/16\,, \tag{55}$$

where $+, -$ stand for spin up or down, respectively and f for free boundary conditions. As the Kac-table of $c_{(4,3)} = 1/2$ consists of exactly these three fields, the one-to-one correspondence is obvious.

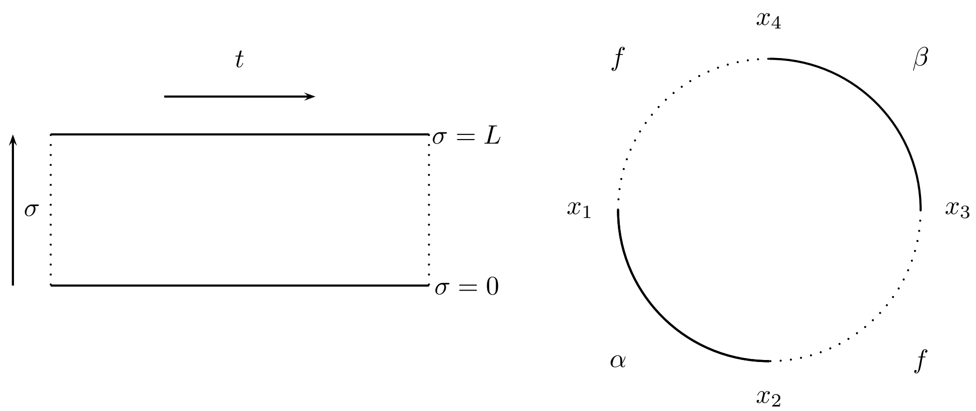

Figure 4: Fixed boundary conditions for one of the two complex directions of $w = t + i\sigma$ at $\sigma = 0, L$ given by α, β mapped onto the upper half plane.

Assuming that any system is invariant under transformations that do not change the boundary conditions, we may construct a strip as in figure 4 and map it back onto the upper half plane or rather the upper half Riemann sphere. The Hamiltonian of the systems depends of course on the boundary conditions. Inserting boundary operators at some point on the real axis means that the vacuum is no longer translational invariant ($L_{-1}|0\rangle \neq 0$) but a new vacuum $|0\rangle_B = \phi_B|0\rangle_{SL(2,\mathbb{C})}$. Additionally it is known that the partition function of a bounded system can be obtained through the free partition function times the correlation function of all boundary operators that have to be inserted to model the conditions:

$$\mathcal{Z}_B = \mathcal{Z}_f \langle \phi_{B_1}(x_1) \ldots \phi_{B_n}(x_n) \rangle \, . \tag{56}$$

A motivation for this interpretation can be found in [14].

1.7. Modular invariance and the partition function

For some purposes it may be interesting to examine CFTs not on a deformed cylinder (punctured complex plane) which is merely the simplest possible choice but on general Riemann surfaces. This approach is for example needed in string theory in order to get interacting strings. Ordering by the genus (g) of the Riemann surface (RS), we have a RS(g) with n tubes attached, one for each interacting string, for non n interacting strings punctures are sufficient to describe their properties by inserting a suitable vertex operator.

For a large class of CFTs it can be shown that the crossing symmetry of correlators on the complex plane and modular invariance of the partition function on the torus is sufficient to require that the CFT is consistent on any Riemann surface. Thus modular invariance is widely believed to be a fundamental requirement for CFT. In fact, it has been proven by Cardy [5] and Nahm [68] that from conformal invariance of a field theory on the two-dimensional sphere, $\mathcal{S}^2$, follows the modular invariance of the partition function on a torus. Flohr [19] conjectured that this statement may be extended to logarithmic conformal field theories as well.

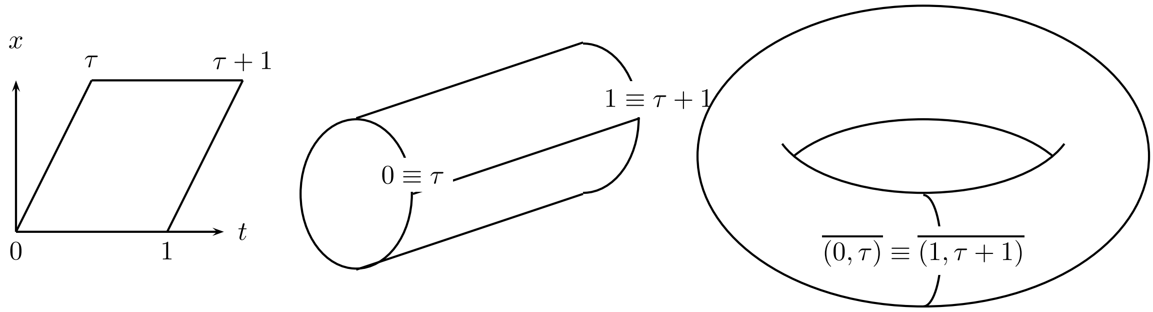

Figure 5: Building a torus.

Modular invariance on the torus means that if we consider a torus constructed by identifying the opposite sides of a parallelogramm on the complex plane with edges at $0, 1, \tau, \tau+1$, there should be a symmetry under

$$T: \quad \tau \quad \mapsto \quad \tau + 1 \tag{57}$$

$$S: \quad \tau \quad \mapsto \quad -\frac{1}{\tau}. \tag{58}$$

T and S span the modular group $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$ which is the set of all matrices M,

$$M := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{59}$$

with unit determinant. Its action is given by $M(\tau) = \frac{a\tau+b}{c\tau+d}$, identifying $M \equiv -M$. Thus modular invariance means invariance under global conformal transformations.

The modular invariant partition function is given by

$$Z(\tau,\overline{\tau}) = (q\overline{q})^{c/24} \text{ tr } \left(q^{L_0}\overline{q}^{L_0}\right), \tag{60}$$

with $q = \exp(2\pi i\tau)$. It can be rewritten in terms of bilinear combinations of the characters (see also (31))

$$\chi_{(h,c)}(\tau) = q^{h-c/24} \prod_{n\geq 1} (1-q^n)^{-1} = q^{h-c/24} \sum_{N\geq 0} P(N) q^N, \tag{61}$$

with $P(N)$ counting the number of states at level N in the Verma module, excluding null states and their descendants. More precisely, we have $Z = \sum_{\bar{h},h} \bar{\chi}_{(\bar{h},c)} M_{\bar{h},h} \chi_{(h,c)}$.

The factor $-c/24$ in (61) emerges from the difference between the stress energy tensor on the complex plane and that on the cylinder

$$T_{cyl} = z^2 T(z) - \frac{c}{24}\mathbb{1} \quad (L_n)_{cyl} = L_n - \frac{c}{24}\delta_n, 0\,. \tag{62}$$

1.8. Logarithmic CFT

1.8.1. Jordan cell structure

The "logarithmic" property of LCFTs emerges due to the existence of indecomposable highest weight representations. Assuming that r such states of the same highest weight generate a non trivial Jordan cell of rank r and $n \in \{0, \ldots, r-1\}$, we have (normalizing the off diagonal entries of the Jordan cell to 1)

$$L_0|h;n\rangle \quad = \quad h|h;n\rangle + (1-\delta_{n,0})|h;n-1\rangle \tag{63}$$

$$L_m|h;n\rangle \quad = \quad 0 \text{ for } m > 0\,. \tag{64}$$

Obviously there is an irreducible subrepresentation for $n = 0$ which belongs to the original primary field whereas all others are called logarithmic partners. Due to the new structure, the Virasoro modes act no longer as linear differential operators as defined in (28) but

$$\mathcal{L}_{-k}\psi_{(h;n)}(w) = \frac{(1-k)h}{(z-w)^k}\psi_{(h;n)}(w) - \frac{1}{(z-w)^{k-1}}\frac{\partial}{\partial w}\psi_{(h;n)}(w) - (1-\delta_{n,0})\frac{(1-k)}{(z-w)^k}\psi_{(h;n-1)}(w)\,. \tag{65}$$

Most basic terms of CFTs can be generalized to LCFT, e.g. the null vectors

$$|\tilde{\chi}^{(n)}_{(h,c)}\rangle = \sum_k \sum_{\{n\}} b_k^{\{n\}} L_{-\{n\}}|h;k\rangle\,, \tag{66}$$

with k denoting their place within the Jordan cell. Thus the original null states may no longer be orthogonal any other state in an LCFT extended model. Therefore we have to include the logarithmic partner states into our considerations to get true LCFT null states. Examples for such states are given in [23].

1.8.2. Logarithms in LCFT

There are essentially two ways to write down a partition function for an LCFT which we will sketch in the following [25]. As already mentioned, in standard CFT the partition function is given by

$$\mathcal{Z} = \sum_{h,\bar{h}} \bar{\chi}_{(\bar{h},c)}(\tau) M_{\bar{h},h} \chi_{(h,c)}(\tau)\,, \tag{67}$$

with

$$\chi_{(h,c)}(\tau) = \mathrm{tr}_{M_{h,c}}\, q^{L_0 - c/24}\,, \qquad q \equiv \exp(2\pi \mathrm{i}\tau)\,. \tag{68}$$

Additionally, we know that the vector space spanned by the characters $\chi_{(c,h)}(\tau)$ is isomorphic to that of the torus amplitudes $\psi_{(c,h)}(\tau)$.

However, in LCFT the situation is slightly different. We have two possibilities to choose a partition function, i.e. $\mathcal{Z}_{red}$ and $\mathcal{Z}_{full}$. While the reduced partition function is a modular invariant that can be expressed solely by traces over modules,

$$\mathcal{Z}_{red} = \sum_{\{h,\bar{h}\}\in I_{red}} \bar{\chi}^{red}_{(\bar{h},c)}(\tau) M_{\bar{h},h} \chi^{red}_{(h,c)}(\tau) \tag{69}$$

with

$$\chi^{red}_{(h,c)}(\tau) = \begin{cases} \mathrm{tr}_{M_{h,c}}\, q^{L_0 - c/24} & \text{if } |h\rangle \text{ generates an irrep } \not\subseteq \text{ indecomposable representation} \\ \mathrm{tr}_{R_{h,c}}\, q^{L_0 - c/24} & \text{if } |h\rangle \text{ is part of an indecomposable representation} \end{cases} \tag{70}$$

In the latter case, the trace runs over the maximal indecomposable representation $R_{(c,h)}$ containing $|h\rangle$ only. Therefore we observe that the vector space spanned by the reduced characters is strictly contained in that of the torus amplitudes obtained from the modular differential equation or, more precisely, is even smaller than it.

Note that for a logarithmic setup, here for a rank two Jordan cell structure, L_0 can be represented by a matrix,

$$L_0 = \begin{pmatrix} L_{(0,0)} & 1 \\ 0 & L_{(0,1)} \end{pmatrix} \quad \Rightarrow \quad q^{L_0} = \begin{pmatrix} q^{L_{(0,0)}} & \log(q) q^{L_{(0,0)}} \\ 0 & q^{L_{(0,1)}} \end{pmatrix}\,. \tag{71}$$

The exact form of q^{L_0} depends on the choice of the basis since in an LCFT the Jordan cell structure is non-trivial and the states are no longer orthogonal to each other. However, a suitable linear combination remedies the problem.

The second possibility is to take the full partition function defined by

$$\mathcal{Z}_{full} = \sum_{\{h,\bar{h}\}\in I_{full}} \bar{\psi}_{(\bar{h},c)}(\tau) M_{\bar{h},h} \psi_{(h,c)}(\tau)\,, \tag{72}$$

where the ψ are the torus amplitudes, i.e. the solutions of the modular differential equation, in a suitable linear combination such that we can rewrite (72) in the rank two case as

$$\begin{aligned}\mathcal{Z}_{full} &= \sum_{irreps} (\bar{\psi}_{(\bar{h},c)}(\tau) M^{irred}_{\bar{h},h} \chi_{(h,c)}(\tau) + h.c.) \\ &= \mathcal{Z}_{red} + \alpha \log(q\bar{q}) \sum_{indec} \bar{\psi}_{(\bar{h},c}(\tau) M^{red}_{\bar{h},h} \psi_{(h,c)}(\tau) \end{aligned}\tag{73}$$

In the case of LCFTs, not all of the torus amplitudes $\psi_{(h,c)}$ can be interpreted as traces over modules. Hence, to obtain the full partition function in an LCFT, the isomorphism between the vector space of the characters and that of the torus amplitudes ceases to exist and logarithms arise in the CFT. The additional coefficient α represents the freedom of choice that we have for this term since we have no restrictions from modular invariance alone to fix it in magnitude while for all torus amplitudes that correspond to characters we get an additional constraint. This emerges from the fact that the character $\chi_{(c,h)}$, as a trace over the module, is counting the states contained therein. Thus by imposing that the lowest term in its expansion should represent the multiplicity of the ground states of the module, the overall factor α can naturally be fixed.

In the two-point functions of fields of the same weight we have for a rank two theory:

$$\langle \psi_{(h;0)}(z)\psi_{(h';0)}(w)\rangle = 0\,, \tag{74}$$

$$\langle \psi_{(h;0)}(z)\psi_{(h';1)}(w)\rangle = \delta_{h,h'}\frac{A}{(z-w)^{2h}}\,, \tag{75}$$

$$\langle \psi_{(h;1)}(z)\psi_{(h';1)}(w)\rangle = \delta_{h,h'}\frac{B-2A\log(z-w)}{(z-w)^{2h}}\,, \tag{76}$$

which means that the property that the vacuum is normalized to zero is a very important feature especially in LCFTs with vanishing central charge.

The logarithmic fields transform under conformal transformations $f(z)$ as follows

$$\psi_{(h;n)}(z) = \left(\frac{\partial f(z)}{\partial z}\right)^h (1+\log(\partial f(z))\delta_{n,0})\psi_{(h;n)}(f(z))\,. \tag{77}$$

Thus $n=0$ reproduces the ordinary primary field while for $n \neq 0$ the fields can not have this property. From these two properties follows that the solutions for differential equations for n-point functions arising from null vector conditions may contain logarithmic divergences as well.

The global conformal Ward identities are affected by the modified action of the Virasoro modes and thus the form of the one-, two- and three-point function is changed, too. They can be found in [22], [21], [19] and [20].

1.8.3. An example: the representation structure of $c = -2$

The detailed structure of Jordan cells in augmented minimal models with central charge $c = c_{(p,q)}$ is only known in sufficient detail for $q = 1$ so far. We will try to illustrate their behavior with the help of the canonical example, $c_{(2,1)} = 0$. Its Kac-table is given by

$$c_{(2,1)} : \begin{array}{|c|c|c|c|c|}\hline 0 & -\frac{1}{8} & 0 & \frac{3}{8} & 1 \\ \hline 1 & \frac{3}{8} & 0 & -\frac{1}{8} & 0 \\ \hline \end{array} . \tag{78}$$

It consists of the pre-logarithmic field $\mu_{h=-\frac{1}{8}}$ residing at $(0,0) \equiv (q,p) \equiv (2q,2p)$ which gives rise to an irreducible representation. It has a corresponding irreducible representation residing at $(q,2p) \equiv (2q,p)$ which differs from μ in its weight by $(p \cdot q)/4$. The three remaining fields residing at $(1,1) \equiv (2,5)$, $(1,3) \equiv (2,3)$ and $(1,6) \equiv (2,1)$ have a more sublte connection. $\phi_{(1,1)}$ generates an irreducible subrepresentation $\mathcal{V}_0$ of the indecomposable representation $\mathcal{R}_0$ based on $\phi_{(1,3)}$. The level of the null vector on both is given by the product $r \cdot s$ of their coordinates in the Kac-table. The former indicates the difference in its weight of the third (irreducible) representation $\mathcal{V}_1$ - $r \cdot s = 1 \cdot 1 = 1$ and thus $h_{(1,5)} = h_{(1,1)} + 1 = 1$. It is a subrepresentation of another indecomposable triplet representation denoted by $\mathcal{R}_1$ which is generated by a $\mathcal{W}$-algebra described below. This new indecomposable representation $\mathcal{R}_1$ is not a generalized highest weight representation (HWR) since the action of L_1 on its generating vector does not vanish. Due to this feature, it does not appear in the Kac-table which only states the HWRs. Schematically, this is denoted by $(\mathcal{V}_0, \mathcal{R}_0, \mathcal{V}_1)$ wherein $\mathcal{R}_1 \cong \mathcal{R}_0$ in the sense that they contain the same number of states.

The pre-logarithmic field is important to compute these relations since in the fusion product with itself the indecomposable representations arise, i.e.

$$\left[-\frac{1}{8}\right] \times \left[-\frac{1}{8}\right] = [\Omega] + [\omega] \,. \tag{79}$$

with Ω and ω standing for the generating fields of $\mathcal{V}_0$ and $\mathcal{R}_0$, respectively.

Additionally, the $c = -2$ model exhibits another symmetry based on the so called $\mathcal{W}$-algebra [18] which is a triplet algebra and precisely referred to as $\mathcal{W}(2,3^3)$. Up to the fields of weight three, the fields generated by the modes of this algebra do not contribute additional fields to those generated by the L_n. For higher weight states, this is no longer the case. Thus we have additional commutation relations

$$[L_m, W^a_n] = (2m-n)W^a_{m+n} \,, \tag{80}$$

$$\begin{aligned}[W^a_m, W^b_n] = \; & g^{ab}\left((2m-n)\Lambda_{m+n} + \frac{m-n}{20}(2m^2+2n^2-nm-8)L_{n+m} \right. \\ & \left. -\frac{m(m^2-1)(m^2-4)}{120}\delta_{n+m}\right) \\ & + f^{ab}_c\left(\frac{5}{14}(2m^2+2n^2-3mn-4)W^c_{m+n} + \frac{12}{5}V^c_{m+n}\right) \,, \end{aligned} \tag{81}$$

with $a, b \in \pm, 0$, $\Lambda(z) = {:}TT{:}(z) - \frac{3}{10}\partial^2 T(z)$ and $V^a(z) = {:}TW^a{:}(z) - \frac{3}{14}\partial^2 W^a(z)$.

The closure of the algebra (i.e. the Jacobi-identity) is only realized if the null vector conditions are imposed on the universal enveloping of the $\mathcal{W}$-algebra. The analysis of the result leaves us with four primary fields, $-\frac{1}{8}, \frac{3}{8}, 0$ and 1 of which the latter two correspond to the irreducible representations $\mathcal{V}_0$ and $\mathcal{V}_1$ whereas the other two irreducible representations do not have special names. More details on this structure can be found in [18].

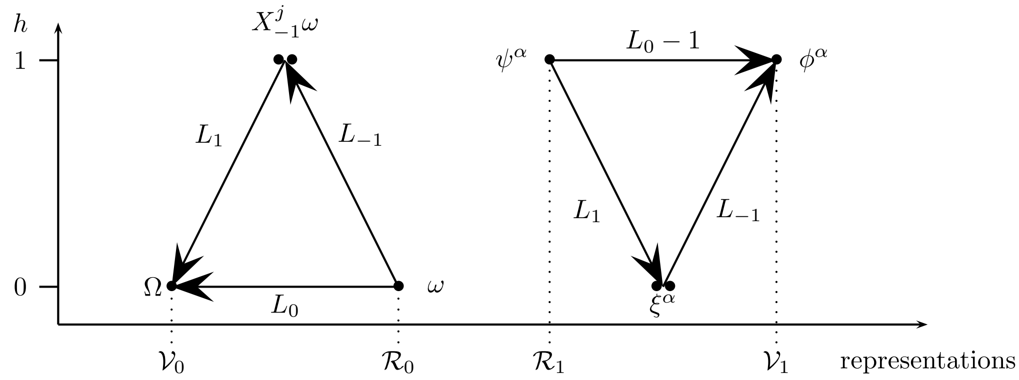

Figure 6: The structure of the representations of $c = -2$. The relations between the generators of the representations are visualized by arrows. The generating states are assigned to their corresponding representation by dotted lines. Every representation is denoted by a point in the sketch.

In figure 6 we therefore have the Jordan block structure

$$\begin{array}{ll} L_0\omega = \Omega\,, & L_0\Omega = 0\,, \\ W_0^a\omega = 0\,, & W_0^a\Omega = 0\,. \end{array} \tag{82}$$

The label X^j_{-1} collectively denotes the four states generated by $L_{-1}\omega$ and $W^\alpha_{-1}\omega$ forming two $su(2)$-doublets.

$\phi^\alpha \equiv \phi^\pm_{h=1}$ is the generator of the indecomposable representation $\mathcal{R}_1$ which contains two ground states, i.e. $\xi^\alpha \equiv \xi^\pm_{h=0}$ and $\psi^\alpha \equiv \psi^\pm_{h=1}$, residing in a Jordan cell generated by L_0. The connections between the fields are given by

$$\begin{array}{ll} L_1\phi^\alpha = -\xi^\alpha\,, & W_1^a\phi^\alpha = t^{a\alpha}_\beta\xi^\beta\,, \\ L_0\phi^\alpha = \phi^\alpha + \xi^\alpha\,, & W_0^a\phi^\alpha = 2t^{a\alpha}_\beta\phi^\beta\,, \\ L_0\xi^\alpha = 0\,, & W_0^a\xi^\alpha = 0\,, \\ L_{-1}\xi^\alpha = \psi^\alpha\,, & W_{-1}^a\xi^\alpha = t^{a\alpha}_\beta\phi^\beta\,, \\ L_0\psi^\alpha = \psi^\alpha\,, & W_0^a\xi^\alpha = 2t^{a\alpha}_\beta\phi^\beta\,. \end{array} \tag{83}$$

The remaining two irreducible representations, $\mathcal{V}_{-1/8}$ and $\mathcal{V}_{3/8}$, are generated by an $su(2)$ singlet μ and an $su(2)$ doublet ν^α, respectively.

2. Percolation

2.1. A phenomenological introduction

2.1.1. What is percolation?

Figure 7: A possible porous stone. [1]

In 1957, Broadbent and Hammersley were the first to formulate the percolation problem asking the question of how probable it is for the center of a porous stone to be wet when laid into a jar of water. The term "percolation" thus refers to the process of random walks through a material depending on the likelihood of ways to be opened or closed. Obviously the probability depends on the size, the shape and the number of open pores.

It is usually modeled based on a lattice, e.g. a subset of $\mathbb{Z}^2$ (the plane square lattice) or the triangular lattice, whose bonds or sites are opened (or closed) with a probability p (or $(1-p)$), $p \in [0,1]$. In the following we will concentrate on bond percolation on the square lattice for easier handling of the problem since the square lattice is dual to itself, making things much easier to calculate (see figure 8). Here the open edges represent open inner passageways of our porous material. Of course, (due to its finite size) a stone may only be represented by a large but finite subset of $\mathbb{Z}^2$, but in physics it is often easier to deal with infinitely large systems or with less dimensions. Thus, in our model, a vertex of the stone will be wet iff there exists a path in $\mathbb{Z}^2$ to some vertex at the boundary running through open bonds. This random subgraph obviously depends on the probability of the bonds to be opened or closed and on the aspect ratio of our two dimensional rectangular stone.

Modeling the system numerically, we find that there exists a critical probability for the bonds or sites to be opened. For an infinite lattice, at this point the probability to cross from one side to another jumps from zero to one. For critical bond percolation on the square lattice, this phase transition occurs at $p_c = 1/2$. At this point the situation changes from having an infinitely large cluster of connected sites (i.e. closed bonds) to non-connected sites (i.e. open bonds). For critical site percolation on the triangular lattices, we encounter the same situation at $p_c = 1/2$. Although bond percolation on the square lattice may be the easiest to compute, site percolation contains a larger number of possible models. Due to its special symmetry, the form of the crossing probabilities can, up to now, only be proven mathematically for the triangular lattice. In general, any bond percolation model has a corresponding model in site percolation but this statement does not hold the other way around.

2.1.2. Phase transitions in physics

Critical behavior is well known from everyday life. In nature, phase transitions do not only occur in the strict sense of thermodynamics between the solid, the fluid and the gaseous phase but for example also in the change from a fluid into a gel. A very simple example would be an egg in boiling water – after five minutes it is solid. What has happened? The molecules, formerly arranged in very small formations group into larger clumps. Another example may be known to the consumers of Pernod. Adding a small amount of water to the beverage does not make any difference, it stays clear. But exceeding the critical portion, the water causes the drink to become opaque.

[0]source: `http://www.terratec.se/minwebbplats/images/sten5.jpg`

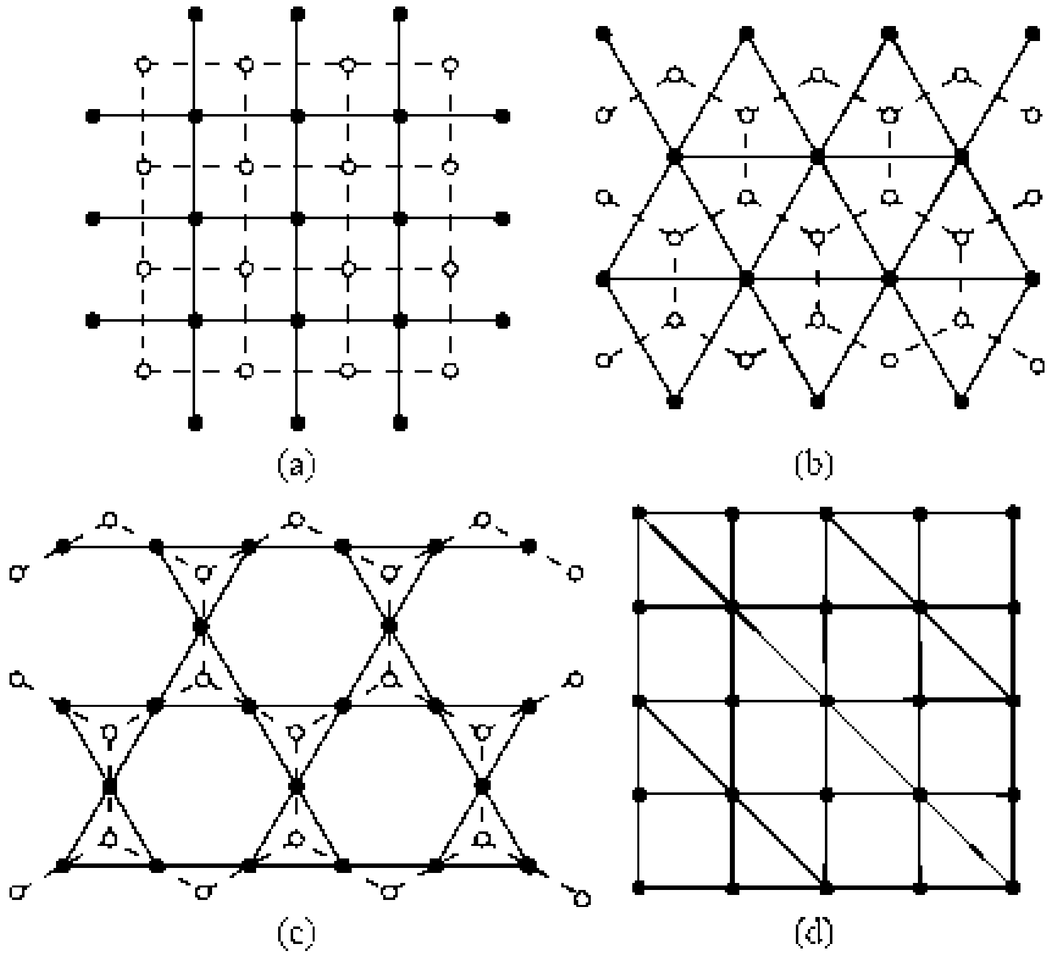

Figure 8: Examples of two dimensional lattices: (a) the self dual square lattice $p_c = 1/2$, (b) the honeycomb lattice (dashed) and its dual, the triangual lattice (solid) $p_c = \sin(\pi/18)$, (c) the hexagonal lattice (dashed) and its covering, the kagome lattice (solid) $p_c = 1 - \sin(\pi/18)$ (d) the bow-tie lattice with $p_c =$ roots of $(1 - p - 6p^2 + 6p^3 - p^5 = 0)$ [37].

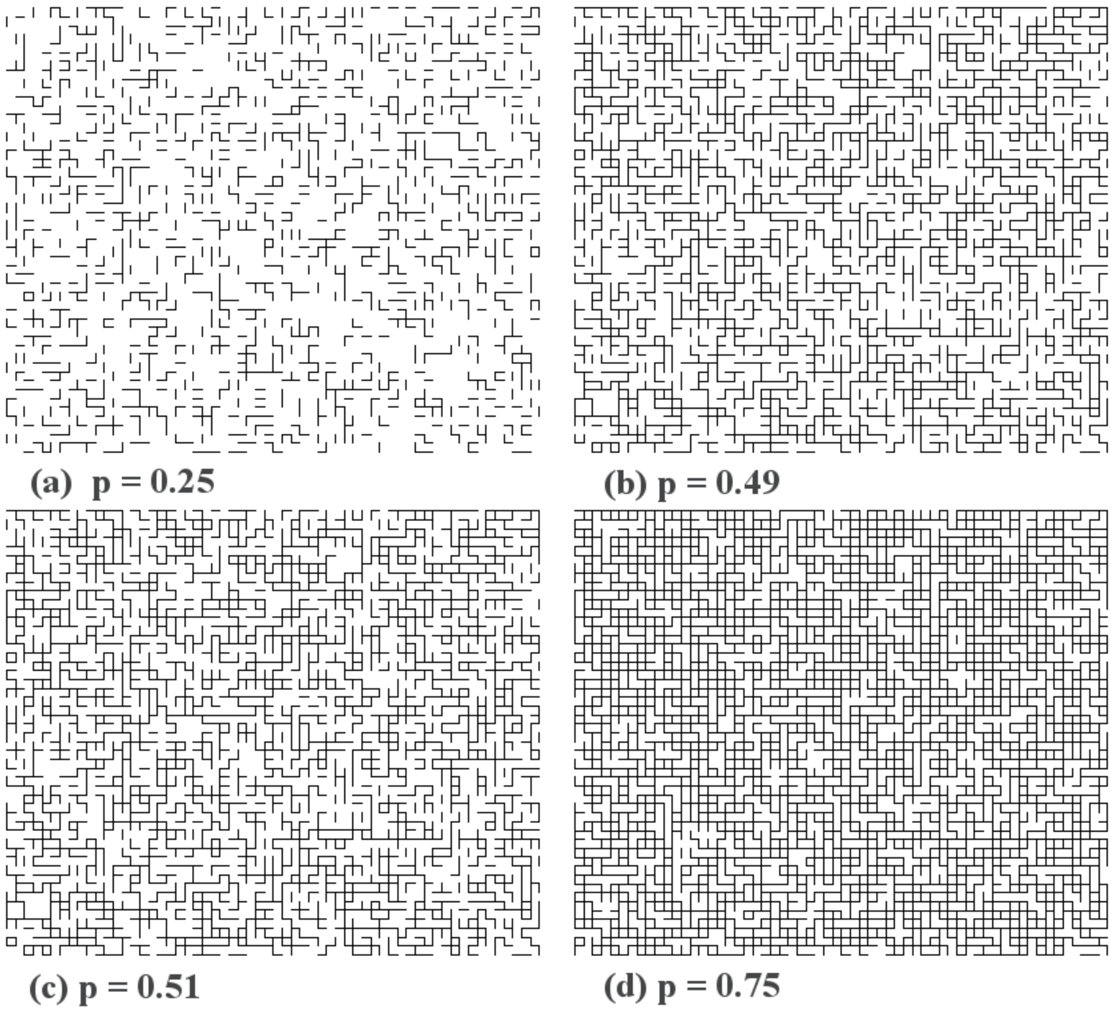

Figure 9: Bond percolation on the square lattice for four different probabilities $p = 0.25, 0.49, 0.51, 0.75$. In this setup, the critical probability is $p_c = 0.5$ [37].

2.1.3. Percolation vs. other disorder models

There are several reasons why we chose percolation as case study and not any other disorder model. First of all it is very easy to formulate as we will see in the following sections. The partition function is especially simple, but not too simple containing a minimal statistical dependence on the probability p of bonds to be open or closed in the bond or site percolation case. Secondly, we have very realistic and qualitatively as well as quantitatively fitting results from percolation models. As an example, there have been numerical simulations on bond percolation on a large rectangular lattice, proving Cardy's formula for the probability of a horizontal crossing from one to the opposite side to be right within an amazing accuracy. Thirdly, percolation is suitable as a playground for more complicated applications such as other disorder models. Last but not least, there are many conjectures concerning percolation, such as Watts' differential equation for the horizontal vertical crossing probability on the rectangular lattice, that remain to be proven and thus provide a huge area of research currently going on. Additionally, percolation can be formulated in many research fields within mathematics and physics, e.g. from a statistical point of view as well as in conformal field theory or in terms of Schramm Loewner Evolution (SLE).

Percolation fits into the greater class of disorder models as a well studied example very much like the harmonic oscillator into quantum mechanics. Complicated problems are often simply being tried to be mapped onto one of the many percolation cases on the different possible lattices with the choice of bond or site percolation. There has been research towards a classification of disorder models categorized by $\mathbb{Z}^d$ with $d \geq 2$ by comparing their behavior to each other and, as said before, often to known percolation models. In various cases, they are known to exhibit the same critical exponents or duality behavior.

Thus, percolation is an important cornerstone for the theory of disordered models and thus a very interesting case to study in $c = 0$ conformal field theories to which those disorder models are widely believed to belong to.

2.2. Applications of percolation

2.2.1. Dynamical percolation

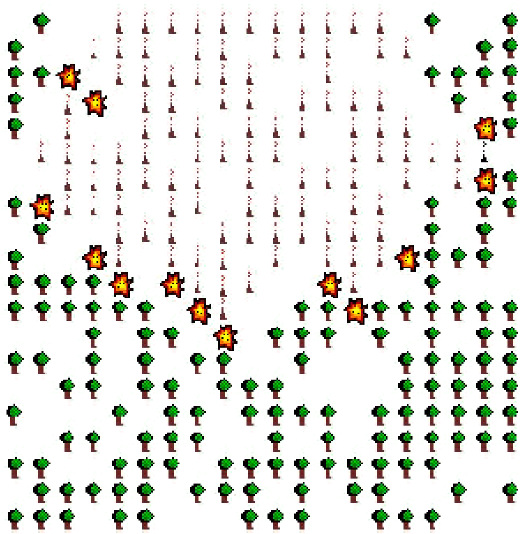

Figure 10: Example of a dynamical percolation process. A forest fire spreads on a square lattice, leaving burnt and intact trees behind. [1]

Percolation is not only interesting as the $Q \to 1$ limit of the Q states Potts model or as an application of $SLE(\kappa, \rho)$ but has also proven to be useful in practice as a model for conductivity of random resistance networks, spreading of diseases and forest fires. Usually, dynamical percolation is needed in most of these cases as introduced by Janssen [46]. The dynamics are often realized as follows: starting with an active site of the lattice, other nearest neighbors in the next step can be activated with a probability p, becoming "infecting" vertices in the next time step and so on. Thus, activation will be spread randomly over the lattice, while any site may die out after some time or has to compete for resources with neighboring sites. After a long time, the overall behavior of the system will depend on the activation rate very much like in the non-dynamical case. Forest fire spreading models are similar to that of epidemics or diseases caused by bacteria or viruses.

[1]source: http://www.ucls.uchicago.edu/people/02/Beckett_Sterner/percolation.gif

2.2.2. Static percolation

An example for static percolation would be disordered electrical networks. In many applications, materials of different effective resistances are mixed, resulting in a composition of unknown resistance. It depends, of course, on the relative proportions of the involved materials, especially in the case of one being a perfect insulator. Regarding an $(n \times n)$ subset of $\mathbb{Z}^2$, connecting the sites with electrical wires of, e.g. 1 Ohm resistance, it would be interesting to know how probable it is for the whole network to be conductive? However, since stone is merely a three-dimensional problem for which CFTs are less restrictive and thus predictive, we will deal with two dimensional problems in the following.

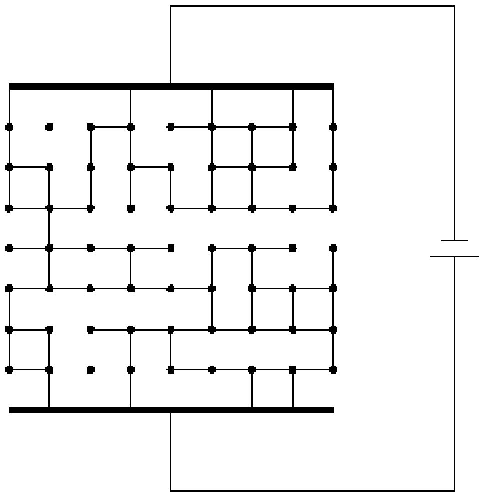

Figure 11: Example of an insulator - conductor composite two dimensional network [37].

One of the most famous examples for site percolation however, is (anti-)ferromagnetism or, as it is usually called, the Ising model. In a magnetic field, the spins inside (anti-)ferromagnetic materials tend to align with the external field. Increasing the external field up to a given value and decreasing it again, leaves the (anti-)ferromagnet in two different types of organization, depending on its temperature. Below the critical (Curie) temperature T_c, in the so called ordered phase, a certain amount of aligned spins will remain whereas above it, no net magnetization can be observed. This phenomenon is called "spontaneous magnetization". The percolation case can be obtained from the Ising model if the neighbors are assumed not to interact. Both belong to a greater set of "generalized percolation models" or random cluster models. More about this link can be found in reference [37].

Another application is the error probability in wafer production. The microchips are usually manufactured on a square grid, of which some are inevitably faulty. For certain applications it would be more useful to take the wafer as a whole without having to check whether the individual chip is intact or not. Thus, in this case, the question is whether there are enough chips for the wafer to work or not.

2.3. Bond percolation and conformal field theory

Since most percolation models are rather similar, this thesis will only deal in detail with the special case of bond percolation on a rectangular lattice of aspect ratio r. Thus we have a subset of $\mathbb{Z}^2$ with bonds connecting nearest neighboring sites with a probability p. Now let $\Pi_h(r)$ be the probability of having a cluster of open bonds spanning from left to right and thus establishing a horizontal crossing through the lattice. In the thermodynamical limit, meaning that the lattice sizes approaches infinity, there exists a critical probability p_c such that $\Pi_h(r) = 0$ for $p < p_c$ and $\Pi_h(r) = 1$ for $p > p_c$. For $p_c = 1/2$ one may find that $\Pi_h(1) = 1/2$.

2.3.1. Percolation as a limit of the Q state Potts model

Bond percolation may also be obtained by taking the $Q \to 1$ limit of the Q states Potts model. On each site of $\mathbb{Z}^2$ we have a spin σ_i attached with $\sigma_i \in Q = \{\sigma_1, \ldots, \sigma_Q\}$. We denote the interaction energy between two nearest neighbor sites i and j having the same spin by J. Thus from statistical mechanics we know that the partition function is given by

$$\begin{aligned} \mathcal{Z} &= \sum_{\{\sigma\}} \prod_{\langle ij \rangle} \left(1 + \exp(-\beta J) \delta_{\sigma_i, \sigma_j}\right) \\ &= \sum_R p^{B(R)} (1-p)^{B-B(R)} Q^{N_C(R)} \end{aligned} \tag{84}$$

with B being the total number of bonds on the lattice of which $B(R)$ are activated, N_C the number of disjoint clusters in R which is a random set of activated bonds. The last factor stands for the number of possibilities of distributing Q colors on $N_C(R)$ clusters and the first part represents the probability of the existence of such a configuration. Obviously, $Q \to 1$ corresponds to percolation.

In case of percolation, the horizontal crossing probability is given by the partition function of only crossings of one color. It can be obtained by the partition function of clusters that include the case of only one color minus the clusters connecting the sides of the lattice of different colors, i.e.

$$\pi_h(r) \propto \lim_{Q \to 1} \left(\mathcal{Z}_{\alpha\alpha} - \mathcal{Z}_{\alpha\beta}\right) . \tag{85}$$

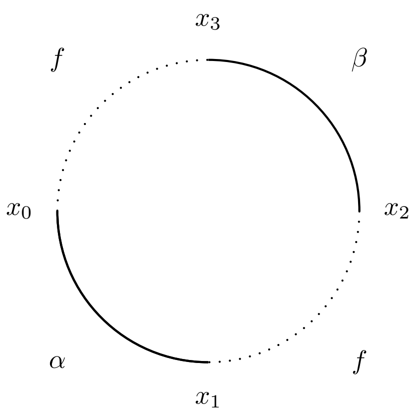

Figure 12: Bounded region with free and two different fixed boundary conditions.

The horizontal crossing probability $\Pi_h(r)$ may be derived in statistical physics as well as in the context of conformal field theory. Taking what we learned from boundary CFT into account, we know that the partition functions of systems with boundary conditions may be expressed by the correlator of the boundary operators inserted at the points at which the conditions change. Mapping the region onto the upper Riemann sphere (including the point at infinity), we can visualize the problem by imagining it on a disc with four boundary conditions as in figure 12.

Now the partition functions of our two cases are given by

$$\begin{aligned} \mathcal{Z}_{\alpha\alpha} &= \mathcal{Z}_f \langle \phi_{(f|\alpha)}(x_0) \phi_{(\alpha|f)}(x_1) \phi_{(f|\alpha)}(x_2) \phi_{(\alpha|f)}(x_3) \rangle \\ \mathcal{Z}_{\alpha\beta} &= \mathcal{Z}_f \langle \phi_{(f|\alpha)}(x_0) \phi_{(\alpha|f)}(x_1) \phi_{(f|\beta)}(x_2) \phi_{(\beta|f)}(x_3) \rangle , \end{aligned}$$

with $\mathcal{Z}_f$ being the partition function for free boundary conditions.

Since we do not yet know which boundary operator we have to insert, we take a look at the Q-state Potts model which is nothing else than the minmal model $\mathcal{M}(m, m-1)$ with $Q = 4\cos^2(\pi/m)$ ($m = 3, 4, 6, \infty$). It turns out that for the horizontal-vertical crossing we have to take $\phi_{(\alpha|\beta)} = \phi_{(1,3)}$ and for the horizontal crossing the correct choice is $\phi_{(\alpha|f)} = \phi_{(1,2)}$. Thus we can make use of the null vector condition for $\phi_{(1,2)}$ and the assumption of scale invariance, $c = 0$, and solve the differential equation,

$$\eta(1-\eta) g'' + \frac{2}{3}(1 - 2\eta) g' = 0 , \tag{86}$$

with η being the anharmonic ratio. The horizontal crossing probability Π_h is now a suitable combination of the two independent solutions, $\eta^{1/3} F(1/3, 2/3, 4/3; \eta)$ and 1. After considering the correct asymptotic behavior, one finds that

$$\Pi_h(r) = \frac{3\Gamma(3/2)}{\Gamma(1/3)^2} \eta^{(1/3)} F(1/3, 2/3, 4/3; \eta) , \tag{87}$$

is the correct solution. Various numerical simulations [56] have proven that this so-called Cardy's Formula is the solution to the critical site percolation problem on the triangular lattice in two dimensions. Since this percolation model exhibits very similar behavior to critical bond percolation on the rectangular lattice, it is widely believed that the result should hold for that case, too.

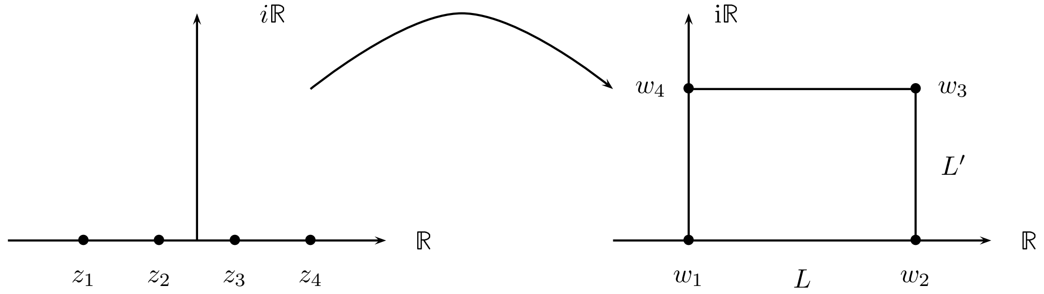

Figure 13: Schwarz Christoffel mapping between the Riemann sphere and the interior of a rectangle of aspect ration $r = L/L'$. The equation for the mapping is given by $w = A\int_0^z \mathrm{d}t\ [(t-z_1)(t-z_2)(t-z_3)(t-z_4)]^{-1/2}$

An especially easy case to compute the exact result occurs when considering a rectangular region on a square grid. By a Schwarz Christoffel mapping we can map the problem of percolation of statistical physics to the correlation of four boundary changing operators in CFT as introduced in the previous chapter. We make the special choice of $(z_1, z_2, z_3, z_4) = (-1/k, -1, 1, 1/k)$ and therefore we can compute the aspect ratio r easily by considering the width and the length of our rectangle

$$L = 2\int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}\,, \tag{88}$$

$$L' = \int_1^{1/k} \frac{\mathrm{d}t}{\sqrt{-(1-t^2)(1-k^2t^2)}}\,, \tag{89}$$

with the aspect ratio being defined by $r := L/L'$. Obviously, these integrals can easily be solved for k. In this parameterization of the z_i the anharmonic ratio is particularly easy, too and is given by

$$\eta = \left(\frac{1-k}{1+k}\right)^2 . \tag{90}$$

Taking Cardy's formula [8], and inserting our previous results, the correctly normalized crossing probability is completely determined

$$\Pi_h(r) = \frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)}\eta^{1/3}{}_2F_1(1/3, 2/3, 4/3; \eta)\,. \tag{91}$$

The dependence on the aspect ratio of the two sides of the rectangle is indirectly present in the solution through the cross ratio η.

Another more elegant solution to the problem has been proposed by Kleban [50], Ziff [82, 83] and Zagier [51] which uses modular forms to describe the crossing probability. Their result is stated in terms of the Dedekind function $\eta = q^{1/24}\prod_n(1-q^n)$:

$$\Pi_h(r) = \frac{2^{7/3}\pi^2}{\sqrt{3}\Gamma(1/3)^2}\int_r^\infty \eta^4(\mathrm{i}r')\,\mathrm{d}r'\,. \tag{92}$$

2.3.2. Percolation and Statistic / Schramm Loewner evolution

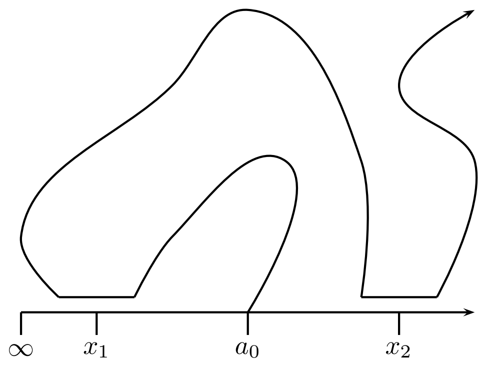

Figure 14: Sketch of SLE. The region between the path and the real axis is an excluded part of the upper half plane. Thus if x_2 gets swallowed before x_1, there can be no crossing between $(x_2, -\infty)$ and (a_0, x_1).

There is another way to approach percolation than starting with a lattice and a bond or site configuration. Stochastic or Schramm Loewner evolution (SLE) has been proven to be statistically equivalent to critical site percolation on the triangular lattice by Smirnov [76] for the horizontal crossing and by Dubedat [16] for the horizontal-vertical crossing. It describes a random walk which leaves i.e. open sites to the right and closed sites to the left, automatically being reflected from itself and the boundary. Usually, SLE is considered on the upper half plane starting at the origin, characterized by a speed κ. Recently, a new type of SLE came up in the literature, characterized additionally by a second parameter $\underline{\rho}$ which includes multiple random curves.

As illustrated in figure 14, a random walk of speed κ starts at a point a_0 on the real axis, randomly running over the upper half complex plane. With $x_1 < a_0 < x_2$, it is obvious that if x_1 is "swallowed" first by the graph, there exists a free path along its outer line, and thus a crossing between the intervals (a_0, x_2) and $(-\infty, x_1)$. If x_2 gets swallowed first, this is not the case. From SLE(κ) follows now that the differential equation for this Bessel process is given by

$$\left(\frac{2}{x_1 - a_0}\frac{\partial}{\partial x_1} + \frac{2}{x_2 - a_0}\frac{\partial}{\partial x_2} + \frac{\kappa}{2}\frac{\partial^2}{\partial a_0^2}\right) P(x_1, x_2; a_0) = 0 \,. \tag{93}$$

which for $\kappa = 6$ has the same solution as the Cardy's famous formula obtained from a level two null vector [7]. Translational invariance enforces $\partial a_0 = -\partial x_1 - \partial x_2$ and conformal invariance ensures that P is a function of the ratio $\eta = \frac{x_2 - a_0}{x_1 - a_0}$.

Another possibility than to write down the equation for the path evolving in time is to look at the conformal mapping $z \to g(z; t)$ which maps the region of the upper half plane which has been excluded by the path onto the whole upper half plane again. The dynamics of the model have now been shifted from the path to the dynamics of such mappings. Now, it has been shown that in the continuum limit, SLE corresponds to percolation via the Loewner equation [73]

$$\frac{\partial g(z; t)}{\partial t} = \frac{2}{g(z; t) - a(t)} \,. \tag{94}$$

The function $a(t)$ is fixed by scaling and locality properties to be a Brownian motion, i.e. $\partial_t a(t) = \xi(t)$, $\xi(t)\xi(t') = \kappa\delta(t - t')$ [74]. The path exhibits different behaviors, depending on κ. For $0 \leq \kappa \leq 4$, it is simple, for $4 < \kappa < 8$ it touches itself and for $\kappa > 8$ it is space filling; in the first two cases, its fractal dimension is given by $1 + \kappa/8$.

For $\kappa = 6$ (as stated above), the differential equation can be led back to Cardy's differential equation for percolation based on a $c_{(3,2)} = 0$ rational CFT. More precisely, the probability of having a crossing of opened sites from $(-\infty, x_1)$ to $(0, x_2)$ is the same as the probability of x_1 getting swallowed by the path before x_2 (see figure 14 for variable assignment).

More on ordinary SLE can be found for example in [72]. Unfortunately it would clearly go beyond the scope of this thesis to introduce the various generalizations of SLE, i.e. SLE($\kappa, \underline{\rho}$) [57],[11],[16] or [3] or multiple SLE(κ) [2] and SLE(κ, b) [60].

3. Watts differential equation and SLE

One of the most important papers for two dimensional percolation in conformal field theory during the last years is that of G. M. T. Watts [78] who motivated a differential equation containing the solutions for the horizontal vertical crossing probability Π_{hv} in two dimensional critical bond percolation. Originally he started with an equation of fifth order based on $c = 0$, which later on has been factorized revealing a level three differential equation. This one turned out to contain the three correct solutions corresponding. In our first paper [28], we presented the first proposal how to interpret the differential equation within a null vector condition action on a four-point correlation function. This section of the thesis will motivate how the ideas of our interpretation of percolation as a rational $c = -24$ CFT evolved.

3.1. A brief review of percolation properties

As already motivated in the second section of this thesis, critical percolation in two dimensions can be described by a conformal field theory. Besides its critical exponents, its partition function should be fixed by the boundary operators which are members of the Kac-table if we assume percolation to be described by a minimal model. From this follows immediately that the three independent crossing probabilities $1, \Pi_h, \Pi_{hv}$ are conformally invariant and, as discussed in the chapter about percolation, should be the solutions of a differential equation arising from a correlator of boundary operators or their descendants [56]. For Π_h, Cardy [7] has already been able to derive an exact solution with the help of boundary conformal field theory which matches the numerical data to a high accuracy.

Five years later, Watts [78] came up with the idea of how to construct boundary operators for Π_{hv} in the context of the $Q \to 1$ limit of the Q-state Potts model. Starting with this model he tried to derive a differential equation within a $c = 0$ theory that agrees with the simulations. As a matter of fact he managed to derive one of fifth order which could be simplified, including the correct third order equation. Additionally to the review on percolation in the second chapter, further details can be looked up in e.g. Kesten [48] or Stauffer and Aharony [77].

3.2. Famous assumptions about percolation

In the previous literature, several arguments have been given to describe the crossing probabilities in two dimensional critical percolation as conformal blocks of a four-point correlation function of ($h = 0$)-operators in a $c = 0$ conformal field theory, using a second (third) level null vector to get Π_h (Π_{hv}). The most prominent are

(1.) (for $c = 0$) the Beraha numbers $Q = 4\cos^2\left(\frac{\pi}{n}\right)$ (with n usually denoted as $m + 1 = 2, 3, 4\ldots$ which in most Potts models are related to the central charge by $c = 1 - \frac{6}{m(m+1)}$ [7]);

(2.) (for $c = 0$) Π_h can be derived mathematically by the Stochastic/Schramm Loewner Evolution (SLE) which strengthens the first argument;

(3.) (for $h = 0$) the proportionality of the partition functions for free boundary conditions to $\mathcal{Z} = 1$ of percolation (as suggested by Cardy [7]);

(4.) (for $c = h = 0$) the interpretation of the central charge as describing the finite size effects of the energy which are believed to be absent.

3.2.1. $c = 0$ from the Beraha numbers

To understand the first point, we give a brief review on the Q-state Potts model (literature for the connection to percolation can be found in [42, 43, 45, 44, 70]). On a simply connected compact region with a piecewise differentiable boundary the horizontal crossing probability Π_h is defined through the partition function. It has originally been derived by Fortuin and Kasteleyn [29, 30] but can also be looked up in e.g. the literature given above or [7, 50, 81].

$$Z = \prod_{(r,r')} \left(1 + x\delta_{s(r),s(r')}\right) = \sum_{G} Q^{N_c} x^{N_b}, \tag{95}$$

where $x = \frac{p}{1-p}$ for $Q \to 1$ and the rightmost sum running over all possible graphs of N_b bonds in N_c clusters. By expanding it in powers of x we can extend the Q-state Potts model to $Q \in \mathbb{R}$.

Π_h describes the probability of having a connection from, e.g., one piece $X = (x_0, x_1)$ of the boundary to another disjoint part $Y = (x_2, x_3)$ where the spins are fixed to values α and β, respectively, while on the rest we have free boundary conditions (for a more detailed introduction see [8]). Hereby any region which can be mapped onto the real axis by a conformal transformation is equivalent (for corners we may get singular behavior but no discontinuities at the corresponding points). For $\alpha \neq \beta$, it is given by [50]

$$\Pi_h(X,Y) = \lim_{Q\to 1}\left(1 - \frac{Z_{\alpha\beta}}{Z_{\alpha\alpha}}\right). \tag{96}$$

In terms of boundary changing operators [4, 6] from free (f) to fixed (α, β) conditions, we get

$$Z_{\alpha,\beta} = Z_f \langle \phi_{(f|\alpha)}(x_0)\phi_{(\alpha|f)}(x_1)\phi_{(f|\beta)}(x_2)\phi_{(\beta|f)}(x_3)\rangle . \tag{97}$$

In the infinite volume limit, these quantities diverge for $Q \neq 1$, but by taking a closer look at the partition function of the Potts Model for $Q \to 1$, we find for a minimal model with central charge $c = 0$ the partition function to be $Z = 1$ in this limit.

For Π_h, the $\phi_{(a|b)}$ are $h_{(1,2)}$ boundary operators, while the results for Π_{hv} contain other boundary operators that can be identified by comparison with known Potts models (i.e. for $Q = 2, 3$) to have weight $h_{(1,3)}$. Another motivation for this ansatz can be found by letting the length of the segment with free boundary conditions tend to zero. Therefore we know from fusion rules that

$$\phi_{(\alpha|f)} \times \phi_{(f|\beta)} \sim \delta_{\alpha\beta} + \phi_{(\alpha|\beta)} , \tag{98}$$

which means that the fusion of two $\phi_{(1,2)}$ boundary operators yields a $\phi_{(1,3)}$ field [7, 50]. Hence we will look out for a rational CFT with a Kac-table which is large enough to contain level three fields (i.e. $\phi_{(1,3)}$ or $\phi_{(3,1)}$).

So far, it seems very reasonable to choose $c = 0$ to describe percolation, but, unfortunately, a minimal model $c_{(3,2)} = 0$ is not very interesting, since its field content only consists of two $h = 0$ fields – $\phi_{(1,1)}$ and $\phi_{(1,2)}$. Thus the $Q \to 1$ limit of the Q-state Potts Model (which corresponds to $c_{(3,2)} = 0$ since both partition functions equal one) does not accommodate Cardy's proposal that boundary operators for the horizontal vertical crossing probability should appear at level $r \cdot s = 3$ in the Kac-table. Thus we might not wish to follow his original approach to the horizontal crossing probability but to reconsider our underlying CFT.

$c = 0$ in an augmented minimal model Naively one could ask why we should not include the $\phi_{(1,3)}$ field into the spectrum. The answer is simple - since the partition function crucially depends on the field content it will no longer be equal to one as suggested by the $Q \to 1$ limit of the Q-state

Potts model procedure. Precisely speaking, including the fields on the boundary of the Kac-table, e.g. $h_{(1,3)} = 1/3$, leads to a so called logarithmic conformal field theory as described in [23, 32].

Taking a closer look on the representation of the boundary condition changing fields, we encounter that it is indecomposable, containing an irreducible subrepresentation with the following character

$$\chi_{(1,3)}(q) = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} (2n+1) q^{3(4n+1)^2/8} , \tag{99}$$

where $\eta(q)$ denotes the Dedekind η-function, $\eta(q) = q^{1/24} \prod_{n\geq 1}(1-q^n)$. As explained in the first part of this thesis, a conformal field theory constructed that way, i.e. by taking the fields on the boundary of the conformal grid and all those necessary to close under fusion, is called augmented minimal model. It is no longer rational in the strict sense but still exhibits the important feature that it contains only finitely many indecomposable and irreducible representations. Additionally, logarithms arise, e.g. in the OPE or the partition function, and therefore it is referred to as an LCFT [39]. It has an effective charge of $c_{\text{eff}} = c - 24h_{\min} = 1$, and thus the correct modular invariant partition function for $c_{(3,2)\text{aug}} = c_{(9,6)}$ is proportional to that of a generic $c = 1$ theory (up to factors like $\log(q\bar{q})$). Indeed, it is given by

$$\mathcal{Z} = \frac{1}{|\eta(q)|^2} \left(|\Theta_{0,6}(q)|^2 + 2 \sum_{\lambda=1}^{5} |\Theta_{\lambda,6}(q)|^2 + |\Theta_{6,6}(q)|^2 \right) , \tag{100}$$

$$\Theta_{\lambda,k}(q) = \sum_{n\in\mathbb{Z}} q^{(2kn+\lambda)^2/4k} . \tag{101}$$

with the radius of compactification given by $2R^2 = 1/(2\cdot 3) = 1/6$. Additionally it is proportional to the partition function of model of the same effective charge, namely $c_{(6,1)} = -24$ which will be important later on.

In contrast to the behavior of the terms arising in ordinary minimal models, the logarithmic corrections are not fixed in magnitude since these contributions can not be interpreted as formal counting of states.

For better comparison, we state the full partition function given by

$$Z_{\text{full}}[\alpha,\beta] = Z + \alpha \frac{\log(q\bar{q})}{|\eta(q)|^2} \sum_{\lambda=1}^{5} |(\partial\Theta)_{\lambda,6}(q)|^2 + \beta \log(q\bar{q})^2 |E_2(q)|^2 , \tag{102}$$

$$(\partial\Theta)_{\lambda,k}(q) = \sum_{n\in\mathbb{Z}} (2kn+\lambda) q^{(2kn+\lambda)^2/4k} , \tag{103}$$

where $E_2(q)$ is the Eisenstein series of modular weight two.

This result can be obtained by solving the modular differential equation. Therefore it suffices to know the vacuum (or any other) character and the conformal weights arising in the representations. This method applied to LCFT can be found explained in detail in [19, 20].

Thus we have shown that in order to interpret the horizontal-vertical crossing probability in a critical bond percolation on a square lattice system using boundary operators, i.e. $\phi_{(1,3)}$, we have to include the boundary of the conformal grid of the $c_{(3,2)} = 0$ minimal model. This results in an enlarged theory with partition function similar to (100), which therefore is definitely not equal to one.

3.2.2. $c = 0$ from SLE

After having excluded the $Q \to 1$ limit of the Q state Potts model as a suitable argument for $c = 0$, we have to take a look at the implications of SLE(κ) on percolation.

Now we will take a look at the second argument for $c = 0$ from Stochastic/Schramm Loewner Evolution (SLE). SLE is based on the orignal work of Loewner [62] and has been applied to Brownian motions e.g. by Lawler, Schramm, Werner and Rhode [60, 61, 72, 73]. These random curves can provide us with another way to formulate the percolation problem (various introductions can be found e.g. in [38, 58, 59, 79, 80]). Unfortunately, up to now it has not been possible to establish a link between Dubedat's [16] proof for Watts' differential equation within an SLE approach and a CFT bond percolation model. Thus we will concentrate on the results for the solution of Cardy's differential equation in the following. Although the issues discussed above concerning the insufficient field content of the minimal model with $c = 0$ do not apply within the SLE setting, we will show that SLE does not necessarily force us to take a CFT with vanishing central charge $c = 0$.

In [12], Cardy gave an elaborate review of how SLE can be applied to calculate crossing probabilities. As already explained in section 2.3.2, a path evolves by a Brownian motion of speed $\kappa = 6$ which repeatedly hits the real axis. In a configuration where the motion starts from a point a_0 on the real axis running all over the complex upper half plane with $x_1 < a_0 < x_2$ being the end points of the crossing intervals, one of the points will be "swallowed" first. For x_1 being the first to be hit by the graph, there obviously exists a free path along the outer line of the graph, for x_2 it is quite as obvious that this is not the case. Thus the probability that there is a crossing between (a_0, x_2) to $(-\infty, x_1)$ is given by a Bessel process, described by a differential equation

$$\left(\frac{2}{x_1 - a_0}\frac{\partial}{\partial x_1} + \frac{2}{x_2 - a_0}\frac{\partial}{\partial x_2} + \frac{\kappa}{2}\frac{\partial^2}{\partial a_0^2}\right) P(x_1, x_2; a_0)\,. \tag{104}$$

From translational invariance we get $\partial a_0 = -\partial x_1 - \partial x_2$ and from conformal invariance, we know that P is a function of the ratio $\eta = \frac{x_2 - a_0}{x_1 - a_0}$. This is exactly the same differential equation one yields with CFT for percolation from a two level null vector [7]. There is also a general expression, relating the speed of the Brownian motion κ to the central charge and thus the highest weight states of the Virasoro algebra (i.e. [1], [12]). For $2 \leq \kappa \leq 8$ this means that

$$c^{\kappa} = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}\,, \tag{105}$$

$$h^{\kappa}_{(r,s)} = \frac{(r\kappa - 4s)^2 - (\kappa - 4)^2}{16\kappa}\,. \tag{106}$$

Hence, $c = 0$ and $h_{(1,2)} = 0$ for $\kappa = 6$ which has been shown to describe Π_h in two dimensional critical site percolation on the triangular lattice [75]. Additionally, Bauer and Bernard [1] stated a direct correspondence between the Q-state Potts model and SLE

$$Q = 4\cos^2\left(\frac{4\pi}{\kappa}\right)\,, \qquad \kappa \geq 4\,, \tag{107}$$

by matching the known value of the dimension of the boundary changing operator for the Q-state Potts model with $h^{\kappa}_{(1,2)}$.

3.2.3. $h = 0$ from the partition function

The third argument makes use of the form of the partition function of the $c = 0$ model. But as we already have shown, the partition function for the augmented $c = 0$ model is not the same as for the minimal $c = 0$ model and thus especially not equal to unity. From this argument, we will show, that we do not longer have to choose $h = 0$ operators as suggested by Cardy [7].

Regarding the problem mentioned above with only a single region with fixed boundary conditions, in the $Q \to 1$ limit, we have

$$Z_\alpha = Z_f \langle \phi_{(f|\alpha)}(x_0)\phi_{(\alpha|f)}(x_1)\rangle = Z_f \times (x_0 - x_1)^{2h} \,. \tag{108}$$

In the minimal model, both partition functions are equal to unity, thus $h = 0$, but in the extended model, we do not know the exact form of Z_f, hence the boundary operator is not a priori fixed in its dimension.

3.2.4. $h = c = 0$ from scaling behavior

The last point addresses the transformation back onto the original region that is described by the formula [7]

$$\langle \phi_0(w_0)\phi_1(w_1)\ldots\rangle = \prod_i |w'(z_i)|^{-h_i} \langle \phi_0(z_0)\phi_1(z_1)\ldots\rangle \,. \tag{109}$$

The expression has a physical meaning in the general non scale invariance of critical systems which picks up a factor $(L/L_0)6ac$ with L being the overall size of the region, L_0 some non universal microscopic scale (i.e. the lattice spacing), c the (effective) central charge and a being dependent on the geometry (i.e. $a = -\pi/\gamma$ if the boundary operator resides in a corner with an interior angle γ, see [7, 49, 50]). Since percolation is assumed to be scale invariant, the effect of the conformal mapping should vanish. But the physical properties of our system only depend on the differential equation arising from null vectors, thus this condition only has to hold in this sense.

In general, this argument considering the finite size scaling effects depends on the asymptotic behavior of the partition function which itself only depends on the central charge modulo 24. Thus, in principle, we can only deduce $c = 0$ up to multiples of 24. Additionally, the correlation functions are invariant in any conformal field theory as long as we take care of the Jacobian determinant arising due to the transformation. As we will see in the following, our solution for Watts' and Cardy's differential equations is merely a quotient of two partition functions and thus the Jacobian transformation factor divides out.

3.3. The Watts differential equation

3.3.1. The original idea

Having explained why we do no longer have to stick to a $c = 0$ theory for percolation, we can take a look at our results for Watts' differential equation [78].

As already mentioned, Watts [78] derived a fifth order differential equation for Π_{hv}, starting from a $c = 0$ theory with $h_{(1,2)} = 0$ boundary changing operators following Cardy's ansatz for Π_h. A priori, as a minimal model $c_{(3,2)} = 0$ we only have two primary fields within the Kac-table, the identity residing at $(1, 1)$ and its duplicated entry at $(1, 2)$. Thus if we assume a null state on the first level $L_{-1}|0\rangle$, we quickly see that from the generic form of the level two null state follows that $L_{-2}|0\rangle = 0$, too, and so on, until the only non-vanishing state is the vacuum itself. Thus, within a true minimal model, there can not be a "direct" null vector on the fifth level whatsoever. Thus, when talking about higher than level two null vectors in a $c = 0$ Kac-table based CFT, we have to add the note that by talking about $c = 0$ we refer to the augmented minimal model, i.e. $c_{(9,6)} = 0$. Whether in this LCFT a null state on the fifth level exists or not remains to be shown. Nevertheless, Watts came up with the correct differential equation for the horizontal-vertical-crossing probability in percolation by motivating a level five null vector which can be interpreted as a level three null vector acting on a level two state as shown in [51]. In a $c = 0$ theory, it seems strange, that in contrary to the results for Π_h, the Π_{hv} boundary operators cannot be identified directly [50]. Considering the asymptotic behavior, one can find the correct expressions for Π_h and Π_{hv} [51] by taking linear combinations of the three physically relevant solutions of

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x(x-1))^{\frac{4}{3}}\frac{\mathrm{d}}{\mathrm{d}x}(x(x-1))^{\frac{2}{3}}\frac{\mathrm{d}}{\mathrm{d}x}F(x)\,, \tag{110}$$

where x is the crossing ratio and F the conformally mapped crossing probability. The equation factorizes into [51]

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x(x-1)) + \frac{1}{2x-1}\frac{\mathrm{d}}{\mathrm{d}x}(2x-1)^2\right)\frac{\mathrm{d}}{\mathrm{d}x}(x(x-1))^{\frac{1}{3}}\frac{\mathrm{d}}{\mathrm{d}x}(x(x-1))^{\frac{2}{3}}\frac{\mathrm{d}}{\mathrm{d}x}F(x)\,, \tag{111}$$

where the rightmost part already provides us with the three expected solutions for the crossing probabilities in percolation.

3.3.2. Interpretation as a level three null vector

Simplifying the third order equation and comparing it to the generic form of a level three null vector in a minimal model, one finds that it has no interpretation in a $c = 0$ theory meaning that we have no level three null vector in $c = 0$ which could give rise to Watts' differential equation.

On the other hand, only for the special choice of the null state $h = h_{(1,3)} = -\frac{2}{3}$, the correlator containing $h_1 = h_2 = h_{(1,3)} = -\frac{2}{3}$ and $h_3 = h_{(1,5)} = -1$ vanishes within a $c_{(p,1)} = -24$ LCFT. More precisely, this means that there is no level three null vector equation from $\phi_{h_{(3,1)}=2}$ or $\phi_{h_{(1,3)}=1/3}$ of this form in $c_{(9,6)} = 0$ whatsoever as we will show in the following.

According to [65], the level three null vector is given by

$$|\chi^{(3)}_{(h,c)}\rangle = \left(L_{-1}^3 - 2(h+1)L_{-2}L_{-1} + h(h+1)L_{-3}\right)|h\rangle\,. \tag{112}$$

In order to get a differential equation we have to remember from the first part of the thesis how the conformal generators correspond to differential operators acting on the correlation function $H(z)$. The calculations can be found in detail in the appendix since the literature often includes mistakes. The differential operators $\mathcal{L}_{-n}$ are defined by

$$\mathcal{L}_{-n}(z) = \sum_i \left(\frac{(n-1)h_i}{(z_i - z)^n} - \frac{1}{(z_i - z)^{n-1}}\partial_{z_i}\right)\,. \tag{113}$$

Letting them act on the four-point function

$$F(z, z_1, z_2, z_3) \equiv \langle\phi_h(z)\phi_{h_1}(z_1)\phi_{h_2}(z_2)\phi_{h_3}(z_3)\rangle\,, \tag{114}$$

yields a quite lengthy expression. Replacing again all derivatives ∂_{z_i} by expressions only containing the derivative ∂_z and finally taking the usual limits $z_1 \to 0$, $z_2 \to 1$ and $z_3 \to \infty$, results in the following ordinary third order differential equation for $F(z) \equiv F(z, 0, 1, \infty)$:

$$\begin{aligned} 0 \;=\;& \frac{\mathrm{d}^3}{\mathrm{d}z^3}F(z) + 2(h+1)\frac{2z-1}{z(z-1)}\frac{\mathrm{d}^2}{\mathrm{d}z^2}F(z) \\ +\;& (h+1)\left(\frac{h-2h_1}{z^2} + \frac{h-2h_2}{(z-1)^2} - 2\frac{h_3-h-h_1-h_2}{z(z-1)} + \frac{h}{z(z-1)}\right)\frac{\mathrm{d}}{\mathrm{d}z}F(z) \\ +\;& h(h+1)\left(-\frac{2h_1}{z^3} - \frac{2h_2}{(z-1)^3} + \frac{(2z-1)(h+h_1+h_2-h_3)}{z^2(z-1)^2}\right)F(z)\,. \end{aligned} \tag{115}$$

Now we have to compare this result to Watts' differential equation in a suitable form [78]

$$\left(\frac{\mathrm{d}^3}{\mathrm{d}z^3} + \frac{5(2z-1)}{z(z-1)}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{4}{3z(z-1)}\frac{\mathrm{d}}{\mathrm{d}z}\right) F(z) = 0\,. \tag{116}$$

Obviously, these equations can not be brought to overlap in this form. Taking advantage of the generic form of the four-point function due to its conformal invariance $F(z) = z^{\mu_{01}}(z-1)^{\mu_{02}}H(z)$ and inserting it into (115) yields a slightly modified differential equation for $H(z)$ for which an appropriate choice of the h, h_1, h_2, h_3 is possible, i.e. $h = h_1 = h_2 = -2/3$ and $h_3 = -1$, meaning that all four weights can be chosen from the Kac-table of one and the same minimal CFT. Additionally, the equation belongs to $c_{(6,1)} = -24$ since the highest weight representation $[-2/3]$ has indeed a third level null vector. Furthermore the special choice of $\mu_{ij} = \frac{1}{3}\sum_k h_k - h_i - h_j$ from $\sum_{j\neq i}\mu_{ij} = -2h_i$, i.e. $(-2/3 - 2/3 - 2/3 - 1)/3 + 2/3 + 2/3 = -1 + 4/3 = 1/3$ holds.

Thus the conformal blocks of the 4-point function

$$\langle \phi_{h=-2/3}(z)\phi_{h_1=-2/3}(0)\phi_{h_2=-2/3}(1)\phi_{h_3=-1}(\infty)\rangle = z^{\mu_{01}}(1-z)^{\mu_{02}}H(z) \tag{117}$$

of the $c = -24$ theory are in one-to-one correspondence with the solutions of Watts' differential equation.

3.4. Properties of the $c = -24$ LCFT

The highest weight that gives rise to the differential equation, $h = -2/3$, generates a reducible but indecomposable representation of the $c_{(6,1)} = -24$ theory. From the study of $c_{(p,1)}$ models we know that the OPE of two fields forming an indecomposable representation contains logarithmic divergences. This is in perfect correspondence to the other two solutions of Watt's original fifth order differential equation. Thus a solution from the augmented minimal models of the type $c_{(3p,3q)}$ is not surprising and it seems to be an interesting application for LCFTs [39, 32, 23]. The logarithmic behavior of such disorder models has already been conjectured before [13] thus the solution fits well into the general expectations.

Taking a look at the fusion rules of $c_{(6,1)} = -24$, we see that $[-2/3] * [-2/3] = [0] + [-2/3] + [1]$ which is in agreement with the three regular solutions of Watts' equation.

Furthermore, this LCFT has, particularly with regards to percolation, a very interesting field content. The entries on the boundary of the conformal grid (which is technically obtained by considering $c_{(18,3)} = c_{(3\cdot 6,3\cdot 1)}$)

0	$-\frac{3}{8}$	$-\frac{2}{3}$	$-\frac{7}{8}$	-1	$-\frac{25}{24}$	-1	$-\frac{7}{8}$	$-\frac{2}{3}$	$-\frac{3}{8}$	0	$\frac{11}{24}$	1	$\frac{13}{8}$	$\frac{7}{3}$	$\frac{25}{8}$	4
4	$\frac{25}{8}$	$\frac{7}{3}$	$\frac{13}{8}$	1	$\frac{11}{24}$	0	$-\frac{3}{8}$	$-\frac{2}{3}$	$-\frac{7}{8}$	-1	$-\frac{25}{24}$	-1	$-\frac{7}{8}$	$-\frac{2}{3}$	$-\frac{3}{8}$	0

contain the critical exponents that are assumed to appear in percolation, shifted by 1, i.e. $h_{(1,2)} = -\frac{3}{8}$ and $h_{(1,4)} = -\frac{7}{8}$. Thus descendants of those fields could describe the physical properties of percolation. It should be mentioned that the $h_{(0,0)} = h_{(1,6)} = -\frac{25}{24}$ field appears in the table which is the so-called pre-logarithmic field whose operator product expansion with itself gives rise to the indecomposable representations [19, 20, 52]. All conformal weights which appear twice in the table above belong to such indecomposable representations.

There is another important remark to state about the rational logarithmic conformal field theory we found to be the solution for the percolation problem. It is not astonishing that our solution, of all possible CFTs, is just this one differing by a multiple of 24 in its central charge. Due to this property its partition function is equivalent to the partition function (102) of the augmented $c = 0$ model discussed before. More precisely, we have [19, 20]

$$Z_{c_{(6,1)}=-24}[\alpha] = Z_{\text{full}}[\alpha, \beta = 0]\,, \tag{118}$$

and thus the non logarithmic parts of the partition functions are identical, meaning that they both count the same states. This is in perfect agreement with our previous observation that most arguments towards which CFT should be taken for percolation depend on conditions that are blind to a change of the central charge by multiples of 24 and thus have the same effective central charge.

3.4.1. Holding on to $c = 0$ in a tensor model

As shown above, the differential equation that provides us with the correct solutions for the horizontal vertical crossing probability points towards a $c = -24$ LCFT. However, for some reasons we might want to stick to a $c = 0$ CFT for percolation. Thus we try a tensor ansatz of two CFTs, one of them being $c = -24$ as needed to satisfy Watts' differential equation and the other being $c = 24$. In this setup, any correlation function, or, more precisely, any field, factorizes into two parts belonging to the two involved CFTs respectively, i.e. $\Phi_H(z) = \Phi_{h,c=-24}(z) \otimes \Phi_{H-h,c=+24}(z)$.

Additionally we assume that the second factor of the third level differential equation

$$G_{c=+24}(z) = \langle \Phi_h(z)\Phi_{h_1}(0)\Phi_{h_2}(1)\Phi_{h_3}(\infty)\rangle_{c=+24} \tag{119}$$

should be trivial since all important information is contained in the first factor given by

$$F_{c=-24}(z) = \langle \Phi_{-2/3}(z)\Phi_{-2/3}(0)\Phi_{-2/2}(1)\Phi_{-1}(\infty)\rangle_{c=-24}\,. \tag{120}$$

A perfect match would be to find

$$H(z) = F_{c=-24}(z)G_{c=+24}(z) \quad\Longrightarrow\quad G_{c=+24}(z) = z^{-1/3}(z-1)^{-1/3}\,. \tag{121}$$

which could come up if $G(z)$ is merely a three-point function, i.e. $\langle \Phi_{1/3}(z)\Phi_{1/3}(0)\Phi_{1/3}(1)\mathbb{I}(\infty)\rangle_{c=+24}$. This result has to be checked within a $c = 24$ CFT on existence and non vanishing.

3.4.2. Cardy's formula and $c = -24$

Obviously, there is a problem now. There are two differential equations numerically "proven" to be correct but arising from two different CFTs, both assumed to describe percolation. But which one is the correct choice? Is there an interpretation of Cardy's formula [7] for Π_h in $c = -24$?

In general, Cardy's formula arises from a level two null vector condition applied to a four-point correlation function,

$$\left(\frac{3}{2(2h+1)}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{2z-1}{z(z-1)}\frac{\mathrm{d}}{\mathrm{d}z} - \frac{h_1}{z^2} - \frac{h_2}{(z-1)^2} + \frac{h+h_1+h_2-h_3}{z(z-1)}\right)F(z) = 0\,. \tag{122}$$

For $c = -24$, we have $\phi_{(1,2)}$ with weight $h = h_{(1,2)} = -\frac{3}{8}$. Now we have to check if the solutions of the differential equation for $c = -24$ span the same solution space as those for $c = 0$, since the latter has already been proven to be correct by numerical simulation of Langlands et. al. [56]. Thus we know that $F(z)$ should be of the form ${}_2F_1(1/3, 2/3, 4/3, z)$. A simple calculation yields $h_1 = h_2 = h_3 = h_{(1,4)} = -\frac{7}{8}$ and $F_1(z) = (z(z-1))^{\frac{1}{4}} \cdot z^{\frac{1}{3}}\ {}_2F_1(1/3, 2/3, 4/3, z)$ as well as $F_2(z) = (z(z-1))^{\frac{1}{4}}$ as the second solution. Hence in comparison to Cardy, the crossing probability for percolation is given by their quotient $\Pi_h \propto F_1/F_2$.

Thus we have shown that we can yield the same horizontal crossing probability as numerically computed by Langlands [56]. It has the desired asymptotic behavior, i.e. vanishes for $z \to 1$ and approaches one for $z \to 0$. Remembering the rectangle whose corners are mapped clockwise in decreasing order to the z_i with $r := (z_3 - z_0)/(z_1 - z_0)$, $r \to 0$ and $r \to \infty$, respectively, with $0 < z < 1$, the correct mapping on the upper complex plane is taking $z_0 \to z$, $z_1 \to 0$, $z_2 \to \infty$ and $z_3 \to 1$.

The normalization is obtained by considering the identity

$$\frac{3\Gamma\left(\frac{2}{3}\right)}{\Gamma^2\left(\frac{1}{3}\right)}\ {}_2F_1(1/3, 2/3, 4/3, z) = 1 - \frac{3\Gamma\left(\frac{2}{3}\right)}{\Gamma^2\left(\frac{1}{3}\right)}(1-z)^{\frac{1}{3}}\ {}_2F_1(1/3, 2/3, 4/3, 1-z)\,. \tag{123}$$

which yields $\frac{3\Gamma\left(\frac{2}{3}\right)}{\Gamma^2\left(\frac{1}{3}\right)}$ as a normalization factor.

Recalling the critical exponents for percolation in $c = -24$, i.e. the level one descendents of $h_{(1,2)} = -\frac{3}{8}$ and $h_{(1,4)} = -\frac{7}{8}$, we see immediately that in principle exactly the two most important fields of percolation play also an important role in the correlation function for the horizontal crossing probability.

3.5. Schramm Loewner evolution and percolation

3.5.1. SLE(κ) and Cardy's percolation formula

Another important thing to be considered are the results of SLE for percolation, showing the equivalence of Cardy's formula and the results for $\kappa = 6$. At first we have to state that the frequently cited proof of Smirnow [75] (or Dubedat [16] as well) only holds for site percolation on a triangular lattice, and according to himself and Werner [76], the method used in [75] can not be applied directly to bond percolation on the square lattice as discussed in this paper. The problem with a proof of bond percolation on the square lattice seems to lie within the properties of the hypergeometric functions which appear to be the solutions of the null vector differential equations. As noted by L. Carleson (we found this mentioned in [8]) the horizontal crossing probability is proportional to

$$\int_0^{\eta} (t(1-t))^{-2/3}\,\mathrm{d}t$$

which is exactly the Schwarz-Christoffel mapping from the upper half plane to a equilateral triangle. Thus, for this special lattice, Π_h becomes very simple which has rigorously been proven by Smirnow as stated above. This problem has been referred to by him as "It seems that $2\pi/3$ rotational symmetry enters in our paper not because of the specific lattice we consider, but rather manifests some symmetry laws characteristic to (continuum) percolation.". For the same reason, Dubedat's proof of Watts' formula [16], using a generalized SLE(κ, ρ), precisely SLE($6, 2, 2$), is only true for the triangular case, too. The connection between SLE and triangular symmetry has also been described by him [15].

Additionally, we know that at one point in the derivation of the differential equation for the SLE(κ)-process, namely the identification evolution operator $\mathcal{A}$ with a level two null vector of a CFT [1], the assumption, that $h_{(1,2)} = 0$ is made. It has consequences on the relation between the coefficients of the differential equation (κ, c and $h_{(1,2)}$) and the evolution operator,

$$\mathcal{A} = -2L_{-2} + \frac{\kappa}{2}L_{-1}^2 \quad vs. \quad L_{-2} - \frac{3}{2(2h_{(1,2)}+1)}L_{-1}^2\,. \tag{124}$$

Hence, we know for $2 \leq \kappa \leq 8$ that

$$\frac{\kappa}{4} = \frac{3}{2(2h_{(1,2)}+1)} . \tag{125}$$

Obviously, this leaves us with $\kappa = 6$ if we restrict ourselves to $h = 0$ in our ansatz for percolation (or equivalently $c_{(3,2)} = 0$ which means $\frac{3}{2(2h_{(1,2)}+1)} = \frac{3}{2}$). But since there are no compulsory conditions to justify this ansatz as explained before, we may question why we should not try $h = -\frac{3}{8}$ and thus $\kappa = 24$ or $h = 4$ and $\kappa = \frac{2}{3}$. We are aware of the fact that if we extend formula (125) to arbitrary values of κ, a solution $\kappa = 24$ is problematic since for this value of κ the curve is space filling. Thus this can be a hint that two dimensional critical bond percolation may have to be formulated in a more complicated setup if it is described by a $c = -24$ LCFT or that for other values of κ relation (125) has to be modified. Therefore we point out that the values of κ for $c = -24$ are exactly four times those of $c = 0$.

3.5.2. SLE($\kappa; b$) and a generalization of Cardy's formula

However, we are left with the question of which of the two choices we should take for the horizontal crossing probability since from original SLE(κ) considerations, we have no data for $c = -24$ (i.e. $\kappa = 24, 2/3$ of which the former is problematic since it yields space-filling curves). But there could be a hint how to see the right path. In 1999, Lawler, Schramm and Werner [60] found a generalization of the SLE(κ) process characterized by a second parameter b. According to them, one can derive a generalized Cardy's formula

$$\Pi_h(b;z) = z^{b+\frac{1}{6}}\,{}_2F_1(\frac{1}{6}+b, \frac{1}{2}+b; 1+2b; z) . \tag{126}$$

which gives us back the known solution for $b = 1/6$. Obviously, this equation can not reproduce all values of b for a single central charge but we could try to find a series of CFTs with four-point functions with central charge c as a function of $b \in \mathbb{Q}$ in order to get minimal models only. Thus we can require all boundary changing operators in the correlator to have weights in the Kac-table.

Matching the general solution of the second-order differential equation (122) arising from a level two null field with (126) yields

$$F(z) = [z(1-z)]^{-\frac{2}{3}h}\Pi_h(b;z) , \qquad h_1 = \tfrac{36b^2-(4h-1)^2}{24(2h+1)} , \qquad h_2 = h_3 = -\tfrac{h(2h-1)}{3(2h+1)} . \tag{127}$$

The question remains whether $h_1, h_2 = h_3$ can be chosen from the Kac-table, too, while h resides in it by construction. This question is non-trivial since c is already fixed by the weight of the field generating the differential equation, $h = \frac{1}{16}(5 - c \pm \sqrt{(c-1)(c-25)})$. In the following, we will express c in this form $c = 13 - 6(t + 1/t)$, introducing a convenient parameter t. Parameterizing $b = p/q > 0$ (p, q coprime) and therefore $h_1 = h_{(r,s)}$ as well as $h_2 = h_3 = h_{(r',s')}$, we see that by inserting the solution for h that

$$s = t\left(r \pm 2\frac{p}{q}\right) , \qquad s' = t\left(r' \pm \frac{1}{3}\right) , \tag{128}$$

with $s, s', r, r' \in \mathbb{N}$ as usual. The parameters of the conformal weights are not entirely fixed but it is obvious, that the famous $c_{(t,1)}$ models with $t = \mathrm{lcm}(3, pq)$ will always be a possible solution such that our requirements are fulfilled and all weights can be taken from the LCFT Kac-table.

Up to now we have overlooked a second condition. The solution $F(z)$ we get by the above considerations is only proportional to $\Pi_h(b;z)$ by a factor $[z(1-z)]^{-\frac{2}{3}h}$. Thus we have to check if we could try the same trick as for $c = -24$, i.e. proving the second solution for the second order differential equation to be exactly this factor. Physically this means that we have a background charge in a free field realization of this CFT which is exactly the charge balance of the fields and thus yielding a trivial second solution. Thus the second condition concerns the charges given by

$$\alpha_{r,s} = \frac{1}{2}(r-1)\sqrt{t} + \frac{1}{2}(1-s)\frac{1}{\sqrt{t}}\,, \qquad \alpha_0 = \frac{1}{2}\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\,. \tag{129}$$

Thus the condition above tells us that $\alpha_{1,2} + \alpha_{r,s} + 2\alpha_{r',s'} = 2\alpha_0$. Since $\alpha_{1,2} + 3\alpha_{r',s'} = 2\alpha_0$ is always fulfilled, we are just left with $\alpha_{r,s} = \alpha_{r',s'}$ which means that $r = r'$ and $s = s'$ and therefore $h_1 = h_2 = h_3$. Hence, for all $b = p/q > 0$ we can chose an LCFT from the augmented minimal models of $c_{(t,1)}$ with $t = \mathrm{lcm}(3, pq)$ to find a solution to the generalized version of Cardy's formula. Defining $t = 3t'$, $t' \in \mathbb{N}$, we have

$$\Pi_h(b;z) = \frac{\langle \phi_{(1,2)}(z)\phi_{(1,3t'(1\pm 2b))}(0), \phi_{(1,2t')}(1)\phi_{(1,2t')}(\infty)\rangle}{\langle \phi_{(1,2)}(z)\phi_{(1,2t')}(0), \phi_{(1,2t')}(1)\phi_{(1,2t')}(\infty)\rangle}\,. \tag{130}$$

As it should be, $3t'(1 \pm 2b)$ is an integer for all choices of rational b, for $b < 1$ the minus sign should be taken as a solution in (128), and for $b > 1$ the positive.

To come to the point, this is a strong hint that we should really take $c = -24$ as a solution for percolation, not only since it provides both differential equations with the correct solutions for Π_h and Π_{hv} for whom a numerical proof exists. But additionally, unlike $c_{(3,2)} = 0$ with $b = 1/6$, it can be extended in a unified fashion to a series of CFTs that for all rational values of b satisfy the generalized version of Cardy's formula.

3.6. Comments on the relation of $c = 0$ and $c = -24$

After having demonstrated how important quantities which can be derived within a $c = 0$ CFT can equally well be deduced within a $c = -24$ rational CFT ansatz, we may ask the question how these two theories are connected besides their effective central charges being the same, as stated above. Therefore let us take a look at the extended Kac-tables for both models.

$c_{(3\cdot 3,3\cdot 2)}$:

0	0	$\frac{1}{3}$	1	2	$\frac{10}{3}$	5	7
$\frac{5}{8}$	$\frac{1}{8}$	$-\frac{1}{24}$	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{35}{24}$	$\frac{21}{8}$	$\frac{33}{8}$
2	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$	1	2
$\frac{33}{8}$	$\frac{21}{8}$	$\frac{35}{24}$	$\frac{5}{8}$	$\frac{1}{8}$	$-\frac{1}{24}$	$\frac{1}{8}$	$\frac{5}{8}$
7	5	$\frac{10}{3}$	2	1	$\frac{1}{3}$	0	0

(131)

and $c_{(3\cdot 6,3\cdot 1)}$:

0	$-\frac{3}{8}$	$-\frac{2}{3}$	$-\frac{7}{8}$	-1	$-\frac{25}{24}$	-1	$-\frac{7}{8}$	$-\frac{2}{3}$	$-\frac{3}{8}$	0	$\frac{11}{24}$	1	$\frac{13}{8}$	$\frac{7}{3}$	$\frac{25}{8}$	4
4	$\frac{25}{8}$	$\frac{7}{3}$	$\frac{13}{8}$	1	$\frac{11}{24}$	0	$-\frac{3}{8}$	$-\frac{2}{3}$	$-\frac{7}{8}$	-1	$-\frac{25}{24}$	-1	$-\frac{7}{8}$	$-\frac{2}{3}$	$-\frac{3}{8}$	0

(132)

Obviously, not by multiplicity but by weight, all fields of the $c = 0$ theory are present in the $c = -24$ as well if we shift them by -1. As already mentioned, the sets of effective conformal weights are thus equivalent. The similarities go further with remarkable consequences when we consider differential equations due to null vectors. For instance, let us take the level two case. For any choice of the other three fields (X, Y, Z) in the four-point function of $c = 0$ Kac-table fields,

$$\langle h^0_{(1,2)} XYZ\rangle \text{ or } \langle h^0_{(2,1)} XYZ\rangle\,, \tag{133}$$

we can find corresponding weights (X',Y',Z') in the Kac-table of $c = -24$ such that there are corresponding four-point functions

$$\langle h^{-24}_{(1,2)} X'Y'Z' \rangle \text{ or } \langle h^{-24}_{(2,1)} X'Y'Z' \rangle \,, \tag{134}$$

which yield the same solutions with respect to the action of the level two null vector operator

$$\frac{3}{2(2h+1)} \mathcal{L}^2_{-1} - \mathcal{L}_{-2}. \tag{135}$$

Since the computation is very easy, we will only state the results: Choosing for the fields X, Y, Z any of the weights H, we have to choose for the fields X', Y', Z' the correpsonding weights H' as given in the two following tables.

$h_{(1,2)}$: (136)

H	$1/8$	0	$-1/24$	$5/8$	1
H'	$-3/8$	$-7/8$	$-25/24$	$13/8$	$25/8$

and analogously

$h_{(2,1)}$: (137)

H	$-1/24$	$1/8$	$5/8$	$33/8$	$21/8$	$35/24$
H'	$-25/24$	-1	$-7/8$	0	$-3/8$	$-2/3$

with $h_{(1,2)}$ being $h = 0$ and $h = -3/8$ and $h_{(2,1)}$ being $h = 5/8$ and $h = 4$ for $c = 0$ and $c = -24$, respectively. Thus it is not surprising that Cardy's formula [7] has also a meaning in $c = -24$.

Additionally, the structure of the Jordan cells of rank two within the two LCFTs is very similar, for any non integer weight we can find triplets corresponding to an irreducible representation which is contained in an indecomposable of the same weight which is isomorphic (with respect to the counting of states) to a hidden indecomposable representation whose subrepresentation is present in the Kac-table and is based on a highest weight differing by an integer from the two other triplet members. Details on this structure can be found explained within the famous $c = -2$ LCFT example [34, 35, 47]. It is present in the $c_{(t,1)}$ series of LCFTs [19] and is conjectured to exist in all augmented minimal models [20]. In the present case, we find such triplets for $c_{(3,2)} = 0$ and $c_{(6,1)} = -24$ respectively as

$$(5/8, 5/8, 21/8) \quad \leftrightarrow \quad (-3/8, -3/8, 13/8) \tag{138}$$
$$(1/3, 1/3, 10/3) \quad \leftrightarrow \quad (-2/3, -2/3, 7/3) \tag{139}$$
$$(1/8, 1/8, 33/8) \quad \leftrightarrow \quad (-7/8, -7/8, 25/8) \tag{140}$$

Unfortunately, the structure of the integer weights (or, more precisely the weights that have previously been inside the Kac-table of the non augmented minimal model) can not be revealed by this analogy since they are assumed to reside in a Jordan cell of rank three [24] which is known not to appear in $c_{(p,1)}$ models. Research on the details is currently going on heading towards a clarification of the representation structure of $c_{(9,6)} = 0$ which will provide us with the necessary knowledge to establish a well-founded link between the two LCFTs rather than just educated guesswork.

4. Percolation as a CFT with vanishing central charge

4.1. General remarks on CFTs with vanishing central charge

During the last decade, interest in $c = 0$ theories rose with regards to percolation and other disorder problems. The problem of vanishing central charge caused a vivid discussion on suitable approaches since the canonical choice of ordinary minimal models does not seem to be sufficient with respect to its field content. There have been several attempts before, i.e. by Cardy [9], Kogan and Nichols [53] or Gurarie and Ludwig [41], which try to explain the deviations of the partition function from one that have been observed by Pearce et. al. [69]. Since the present approaches in the literature so far involve assumptions and extensions which are not necessary for $c \neq 0$ theories, we will concentrate solely on known techniques to fit $c = 0$ into ordinary (L)CFTs.

4.1.1. Problems at $c = 0$

After the introduction of conformal field theories by Belavin, Polyakov and Zamolodchikov [4] twenty years ago and the discovery of logarithmic behavior by Gurarie in 1993 [39] which led to the investigation of so called logarithmic CFT, the understanding of most (L)CFTs, especially the minimal models characterized by the two parameters (p, q) with $q, p \in \mathbb{N}$, $c_{(p,q)} = 1 - 6\frac{(p-q)^2}{pq}$ improved continually. As to CFTs whose field content can not be described solely by the Kac-table, i. e. a non-trivial $c_{(3,2)} = 0$ model, this is not the case. There is still a controversial discussion going on about different approaches to (L)CFTs with vanishing central charge which we will try to elucidate in this paper.

For $c = 0$ as an ordinary minimal model, we have $(p, q) = (3, 2)$ and thus a Kac-table which consists only of one field, the identity. Keeping the vanishing of the central charge in mind, we know that $L_n|0\rangle = 0$ for all $n \in \mathbb{Z}$ and thus the theory is trivial. But from concrete models, e. g. by Pearce and Rittenberg [69], we can deduce that the partition function differs from one and therefore there have to be more fields involved. More concretely, they identify an $h = 1/3$ primary field.

A similar problem occurred in the study of the $c_{(p,1)}$ models whose Kac-tables a priori are empty. Following the procedure which is usually applied to this kind of minimal models, i. e. including the operators on the boundary of the conformal grid into the theory, we get a non trivial $c = 0$ CFT. Additionally it can be shown that this procedure generates indecomposable representations which lead to logarithms in the OPEs of some of these fields [33]. The main advantage of this procedure is that we maintain the properties of all finite Kac-table based CFTs, e. g. the existence of an infinite set of null vectors, thus a rather small field content and the possibility of additional symmetries. Remarkably, up to now in all known logarithmic CFTs, i. e. the $c_{(p,1)}$ models, the identity has a logarithmic partner. Taking the well known formula

$$h_{r,s}(c_{(p,q)}) = \frac{(pr - qs)^2 - (p - q)^2}{4pq} \tag{141}$$

for $q = 1$ and $1 \leq s < 3p$, $1 \leq r < 3q$, i. e. the weights of the operators on the boundary of the conformal grid and those needed for closure under fusion, we always have at least two solutions for $h = 0$: $s_{\pm} = rp \pm (p-1)$. In a logarithmic theory, these cannot be identified with each other. Thus taking a similar approach to construct a non-trivial $c = 0$ LCFT, we would expect it to contain a degenerate vacuum as well.

There exists a variety of proposals on how to approach $c = 0$. Apart from the suggestions of other LCFTs as discussed above, Cardy [9] tried a general replica ansatz in order to find a loophole to the divergences arising in the OPE of primary fields at $c = 0$.

For any conformal field theory (for the time being we will restrict ourselves to non degenerate vacua and the holomorphic parts), we can write down the OPE of a primary scalar field $\phi(z)$ with conformal weight h,

$$\phi_h(z)\phi_h^\dagger(0) \sim \frac{C_{\Phi\Phi}^{\mathbb{1}}}{z^{2h}}\left(1 + \frac{2h}{c}z^2T(0) + \dots\right) + \dots\,, \tag{142}$$

with $C_{\Phi\Phi}^{\mathbb{1}}$ being the coefficient of the three point function usually normalized to 1 or $\frac{c}{h}$ for $h \neq 0$. For the ordinary minimal model $c_{(3,2)} = 0$ the expression is not problematic since the only possible choice for ϕ is $\mathbb{1}$ and thus $h = 0$. Although, if we seek to describe a model as found by [69], we have to assume additional fields to the identity for which the division by the central charge is not well defined.

4.1.2. Suggestions how to treat $c = 0$ properly

According to Cardy [9], there are basically three ways out of the problem as the central charge approaches zero in the OPE of primary fields as given in (142).

(I) $(h, \bar{h}) \to 0$ as $c \to 0$.

(II) $C_{\Phi\Phi}^{\mathbb{1}} \to 0$ as $c \to 0$.

(III) Other operators arise in the OPE, canceling the divergencies.

Thus the first case can be applied to the ordinary minimal model with the Kac-table

$$c_{(2,3)} = 0 \quad : \quad \boxed{0 \,|\, 0}\,. \tag{143}$$

The second case has to be taken if we restrict ourselves to the extended Kac-table as for the $c_{(p,1)}$-models. In this case we have to normalize our three-point functions to $\frac{c}{h_\phi}$ and thus the condition $C_{\Phi\Phi}^{\mathbb{1}} \to 0$ as $c \to 0$ is satisfied trivially and the OPE

$$\phi_h(z)\phi_h^\dagger(0) \sim \frac{c}{hz^{2h}} + \frac{2}{z^{2h-2}}T(0) + \dots \tag{144}$$

stays regular. As discussed above, we expect the identity to have a partner field in these theories, and thus we have to modify the OPE of primary fields.

The third case has been chosen by Kogan and Nichols [53] as well as by Gurarie and Ludwig [41]. It includes a new concept of LCFTs which is structurally different from that of $c_{(p,1)}$ models. Introducing the limiting procedure to get a logarithmic partner of the stress energy tensor, fields outside of the Kac-table arise in the OPE of primary fields. We do not have any knowledge about their behavior in OPEs among themselves and thus there is no a priori limit on the number of fields available in the emerging CFT. Nothing of what is known for Kac-table based models as null states, symmetries or representation properties can be assumed to be extended to this kind of $c = 0$ theory. Furthermore, this approach introduces fields that have no known direct physical meaning at all since in all known applications for $c = 0$ the critical exponents of physical quantities are expected to be out of the Kac-table. Thus it seems more natural to stay within this framework, or, more precisely, include the operators on the boundary of the Kac-table. For example, in the work of Pearce et al. [69], the representation belonging to $h_{(1,3)} = 1/3$ is expected to play an important role.

These are the crucial points where we do not agree with the approach of [9] who suggested that the logarithmic partners of Kac-operators always reside outside of the Kac-table. In contrary, we favor the ansatz of sticking to the restrictive structure of an augmented minimal model following e. g. [20] taking the Kac-table of $c_{(9,6)}$ as a basis to describe a $c = 0$ LCFT.

4.2. OPEs in the augmented minimal model

4.2.1. The standard assumptions for two point functions

In the following we will show how our approach differs from the usual constructions. In contrary to our ansatz it is usually assumed that the Jordan cell exists on the $h = 2$ level and *not* on the identity level as well. But in our opinion any theory with arbitrary central charge $c \neq 0$, if extended to a logarithmic CFT, has to possess a global Jordan cell structure. In standard LCFT, primary fields and their logarithmic partners form Jordan cells with respect to L_0. The identity always resides in such a Jordan cell and, particularly, there can not be a Jordan cell structure at the second level without having this structure in the vacuum sector, although this is the basis for the calculations of [41, 53, 67, 66]. At least we do not know of any LCFT whose identity does not reside in an indecomposable representation.

We will discuss the problem in a general ansatz for arbitrary values of the central charge c. Thus the following statements hold for any such LCFTs and, particularly, for vanishing central charge (see e.g. [71] or [22] for an elaborate treatment). The two-point functions for the $h = 0$ sector in a rank two Jordan cell setup are usually assumed to be given by:

$$\begin{aligned} \langle 0|0\rangle = \langle 0|\mathbb{1}|0\rangle &= 0\,, &(145)\\ \langle\tilde{0}|0\rangle = \langle 0|\tilde{0}\rangle = \langle 0|\tilde{\mathbb{1}}(z)|0\rangle &= 1\,, &(146)\\ \langle 0|\tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w)|0\rangle &= -2\log(z-w)\,. &(147)\end{aligned}$$

From the first vacuum expectation value follows directly, that any n-point function containing only proper primary fields and their descendants, i.e. no logarithmic partner fields, vanishes. This is also valid for the two point function of the stress energy tensor with itself, thus $\langle TT\rangle$ vanishes in any LCFT since its central term is proportional to the identity which can easily be seen in the OPE given by

$$T(z)T(w) = \frac{c/2}{(z-w)^4}\mathbb{1} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial T(w)\,. \tag{148}$$

Denoting the logarithmic partner of the stress energy tensor by $t(z)$ it follows from the standard LCFT procedures, that the OPE between the stress energy tensor $T(z)$ and his partner reads as

$$T(z)t(w) = \frac{c/2}{(z-w)^4}\tilde{\mathbb{1}} + \frac{\mu}{(z-w)^4}\mathbb{1} + \frac{2}{(z-w)^2}t(w) + \frac{\lambda}{(z-w)^2}T(w) + \frac{1}{(z-w)^1}\partial t(w)\,. \tag{149}$$

where the Jordan cell becomes apparent in the extra $(z-w)^{-2}$ term with the normalization of the off diagonal entries, λ, usually set to one. A very simple construction of such a logarithmic partner field is $t(z) = {:}\tilde{\mathbb{1}}T{:}(z)$ which we will use in our calculations later on.

In the latter OPE, the central term is made up of two contributions, proportional to $\tilde{\mathbb{1}}$ and $\mathbb{1}$, respectively. Taking the vacuum expectation value, we quickly see that the two-point function vanishes for $c = 0$. In general, the two-point functions of the stress energy tensor pair are given by

$$\begin{aligned} \langle T(z)T(w)\rangle &= 0\,, &(150)\\ \langle T(z)t(w)\rangle &= \frac{b}{(z-w)^4}\,, &(151)\\ \langle t(z)t(w)\rangle &= \frac{1}{(z-w)^4}\left(\theta - 2b\log(z-w)\right)\,, &(152)\end{aligned}$$

where the normalization of the two point function $\langle Tt\rangle$ appears again twice in the vacuum expectation value of $\langle tt\rangle$.

4.2.2. The generic form of the OPE

As already explained in the first part of the thesis, the form of the OPE can be deduced from global conformal invariance alone including the form of the two and three point functions and their symmetry properties following [22, 26]. For LCFTs we have the following OPE:

$$\phi_{h_i}(z)\phi_{h_j}(w) = \sum_k C_{ij}^k (z-w)^{h_k} \left(\phi_{h_k} + \sum_{\{n\}} \beta_{ij}^{k,\{n\}} (z-w)^{|\{n\}|} \phi_{h_k}^{(-\{n\})}(w) \right) \tag{153}$$

where the coefficients $\beta_{ij}^{k,\{n\}}$ of the descendant contributions,

$$\phi_{h_k}^{(-\{n\})} = L_{(-\{n\})}\phi_{h_k} = L_{-n_1}L_{-n_2}\dots L_{-n_l}\phi_{h_k}\,, \tag{154}$$

are fixed by conformal covariance. The structure “constants” C_{ij}^k (which in an LCFT can no longer referred to as constants since they partly become functions containing logarithms) can be derived through the two- and three-point functions, i. e. $C_{ij}^k = C_{ijl}D^{lk}$, with

$$\begin{aligned} D_{ij} &= \langle \phi_{h_i}(\infty)\phi_{h_j}(0)\rangle \propto \delta_{h_i,h_j}\,, & (155)\\ C_{ijk} &= \langle \phi_{h_i}(\infty)\phi_{h_j}(1)\phi_{h_k}(0)\rangle\,. & (156) \end{aligned}$$

Note that in our case of an LCFT the metric is no longer diagonal but for $h \equiv h_i = h_j$ looks like

$$D_{(i,j)} = \begin{pmatrix} 0 & D_{\Phi\Phi}^{(0)} \\ D_{\Phi\Phi}^{(0)} & D_{\Phi\Phi}^{(1)} - 2D_{\Phi\Phi}^{(0)}\log(z-w) \end{pmatrix} (z-w)^{-2h} \tag{157}$$

in the notation following below.

Taking a general ansatz, the two point functions are given by

$$\begin{aligned} \langle \Phi(z)\Phi(w)\rangle &= 0\,, & (158)\\ \langle \Phi(z)\tilde\Phi(w)\rangle = \langle \tilde\Phi(z)\Phi(w)\rangle &= D_{\Phi\Phi}^{(0)}(z-w)^{-2h}\,, & (159)\\ \langle \tilde\Phi(z)\tilde\Phi(w)\rangle &= \left(D_{\Phi\Phi}^{(1)} - 2\log(z-w)D_{\Phi\Phi}^{(0)}\right)(z-w)^{-2h}\,, & (160) \end{aligned}$$

with $D_{\Phi\Phi}^{(0)} = D_{\Phi\tilde\Phi} = D_{\tilde\Phi\Phi}$ and $D_{\Phi\Phi}^{(1)} = D_{\tilde\Phi\tilde\Phi}$. Here Φ denotes a primary field and $\tilde\Phi$ its logarithmic partner.

The form of the three point functions is more complicated, thus omitting the dependence on the coordinates, z_1, z_2, z_3 we have only to pay attention to those three point functions that at least contain a logarithmic field $\tilde\Phi$, since terms containing only Φ vanish.

$$\begin{aligned} \langle TT\Phi\rangle &= 0\,, & (161)\\ \langle TT\tilde\Phi\rangle &= C_{TT\Phi}^{(0)} z_{12}^{h-4} z_{13}^{-h} z_{23}^{-h}\,, \\ &= \langle tT\Phi\rangle = \langle Tt\Phi\rangle & (162)\\ \langle tt\Phi\rangle &= \left(C_{TT\Phi}^{(1)} - 2\log(z_{12})C_{TT\Phi}^{(0)}\right) z_{12}^{h-4} z_{13}^{-h} z_{23}^{-h}\,, & (163)\\ \langle tT\tilde\Phi\rangle &= \left(C_{TT\Phi}^{(1)} - 2\log(z_{13})C_{TT\Phi}^{(0)}\right) z_{12}^{h-4} z_{13}^{-h} z_{23}^{-h}\,, & (164)\\ \langle Tt\tilde\Phi\rangle &= \left(C_{TT\Phi}^{(1)} - 2\log(z_{23})C_{TT\Phi}^{(0)}\right) z_{12}^{h-4} z_{13}^{-h} z_{23}^{-h}\,, & (165)\\ \langle tt\tilde\Phi\rangle &= \Big(C_{TT\Phi}^{(2)} - C_{TT\Phi}^{(1)}(\log(z_{12}) + \log(z_{13}) + \log(z_{23})) \\ &\quad - C_{TT\Phi}^{(0)}(\log^2(z_{12}) + \log^2(z_{13}) + \log^2(z_{23}) - 2\log(z_{12})\log(z_{13}) \\ &\quad - 2\log(z_{12})\log(z_{23}) - 2\log(z_{13})\log(z_{23}))\Big) z_{12}^{h-4} z_{13}^{-h} z_{23}^{-h}\,. & (166) \end{aligned}$$

With these results we can compute the structure constants in the OPE since theses three-point functions, formerly denoted by C_{ijk} with $i, j, k \in \{T, t, \Phi, \tilde{\Phi}$, are connected to the structure constants through the metric as stated before, i.e. $C_{ij}^{k} = C_{ijl}D^{lk}$. Inserting these results into (153), we get the OPE.

As already stated for the case of the identity, the proportionality factors coming with the logarithmic terms are always multiples of the respective correlator of fields of the same block but with less level in the same Jordan cell. They depend on the total number of logarithmic partner fields within the correlator. In our case, we have to look out for the contributions of the two lowest weight fields, meaning that $\Phi, \tilde{\Phi}$ are either $\mathbb{1}, \tilde{\mathbb{1}}$ or $T(z), t(z)$, which will give us the most singular terms of the OPE of $t(z)t(w)$.

In detail, the OPE for the LCFT case stated in (153) in its explicit form for the three interesting cases, $T(z)T(0)$, $T(z)t(0)$ and $t(z)t(0)$ is given by

$$T(z)T(0) = z^{h-4}\frac{C^{(0)}_{TT\Phi}}{D^{(0)}_{\Phi\Phi}}\Phi(0)\,, \tag{167}$$

$$T(z)t(0) = z^{h-4}\left(\frac{C^{(0)}_{TT\Phi}}{D^{(0)}_{\Phi\Phi}}\tilde{\Phi}(0) + \frac{C^{(1)}_{TT\Phi}D^{(0)}_{\Phi\Phi} - C^{(0)}_{TT\Phi}D^{(1)}_{\Phi\Phi}}{(D^{(0)}_{\Phi\Phi})^2}\Phi(0)\right)\,, \tag{168}$$

$$\begin{aligned} t(z)t(0) = z^{h-4}\Bigg[&\left(\frac{C^{(1)}_{TT\Phi}}{D^{(0)}_{\Phi\Phi}} - 2\log(z)\frac{C^{(0)}_{TT\Phi}}{D^{(0)}_{\Phi\Phi}}\right)\tilde{\Phi}(0) + \left(\frac{C^{(2)}_{TT\Phi}D^{(0)}_{\Phi\Phi} - C^{(1)}_{TT\Phi}D^{(1)}_{\Phi\Phi}}{(D^{(0)}_{\Phi\Phi})^2}\right. \\ &\left. - \log(z)\frac{C^{(1)}_{TT\Phi}D^{(0)}_{\Phi\Phi} - 2C^{(0)}_{TT\Phi}D^{(1)}_{\Phi\Phi}}{(D^{(0)}_{\Phi\Phi})^2} - \log^2(z)\frac{C^{(0)}_{TT\Phi}}{D^{(0)}_{\Phi\Phi}}\right)\Phi(0)\Bigg] \end{aligned} \tag{169}$$

Taking into account what we already know about the OPE of $T(z)T(0)$ and $T(z)t(0)$, we can fix most of the free parameters that remain in the equations above.

First of all we know that usually the OPE between the two true identities is fixed to one, and thus $D^{(0)}_{\mathbb{1}\mathbb{1}} = 1$. The link to its logarithmic partner is not fixed by any condition and thus may be left free, for simplicity denoted by $D^{(1)}_{\mathbb{1}\mathbb{1}} = d$. To be consistent with our ansatz, we have to fix the normalization for the stress energy operator contributions to $D^{(0)}_{TT} = b$ and $D^{(1)}_{TT} = \theta$. From the equations (148) and (149) we know that by comparing the coefficients, we have to put $C^{(0)}_{TTT} = 2b$, $C^{(1)}_{TTT} = \lambda b + 2\theta$ and $C^{(0)}_{TT\mathbb{1}} = c/2$, $C^{(1)}_{TT\mathbb{1}} = \mu + cd/2$.

But we can not chose the normalization of the two-point functions of $T(z)$ and $t(z)$ independently of the central charge of the given LCFT. Taking advantage of the fact that the vacuum expectation values do not change if we extend any two-point function to a three-point function by inserting the identity as a third field, we get relations among the undetermined constants so far. Thus the OPE will only be consistent if we chose $D_{Tt} = C_{Tt\mathbb{1}} = C_{TT\tilde{\mathbb{1}}}$ and $D_{tt} = C_{tt\mathbb{1}} = C_{Tt\tilde{\mathbb{1}}}$ and hence $D^{(0)}_{TT} = C^{(0)}_{TT\mathbb{1}}$ from which follows directly that $b = c/2$. Furthermore, we have $D^{(1)}_{TT} = C^{(1)}_{TT\mathbb{1}}$ and thus $\mu = \theta - cd/2$.

This leaves us with only three free parameters – the central charge c, the normalization d of the two point function $\langle\mathbb{1}\tilde{\mathbb{1}}\rangle$ and θ of $\langle\tilde{\mathbb{1}}\tilde{\mathbb{1}}\rangle$ which can also be regarded as the central extension of the algebra between the modes of T and t, μ which differs from θ only by a linear combination of the other free parameters, i.e. $\mu = \theta - cd/2$ as stated above.

Inserting the our knowledge into the general OPE, we see that $t(z)t(0)$ has the following structure:

$$\begin{aligned} t(z)t(0) &= z^{-4}\left(\theta - \log(z)c\right)\tilde{\mathbb{1}}(0) \\ &+ z^{-4}\left(C^{(2)}_{TT\mathbb{1}} - \theta d + \log(z)(cd-\theta) - \log^2(z)\frac{c}{2}\right)\mathbb{1} \\ &+ z^{-2}\left(1 + \frac{4\theta}{c} - 4\log(z)\right)t(0) \\ &+ z^{-2}\left(2\frac{C^{(2)}_{TTT} - \theta}{c} - \frac{4\theta^2}{c^2} - \log(z)(1 - \frac{4\theta}{c}) - 2\log^2(z)\right)T(0)\,. \end{aligned} \tag{170}$$

The choice $\theta = 0$ is similar to the result of Gurarie and Ludwig although the ansatz is quite different. In the approach chosen above, we only see the primary fields and the logarithmic partners, but none of the descendants. This is why our formula misses the canonical terms proportional to ∂t, ∂T and $\partial\tilde{\mathbb{1}}$. Particularly, the formula given above does not account for any descendants that could affect the multiplicities of t and T and their descendants since both are level two descendants of the identity and its logarithmic partner, respectively, i.e. $L_{-2}\mathbb{1} = T$ and $L_{-2}\tilde{\mathbb{1}} = t$, and both arise in the most singular term proportional to z^{-4}.

The result is also very similar to the one derived in [53] (but only for $\theta = 0$, too). Allowing for the terms containing $\tilde{\mathbb{1}}$ to differ, we see that the two characteristic parameters can be described by the central charge and $C^{(2)}_{TTT}$. But $\theta = 0$ implies a vanishing two-point function for $\langle Tt\rangle$ and $\langle tt\rangle$, at least for our ansatz of a Jordan cell structured identity sector with $b = \frac{c}{2} = 0$. Thus if our ansatz is correct we should try to find a realization where this crucial parameter is fixed to some non vanishing constant.

Most of the results concerning the stress energy tensor have also been derived in [65], especially $b = \frac{c}{2}$ and the normalization of the Jordan cell of the stress energy tensor.

4.2.3. Consequences on the $c \to 0$ catastrophe

The impact on the OPE of primary fields of the results derived in the previous section is immense.

Therefore let us recall what we know about the general form of the OPE in equations (153ff). Inserting the fact that for our ansatz (and $h = 0$) we have $D^{(0)}_{\Phi\Phi} \propto \langle \frac{c}{2}\tilde{\mathbb{1}}(w) + \mu\mathbb{1}\rangle = 0$, we run into a problem inverting the matrix of two-point functions which is needed to raise indices, since $D^{ij} = (D_{ij})^{-1}$. Hence the OPE of two primary fields in a $c = 0$ theory with a Jordan cell structure on the $h = 0$ level and $T(z), t(z)$ being descendants of the $h = 0$ fields, remains ill defined. The only loophole to this could be to define the normalization of the three-point functions to $\frac{c}{h_\phi}$. Thus for $h \neq 0$, the metric would be invertible again following the suggestion of Cardy [9].

Since for $h = 0$ the problem is not solved yet, this brings up the question whether for $c = 0$ we can still stick to our usual definition of vacuum expectation values or if we should simply redefine the vev to be proportional to

$$\langle\cdot\rangle := \langle 0|\cdot|\tilde{0}\rangle + \langle\tilde{0}|\cdot|0\rangle\,, \tag{171}$$

leaving us with the problem of how expressions like the vev of the OPE of $t(z)t(w)$ may be dealt with. A motivation for this behavior may be found in [40] where the vanishing of the fermionic path integral in the $c = -2$ LCFT is discussed.

4.3. A bosonic free field construction

4.3.1. Ansatz

To illustrate the results obtained from the most singular term of the OPE by global conformal invariance, we take a free field construction with arbitrary central charge for the stress-energy tensor and its logarithmic partner field:

$$
\begin{aligned}
T(z) &= -\frac{1}{2}{:}\partial\phi(z)\partial\phi(z){:} + i\sqrt{2}\alpha_0{:}\partial^2\phi(z){:}\,, && (172)\\
t(z) &= {:}\lambda\phi(z)\tfrac{\exp\left(i\sqrt{2}a\phi(z)\right)}{i\sqrt{2}\alpha_0}T(z){:}\,. && (173)
\end{aligned}
$$

For the logarithmic partner of the identity we chose a vertex operator ansatz with conformal weight $h(a) = a^2 - 2a\alpha_0 = 0$ which means that we have two possible weights for the Vertex operator, $h(a = 0)$ for the true identity and $h(a = 2\alpha_0)$ for the second. Thus we expect another vertex operator to appear in the OPE behaving like the identity in correlators. Hence we define with $a = 2\alpha_0$:

$$
\begin{aligned}
\tilde{\mathbb{1}}(z) &= \lambda\phi(z)\frac{\exp\left(i\sqrt{2}a\phi(z)\right)}{i\sqrt{2}\alpha_0}\,, && (174)\\
\mathbb{1}'(z) &= \exp(\mathrm{i}\sqrt{2}a\phi(z)) \ \equiv \mathbb{1}\,. && (175)
\end{aligned}
$$

Similar considerations can be found within the Coulomb gas formalism used in [52].

Note that there is a subtlety here. This ansatz can not be directly compared to the general formula, especially not for $t(z)t(w)$ as in (170) since the propagator is not of the standard form. We do not only have the identity $\mathbb{1}$ and its logarithmic partner field $\tilde{\mathbb{1}}$ but in addition also a field that is conjugated to the identity, $\mathbb{1}' = \exp(\mathrm{i}\sqrt{2}a\phi)$ with $a = 2\alpha_0$. Thus contributions by this field have to be taken into account, too. This leads to different prefactors and changes in signs. Additionally, it is not surprising that we are not able to get the coefficients of $T(w)$ on the rhs of $T(z)t(w)$ and those of the most singular terms in the OPE of $t(z)t(w)$ to overlap. This is simply due to the fact that the normalization of the Jordan cell of the stress energy tensor is already fixed by that of the identity. This is the reason why some factors appear twice as often as expected when compared to the OPE derived by Gurarie and Ludwig [41] or Kogan and Nichols [53] apart from the simple fact that their whole ansatz lacks a logarithmic partner for the identity.

The reader may ask the question why we did not choose a better matching ansatz, but the answer is simple. We want to have logarithmic terms of orders not higher than two and a simple constructed $t(z) =: T(z)\tilde{\mathbb{1}}(z) :$ in a bosonic free field construction. As $\tilde{\mathbb{1}}(z)$ is a weight $h = 0$ field, it may only consist of a linear combination of fields ϕ and not their derivatives. Taking the general Ansatz for $t(z) = \sum_{n,m=0}^{\infty} a_n : \phi^n(z)T(z) :$ and computing the OPE $T(z)t(w)$ which is known to be of the following form

$$
T(z)t(w) \sim \frac{\frac{c}{2}\tilde{\mathbb{1}}(w)}{(z-w)^4} + \frac{2t(w) + \lambda T(w)}{(z-w)^2} + \frac{\partial t(w)}{z-w}\,, \qquad (176)
$$

it is easy to check that we end up with a vertex operator ansatz for $\tilde{\mathbb{1}}$ since otherwise we can not bring the OPE to overlap with (176). Thus, for a bosonic free field construction, there is no way around these vertex operator ansatz but it may certainly be that within a fermionic approach, things will turn out comparable to the general form discussed in the previous section. Additionally, it would have a natural truncation feature due to the nilpotence of its fields such that logarithmic terms of higher order than $\log^2$ are not able to appear by construction.

4.3.2. The non logarithmic OPEs

Of course, the OPE of the stress energy tensor with itself is as usual,

$$T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,, \tag{177}$$

thus, due to the vanishing vacuum expectation value (vev) of $\mathbb{1}$, we have

$$\langle T(z)T(w)\rangle = 0\,. \tag{178}$$

The first different result appears in the correlator of the stress energy tensor with its logarithmic partner:

$$T(z)t(w) \sim \frac{\frac{c}{2}\tilde{\mathbb{1}}(w)}{(z-w)^4} + \frac{2t(w)+\lambda T(w)}{(z-w)^2} + \frac{\partial t(w)}{z-w}\,, \tag{179}$$

where λ depends on the normalization of the off-diagonal entries of the Jordan cell between $T(z)$ and $t(z)$, i. e. $L_0 t(z) = 2t(z) + \lambda T(z)$. These two results are exactly of the form that we derived within our general ansatz. Since the vev of $\tilde{\mathbb{1}}$ does not vanish in an LCFT, we are left with a vacuum expectation value of

$$\langle T(z)t(w)\rangle = \frac{\frac{c}{2}}{(z-w)^4} = 0\,. \tag{180}$$

However, for $c = 0$ this is equal to zero, too.

4.3.3. The logarithmic OPE

The OPE of $t(z)$ with itself, however, is more complicated as one would expect from the derivation from its most singular term due to the appearance of descendants of $\mathbb{1}$ and $\tilde{\mathbb{1}}$ as already mentioned above. Another point to bear in mind are the divergences of lowest order, $\log(z-w)(z-w)^0$, which have been omitted in the literature before.

To keep things as simple as possible, we will just state our result for the case that $4-2a^2 > 0$ and omit the terms that are dispensable for the comparison with the results of [41] and our general calculation (170) which means that we will restrict ourselves to the contributions of $T, t, \mathbb{1}$ and $\tilde{\mathbb{1}}$ up to first order without logarithmic divergences or composite fields. The full results will be given in the appendix. Choosing $\lambda = \frac{1}{2}$ and for $c = 1-24\alpha_0^2 = 0$, $\alpha_0^2 = \frac{1}{24}$ we get:

$$\begin{aligned} t(z)t(w) \quad \sim \quad & \left(-2\log(z-w)\tilde{\mathbb{1}}(w) + \frac{1}{2c} + \log^2(z-w) + 3\log(z-w)\right)\frac{\frac{c}{2}}{(z-w)^4} \\ & + \frac{(1-4\log(z-w))t(w) + \left(3\log(z-w) + 2\log^2(z-w)\right)T(w)}{(z-w)^2} \\ & + \frac{(1-4\log(z-w))\partial t(w) + \left(3\log(z-w) + 2\log^2(z-w)\right)\partial T(w)}{2(z-w)}\,. \end{aligned} \tag{181}$$

Of course, the first line vanishes for $c = 0$ (due to $\langle 0|\mathbb{1}|0\rangle = 0$) but since these terms may be interesting for the examination of LCFTs with arbitrary central charge, we stated them in spite of that. Additionally we should mention that it is obviously possible to chose a quasi-primary $\tilde{\mathbb{1}}$ since its OPE with the stress energy tensor looks like

$$T(z)\tilde{\mathbb{1}}(w) \sim \frac{\lambda\mathbb{1}}{(z-w)^2} + \frac{\partial_w\tilde{\mathbb{1}}(w)}{(z-w)}\,. \tag{182}$$

Recapitulating, we have shown that as in [41], we have $\theta = 0$, which means that we would have vanishing vacuum expectation values for $c = 0$. These results suggest that we can not take a naive free field construction to describe the problem, since all vacuum expectation values vanish. In order to get the necessary central extensions, more sophisticated constructions should be considered, such as several free fields or deformations of the stress energy tensor similar to the ones introduced in [17]. In contrast to their assumption, we found a number of more complicated fields than $t(w)$, $T(w)$ or their descendants although there are no other primaries involved.

Therefore we tried to find a way out by searching for $\sum_{\{(p,q)\}} c_{p,q} = 0$ to construct a $c = 0$ theory out of tensorized minimal models to get a non trivial CFT with vanishing central charge in chapter seven.

Another possibility would be to find a fermionic theory with vanishing central charge.

4.4. Discussion of the two LCFT approaches

As already stated in section 4.1.2, we have a third possible loophole to avoid the $c \to 0$ catastrophe in the OPEs of primary fields. In the following we will explain why the ansatz we chose, i.e. the one based on the augmented minimal model, may be a more natural solution.

Therefore we will give a brief overview on the ansatz of Kogan and Nichols [53] followed by our comments on their approach. Additionally we will state some facts about the $c = 0$ case including implications for percolation and a discussion of current research on augmented $c_{(p,q)}$ models with $q > 1$ which have not been treated in the literature so far, focusing on $p = 3, q = 2$.

4.4.1. The replica approach to vanishing central charge

Following the replica approach of [9], Kogan and Nichols [53] introduced another field $\tilde{T}$ with dimension $h = 2 + \alpha(c)$ which satisfies $\alpha(c) \to 0$ for $c \to 0$ being normalized to

$$\langle \tilde{T}(z)\tilde{T}(0)\rangle = \frac{1}{c}\frac{B(c)}{z^{4+2\alpha(c)}} \tag{183}$$

with $B(c) = -\frac{h^2}{2} + B_1 c + \dots$. Thus for $c \to 0$ this vacuum expectation value diverges.

Then, after a small c expansion, the OPE of our primary field looks like

$$\begin{aligned} \phi_h(z)\phi_h^\dagger(0) \quad &\sim \quad \frac{1}{z^{2h}}\left(1 + \frac{2h}{c}z^2 T(0) + 2z^{2+\alpha(c)}\tilde{T}(0) + \dots\right) + \dots \\ &\sim \quad \frac{1}{z^{2h}} + \frac{1}{z^{2h}}z^{2+\alpha(c)}\left(\frac{2h}{c}\left(1 - \alpha(c)\log(z)\right)T(0) + 2\tilde{T}(0) + \dots\right) + \dots \quad , \end{aligned} \tag{184}$$

which again is well defined if the parameter μ defined by

$$\mu^{-1} \equiv \lim_{c\to 0} -\frac{2\alpha(c)}{c} = -2\alpha'(c)\,, \tag{185}$$

is not equal to zero.

The logarithmic partner field can now be identified with a linear combination of the stress energy tensor and the new $h = 2 + \alpha(c)$-field,

$$\frac{h}{\mu}t = \frac{2h}{c}T + 2\tilde{T}\,, \tag{186}$$

satisfying

$$L_0 T = 2T \quad L_0 t = 2t + T. \tag{187}$$

This means that $t(z)$ is a field of the same conformal weight living in a Jordan cell due to L_0 being non-diagonalizable. Thus the OPE becomes

$$\phi_h(z)\phi_h^\dagger(0) \sim \frac{1}{z^{2h}}\left(1 + \frac{h}{\mu}\left(t(0) - \log(z)T(0)\right) + \dots\right) + \dots \quad , \tag{188}$$

which yields the following vevs after redefining $t \to t + \gamma T$ with a suitable choice of γ:

$$\langle T(z)T(0)\rangle \sim 0\,, \tag{189}$$

$$\langle T(z)t(0)\rangle \sim \frac{b}{z^4}\,, \tag{190}$$

$$\langle t(z)t(0)\rangle \sim \frac{-2b\log z}{z^4}\,. \tag{191}$$

This result can only be obtained by assuming that we are dealing with non-degenerate vacua, which means, that the vacuum expectation value of the identity operator does not vanish and we have only one $h = 0$ field contributing to the OPE. It is based on the following algebra between the modes of $T(z)$ and $t(z)$:

$$[L_n, t(z)] = z^n\left\{\left(z\frac{\mathrm{d}}{\mathrm{d}z} + z(n+1)\right)t(z) + (n+1)T(z)\right\} + \frac{\mu\mathbb{1}}{6}n(n^2+1)z^{n-2}\,, \tag{192}$$

where for their ansatz, we have $\mu = b$ meaning that the central term of the algebra between the modes of $t(w)$ and $T(z)$ is proportional to the vacuum expectation value of the OPE of these fields.

Thus, since $\langle\mathbb{1}\rangle \neq 0$, the most singular part of the OPE yields the vacuum expectation value as stated in (190). This is only true if we assume $L_{-2}|0\rangle = T(0)|0\rangle$ not to be zero by construction since the action of the conformal generators on the vacuum vanish in a $c = 0$ CFT but to be some kind of generalized null state on which $l_{-2}|0\rangle = t(0)|0\rangle$ is non orthogonal. To keep this assumption it is crucial not to have a logarithmic partner of the identity and thus $\langle\mathbb{1}\rangle \neq 0$.

4.4.2. Comments on the replica approach

However, we have a few comments on this non-rational ansatz for $c = 0$ LCFTs.

A rather small one is about the fact that for all $c \neq 0$ μ may be set to zero by a redefinition $l_m \to l_m - \frac{2\mu}{c}L_m$. Hence it is not obvious how this limit may equal the value of μ^{-1} for $c = 0$ due to the discontinuity of being free to choose $\mu = 0$ for $c \neq 0$ but staying with fixed μ for $c = 0$. Furthermore, there is no physical quantity known to correspond to this arbitrary parameter μ, thus it is rather awkward that it may show up with such a significant role in our (L)CFT. Furthermore it is questionable whether we can define $\tilde{T}$ and t in such a way that they are divergent for $c \to 0$. More on suitable choices of μ will be given in section 4.5.2.

Additionally, we doubt that it is possible to choose the vacuum as a “stand alone” irreducible representation not contained in an indecomposable one based on a second $h = 0$ state, called $\tilde{\mathbb{1}}$. But as already mentioned, in a Kac table based (L)CFT ansatz for $c = 0$ we have indeed three $h = 0$ fields which seem to belong to rank three Jordan cell structures whose details are not yet clarified.

For Kogan and Nichols [53], the term proportional to the identity in the central extension of the algebra between the Laurent modes of $t(z)$ and $T(z)$, μ, is the same as the proportionality factor of

$\langle Tt \rangle$. In our calculations, however, they are different since we assume a Jordan cell on the identity level, yielding

$$[L_n, t(z)] = z^n \left\{ \left(z\frac{\mathrm{d}}{\mathrm{dz}} + z(n+1) \right) t(z) + (n+1)T(z) \right\} + \frac{\mu\mathbb{1} + \frac{c}{2}\tilde{\mathbb{1}}}{6} n(n^2+1)z^{n-2} \,, \qquad (193)$$

or, equivalently $T(z)t(0) \sim (\mu\mathbb{1}+b\tilde{\mathbb{1}})z^{-4}+\ldots$. Thus $b = \frac{c}{2}$ in our case and $\langle Tt \rangle$ has to vanish, too. A priori, as already discussed above, there is no constraint on the choice of μ. Following Gurarie and Ludwig [41], we will show how various values of μ affect the theory in the next section.

Thus we have motivated that in a $c_{(9,6)} = 0$ (L)CFT, there is no level two state which is non orthogonal to $T(0)|0\rangle$, especially not $t(0)|0\rangle$ since the two-point function has to vanish. Thus in this setup, we can keep the full (and not only global) conformal invariance of the vacuum. This may lead to consequences on the construction of the stress energy tensor.

4.5. More on the $c_{(9,6)} = 0$ augmented minimal model

4.5.1. Consequences of full conformal invariance

If we restrict ourselves to the case of not having a Jordan cell structure at the ($h = 0$)-level, we encounter the fact that any two-point function involving T has to vanish. This follows directly from the behavior of the identity sector in a ($c = 0$)-theory unless we introduce a non orthogonal state to $L_{-2}|0\rangle$. We know that by global conformal invariance and the highest weight condition, we have $L_n\mathbb{1} = L_n|0\rangle = 0$ for all $n \geq -1$. In the following, let n be > 0. Starting with a vanishing central charge and $h = 0$, we know

$$\begin{aligned} 0 &= 2nL_0|0\rangle + \frac{c}{12}n(n^2-1)|0\rangle \\ &= [L_n, L_{-n}]|0\rangle \\ &= L_nL_{-n}|0\rangle - L_{-n}L_n|0\rangle \\ &= L_nL_{-n}|0\rangle \,, \end{aligned} \qquad (194)$$

and thus we have $L_{-n}|0\rangle = 0$ for all $n \in \mathbb{Z}$. This means that if we expand $T(z)$ in powers of z, i. e.

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \qquad (195)$$

we clearly see that $\langle 0|T(z) = T(z)|0\rangle = 0$ if we impose full conformal invariance. This is possible if all states are orthogonal to $L_n|0\rangle$ which is the case for the minimal model $c_{(3,2)} = 0$. More precisely: the null vector is present in the irreducible vacuum representation but may disappear in the full indecomposable representation based on $|\tilde{0}\rangle$. Note that if we include fields outside the Kac-table without assuming a Jordan cell structure for the identity level with $L_0\tilde{\mathbb{1}} = \mathbb{1}$, non orthogonal states can be constructed since there are no constraints on their properties.

Nevertheless in our ansatz (the $c_{(9,6)} = 0$ augmented minimal model), the state usually identified with the stress energy tensor seems to decouple completely from the theory since it is even orthogonal to $l_{-2}|0\rangle$ (there may be additional $h = 2$ fields present in the theory which are non orthogonal but we do not know about any of them up to now). However, this also forces any two point function involving T to vanish (as long as we do not modify the theory as touched in section 3.3). Thus the first two-point function not to vanish is $\langle tt \rangle$.

If we assume $L_{-2}|0\rangle = T(0)|0\rangle$ to be just an ordinary null state, $|\chi^{(2)}_{(h,c)}\rangle$, and not a fundamental property of the vacuum at $c_{(3,2)} = 0$, we can obtain different results.

What we already know is that $L_{-2}|0\rangle$ is a null state with respect to the action of all L_n. Thus we only have to check whether this holds for the action of the l_n, too. Taking a look at

$$\langle 0|[l_2, L_{-2}]|0\rangle = \langle 0|4l_0|0\rangle + \langle 0|\mu|0\rangle = \mu\,, \tag{196}$$

we see that it is consistent to assume a non-orthogonal state to $L_{-2}|0\rangle$ if we exclude a Jordan cell for the identity. Even the Jordan cell relation between the usual state associated with the stress energy tensor and its logarithmic partner turns out to be as expected -

$$L_0 t(z)|0\rangle \equiv L_0 l_{-2}|0\rangle = 2l_{-2}|0\rangle + L_{-2}|0\rangle = 2t(0)|0\rangle + T(0)|0\rangle\,. \tag{197}$$

Once more we stress that $t(z)$ can only be non-orthogonal in a non Kac based approach to $c = 0$ since otherwise we know that we have a Jordan cell connection between the identity and other states which would cause the two-point function to vanish:

$$\langle 0|[l_2, L_{-2}]|0\rangle = 4\langle 0|l_0|0\rangle + \langle 0|\mu\mathbb{1} + \frac{c}{2}\tilde{\mathbb{1}}|0\rangle = 0\,. \tag{198}$$

4.5.2. Null vectors in a Kac-table based $c = 0$ theory

Having agreed upon the proposal that the Kac-table of the augmented $c = 0$ should be taken, we know that under certain circumstances we can have null vectors in our theory. The assumption that no other fields than those of the Kac-table and their descendants may arise is crucial to this calculation since we do not have any knowledge on the properties of non-Kac fields. Thus we point out that there are problems with any arguments based on the assumption of null vectors in a non strictly Kac-based theory.

Assuming $t(z)|0\rangle$ to have a mode expansion like $T(z)|0\rangle$, i. e.

$$t(z)|0\rangle = \sum_{n\in\mathbb{Z}} l_n z^{-n-2}|0\rangle\,, \tag{199}$$

and following the idea of Gurarie and Ludwig [41], we will try to construct universal null vectors that do not only vanish under the action of all L_n for $n > 0$ but also after the application of l_m for $m > 0$. We have to emphasize that this simple expansion of $t(z)$ only holds when acting on a highest weight state.

The ordinary level two null vector Now let us have a look at the ordinary null vector on the second level

$$|\chi^{(2)}_{(h,c)}\rangle = \left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\right)|h\rangle\,, \tag{200}$$

$$h = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right)\,. \tag{201}$$

What we already know is that $L_{\{n\}}|\chi^{(2)}_{(h,c)}\rangle = 0$ for all $|\{n\}| > 0$ with $\{n\} = \{n_1, n_2 \ldots, n_k\}$ and $|\{n\}| = \sum_i n_i$. But what about the action of $l_{\{n\}}$ on $|\chi^{(2)}_{(h,c)}(0)\rangle$? For $|\{n\}| > 3$ this is obviously trivial since commuting the $l_{\{n\}}$ to the right will leave us with some linear combination of $l_{\{m\}}$ and $L_{\{m'\}}$ with $|\{m\}|, |\{m'\}| > 0$ which vanishes. Thus the interesting cases are the application of l_2 and l_1^2.

Therefore we have to use the algebra of the L_n and l_n. The algebra between the modes of $T(z)$,

$$[L_n, L_m] = (n-m)L_{n+m}\,, \tag{202}$$

is just the same as in any ordinary CFT in spite of lacking the central extension due to the vanishing central charge. The mixed commutator is given by

$$[L_n, l_m]|0\rangle = (n-m)l_{n+m}|0\rangle + (n+1)L_{n+m}|0\rangle + \frac{\mu}{6}n(n^2-1)\delta_{n+m,0}|0\rangle\,. \tag{203}$$

Now we can test the known level two null vector from the ordinary theory by applying the generators of $t(z)$:

$$l_2|\chi^{(2)}_{(h,c)}\rangle = \left[\left(4 - \frac{18}{2(2h+1)}\right) l_0 + h + \mu\right]|h\rangle\,. \tag{204}$$

This result raises the question what the action of l_0 on some state of weight h might be. According to Gurarie and Ludwig [41], it can be chosen to be equal to zero, $l_0|h\rangle = 0$, but this statement contradicts the level one null vector assumption, i. e.

$$\left(l_{-1} - \frac{1}{2}L_{-1}\right)|h\rangle = 0 \tag{205}$$

for $h \neq 0$. To be sure, we prove this statement here. Thus let us have a look at this mixed level one null state from a general point of view, i. e. $|\tilde{\chi}^{(1)}_{(h,c)}\rangle = (al_{-1} + bL_{-1})|h\rangle$ for some suitable $|h\rangle$. Claiming $L_n|\tilde{\chi}^{(1)}_{(h,c)}\rangle = 0$ for all $n > 0$ it follows that

$$(2al_0 + 2h(a+b))|h\rangle = 0\,. \tag{206}$$

Here we have to distinguish between different cases:

(a) $h \neq 0$:

$$\begin{array}{rclcrcl} l_0|h\rangle & = & 0 & \Rightarrow & a & = & -b\,, \\ l_0|h\rangle & = & h|h\rangle & \Rightarrow & a & = & -\frac{1}{2}b\,. \end{array} \tag{207}$$

Obviously the first result of (207) contradicts (205) but the second result is ok to agree upon.

(b) $h = 0$, and thus

$$L_n|\tilde{\chi}^{(1)}_{(h,c)}\rangle = aL_n l_{-1}|0\rangle = -a(n+1)l_{n-1}|0\rangle \tag{208}$$

which vanishes for $n > 0$.

Thus with reservation regarding the action of l_n on the mixed level one null state, $|\tilde{\chi}^{(1)}_{(h,c)}\rangle$ is a null vector for all h, but only with the special choice of $l_0|h\rangle = h|h\rangle$ [41]. Conversely, if we say $l_0|h\rangle = 0$, we do not have the special null vector (205) (independent on what the commutator of the modes of $t(z)$ alone might be). However, for $h = 0$, $|\tilde{\chi}^{(1)}_{(h,c)}\rangle$ is obviously a null state for both choices since in spite of the fact that we do not know the exact form of the commutator $[l_n, l_m]$, we can conclude that $[l_1, l_{-1}] \propto al_0 + bL_0 + \phi$ with ϕ being a linear combination of other primaries of weight zero. Thus for $h = 0$ and the action of l_0 on $|h\rangle$ either being 0 or h, we can conclude that $l_{-1}|0\rangle = 0$ since

$$l_1|\tilde{\chi}^{(1)}_{(h,c)}\rangle = al_1 l_{-1}|0\rangle + bl_1 L_{-1}|0\rangle = a[l_1, l_{-1}]|0\rangle + 2bl_0|0\rangle = 0. \tag{209}$$

and thus $l_{-1}|0\rangle \propto L_{-1}|0\rangle = 0$. Therefore we can show that $l_1^2|\chi^{(2)}_{(h,c)}\rangle = 0$ for $h = 0$ and thus $l_0|0\rangle = 0$:

$$\begin{aligned} l_1|\chi^{(2)}_{(h,c)}\rangle &= l_1\left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\right)|0\rangle \\ &= (3l_{-1} + L_{-1})|h\rangle - \frac{3}{2(2h+1)}(2L_{-1}l_0 + 2l_{-1})|h\rangle \\ &= 0\,. \end{aligned} \tag{210}$$

Unfortunately, there is no criterion to draw the right conclusion which is the right choice of the action of l_0 on a state. Thus we will state or results of equation (204) for both choices of l_0, i.e. $l_0|h\rangle = h|h\rangle$ or $= 0$, we found $\mu = 0$ for $h = 0$ and $\mu = -\frac{5}{8}$ for $h = \frac{5}{8}$.

Although it is always consistent to assume the existence of the special null vector (205) if we choose $l_0|h\rangle = h|h\rangle$, we do not know whether for $h \neq 0$ we have $l_1|\tilde{\chi}^{(1)}_{(h,c)}\rangle = 0$. But we have to be careful since this is only a circular reasoning.

The level three null vector We could try the same procedure on the level three null state

$$|\chi^{(3)}_{(h,c)}\rangle = \left(L_{-1}^3 - 2\,(h+1)\,L_{-2}L_{-1} + h\,(h+1)\,L_{-3}\right)\phi_h\,, \tag{211}$$

$$h = \frac{1}{6}\left(7 - c \pm \sqrt{(c-1)(c-25)}\right)\,. \tag{212}$$

Testing the level three null vector for consistency by applying l_3 to $|\chi^{(3)}_{(h,c)}\rangle$, we find that for both choices of l_0, i.e. $l_0|h\rangle = h|h\rangle$ or $= 0$, we have $\mu = 0$ for $h = 2$ and $\mu = \frac{5}{6}$ for $h = \frac{1}{3}$. However, we have to bear in mind that these are only necessary conditions, we did not check whether the action of $l_2 l_1$ and l_1^3 supports this result or gives a contradiction, meaning that there is no level three null state in the theory any more.

Comments on percolation as an augmented $c = 0$ model Independently we can conclude that we do not only have different theories for different values of μ but also that any given $c = 0$ theory splits up in certain subsets of primary operators which "cannot give rise to [...] differential equations simultaneously in the same theory" [41].

However, even only from the necessary conditions for the values of μ we see that for $\mu \neq 0$ we can not have a level three and level two null vector differential equation in the augmented $c = 0$ model. Moreover, since the algebra between the l_n and the L_m is the same as for the paper of Gurarie and Ludwig [41], we have shown that if $c = 0$ should be a model for percolation which does not exhibit the divergence problem, we have to take $\mu = 0$ and thus can not take an ansatz without a Jordan cell for the identity. Again it is very interesting, that the unphysical parameter μ seems to disappear if we want to have null states for the whole theory, this means also that there is no central term in the algebra between the l_n and L_m and all vacuum expectation values of the stress energy tensor and its logarithmic partner vanish since $\mu = \theta$ for $c = 0$. .

4.5.3. The field content of a $c_{(9,6)} = 0$ augmented minimal model

After having talked so much about the augmented $c_{(9,6)} = 0$ model, we should give at least a brief overview on its features since there has been not much literature published about generalized augmented $c_{(p,q)}$ models with $q > 1$ so far. Following the ideas of [20], we know that the smallest closing set of modular functions larger than the $\frac{1}{2}(p-1)(q-1)$ characters for the minimal $c_{(p,q)}$ model contains $\frac{1}{2}(3p-1)(3q-1)$ individual functions which stay in some suitable linear combination in direct correspondence to the number of highest weight representations or fields in the augmented Kac-table. The modular functions can be found by solving the modular differential equation as introduced in [64, 63]. The generalization of this method towards LCFT can be found in [25]. In our example, the $c_{(9,6)}$ model, twenty torus amplitudes can be matched with the twenty representations of the modular group being present in the Kac table of $c_{(9,6)} = 0$ [24]. Closed sets of such functions can only be obtained considering an odd multiple of (p, q) thus usually one tries to get along with the smallest set, i. e. $(3p, 3q)$.

Thus contrary to the minimal model $c_{(p,q)}$ we technically have to deal with an extended Kac-table of $c_{(3p,3q)}$:

$$c_{(9,6)} \quad : \quad \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 0 & \frac{1}{3} & 1 & 2 & \frac{10}{3} & 5 & 7 \\ \hline \frac{5}{8} & \frac{1}{8} & -\frac{1}{24} & \frac{1}{8} & \frac{5}{8} & \frac{35}{24} & \frac{21}{8} & \frac{33}{8} \\ \hline 2 & 1 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 1 & 2 \\ \hline \frac{33}{8} & \frac{21}{8} & \frac{35}{24} & \frac{5}{8} & \frac{1}{8} & -\frac{1}{24} & \frac{1}{8} & \frac{5}{8} \\ \hline 7 & 5 & \frac{10}{3} & 2 & 1 & \frac{1}{3} & 0 & 0 \\ \hline \end{array} \quad . \tag{213}$$

As it is always the case for $c_{(3p,3q)}$ augmented models, we have 3×2 fields in the Kac-table which are of weight $h = 0$ and lie within the upper left and lower right corners of the replicated minimal Kac-tables on the diagonal. It is conjectured [24] that all fields inside the boundary of the replicated minimal Kac-table belong to rank 3 Jordan cells whose detailed structure is not yet known.

Fields on the boundary of the replicated minimal Kac-table show up with a multiplicity of 2×2 and belong to rank 2 Jordan cells. The corresponding representation of weight $h_{(r,s)} + rs$ is present 1×2 times as expected, too. Additionally, the fields on the edges of the boundaries show up only 1×2 times as well, with their corresponding representations of weight $h_{(p,q)} + pq/4$ showing up at the anti-diagonal edges.

Thus in the special case of $c = 0$ we have two highest weights which do not form Jordan cells, i.e. $-\frac{1}{24}, \frac{35}{24}$ while the other operators of the boundary of the conformal grid are arranged in triplets of which two states of the same weight form an indecomposable representation and one belongs to an irreducible representation which is differing by an integer in its weight (more precisely rs), i.e.

$$\begin{aligned} \left(\frac{5}{8}, \frac{5}{8}, \frac{21}{8}\right) &= \left(\frac{5}{8}, \frac{5}{8}, 2 + \frac{5}{8}\right) , \\ \left(\frac{1}{3}, \frac{1}{3}, \frac{10}{3}\right) &= \left(\frac{1}{3}, \frac{1}{3}, 3 + \frac{1}{3}\right) , \\ \left(\frac{1}{8}, \frac{1}{8}, \frac{33}{8}\right) &= \left(\frac{1}{8}, \frac{1}{8}, 4 + \frac{1}{8}\right) . \end{aligned} \tag{214}$$

Due to these indecomposable representations, logarithms arise in the OPEs and especially in the fusion product of the pre-logarithmic field $\phi_{-\frac{1}{24}}$ with itself.

The sector containing the $h = 0$ fields has a more complicated structure. We have three multiple weights $(0, 0, 0), (1, 1)$ and $(2, 2)$ but we do not yet know how they are arranged among the other two fields of weights 5 and 7, respectively. As stated above, it is conjectured [24] that they may form a rank three Jordan cell structure whose details are currently being worked out. Additionally, we can not exclude exotic behavior such as Jordan cells with respect to other generators than L_0, e. g. $\mathcal{W}$-algebra zero modes. Even worse, there might exist indecomposable structures with respect to L_n, $n \neq 0$, as in [55].

As far as we know there has not been any research concerning this issue before. It seems reasonable to assume a structure related to that of the $c_{(p,1)}$ models which has already been discussed in detail [34], [35], [47], but obviously at least for the integer weights it can not be the whole story.

If we accept that the Kac-table of $c = 0$ has to be extended beyond its minimal truncation, we immediately encounter a problem. The field corresponding to the entry $(2, 3)$ in the Kac-table has a negative conformal weight $h_{2,3} = -1/24$. Hence, the theory cannot be unitary. Furthermore, the effective central charge $c_{\text{eff}} = c - 24h_{\min}$ with $h_{\min}$ the minimal eigen value of L_0 is then given by $c_{\text{eff}} = (c = 0) - 24(h = -1/24) = 1$. It follows that such a theory cannot be rational with respect to the Virasoro algebra alone, but only quasi-rational. However, there presumably exists an extended chiral symmetry algebra, $\mathcal{W}(2,15,15,15)$ under which the theory is rational [24]. Fortunately, most of the structures which will interest us in this paper can be studied from the the perspective of the Virasoro algebra.

As a concluding remark, let us note that there seems to be a connection to $c_{(6,1)} = -24$ which is the only rational (L)CFT with equal central charge modulo 24 and thus exhibiting the same modular properties. This theory also has effective central charge one. Unfortunately, the analogies only hold for the boundary of the Kac table and therefore we can only deduce the properties for the representations from the boundary of the Kac-table of the $c = 0$ model and not for the integer weight states.

4.6. The forgotten loophole

4.6.1. Tensorized (L)CFTs with $c = 0$

Obviously there is a fourth way out of the dilemma. Taking two non-interacting CFTs with central charges c_1 and $c_2 = -c_1$, respectively, and tensorizing them, we get a CFT with vanishing central charge again but the OPE (142) looks like

$$\phi_h(z)\phi_h^\dagger(0) \sim \frac{C_{\Phi\Phi}^{\mathbb{1}}}{z^{2h}}\left(1 + \frac{2h}{c_1}z^2(T_{c_1}(0) - T_{-c_2}(0)) + \dots\right) + \dots \quad , \tag{215}$$

which is perfectly well defined for $c = 0$ if $c_1 \neq 0$.

But the result comes with a price, too: we have to introduce a new field $t(z) := T_{c_1}(z) - T_{-c_2}(z)$ which can be shown to satisfy the following OPEs with the stress energy tensor [41]

$$T(z)T(0) \sim \frac{2T(0)}{z^2} + \frac{T'(0)}{z} + \dots \quad , \tag{216}$$

$$T(z)t(0) \sim \frac{c_1}{z^4} + \frac{2t(0)}{z^2} + \frac{t'(0)}{z} + \dots \quad , \tag{217}$$

$$t(z)t(0) \sim \frac{2T(0)}{z^2} + \frac{T'(0)}{z} + \dots \quad . \tag{218}$$

The OPE of the tensorized $c = c_1 + c_2 = 0$ LCFT model consists of an ordinary CFT part from the c_2-sector and a LCFT part from the c_1 sector. Thus we would get a $c = 0$ theory with logarithmic operators without vanishing two-point function.

Operators in the full tensorized theory therefore are just direct products $\phi_h^{(0)} = \phi_{h_1}^{(1)} \otimes \phi_{h_2}^{(2)}$ whose weights are given by the sum of both parts $h = h_1 + h_2$. Thus the OPE of a primary field is given by (see [53])

$$\begin{aligned}
\phi_h^{(0)}(z)\phi_h^{(0)}(0) &= \phi_{h_1}^{(1)}(z)\phi_{h_1}^{(1)}(0) \otimes \phi_{h_2}^{(2)}(z)\phi_{h_2}^{(2)}(0) \\
&\sim \frac{1}{z^{2h_1}}\left(\mathbb{1}^{(1)} + z^2\frac{2h_1}{c_1}T^{(1)}(0) + \dots\right) \\
&\qquad \times \frac{1}{z^{2h_2}}\left(\mathbb{1}^{(2)} + z^2\frac{2h_2}{c_2}T^{(2)}(0) + \dots\right) + \dots \\
&\sim \frac{1}{z^{2h}}\left(1 + z^2\left(\frac{2h_1}{c_1}T^{(1)}(0) + \frac{2h_2}{c_2}T^{(2)}(0)\right)\right) ,
\end{aligned} \tag{219}$$

which is well defined since the $c_i \neq 0$ and the theories by themselves are regular.

4.6.2. The general case

In some cases we may not be able to choose a (bosonic) free field construction for the stress-energy-tensor. Thus we have to take a look at the general OPEs for a tensorized theory of an LCFT with central charge c_1 and an ordinary CFT with $c_2 = -c_1$. We start with the known OPEs

$$\begin{aligned} T^{(i)}(z)T^{(i)}(w) &= \frac{\frac{c_i}{2}}{(z-w)^4} + \frac{2T^{(i)}(w)}{(z-w)^2} + \frac{\partial_w T^{(i)}(w)}{(z-w)}\,, \\ \tilde{\mathbb{1}}^{(1)}(z)\tilde{\mathbb{1}}^{(1)}(w) &= \log^2(z-w)\mathbb{1}^{(1)} + 2\log(z-w)\tilde{\mathbb{1}}^{(1)}(w)\,, \\ T^{(1)}(z)\tilde{\mathbb{1}}^{(1)}(w) &= \frac{\mathbb{1}^{(1)}}{(z-w)^2} + \frac{\partial_w \tilde{\mathbb{1}}^{(1)}(w)}{(z-w)}\,, \end{aligned} \tag{220}$$

and we define

$$t^{(1)}(w) \;:=\; :T^{(1)}\tilde{\mathbb{1}}^{(1)}:(w)\,, \tag{221}$$

$$t^{(0)}(w) \;:=\; t^{(1)}(w)\otimes\mathbb{1}^{(2)} + (\alpha\mathbb{1}^{(1)} + \beta\tilde{\mathbb{1}}^{(1)}(w))\otimes T^{(2)}(w)\,. \tag{222}$$

To obtain the two point functions, we make the ansatz:

$$\begin{aligned} T^{(0)}(z) &= T^{(1)}(z)\otimes\mathbb{1}^{(2)}(z) + \mathbb{1}^{(1)}(z)\otimes T^{(2)}(z) \\ t^{(0)}(z) &= t^{(1)}(z)\otimes\mathbb{1}^{(2)}(z) + (\alpha\mathbb{1}^{(1)}(z) + \beta\tilde{\mathbb{1}}^{(1)}(z))\otimes T^{(2)}(z)\,. \end{aligned} \tag{223}$$

This leaves us with the following results for $T^{(0)}(z)T^{(0)}(w)$ and $T^{(0)}(z)t^{(0)}(w)$:

$$\begin{aligned} T^{(0)}(z)T^{(0)}(w) &= T^{(1)}(z)T^{(1)}(w) + T^{(2)}(z)T^{(2)}(w) \\ &\sim \frac{\frac{c_1}{2}+\frac{c_2}{2}}{(z-w)^4} + \frac{2(T^{(1)}+T^{(2)})(w)}{(z-w)^2} + \frac{\partial_w(T^{(1)}+T^{(2)})(w)}{(z-w)} \\ &= \frac{2T^{(0)}(w)}{(z-w)^2} + \frac{\partial_w T^{(0)}(w)}{(z-w)}\,, \end{aligned} \tag{224}$$

whereas the OPE with its logarithmic partner

$$\begin{aligned} T^{(0)}(z)t^{(0)}(w) &= T^{(1)}(z)t^{(1)}(w)\otimes\mathbb{1}^{(2)} + (\alpha\mathbb{1}^{(1)} + \beta\tilde{\mathbb{1}}^{(1)})\otimes T^{(2)}(z)T^{(2)}(w) \qquad (225) \\ &\sim \frac{\frac{c_1}{2}\left((1-\beta)\tilde{\mathbb{1}}^{(1)} - \alpha\mathbb{1}^{(1)}\right)\otimes\mathbb{1}^{(2)}}{(z-w)^4} + \frac{2t^{(0)}(w) + T^{(1)}(w)\otimes\mathbb{1}^{(2)}}{(z-w)^2} + \frac{\partial_w t^{(0)}(w)}{(z-w)} \end{aligned}$$

yields a non-vanishing vev with a modified b-term:

$$\langle T^{(0)}(z)t^{(0)}(w)\rangle \;=\; \frac{\frac{c_1}{2}(1-\beta)}{(z-w)^4}\,. \tag{226}$$

For the OPE of the logarithmic partner fields, we get

$$\begin{aligned} t^{(0)}(z)t^{(0)}(w) \;\sim\; & \frac{1}{(z-w)^4}\Bigg(\left(1+\alpha^2\frac{c_2}{2}\right) + \left(\frac{c_1}{2}+\beta^2\frac{c_2}{2}\right)\log^2(z-w) \\ & -2\left(\frac{c_1}{2}+\beta^2\frac{c_2}{2}\right)\log(z-w)\tilde{\mathbb{1}}(w) + \alpha\beta\tilde{\mathbb{1}}c_2 + \left(\frac{c_1}{2}+\beta^2\frac{c_2}{2}\right):\tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}_1(w):\Bigg) \\ & +\frac{1}{(z-w)^2}\Bigg(2\left(T^{(0)}(w) - (1-\beta^2)T^{(2)}(w)\right)\log^2(z-w) \\ & -4\left(t^{(0)}(w) - \alpha\beta\tilde{\mathbb{1}}T^{(2)}(w)\right)\log(z-w) + 2t^{(0)}(w) + 2\alpha\beta\tilde{\mathbb{1}}T^{(2)}(w) \\ & +\left(T^{(0)}(w) - (1-\beta^2)T^{(2)}(w)\right):\tilde{\mathbb{1}}(w)\tilde{\mathbb{1}}(w):\Bigg) \\ & +\frac{1}{(z-w)}\Bigg(\left(\partial T^{(0)}(w) - (1-\beta^2)\partial T^{(2)}\right)\log^2(z-w) \\ & -2\left(\partial t^{(0)}(w) - \alpha\beta\tilde{\mathbb{1}}\partial T^{(2)}(w)\right)\log(z-w) \\ & +\partial t^{(0)}(w) + \alpha\beta\tilde{\mathbb{1}}\partial T^{(2)}(w) + \partial\left(\left(T^{(0)}(w) - (1-\beta^2)T^{(2)}(w)\right):\tilde{\mathbb{1}}(w)\tilde{\mathbb{1}}(w):\right)\Bigg) \\ & + \left(\log^2(z-w) - 2\log(z-w)\tilde{\mathbb{1}}(w)\right) \\ & \qquad \cdot\left(:T^{(0)}(w)T^{(0)}(w): - (1-\beta^2):T^{(2)}(w)T^{(2)}(w):\right)\,, \end{aligned} \tag{227}$$

where we suppressed the labels for the tensor factors as they are clear from the context. Obviously the only possibility to get nothing but "zero charge" quantities on the rhs is to put $\alpha = 0$ and $\beta = 1$ which means, that we are left with vanishing vevs for $\langle TT\rangle$ and $\langle Tt\rangle$. In that case the equations would be of the same form as for the ordinary $c = 0$ LCFT and our construction would be useless. To be exhaustive, we will give the vev of this calculation, too,

$$\langle t^{(0)}(z)t^{(0)}(w)\rangle = c_2\frac{\alpha\beta + \left(1-\beta^2\right)\log(z-w)}{(z-w)^4}\,. \tag{228}$$

One of many possible applications is a tensor product of a $c = -2$ theory and four Ising models. This ansatz has many advantages, e. g. a logarithmic pair in the identity sector of the part with $c = -2$ and the closure under fusion of a small subset of the fields.

As stated in [54] and [53], this corresponds to an $SU(2)_0$ or $OSp(2|2)_{-2}$ model, where the logarithmic structure appears in the $c = -2$ part.

The Ising Model Remembering the Kac-table for the Ising model (35), we can compute the fusion rules:

$$\sigma\times\sigma = \mathbb{1}+\varepsilon\,, \tag{229}$$
$$\sigma\times\varepsilon = \sigma\,, \tag{230}$$
$$\varepsilon\times\varepsilon = \mathbb{1}\,. \tag{231}$$

For $c = c_{2,1} = -2$, we will use the same notation as in the first part of this thesis, i.e. for the indecomposable representation of the $h = 0$ sector $\mathcal{R}_{\mathbb{1}}$, consisting of two fields with $h = 0$ whose details are not important for our further discussion and two others, i.e. μ with $h = -\frac{1}{8}$ and ν with $h = \frac{3}{8}$. These fields obey

$$\mu\times\mu = \mu\times\nu = \nu\times\nu = \mathcal{R}_{\mathbb{1}}\,, \tag{232}$$
$$\mu\times\mathcal{R}_{\mathbb{1}} = \nu\times\mathcal{R}_{\mathbb{1}} = \mu+\nu\,, \tag{233}$$
$$\mathcal{R}_{\mathbb{1}}\times\mathcal{R}_{\mathbb{1}} = 2\mathcal{R}_{\mathbb{1}}\,. \tag{234}$$

It is easy to check, that the symmetrized fields of the four Ising models $\mathbb{1}, E_1, E_2, E_3, E_4$ and S (where E_i denotes the totally symmetric tensor product of i fields ε and $4-i$ fields $\mathbb{1}$ and $S = \otimes^4\sigma \equiv (\sigma,\sigma,\sigma,\sigma)$) close under fusion. From these fields tensorized with those of $c = -2$ we can choose a consistent subset $(\mathcal{R}_{\mathbb{1}}, \mathbb{1})$, $(\mathcal{R}_{\mathbb{1}}, E_i)$, (μ, S) and (ν, S). Obviously, (ν, S) has conformal weight $h = \frac{5}{8}$ and (μ, S) has conformal weight $h = \frac{1}{8}$ which are fields assumed to appear in percolation. However, if percolation can be described by a $c = 0$ model such as $(c = 2)\otimes(c = -2)$, the question remains how Watts' differential equation [78] can be derived through a level three null vector condition acting on a four point function of boundary changing operators in this theory [28].

The operator product expansion The OPE of the tensorized $c = 0$ model, consists of an ordinary CFT part from the $c = 2$ sector and and LCFT part from the $c = -2$ sector. To obtain the two point functions, we make the same ansatz as before (223)

$$T(z) = {:}\partial\theta^+(z)\partial\theta^-(z){:}\,, \tag{235}$$
$$\tilde{\mathbb{1}}(z) = {:}\theta^-(z)\theta^+(z){:}\,, \tag{236}$$
$$t(z) = {:}T(z)\tilde{\mathbb{1}}(z){:}\,. \tag{237}$$

The results for $\langle Tt\rangle$ and $\langle Tt\rangle$ are exactly the same as for the general case. Since the OPE of $t(z)t(w)$ is relatively short, we will state all terms:

$$\begin{aligned}
& t^{(0)}(z)t^{(0)}(w) \\
= \;& t^{(1)}(z)t^{(1)}(w) + \left(\alpha + 2\alpha\beta\tilde{\mathbb{1}} + \beta^2\tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w)\right) T^{(2)}(z)T^{(2)}(w) \\
\sim \;& \frac{1}{(z-w)^4}\left(\log^2(z-w) + 2\log(z-w)\tilde{\mathbb{1}}(w) + 1\right) \\
& + \sum_{i=0}^{3} \frac{\log(z-w)\partial^i\tilde{\mathbb{1}}(w)}{i!(z-w)^{4-i}} \\
& + \frac{1}{(z-w)^2}\left(\left[\log(z-w) - 2\log^2(z-w)\right] T^{(1)}(w) + \left[2 - 4\log(z-w)\right] t^{(1)}(w)\right) \\
& + \frac{1}{2(z-w)}\left(\left[\log(z-w) - 2\log^2(z-w)\right] \partial T^{(1)}(w) + \left[2 - 4\log(z-w)\right] \partial t^{(1)}(w)\right) \\
& + \frac{\frac{c_2}{2}(\alpha^2 + 2\alpha\beta\tilde{\mathbb{1}} - \beta^2(2\log(z-w)\tilde{\mathbb{1}} + \log^2(z-w))}{(z-w)^4} \\
& + \frac{2(\alpha^2 + 2\alpha\beta\tilde{\mathbb{1}} - \beta^2(2\log(z-w)\tilde{\mathbb{1}} + \log^2(z-w))T^{(2)}(w)}{(z-w)^2} \\
& + \frac{\partial_w[(\alpha^2 + 2\alpha\beta\tilde{\mathbb{1}} - \beta^2(2\log(z-w)\tilde{\mathbb{1}} + \log^2(z-w))T^{(2)}(w)]}{(z-w)}\,.
\end{aligned} \tag{238}$$

Note that since the θ anti-commute, $:\tilde{\mathbb{1}}(w)\tilde{\mathbb{1}}(w):$ vanishes.

Obviously here, too, it is not possible to reduce the rhs of the equation to terms only consisting of the "neutral" operators, since it would be necessary to set $\alpha = 0$ and $\beta = 1$ which means that the OPEs of $T^0(z)t^0(w)$ would vanish. Even the vev of the two-point function of the logarithmic partner vanishes in this case:

$$\langle t^{(0)}(z)t^{(0)}(w)\rangle \;=\; c_2\frac{\alpha\beta + 2(1-\beta^2)\log(z-w)}{(z-w)^4}\,. \tag{239}$$

5. Conclusions

5.1. Watts' differential equation

In our first paper about Watts' differential equation [78] we have shown that the only possible way to interpret this equation within an (L)CFT null vector condition is the $c_{(6,1)} = -24$ LCFT of the triplet series which excludes any Kac-table based $c = 0$ model. Thus if the differential equation for the (numerically proven) horizontal-vertical crossing probability Π_{hv} has a meaning in CFT, it arises from a third level null vector differential equation acting on the four point function of boundary operators $\langle\phi_{h_{1,3}=-2/3}(z)\phi_{-2/3}(0)\phi_{-2/3}(1)\phi_{-1}(\infty)\rangle$ in an $c_{(6,1)} = -24$ LCFT.

The LCFT solution also provides the horizontal crossing probability Π_h whose exact form has already been provided by Cardy [7]. These results raise the question whether we should rather model percolation via an $c = -24$ ansatz in LCFT than with vanishing central charge. The new interpretation is suggested by several arguments which show that $c = -24$ fits more naturally on the set of problems containing percolation, i.e. the generalized Cardy's formulas within an extended Schramm Loewner evolution usually called SLE(κ, b). Since the commonly used arguments in favor of $c = 0$ (and thus null vector conditions based on $h = 0$ fields) only fix the central charge modulo 24, there are no strict arguments left that contradict our solution. Moreover, some of the previous assumptions even turn out to be problematic such as the minimal model (with partition function $\mathcal{Z} = 1$) excluding the boundary operator at position $(1, 3)$ or $(3, 1)$ from the Kac-table needed for the horizontal-vertical crossing setup.

Providing a proof for our proposal would clearly go beyond the scope of this thesis, since it already turned out to be impossible to use the known techniques developed by Smirnov for the horizontal crossing probability for bond percolation.

Some questions still remain, concerning the connection to the various SLE approaches. The crossing probabilities Π_h and Π_{hv} have already been proven by Smirnov [75] via an ordinary SLEκ and Dubedat [16] via an extended SLE$(\kappa, \underline{\rho})$, respectively, but only for site percolation on the triangular lattice. Thus it would be interesting how those results can be extended to bond percolation on the square lattice and if they support or obstruct the $c = -24$ proposal.

Besides the discussion whether one or the other ansatz is correct, another important issue is to investigate in more detail the close relationship between conformal field theories whose central charges differ by multiples of 24, especially why $c = -24$ and $c = 0$ have so many similar properties concerning percolation.

5.2. $c = 0$ and percolation

In our investigation of the structure of (L)CFTs with vanishing central charge we chose a new approach based on the augmented minimal model $c_{(9,6)} = 0$, including a Jordan cell structure on the identity level with respect to L_0, i.e. $L_0\tilde{\mathbb{1}} = \mathbb{1}$. From this assumption follows immediately the Jordan cell connection of the level two descendants of $\mathbb{1}$ and its logarithmic partner $\tilde{\mathbb{1}}$, i.e. $L_{-2}\mathbb{1} \equiv T(0)$ and $L_{-2}\tilde{\mathbb{1}} \equiv t(0)$, being connected by $L_0 t(z) = 2t(z) + \lambda T(z)$. A special feature of this setup is the vanishing of any two-point function involving $T(z)$. Depending on the different resolutions of this puzzle, one is forced to take certain consequences into account. If we stick to taking $T(z)|0\rangle \equiv 0$ in the irreducible vacuum representation of the Virasoro algebra, we might reconsider the field-state-isomorphism for $c = 0$. Also, the assumption of a logarithmic partner of the identity naturally leads to a vanishing of all correlation functions which only involve proper primary fields. The stress energy tensor is a proper primary field in the case of vanishing central charge.

A possible way out of this dilemma is to interpret the vanishing of correlators as being due to the presence of certain zero modes. Such behavior is well known in fermionic theories as ghost systems, particularly in $c = -2$. Only when the zero modes are canceled due to certain field insertions we get non-vanishing results. It would be most tempting to try to construct a free field realization of a $c = 0$ theory as a Kac-table based theory with anti-commuting fields.

We presented several arguments why the Kac table based ansatz is more promising than the replica approach introduced by Gurarie and Ludwig [41] or Kogan and Nichols [53], especially with respect to its interpretation in physics and determination of the field content. Furthermore, in addition to Cardy's proposals [9], we gave a fourth loophole to the $c \to 0$ catastrophe within an (L)CFT setup and gave examples for both approaches to $c = 0$.

Since none of the approaches to $c = 0$ currently seems to be able to fulfill all desired features at a time we suggest deeper going investigation of the subjects related to the problems discussed in this theis. This includes a fermionic realization of the augmented minimal model and, above all, general research on the representation theory of the augmented minimal $c_{(9,6)} = 0$ model with extended Kac table [24], or, in general, the extension of the formalism for $c_{(p,q=1)}$ models to arbitrary q.

5.3. General remarks

We believe that the results of this paper have been a small but important step towards the implementation of percolation models within an LCFT approach. We have shown that we should reconsider widely accepted assumptions such as the postulation of $c = 0$ for percolation and Jordan cell structures on higher levels without the same structure for the identity in the $c_{(9,6)} = 0$ theory. More precisely, we proved that there can not be a level three and level two null vector condition in a $c = 0$ augmented minimal model simultaneously for a standard assumption of the action of l_0, the zeroth mode of the logarithmic partner of the stress energy tensor $t(0)$. This would exclude either Watts' or Cardy's differential equation for the two respective crossing probabilities in percolation within a Kac table based ansatz for $c = 0$.

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A. Appendix

A.1. Differential equations from null vector conditions

Since the calculation given in Di Francesco [14] has several printing errors although the result is correct, the calculations of the level two and three null vectors will be given here.

A.1.1. The level two null vector differential equation

In its general form, the differential equation for a four-point function generated by the the level two null state $\left(L_{-1}^2 - \frac{3}{2(h+1)}L_{-2}\right)|h\rangle = 0$, is given by

$$\left[\partial_{z_0}^2 - \frac{3}{2(h+1)}\left(\sum_{i=1,2,3}\frac{h_i}{z_{i0}^2} - \frac{1}{z_{i0}}\partial_{z_i}\right)\right]\prod_{0\leq i<j\leq 3} z_{ij}^{\mu_{ij}}G(z) = 0\,, \tag{240}$$

where the four-point function $\langle\phi_0(z_0)\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle$ can be written as $\prod_{0\leq i<j\leq 3} z_{ij}^{\mu_{ij}}G(z) \equiv f(\tilde{z})G(z)$ with z being the anharmonic ratio, $\tilde{z} = \{z_0, z_1, z_2, z_3\}$ and $\mu_{ij} = \frac{1}{3}\left(\sum_k h_k\right) - h_i - h_j$. In the following, we will pull the factor before $G(z)$ through and therefore introduce a short hand notation for the differential equation

$$\left[D_{z_0}^2 - \frac{3}{2(h+1)}\left(C_{z_i} + D_{z_i}\right)\right] f(\tilde{z})G(z) = 0\,. \tag{241}$$

Taking a look at the different contributions,

$$\begin{aligned} D_{z_0}^2 f(\tilde{z})G(z) &= (D_{z_0}^2 f(\tilde{z}))G(z) + 2(D_{z_0}f(\tilde{z}))(D_{z_0}G(z)) + f(\tilde{z})D_{z_0}^2 G(z)\,, & (242)\\ C_{z_i}f(\tilde{z})G(z) &= f(\tilde{z})C_{z_i}G(z)\,, & (243)\\ D_{z_i}f(\tilde{z})G(z) &= (D_{z_i}f(z))G(z) + f(\tilde{z})D_{z_i}G(z)\,, & (244) \end{aligned}$$

we have to compute

$$\begin{aligned} D_{z_0}f(\tilde{z}) &= f(z)\sum_{i=1,2,3}\frac{\mu_{0i}}{(z_0 - z_i)}\,, & (245)\\ D_{z_0}^2 f(\tilde{z}) &= f(z)\left(\left(\sum_{i=1,2,3}\frac{\mu_{0i}}{(z_0 - z_i)}\right)^2 - \sum_{i=1,2,3}\frac{\mu_{0i}}{(z_0 - z_i)^2}\right)\,, & (246)\\ D_{z_i}f(\tilde{z}) &= f(z)\left(\sum_{\substack{i\neq j=1,2,3\\ j=0,1,2,3}}\frac{\mu_{ij}}{z_{ij}}\right)\,. & (247) \end{aligned}$$

Taking advantage of the property of primary fields, that they transform with a Jacobian factor only when executing the limit $z_0 \to z$, $z_1 \to 0$, $z_2 \to 1$ and $z_3 \to \infty$, we have to substitute the differentiation with respect to the z_i as well, i.e. $\partial_{z_i} = \frac{\partial z}{\partial z_i}\partial_z$ and $\partial_{z_0}^2 = \frac{\partial^2 z}{\partial z_0^2}\partial z + \left(\frac{\partial z}{\partial z_0}\right)^2\partial_z^2$. Thus we have to compute the respective terms, i.e.

$$\begin{aligned} \frac{\partial z}{\partial z_0} &= \frac{(z_1 - z_3)(z_2 - z_3)}{(z_2 - z_1)(z_0 - z_3)^2} \to 1\,, & (248)\\ \frac{\partial z}{\partial z_1} &= \frac{(z_0 - z_1)(z_2 - z_3)}{(z_0 - z_3)(z_2 - z_1)^2} \to z\,, & (249)\\ \frac{\partial z}{\partial z_2} &= \frac{(z_3 - z_1)(z_0 - z_1)}{(z_0 - z_3)(z_2 - z_1)^2} \to 1 - z\,, & (250)\\ \frac{\partial z}{\partial z_3} &= \frac{(z_0 - z_2)(z_0 - z_1)}{(z_2 - z_1)(z_0 - z_3)^2} \to 0\,, & (251) \end{aligned}$$

and thus $\partial_{z_0}^2 \to \partial_z^2$ as well. Thus the differential equation becomes

$$\left[\frac{\mu_{01}(\mu_{01}-1)}{z^2} + \frac{\mu_{02}(\mu_{02}-1)}{(z-1)^2} + 2\frac{\mu_{01}\mu_{02}}{z(z-1)} + \frac{3}{2(h+1)}\left(\frac{\mu_{01}-h_1}{z^2} + \frac{\mu_{02}-h_2}{(z-1)^2}\right)\right.$$
$$\left. -\frac{\mu_{12}}{z}(z-1) + \frac{3}{2(h+1)}\left(\frac{2z-1}{z(z-1)} + 2\frac{\mu_{01}}{z} + 2\frac{\mu_{02}}{z-1}\right)\partial_z + \partial_z^2\right] \quad G(z) \quad = 0\,. \tag{252}$$

Now we take advantage of the special form of $G(z) = z^{-\mu_{01}}(z-1)^{-\mu_{02}}H(z)$. Pulling the prefactor of $H(z)$ through the differential equation again, we get the final form of the level two null vector differential equation:

$$\left(\frac{2(h+1)}{3}\partial_z^2 + \frac{2z-1}{z(z-1)}\partial_z - \frac{h_1}{z^2} - \frac{h_2}{(z-1)^2} + \frac{h_0+h_1+h_2-h_3}{z(z-1)}\right) H(z) \quad = \quad 0\,. \tag{253}$$

A.1.2. Level three from global conformal invariance

This time, we will try to use a simpler procedure since the pulling through of the prefactor $f(z)$ is quite complicated due to the presence of third order derivatives. Thus we impose the constraints following from the conformal Ward identities trying to find expressions for the derivatives with respect to the z_i.

Firstly, we take the conformal Ward identity for $\mathcal{L}_{-1}$, solving for ∂_{z_1}:

$$\sum_{i=0}^{3} \partial_{z_i} \langle \phi_0(z_0)\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = 0 \quad \Rightarrow \quad \partial_{z_1} = -\partial_{z_0} - \partial_{z_2} - \partial_{z_3}\,. \tag{254}$$

Secondly, we take the respective identity for $\mathcal{L}_0$ and solved it for ∂_{z_2}, inserting the known expression of ∂_{z_1}:

$$\left(\sum_{i=0}^{3} z_i\partial_{z_i} + h_i\right) \langle \phi_0(z_0)\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = 0\,. \tag{255}$$

$$(z_0\partial_{z_0} + h_0 - z_1(\partial_{z_0} + \partial_{z_2} + \partial_{z_3}) + h_1 + z_2\partial_{z_2} + h_2 + z_3\partial_{z_3} + h_3)\, \langle \phi_0(z_0)\ldots\rangle \quad = \quad 0 \tag{256}$$

$$\left((z_0-z_1)\partial_{z_0} + (z_2-z_1)\partial_{z_2} + (z_3-z_1)\partial_{z_3} + \sum_{i=0}^{3} h_i\right) \langle \phi_0(z_0)\ldots\rangle \quad = \quad 0 \tag{257}$$

Thus the final result for ∂_{z_2} is given by:

$$\partial_{z_2} = -\frac{(z_0-z_1)}{(z_2-z_1)}\partial_{z_0} - \frac{(z_3-z_1)}{(z_2-z_1)}\partial_{z_3} - \frac{\sum_{i=0}^{3} h_i}{(z_2-z_1)}\,. \tag{258}$$

Thirdly, we do the same thing for $\mathcal{L}_1$, solving for ∂_{z_3}:

$$\left(\sum_{i=0}^{3} z_i^2\partial_{z_i} + 2z_ih_i\right) \langle \phi_0(z_0)\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = 0\,. \tag{259}$$

$$\begin{aligned} 0 \quad = \quad & \left\{ z_0^2\partial_{z_0} - z_1^2\left(\partial_{z_0} - \frac{(z_0-z_1)}{(z_2-z_1)}\partial_{z_0} - \frac{(z_3-z_1)}{(z_2-z_1)}\partial_{z_3} - \frac{\sum_{i=0}^{3} h_i}{(z_2-z_1)} + \partial_{z_3}\right)\right. \\ & \left. -z_3^2\left(\frac{(z_0-z_1)}{(z_2-z_1)}\partial_{z_0} + \frac{(z_3-z_1)}{(z_2-z_1)}\partial_{z_3} + \frac{\sum_{i=0}^{3} h_i}{(z_2-z_1)}\right) + z_3^2\partial_{z_3} + 2\sum_{i=0}^{3} z_ih_i\right\} \langle \phi_0(z_0)\ldots\rangle \\ = \quad & \left\{\left(z_0^2 - z_1^2 + z_1^2\frac{(z_0-z_1)}{(z_2-z_1)} - z_2^2\frac{(z_0-z_1)}{(z_2-z_1)}\right)\partial_{z_0}\left(z_3^2 - z_1^2 + z_1^2\frac{(z_3-z_1)}{(z_2-z_1)} - z_2^2\frac{(z_3-z_1)}{(z_2-z_1)}\right)\partial_{z_3}\right. \\ & \left. +\frac{(z_1^2-z_2^2)\sum_{i=0}^{3} h_i}{(z_2-z_1)} + 2\sum_{i=0}^{3} z_ih_i\right\} \langle \phi_0(z_0)\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle\,, \end{aligned} \tag{260}$$

with

$$(z_1^2 - z_2^2)\frac{(z_3 - z_1)}{(z_2 - z_1)} = -(z_3 - z_1)(z_1 + z_2) \tag{261}$$

$$\begin{aligned} z_3^2 - z_1^2 - (z_3 - z_1)(z_1 + z_2) &= (z_3 - z_1)(z_3 + z_1 - z_1 - z_2) \\ &= (z_3 - z_1)(z_3 - z_2)\,, \end{aligned} \tag{262}$$

and analogously for $z_3 \to z_1 : (z_0 - z_1)(z_0 - z_2)$:

$$\partial_{z_3} = -\frac{(z_0 - z_1)(z_0 - z_2)}{(z_3 - z_1)(z_3 - z_2)}\partial_{z_0} + \frac{(z_1 + z_2)\sum_{i=0}^3 h_i}{(z_3 - z_1)(z_3 - z_2)} - \frac{2\sum_{i=0}^3 z_i h_i}{(z_3 - z_1)(z_3 - z_2)}\,. \tag{263}$$

Inserting the outcome into our previous results, we get the equations for ∂_{z_2} and ∂_{z_1}.

$$\begin{aligned} \partial_{z_2} &= -\frac{(z_0 - z_1)}{(z_2 - z_1)}\partial_{z_0} - \frac{(z_3 - z_1)}{(z_2 - z_1)}\partial_{z_3} - \frac{\sum_{i=0}^3 h_i}{(z_2 - z_1)} \\ &= -\left(\frac{(z_0 - z_1)}{(z_2 - z_1)} - \frac{(z_3 - z_1)}{(z_2 - z_1)}\frac{(z_0 - z_1)(z_0 - z_2)}{(z_3 - z_1)(z_3 - z_2)}\right)\partial_{z_0} \\ &\quad -\left(\frac{(z_3 - z_1)}{(z_2 - z_1)}\frac{(z_1 + z_2)}{(z_3 - z_1)(z_3 - z_2)} + \frac{1}{(z_2 - z_1)}\right)\sum_{i=0}^3 h_i + \frac{(z_3 - z_1)}{(z_2 - z_1)}\frac{2\sum_{i=0}^3 z_i h_i}{(z_3 - z_1)(z_3 - z_2)} \\ &= -\frac{(z_0 - z_1)(z_0 - z_3)}{(z_2 - z_1)(z_2 - z_3)}\partial_{z_0} + \frac{(z_1 + z_3)\sum_{i=0}^3 h_i}{(z_2 - z_1)(z_2 - z_3)} - \frac{2\sum_{i=0}^3 z_i h_i}{(z_2 - z_1)(z_2 - z_3)} \end{aligned} \tag{264}$$

$$\begin{aligned} \partial_{z_1} &= -\partial_{z_0} - \partial_{z_2} - \partial_{z_3} \\ &= \left(\frac{(z_0 - z_1)(z_0 - z_3)}{(z_2 - z_1)(z_2 - z_3)} + \frac{(z_0 - z_1)(z_0 - z_2)}{(z_3 - z_1)(z_3 - z_2)} - 1\right)\partial_{z_0} \\ &\quad -\frac{(z_1 + z_3)\sum_{i=0}^3 h_i}{(z_2 - z_1)(z_2 - z_3)} + \frac{2\sum_{i=0}^3 z_i h_i}{(z_2 - z_1)(z_2 - z_3)} - \frac{(z_1 + z_2)\sum_{i=0}^3 h_i}{(z_3 - z_1)(z_3 - z_2)} + \frac{2\sum_{i=0}^3 z_i h_i}{(z_3 - z_1)(z_3 - z_2)} \\ &= -\frac{(z_0 - z_3)(z_0 - z_2)}{(z_2 - z_1)(z_3 - z_1)}\partial_{z_0} \\ &\quad -\frac{((z_3 - z_1)(z_1 + z_3) - (z_2 - z_1)(z_1 + z_2))\sum_{i=0}^3 h_i}{(z_2 - z_1)(z_2 - z_3)(z_3 - z_1)} + \frac{((z_2 - z_1) - (z_3 - z_1))2\sum_{i=0}^3 z_i h_i}{(z_2 - z_1)(z_2 - z_3)(z_3 - z_1)} \\ &= -\frac{(z_0 - z_3)(z_0 - z_2)}{(z_2 - z_1)(z_3 - z_1)}\partial_{z_0} + \frac{(z_3 + z_2)\sum_{i=0}^3 h_i}{(z_2 - z_1)(z_3 - z_1)} - \frac{2\sum_{i=0}^3 z_i h_i}{(z_2 - z_1)(z_3 - z_1)}\,. \end{aligned} \tag{265}$$

Thus for $i \neq j \neq k$ we get:

$$\partial_{z_i} = -\frac{(z_0 - z_j)(z_0 - z_k)}{(z_j - z_i)(z_k - z_i)}\partial_{z_0} + \frac{(z_j + z_k)\sum_{i=0}^3 h_i}{(z_j - z_i)(z_k - z_i)} - \frac{2\sum_{i=0}^3 z_i h_i}{(z_j - z_i)(z_k - z_i)}\,. \tag{266}$$

The differential equation for the third level null vector is given by:

$$\mathcal{L}_{-1}^3 - 2(h+1)\mathcal{L}_{-1}\mathcal{L}_{-2} + (h+1)(h+2)\mathcal{L}_{-3}\,, \tag{267}$$

which is equivalent to

$$\mathcal{L}_{-1}^3 - 2(h+1)\mathcal{L}_{-2}\mathcal{L}_{-1} + h(h+1)\mathcal{L}_{-3}\,. \tag{268}$$

Since nothing is acting on $\mathcal{L}_{-1}$ and $\mathcal{L}_{-3}$; we can take the limit $z_0 \to z$, $z_1 \to 0$, $z_2 \to 1$ and $z_3 \to \infty$ directly. Additionally we get a further contribution from $\partial_{z_i}\partial_{z_0}H(z)$ since the ∂_{z_i} are functions of ∂_{z_0} and $\partial_{z_0}\left(\frac{\partial z}{\partial z_0}\right)$ vanishes but not $\partial_{z_i}(\partial_{z_0})$ which gives us an additional term of $\frac{2(h+1)}{z(z-1)}\partial_z$ when $\mathcal{L}_{-2}$ is applied to $\mathcal{L}_{-1}$.

$$\partial_{z_i}\partial_{z_0} z = (\partial_{z_i}(\partial_{z_0} z))\partial_z + (\partial_{z_0} z)(\partial_{z_i} z)\partial_z^2 \quad\longrightarrow\quad \begin{cases} \partial_z + (z-1)\partial_z^2 & \text{for } i = 1 \\ -\partial_z - z\partial_z^2 & \text{for } i = 2\,. \\ 0 & \text{for } i = 3 \end{cases} \tag{269}$$

Thus we will get two additional terms to the expected ∂_z^2 term which add up to: $\frac{-2}{z}\partial_z + \frac{2}{(z-1)}\partial_z = \frac{2}{z(z-1)}\partial_z$.

$$\begin{aligned}
\partial_{z_1} &= -\frac{(z_0-z_3)(z_0-z_2)}{(z_2-z_1)(z_3-z_1)}\partial_{z_0} + \frac{(z_3+z_2)\sum_{i=0}^3 h_i}{(z_2-z_1)(z_3-z_1)} - \frac{2\sum_{i=0}^3 z_i h_i}{(z_2-z_1)(z_3-z_1)} && (270)\\
&\to (z-1)\partial_z + h_0 + h_1 + h_2 - h_3\,, && (271)
\end{aligned}$$

and

$$\begin{aligned}
\partial_{z_2} &= -\frac{(z_0-z_1)(z_0-z_3)}{(z_2-z_1)(z_2-z_3)}\partial_{z_0} + \frac{(z_1+z_3)\sum_{i=0}^3 h_i}{(z_2-z_1)(z_2-z_3)} - \frac{2\sum_{i=0}^3 z_i h_i}{(z_2-z_1)(z_2-z_3)} && (272)\\
&\to -z\partial_z - h_0 - h_1 - h_2 + h_3\,. && (273)
\end{aligned}$$

It is interesting to note that all contributions from ∂_{z_3} vanish in the $z_3 \to \infty$ limit due to their order in z_3. Thus we are left with

$$\begin{aligned}
\mathcal{L}_{-1} &= \partial_z\,, && (274)\\
\mathcal{L}_{-2} &= \sum_{i=0}^{3} \frac{h_i}{(z_i-z_0)^2} - \frac{\partial_{z_i}}{(z_i-z_0)}\\
&= \frac{h_1}{z^2} + \frac{h_2}{(z-1)^2} + \frac{h_3-h_0-h_1-h_2}{z(z-1} - \frac{2z-1}{z(z-1)}\partial_z\,, && (275)\\
\mathcal{L}_{-3} &= \sum_{i=0}^{3} \frac{2h_i}{(z_i-z_0)^3} - \frac{\partial_{z_i}}{(z_i-z_0)^2}\\
&= -\frac{2h_1}{z^3} - \frac{2h_2}{(z-1)^3} + \frac{(2z-1)(h_0+h_1+h_2-h_3)}{z^2(z-1)^2} + \frac{z^3-(z-1)^3}{z^2(z-1)^2}\partial_z\,, && (276)
\end{aligned}$$

Hence our differential operator, which shall act on $G(z)$, becomes

$$\begin{aligned}
\Delta^G_{(1,3)} &= \partial_z^3 - 2(h+1)\left(\frac{h_1}{z^2} + \frac{h_2}{(z-1)^2} + \frac{h_3-h_0-h_1-h_2}{z(z-1)} - \frac{2z-1}{z(z-1)}\partial_z - \frac{1}{z(z-1)}\right)\partial_z\\
&\quad + h(h+1)\left(-\frac{2h_1}{z^3} - \frac{2h_2}{(z-1)^3} + \frac{(2z-1)(h_0+h_1+h_2-h_3)}{z^2(z-1)^2} + \frac{z^3-(z-1)^3}{z^2(z-1)^2}\partial_z\right)\\
&= \partial_z^3 + 2(h+1)\frac{2z-1}{z(z-1)}\partial_z^2\\
&\quad + (h+1)\left(\frac{h-2h_1}{z^2} + \frac{h-2h_2}{(z-1)^2} - 2\frac{h_3-h_0-h_1-h_2}{z(z-1)} + \frac{h+2}{z(z-1)}\right)\partial_z\\
&\quad + h(h+1)\left(-\frac{2h_1}{z^3} - \frac{2h_2}{(z-1)^3} + \frac{(2z-1)(h_0+h_1+h_2-h_3)}{z^2(z-1)^2}\right)\,. && (277)
\end{aligned}$$

A.1.3. Level three following Di Francesco

Since we applied the method above for the first time, we checked it using the known one already shown for the case of the level two null vector differential equation.

A level three null vector is defined through the differential operator

$$\mathcal{L}_{-1}^3 - 2(h+1)\mathcal{L}_{-1}\mathcal{L}_{-2} + (h+1)(h+2)\mathcal{L}_{-3} =: \Delta_{(1,3)}\,, \qquad (278)$$

which is fixed by global conformal invariance. The L_i are given by

$$\mathcal{L}_{-r}(z_0) = \sum_{i\geq 1} \frac{(r-1)h_i}{(z_i-z_0)^r} - \frac{1}{(z_i-z_0)^{r-1}}\partial_{z_i}\,. \qquad (279)$$

The four-point function can be expressed by

$$\langle\phi_0(z_0)\cdots\phi_3(z_3)\rangle = \prod_{i<j}^{3}(z_i - z_j)^{\mu_{ij}}G(z) =: f(z)G(z) =: H(z)\,, \tag{280}$$

with z being the anharmonic ratio.

Thus we have

$$\mathcal{L}_{-1} = \partial_{z_0}\,, \tag{281}$$

$$\mathcal{L}_{-2} = \sum_{i\geq 1}\frac{h_i}{(z_i - z_0)^2} - \frac{1}{(z_i - z_0)}\partial_{z_i}\,, \tag{282}$$

$$\mathcal{L}_{-3} = \sum_{i\geq 1}\frac{2h_i}{(z_i - z_0)^3} - \frac{1}{(z_i - z_0)^2}\partial_{z_i}\,. \tag{283}$$

Now we will have a look at the mixed term. For our considerations, it is easier, to simplify Δ with the help of the Virasoro commutation relation. This provides us with an extra term $-2(h+2)\mathcal{L}_{-3}$.

$$\begin{aligned}
\Delta_{(1,3)}^{-2,-1}H(z) &= \left(\sum_{i\geq 1}\frac{h_i}{(z_i - z_0)^2} - \frac{1}{(z_i - z_0)}\partial_{z_i}\right)\partial_{z_0}H(z)\\
&= \sum_{i\geq 1}\frac{h_i}{(z_i - z_0)^2}\partial_{z_0}H(z) - \sum_{i\geq 1}\frac{1}{(z_i - z_0)}\partial_{z_i}\partial_{z_0}H(z)\\
&= \sum_{i\geq 1}\frac{h_i}{(z_i - z_0)^2}\partial_{z}H(z) - \sum_{i\geq 1}\frac{1}{(z_i - z_0)}\partial_{z_i}(\partial_{z_0}z)\partial_z H(z)\\
&= \sum_{i\geq 1}\frac{h_i}{(z_i - z_0)^2}\partial_{z}H(z) - \sum_{i\geq 1}\frac{1}{(z_i - z_0)}\left((\partial_{z_i}(\partial_{z_0}z))\partial_z + (\partial_{z_i}z)(\partial_{z_0}z))\partial_z\partial_z\right)H(z)\,.
\end{aligned} \tag{284}$$

Next we have to translate our derivatives with respect to the z_i into derivatives with respect to the anharmonic ratio $z = \frac{(z_0-z_1)(z_2-z_3)}{(z_0-z_3)(z_2-z_1)}$. Afterwards, we take the limits $z_0 \to z$, $z_1 \to 0$, $z_2 \to 1$ and $z_3 \to \infty$. Most of this has been stated before for the second level null state (see equations (248) to (251)). Analogously, $\partial_{z_0}^2 \to \partial_z^2$ and $\partial_{z_0}^3 \to \partial_z^3$ within this limiting procedure. The mixed derivatives however are a bit more complicated as we have seen in (269) With these results, we can write:

$$\begin{aligned}
\mathcal{L}_{-1} &= \partial_{z_0}\\
&\to \partial_z\,,
\end{aligned} \tag{285}$$

$$\begin{aligned}
\mathcal{L}_{-2} &= \sum_{i\geq 1}\frac{h_i}{(z_i - z_0)^2} - \frac{1}{(z_i - z_0)}\partial_{z_i}\\
&\to \frac{h_1}{z^2} + \frac{h_2}{(z-1)^2} + \frac{(z-1)}{z}\partial_z - \frac{z}{(z-1)}\partial_z\,,
\end{aligned} \tag{286}$$

$$\begin{aligned}
\mathcal{L}_{-3} &= \sum_{i\geq 1}\frac{2h_i}{(z_i - z_0)^2} - \frac{1}{(z_i - z_0)^3}\partial_{z_i}\\
&\to \frac{-2h_1}{z^3} + \frac{-2h_2}{(z-1)^3} - \frac{(z-1)}{z^2}\partial_z + \frac{z}{(z-1)^2}\partial_z\\
&= \frac{-2h_1}{z^3} + \frac{-2h_2}{(z-1)^3} + \left(\frac{1}{z^2} + \frac{1}{(z-1)^2} + \frac{1}{z(z-1)}\right)\partial_z\,,
\end{aligned} \tag{287}$$

$$\begin{aligned}
\mathcal{L}_{-2}\mathcal{L}_{-1} &= \left(\sum_{i\geq 1}\frac{h_i}{(z_i - z_0)^2} - \frac{1}{(z_i - z_0)}\partial_{z_i}\right)\partial_{z_0}\\
&\to \left(\frac{h_1}{z^2} + \frac{h_2}{(z-1)^2}\right)\partial_z + \frac{1}{z}\left(\partial_z + (z-1)\partial_z^2\right) + \frac{1}{(z-1)}\left(-\partial_z - z\partial_z^2\right)\,.
\end{aligned} \tag{288}$$

To compute the differential operator $\Delta_{1,3}$ for the level three null vector condition, it is easier to replace $\mathcal{L}_{-1}\mathcal{L}_{-2} = \mathcal{L}_{-2}\mathcal{L}_{-1} + \mathcal{L}_{-3}$. Inserting the expressions we have derived so far, it can be written down as:

$$\begin{aligned}
\Delta^H_{(1,3)} &= \mathcal{L}^3_{-1} - 2(h+1)\mathcal{L}_{-1}\mathcal{L}_{-2} + (h+1)(h+2)\mathcal{L}_{-3} \\
&= \mathcal{L}^3_{-1} - 2(h+1)\mathcal{L}_{-2}L_{-1} + [(h+1)(h+2) - 2(h+1)]\mathcal{L}_{-3} \\
&= \partial_z^3 + 2(h+1)\frac{2z-1}{z(z-1)}\partial_z^2 \\
&\quad + \frac{(h+1)}{z^2(z-1)^2}\left((3h+2-2h_1-2h_2)z^2 + (-2-3h+4h_1)z + h - 2h_1\right)\partial_z \\
&\quad + \frac{h(h+1)}{z^3(z-1)^3}\left(-2(h_1+h_2)z^3 + 6h_1z^2 - 6h_1z + 2h_1\right) .
\end{aligned} \tag{289}$$

Remembering the result from global conformal invariance alone as in the previous subsection following [28], we find the following expression for the differential operator acting on the four-point function

$$\begin{aligned}
\Delta^G_{(1,3)} &= \partial_z^3 + 2(h+1)\frac{2z-1}{z(z-1)}\partial_z^2 \\
&\quad + \frac{(h+1)}{z^2(z-1)^2}\left((5h+2-2h_3)z^2 + (-5h-2h_2-2+2h_3+2h_1)z - 2h_1 + h\right)\partial_z \\
&\quad + \frac{h(h+1)}{z^3(z-1)^3}\left(2(h-h_3)z^3 + 3(h_1+h_3-h-h_2)z^2 + (h+h_2-h_3-5h_1)z + 2h_1\right) .
\end{aligned} \tag{290}$$

If we compare the results directly, we would end up with the condition $h + h_1 + h_2 - h_3 = 0$. But in that case we would have overlooked that they act on functions of z differing by a scaling prefactor which after being pulled through the differential operator yields exactly the missing terms. This occurs since in the ansatz of [28], the correlator is taken directly at the points $0, 1, z, \infty$ and is not obtained via the limiting procedure, thus the results differ by the Jacobian factor.

A.2. Mode expansions of logarithmic partner fields and the mixed algebra

A.2.1. The algebra between the modes of $T(z)$ and $t(z)$

We use the general ansatz for a logarithmic field,

$$\tilde{\phi}_h(z) = \sum_{m\in\mathbb{Z}, q\in\mathbb{N}_0} \phi_{m,q}\log^q(z) z^{-m-h}, \tag{291}$$

to calculate the commutator by comparison of the powers of $\log(w)$ and w on both sides of the equation $[\cdot,\cdot] = \ldots$. Note that this method is only applicable to the mixed algebra and the ordinary between the L_n alone since otherwise the residue theorem would have to be applied to the non analytic function $\log(w)$.

$$\begin{aligned}
[L_n, t(w)] &= \sum_{\substack{m\in\mathbb{Z}\\ q\in\mathbb{N}_0}} [L_n, l_{m,q}]\log^q(w) w^{-m-2} \\
&= \oint_0 \mathrm{d}z\, z^{n+1} T(z)t(w) \\
&= \oint_0 \mathrm{d}z\, z^{n+1}\left(\frac{c/2\tilde{\mathbb{1}} + \mu\mathbb{1}}{(z-w)^4} + \frac{2t(w) + \lambda T(w)}{(z-w)^2} + \frac{\partial t(w)}{(z-w)^1}\right) \\
&= w^{n-2}\frac{n(n^2-1)}{6}\left(c/2\tilde{\mathbb{1}} + \mu\mathbb{1}\right) + (n+1)w^n\left(2t(w) + T(w)\right) + w^{n+1}\partial_w t(w) \\
&= \sum_{\substack{m\in\mathbb{Z}\\ q\in\mathbb{N}_0}} \log^q(w) w^{-m-2}\Bigg[\left(\frac{n(n^2-1)}{6}\left(c/2\tilde{\mathbb{1}} + \mu\mathbb{1}\right)\delta_{n+m,0} + (n+1)L_{n+m}\right)\delta_{q,0} \\
&\qquad (n-m)l_{n+m,q} + (q+1)l_{n+m,q+1}\ \Bigg] .
\end{aligned} \tag{292}$$

from which we may extract the commutator

$$[L_n, l_{m,q}] = \left(\frac{n(n^2-1)}{6}\left(c/2\tilde{\mathbb{1}} + \mu\mathbb{1}\right)\delta_{n+m,0} + (n+1)L_{n+m}\right)\delta_{q,0} + (n-m)l_{n+m,q} + (q+1)l_{n+m,q+1}\,. \tag{293}$$

A rather practical than elegant way out of the problem of complicated commutators as (293) is the application of the whole thing to the vacuum (or any other highest weight state). Imposing regularity at $w \to 0$ we can conclude, that all modes with $q \neq 0$ have to vanish in that case. Thus we are left with an analytic expression for $t(w)$, i. e.

$$t(w)|0\rangle = \sum_{m\in\mathbb{Z}} l_m w^{-m-2}|0\rangle\,, \tag{294}$$

and we could even calculate the OPE as in the usual way for non logarithmic fields,

$$\begin{aligned}
[L_n, l_m]|0\rangle &= \frac{1}{(2\pi\mathrm{i})^2}\oint_0 \mathrm{d}w\, w^{m+1}\oint_0 \mathrm{d}z\, z^{n+1}\left(\frac{c/2\tilde{\mathbb{1}}+\mu\mathbb{1}}{(z-w)^4} + \frac{2t(w)+\lambda T(w)}{(z-w)^2} + \frac{\partial t(w)}{(z-w)^1}\right)|0\rangle \\
&= \frac{1}{2\pi\mathrm{i}}\oint_0 \mathrm{d}w\, w^{m+n-1}\frac{n(n^2-1)}{6}(\frac{c}{2}\tilde{\mathbb{1}}+\mu\mathbb{1})|0\rangle \\
&\quad + \frac{1}{2\pi\mathrm{i}}\oint_0 \mathrm{d}w\, w^{m+n+1}(n+1)(2t(w)+\lambda T(w))|0\rangle + \frac{1}{2\pi\mathrm{i}}\oint_0 \mathrm{d}w\, w^{m+n+2}\partial t(w)|0\rangle \\
&= \left(\frac{n(n^2-1)}{6}(\frac{c}{2}\tilde{\mathbb{1}}+\mu\mathbb{1})\delta_{n,-m} + (n+1)(2l_{n+m}+\lambda L_{n+m}) - (n+m+2)l_{n+m}\right)|0\rangle\,,
\end{aligned} \tag{295}$$

with $l_{m,0} \equiv l_m$ and therefore

$$[L_n, l_m]|0\rangle = (n-m)l_{n+m}|0\rangle + (n+1)\lambda L_{n+m}|0\rangle + \frac{n(n^2-1)}{6}\mu\delta_{n+m,0}|0\rangle\,, \tag{296}$$

with $T(z)t(w)$ as given in (149).

A.2.2. Mode expansion of $t(z)$ in $c = -2$

As a concrete example for a non trivial mode expansion containing logarithms, we chose the $c = -2$ CFT which is known to have a special realization in terms of the two free fields

$$\theta^{\pm} = \theta_0^{\pm}\log(z) + \xi^{\pm} + \sum_{n\neq 0}\theta_n^{\pm} z^{-n}\,, \tag{297}$$

with the modes of $\theta^{\pm}$ obeying the canonical anti-commutation relations:

$$\{\theta_n^{\pm}, \theta_m^{\mp}\} = \frac{1}{n}\delta_{m+n,0}\,, \tag{298}$$

$$\{\xi^{\pm}, \theta_0^{\mp}\} = \pm 1\,. \tag{299}$$

The logarithmic partner of the identity is given by

$$\tilde{\mathbb{1}}(z) := :\theta^-\theta^+:(z) = \sum_{n\in\mathbb{Z}}\left(\imath_n + \log(z)\tilde{\imath}_n + \log^2(z)\hat{\imath}_n\right)z^{-n}\,. \tag{300}$$

Inserting (297), we observe

$$\begin{aligned}
\tilde{\mathbb{1}}(z) &:= :\theta^-\theta^+:(z) \\
&= :\Big(\theta_0^- \log(z) + \xi^- + \textstyle\sum_{n\neq 0}\theta_n^- z^{-n}\Big)\Big(\theta_0^+ \log(z) + \xi^+ + \textstyle\sum_{m\neq 0}\theta_m^+ z^{-m}\Big): \\
&= \log^2(z):\theta_0^-\theta_0^+: + \log(z)\left[\left(:\theta_0^-\xi^+: + :\xi^-\theta_0^+:\right) + \left(\sum_{n\neq 0}\left(:\theta_n^-\theta_0^+: + :\theta_0^-\theta_n^+:\right) z^{-n}\right)\right] \\
&\quad + :\xi^-\xi^+: + \sum_{n\neq 0}\left[:\theta_n^-\theta_{-n}^+: + \left(:\xi^-\theta_n^+: + :\theta_n^-\xi^+:\right) z^{-n}\right] + \sum_{\substack{m,n\neq 0\\ n\neq m}} :\theta_n^-\theta_{-m}^+:z^{m-n}\,. \qquad (301)
\end{aligned}$$

Now we can identify the terms. For $n \neq 0$ the modes of $\tilde{\mathbb{1}}(z)$ are

$$\imath_0 = :\xi^-\xi^+: + \sum_{n\neq 0} :\theta_n^-\theta_{-n}^+:\,, \qquad (302)$$

$$\imath_n = \sum_{n\neq 0}\left(:\xi^-\theta_n^+: + :\theta_n^-\xi^+:\right) z^{-n} + \sum_{n,m\neq 0, n\neq m} :\theta_n^-\theta_{-m}^+:z^{m-n}\,, \qquad (303)$$

$$\tilde{\imath}_0 = :\theta_0^-\xi^+: + :\xi^-\theta_0^+:\,, \qquad (304)$$

$$\tilde{\imath}_n = \sum_{n\neq 0}\left(:\theta_n^-\theta_0^+: + :\theta_0^-\theta_n^+:\right)\,, \qquad (305)$$

$$\hat{\imath}_0 = :\theta_0^-\theta_0^+:\,, \qquad (306)$$

$$\hat{\imath}_n = 0\,. \qquad (307)$$

Since $t(z) = :T(z)\tilde{\mathbb{1}}(z):$ with $T(z) = :\partial\theta^+(z)\partial\theta^-(z):$, we have to check the mode expansion of $T(z)$. Taking the derivative of (297) with respect to z, we see that the logarithm and the ξ modes vanish. Thus, taking the normal ordered product $:T(z)\tilde{\mathbb{1}}(z):$ and expanding it by modes yields the same structure as in (300). Eventually, some of the modes which vanished for $\tilde{\mathbb{1}}$ may not vanish for $t(z)$, i. e. in general the $\hat{l}_n$ may differ from zero, where $\hat{l}_n = l_{n,2}$ in the notation of (291).

A.3. Operator product expansions

A.3.1. The bosonic free field vertex operator construction

Ansatz For the fields involved in the OPEs, we take the following ansatz:

$$T(z) = -\frac{1}{2}:\partial\phi(z)\partial\phi(z): + i\sqrt{2}\alpha_0:\partial^2\phi(z):\,, \qquad (308)$$

$$\tilde{\mathbb{1}}(z) = \lambda\phi(z)\frac{\exp\left(i\sqrt{2}a\phi(z)\right)}{i\sqrt{2}\alpha_0}\,, \qquad (309)$$

$$\mathbb{1} = \exp(i\sqrt{2}a\phi) \text{ in a correlator}\,, \qquad (310)$$

$$t(z) = :\lambda\phi(z)\frac{\exp\left(i\sqrt{2}a\phi(z)\right)}{i\sqrt{2}\alpha_0}T(z):\,. \qquad (311)$$

In contrary to other papers, i.e. [41], we will give terms up to $(z-w)^{-1}$ and $\log(z-w)\cdot(z-w)^0$ (which is still divergent for $z \to w$), respectively, while

$$\lim_{z\to w}\log(z-w)(z-w) = \lim_{z\to w}\frac{\log(z-w)}{\frac{1}{(z-w)}} = \lim_{z\to w}\frac{\frac{1}{(z-w)}}{-\frac{1}{(z-w)}^2} = \lim_{z\to w} -(z-w) = 0$$

is non divergent and will therefore be the first order to be omitted.

Additionally, as already stated in the section about vertex operators, the identity and its logarithmic partner field, $\mathbb{1}$ and $\tilde{\mathbb{1}}$ are both proportional to $\exp(i\sqrt{2}a\phi(z))$ and thus the two have weights corresponding to the two roots of $h(\tilde{\mathbb{1}}) = h(\mathbb{1}) = a^2 - 2a\alpha_0 = 0$, i.e. $a(\mathbb{1}) = 0$ and $a(\tilde{\mathbb{1}}) = 2\alpha_0$. However, in a correlation function, they can be shifted by taking advantage of screening charges as explained in the first chapter.

How the wick theorem applies to vertex operators can be found in [14]:

$$\overline{A(z)e}^{B(w)} = \overline{A(z)B}(w)\, e^{B(w)}\,,$$

and:

$$\overline{e^{A(z)}e}^{B(w)} = e^{\overline{A(z)B}(w)} :e^{A(z)}e^{B(w)}: \,.$$

The usual OPE - $T(z)T(w)$ The ordinary OPE between the stress energy tensor and itself is simple and that of the ordinary free boson in a CFT with background charge α_0:

$$\begin{aligned} T(z)T(w) &\sim (-\frac{1}{2}{:}\partial\phi(z)\partial\phi(z){:} + i\sqrt{2}\alpha_0{:}\partial^2\phi(z){:})(-\frac{1}{2}{:}\partial\phi(w)\partial\phi(w){:} + i\sqrt{2}\alpha_0{:}\partial^2\phi(w){:}) \\ &\sim \frac{1}{4}\left(2(\overline{\partial\phi(z)\partial}\phi(w))^2 + 4(\overline{\partial\phi(z)\partial}\phi(w)){:}\partial\phi(z)\partial\phi(w){:}\right) - 2\alpha_0^2(\overline{\partial^2\phi(z)\partial^2}\phi(w)) \\ &\quad -i\sqrt{2}\alpha_0\left((\overline{\partial\phi(z)\partial^2}\phi(w)){:}\partial\phi(z){:} + (\overline{\partial^2\phi(z)\partial}\phi(w)){:}\partial\phi(w){:}\right) \\ &\sim \frac{\frac{1-24\alpha_0^2}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,. \end{aligned} \tag{312}$$

The new OPE part I - $T(z)t(w)$ - auxiliary calculations For the mixed OPE we have to do some auxiliary calculations since we apply the wick theorem on the two parts of $t(w)$, i.e. $\tilde{\mathbb{1}}(w)$ and $T(w)$. This is easier to calculate since we can use some of the results from above. Thus we have

$$T(z)\tilde{\mathbb{1}}(w) \sim (T(z)\phi(w))\,\lambda\frac{\exp\left(i\sqrt{2}a\phi(w)\right)}{i\sqrt{2}\alpha_0} + \left(T(z)\lambda\frac{\exp\left(i\sqrt{2}a\phi(w)\right)}{i\sqrt{2}\alpha_0}\right)\phi(w)\,, \tag{313}$$

with

$$T(z)\phi(w) \sim -\frac{1}{2}2\frac{-1}{z-w}\partial\phi(z) + \frac{i\sqrt{2}\alpha_0}{(z-w)^2} = \frac{i\sqrt{2}\alpha_0}{(z-w)^2} + \frac{1}{z-w}\partial\phi(w)\,, \tag{314}$$

and

$$\begin{aligned} &T(z)\lambda\frac{\exp\left(i\sqrt{2}a\phi(w)\right)}{i\sqrt{2}\alpha_0} \\ \sim\; & -\frac{1}{2}2\partial\phi(z)\,\overline{\partial\phi(z)\frac{e^{\mathrm{i}a\sqrt{2}}}{i\sqrt{2}\alpha_0}}\phi(w) - \frac{1}{2}(\overline{\partial\phi(z)\phi}(w))^2\lambda\frac{\exp\left(i\sqrt{2}a\phi(w)\right)}{i\sqrt{2}\alpha_0} + i\sqrt{2}\alpha_0\,\overline{\partial^2\phi(z)\frac{e^{\mathrm{i}a\sqrt{2}}}{i\sqrt{2}\alpha_0}}\phi(w) \\ \sim\; & \frac{\mathrm{i}a\sqrt{2}}{z-w}\partial\phi(z)\lambda\frac{\exp\left(i\sqrt{2}a\phi(w)\right)}{i\sqrt{2}\alpha_0} + \frac{a^2}{(z-w)^2}\lambda\frac{\exp\left(i\sqrt{2}a\phi(w)\right)}{i\sqrt{2}\alpha_0} - \frac{2\alpha_0 a}{(z-w)^2}\lambda\frac{\exp\left(i\sqrt{2}a\phi(w)\right)}{i\sqrt{2}\alpha_0}\,. \end{aligned} \tag{315}$$

Additionally, we get the second part of the OPE:

$$\begin{aligned} T(z)\tilde{\mathbb{1}}(w) &\sim (T(z)\phi(w))\,\lambda\frac{\exp\left(i\sqrt{2}a\phi(w)\right)}{i\sqrt{2}\alpha_0} + \left(T(z)\lambda\frac{\exp\left(i\sqrt{2}a\phi(w)\right)}{i\sqrt{2}\alpha_0}\right)\phi(w) \\ &\sim \frac{\partial\tilde{\mathbb{1}}(w)}{z-w} + \frac{h(a)}{(z-w)^2}\tilde{\mathbb{1}}(w) + \frac{\lambda\mathbb{1}(w)}{(z-w)^2} \\ &\sim \frac{\partial\tilde{\mathbb{1}}(w)}{z-w} + \frac{\lambda\mathbb{1}(w)}{(z-w)^2}\,. \end{aligned} \tag{316}$$

The new OPE part I - $T(z)t(w)$ With the help of the calculations above we can construct the OPE between the energy stress tensor and its logarithmic partner field:

$$\begin{aligned} T(z)t(w) &\sim (T(z)T(w))\,\tilde{\mathbb{1}}(w) + \left(T(z)\tilde{\mathbb{1}}(w)\right)T(w) \\ &\sim \left(\frac{\frac{1-24\alpha_0^2}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\right)\tilde{\mathbb{1}}(w) + \left(\frac{\partial\tilde{\mathbb{1}}(w)}{z-w} + \frac{h(a)}{(z-w)^2}\tilde{\mathbb{1}}(w) + \frac{\lambda\mathbb{1}(w)}{(z-w)^2}\right)T(w) \\ &\sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2t(w)+\lambda T(w)}{(z-w)^2} + \frac{\partial t(w)}{z-w}\,. \end{aligned} \tag{317}$$

Remark: This result can only be obtained within a correlation function since the vertex operator for the identity, $\mathbb{1}'$, does only behave like the identity but is strictly another field. We can not obtain the same coefficients as in the theoretical calculations since we have three $h=0$ fields - $\mathbb{1}$, $\tilde{\mathbb{1}}$ and $\exp(\mathrm{i}\sqrt{2}2\alpha_0)$ with the second root of $h(a)=0$ as weight. Thus the metric changes and some multiplicities change. But in principle the OPEs should be of the same form.

In the following it will be interesting to know the form of the reversed OPE. Thus we will give the calculations and the result here:

$$\tilde{\mathbb{1}}(z)T(w) \sim (\phi(z)T(w))\,\lambda\frac{\exp\left(i\sqrt{2}a\phi(z)\right)}{i\sqrt{2}\alpha_0} + \left(\lambda\frac{\exp\left(i\sqrt{2}a\phi(z)\right)}{i\sqrt{2}\alpha_0}T(w)\right)\phi(z)\,, \tag{318}$$

wherein

$$\phi(z)T(w) \sim -\frac{1}{2}2\frac{1}{(z-w)}\partial\phi(w) + \frac{\mathrm{i}\sqrt{2}\alpha_0}{(z-w)^2} = -\frac{\partial\phi(w)}{z-w} + \frac{\mathrm{i}\sqrt{2}\alpha_0}{(z-w)^2}\,, \tag{319}$$

and

$$\lambda\frac{\exp\left(i\sqrt{2}a\phi(z)\right)}{i\sqrt{2}\alpha_0}T(w) \sim \left(-\frac{\mathrm{i}a\sqrt{2}}{z-w}\partial\phi(w) + \frac{h(a)}{(z-w)^2}\right)\lambda\frac{\exp\left(i\sqrt{2}a\phi(z)\right)}{i\sqrt{2}\alpha_0}\,. \tag{320}$$

Thus we have for the complete reversed OPE

$$\begin{aligned} \tilde{\mathbb{1}}(z)T(w) &\sim (\phi(z)T(w))\,\lambda\frac{\exp\left(i\sqrt{2}a\phi(z)\right)}{i\sqrt{2}\alpha_0} + \left(\lambda\frac{\exp\left(i\sqrt{2}a\phi(z)\right)}{i\sqrt{2}\alpha_0}T(w)\right)\phi(z) \\ &\sim \lambda\frac{\exp\left(i\sqrt{2}a\phi(z)\right)}{i\sqrt{2}\alpha_0}\left(-\frac{\partial\phi(w)}{z-w} + \frac{\mathrm{i}\sqrt{2}\alpha_0}{(z-w)^2} - \frac{\mathrm{i}a\sqrt{2}}{z-w}\partial\phi(w)\phi(w)\right) \\ &\sim -\frac{\partial\tilde{\mathbb{1}}}{z-w} + \frac{\lambda\mathbb{1}}{(z-w)^2}\,. \end{aligned} \tag{321}$$

The new OPE part II - $t(z)t(w)$ - auxiliary calculations To calculate the OPE of two logarithmic partner fields of the stress energy tensor we will try the ansatz

$$\begin{aligned} t(z)t(w) &= :\tilde{\mathbb{1}}(z)T(z)::\tilde{\mathbb{1}}(w)T(w): \\ &\sim :(\tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w))(T(z)T(w)): + :(\tilde{\mathbb{1}}(z)T(w))(T(z)\tilde{\mathbb{1}}(w)): \\ &\quad +:(\tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w)):T(z)T(w):: + :(T(z)T(w)):\tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w):: \\ &\quad +:(\tilde{\mathbb{1}}(z)T(w)):T(z)\tilde{\mathbb{1}}(w):: + :(T(z)\tilde{\mathbb{1}}(w)):\tilde{\mathbb{1}}(z)T(w)::\,. \end{aligned} \tag{322}$$

Since some of the parts have already been calculated before, we will only give the calculations for the new ones here. We will not state the terms up to the highest order needed for the OPE of the partner of the stress energy tensor with itself but leave it up to the reader to expand the terms on the rhs of the OPE depending on z around w.

•

$$
\begin{aligned}
& \tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w) \\
= \quad & \lambda\phi(z)\frac{\exp\left(i\sqrt{2}a\phi(z)\right)}{i\sqrt{2}\alpha_0}\lambda\phi(w)\frac{\exp\left(i\sqrt{2}a\phi(w)\right)}{i\sqrt{2}\alpha_0} \\
\sim \quad & \lambda^2\, \overline{\phi(z)\phi}(w)\, \frac{e^{\mathrm{i}a\sqrt{\overline{2}\phi(z)}}}{i\sqrt{2}\alpha_0}\frac{e^{\mathrm{i}a\sqrt{2}\phi(w)}}{i\sqrt{2}\alpha_0} + \lambda^2\, \phi(z)\frac{e^{\mathrm{i}a\sqrt{2}\phi(w)}}{i\sqrt{2}\alpha_0}\, \frac{e^{\mathrm{i}a\sqrt{2}\phi(z)}}{i\sqrt{2}\alpha_0}\phi(w) \\
& + \overline{\phi(z)\phi}(w)\, {:}\lambda\frac{\exp\left(i\sqrt{2}a\phi(z)\right)}{i\sqrt{2}\alpha_0}\lambda\frac{\exp\left(i\sqrt{2}a\phi(w)\right)}{i\sqrt{2}\alpha_0}{:} + \lambda^2\, \frac{e^{\mathrm{i}a\sqrt{2}\phi(z)}}{\mathrm{i}a\sqrt{2}}\, \frac{e^{\mathrm{i}a\sqrt{2}\phi(w)}}{i\sqrt{2}\alpha_0}\, {:}\phi(z)\phi(w){:} \\
& + \lambda\, \phi(z)\frac{e^{\mathrm{i}a\sqrt{2}\phi(w)}}{i\sqrt{2}\alpha_0}\, {:}\lambda\frac{\exp\left(i\sqrt{2}a\phi(z)\right)}{i\sqrt{2}\alpha_0}\phi(w){:} + \lambda\, \frac{e^{\mathrm{i}a\sqrt{2}\phi(z)}}{i\sqrt{2}\alpha_0}\phi(w)\, {:}\phi(z)\lambda\frac{\exp\left(i\sqrt{2}a\phi(w)\right)}{i\sqrt{2}\alpha_0}{:} \\
\sim \quad & -\frac{\lambda^2}{2\alpha_0^2}\left(-\log(z-w)(z-w)^{2a^2}\exp(i2a\sqrt{2}\phi(w)) - 2a^2\log^2(z-w) - \log(z-w)\right) \\
& + (z-w)^{2a^2}{:}\lambda\frac{\exp\left(i\sqrt{2}a\phi(z)\right)}{i\sqrt{2}\alpha_0}\lambda\frac{\exp\left(i\sqrt{2}a\phi(w)\right)}{i\sqrt{2}\alpha_0}{::}\phi(z)\phi(w){:} \\
& + 2\lambda\left(-2\log(z-w)\tilde{\mathbb{1}}(w) - \log(z-w)(z-w)\partial\tilde{\mathbb{1}}(w) - \frac{1}{2}\log(z-w)(z-w)^2\partial^2\tilde{\mathbb{1}}(w) + \dots\right).
\end{aligned}
\tag{323}
$$

Hereby the conformal weight of $\exp(i2a\sqrt{2}\phi(w))$ is exactly $h(2a) = 4a^2 - 2(2a)\alpha_0 = 2a^2$ to fit the dimensions of the prefactor $(z-w)^{2a^2}$.

•

$$
T(z)T(w) \sim \frac{\frac{1-24\alpha_0^2}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(z)}{(z-w)}\,. \tag{324}
$$

•

$$
T(z)\tilde{\mathbb{1}}(w) \sim \frac{\partial\tilde{\mathbb{1}}(w)}{z-w} + \frac{\lambda\mathbb{1}(w)}{(z-w)^2} + \lambda\frac{\partial^2\phi(z)}{i\sqrt{2}\alpha_0}\,. \tag{325}
$$

•

$$
\tilde{\mathbb{1}}(z)T(w) \sim -\frac{\partial\tilde{\mathbb{1}}}{z-w} + \frac{\lambda\mathbb{1}}{(z-w)^2}\,. \tag{326}
$$

Now we have all the components we need to calculate the terms of the full OPE:

- Since the charge of a vertex operator is always real, we may assume that $a^2 > 0$. Thus we do not have to pay attention to some terms any more since they are regular due to this condition:

$$
\lim_{z\to w}\log(z-w)(z-w)^{2a^2} = \lim_{z\to w}\frac{\frac{1}{z-w}}{\frac{-2a^2}{(z-w)^{2a^2+1}}} = \lim_{z\to w} -2a^2(z-w)^{2a^2} = 0\,. \tag{327}
$$

  Of course, $a \neq 0$ since $a = 0$ is the identity.

$$
\begin{aligned}
& \left(\tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w)\right)(T(z)T(w)) \\
\sim\quad & \lambda^2\left(4\log^2(z-w)+\frac{\log(z-w)}{2\alpha_0^2}\right)\frac{\frac{1-24\alpha_0^2}{2}}{(z-w)^4} \\
& +\lambda^2\log(z-w)\ \frac{\frac{1-24\alpha_0^2}{2}}{(z-w)^{4-2a^2}}\frac{\exp(i2a\sqrt{2}\phi(w))}{2\alpha_0^2}\left[\sum_{i=0}^{i<4-2a^2}\frac{(z-w)^i}{i!}\left(ia\sqrt{2}\partial\phi(z)+\partial_z\right)^i\right]\Bigg|_{z=w} \\
& -\lambda^2\ \frac{\frac{1-24\alpha_0^2}{2}}{(z-w)^{4-2a^2}}\frac{\exp(i2a\sqrt{2}\phi(w))}{2\alpha_0^2}\left[\sum_{i=0}^{i<4-2a^2}\frac{(z-w)^i}{i!}\left(ia\sqrt{2}\partial\phi(z)+\partial_z\right)^i\right]:\phi(z)\phi(w):\Bigg|_{z=w} \\
& -\log(z-w)\frac{\frac{1-24\alpha_0^2}{2}}{(z-w)^4}\sum_{i=0}^{3}(z-w)^i\partial^i\tilde{\mathbb{1}}(w)-2\lambda\log(z-w)\partial[\partial\tilde{\mathbb{1}}(w)T(w)] \\
& +\lambda^2\frac{\frac{\exp(i2a\sqrt{2}\phi(w))}{\alpha_0^2}\left(\log(z-w)T(w)-2T(w){:}\phi(w)\phi(w){:}\right)}{(z-w)^{2-2a^2}} \\
& +\lambda^2\frac{\partial\left[\frac{\exp(i2a\sqrt{2}\phi(w))}{\alpha_0^2}\left(\log(z-w)T(w)-2T(w){:}\phi(w)\phi(w){:}\right)\right]}{2(z-w)^{1-2a^2}} \\
& +\frac{(\frac{\lambda^2}{\alpha_0^2}\log(z-w)+8\lambda^2\log^2(z-w))T(w)-8\lambda\log(z-w)t(w)}{(z-w)^2} \\
& +\frac{(\frac{\lambda^2}{\alpha_0^2}\log(z-w)+8\lambda^2\log^2(z-w))\partial T(w)-8\lambda\log(z-w)\partial t(w)}{2(z-w)}\,.
\end{aligned}
\tag{328}
$$

•

$$
\begin{aligned}
& (\tilde{\mathbb{1}}(z)T(w))(T(z)\tilde{\mathbb{1}}(w)) \\
\sim\quad & \frac{\lambda^2}{(z-w)^4}+\frac{1}{(z-w)^2}\left(-\partial\tilde{\mathbb{1}}(w)\partial\tilde{\mathbb{1}}(w)-2\lambda^2\partial\phi(w)\partial\phi(w)+\left(\frac{\lambda^2}{i\sqrt{2}\alpha_0}+\lambda ia\sqrt{2}\tilde{\mathbb{1}}(w)\right)\partial^2\phi(w)\right) \\
& +\frac{1}{(z-w)}\left(\partial\tilde{\mathbb{1}}(w)\left(-2\lambda\partial\phi(w)\partial\phi(w)-\left(\frac{\lambda}{i\sqrt{2}\alpha_0}+\mathrm{i}a\sqrt{2}\tilde{\mathbb{1}}(w)\right)\partial^2\phi(w)\right)\right. \\
& \left.+\left(\frac{\lambda^2}{2i\sqrt{2}\alpha_0}+\frac{\lambda\mathrm{i}a\sqrt{2}}{2}\tilde{\mathbb{1}}(w)\right)\partial^3\phi(w)-\lambda^2\partial\phi(w)\partial^2\phi(w)\right)\,.
\end{aligned}
\tag{329}
$$

•

$$
\begin{aligned}
& (\tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w)){:}T(z)T(w){:} \\
\sim\quad & \left(\frac{\lambda^2}{2\alpha_0^2}\log(z-w)(z-w)^{2a^2}\exp(i2a\sqrt{2}\phi(w))+4\lambda^2\log^2(z-w)+\frac{\lambda^2}{2\alpha_0^2}\log(z-w)\right. \\
& \left.-\frac{\lambda^2}{2\alpha_0^2}(z-w)^{2a^2}\exp(i2a\sqrt{2}\phi(w)){:}\phi(w)\phi(w){:}-4\lambda\log(z-w)\tilde{\mathbb{1}}(w)\right){:}T(w)T(w){:}\,.
\end{aligned}
\tag{330}
$$

These terms are uninteresting for comparison with the previous calculations, since divergences proportional to $\log(z-w)$ have been neglected in the literature before.

•

$$
\begin{aligned}
(T(z)T(w)){:}\tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w){:}\quad \sim\quad & \left(\frac{\frac{1-24\alpha_0^2}{2}}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{(z-w)}\right){:}\tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w){:} \\
=\quad & 4\lambda^2\frac{\frac{1-24\alpha_0^2}{2}}{(z-w)^4}\left[\sum_{i=0}^{i<4-2a^2}\frac{(z-w)^i}{i!}\left(ia\sqrt{2}\partial\phi(z)+\partial_z\right)^i\right]{:}\phi(z)\phi(w){:}\Bigg|_{z=w} \\
& +\frac{2{:}T(w)\tilde{\mathbb{1}}(w)\tilde{\mathbb{1}}(w){:}}{(z-w)^2}+\frac{\partial{:}T(w)\tilde{\mathbb{1}}(w)\tilde{\mathbb{1}}(w){:}}{(z-w)}\,.
\end{aligned}
\tag{331}
$$

-

$$(\tilde{\mathbb{1}}(z)T(w)):T(z)\tilde{\mathbb{1}}(w): \sim \left(-\frac{\partial\tilde{\mathbb{1}}(w)}{z-w}+\frac{\lambda\mathbb{1}(w)}{(z-w)^2}\right):T(z)\tilde{\mathbb{1}}(w):$$
$$= -\frac{\left(\partial\tilde{\mathbb{1}}(w)\right)t(w)}{z-w}+\frac{\lambda t(w)}{(z-w)^2}+\frac{\lambda\left(\partial T(w)\right)\tilde{\mathbb{1}}(w)}{(z-w)}. \quad (332)$$

-

$$(T(z)\tilde{\mathbb{1}}(w)):\tilde{\mathbb{1}}(z)T(w): \sim \left(\frac{\partial\tilde{\mathbb{1}}(w)}{z-w}+\frac{\lambda\mathbb{1}(w)}{(z-w)^2}\right):\tilde{\mathbb{1}}(z)T(w):$$
$$= \frac{\left(\partial\tilde{\mathbb{1}}(w)\right)t(w)}{z-w}+\frac{\lambda t(w)}{(z-w)^2}+\frac{\lambda\left(\partial\tilde{\mathbb{1}}(w)\right)T(w)}{(z-w)}. \quad (333)$$

Obviously, the two last parts of $t(z)t(w)$ can be added up to

$$(\tilde{\mathbb{1}}(z)T(w)):T(z)\tilde{\mathbb{1}}(w): + (T(z)\tilde{\mathbb{1}}(w)):\tilde{\mathbb{1}}(z)T(w): = \frac{2\lambda t(w)}{(z-w)^2}+\frac{\lambda\partial t(w)}{(z-w)}. \quad (334)$$

The new OPE part II - $t(z)t(w)$ - simplified We will not state the terms added up for the full OPE but just a simpler version to keep track of the most important terms. Thus we consider the assumption $2a^2 > 4$, i.e. $a > \sqrt{2}$ to let the terms of order $(z-w)^a$ vanish. Thus we choose the ansatz $\alpha_0 > \frac{1}{\sqrt{2}}$. Additionally we leave out the $\log(z-w)$ poles. Note that this ansatz is not valid for $c = 0$ but e.g. for $c = -24$.

$$\begin{aligned}
&t(z)t(w)\\
\sim\quad & -4\lambda\log(z-w)\tilde{\mathbb{1}}(w)\frac{\frac{c}{2}}{(z-w)^4}+\frac{\lambda^2}{(z-w)^4}\\
&+\frac{2\lambda(1-4\log(z-w))t(w)+\left(\frac{\lambda^2}{\alpha_0^2}\log(z-w)+8\lambda^2\log^2(z-w)\right)T(w)}{(z-w)^2}\\
&+\frac{2\lambda(1-4\log(z-w))\partial t(w)+\left(\frac{\lambda^2}{\alpha_0^2}\log(z-w)+2\log^2(z-w)\right)\partial T(w)}{2(z-w)}\\
&+\left(4\lambda^2\log^2(z-w)+\frac{\lambda^2}{2\alpha_0^2}\log(z-w)\right)\frac{\frac{c}{2}}{(z-w)^4}\\
&+\frac{c}{2}\left(\frac{1}{(z-w)^4}:\tilde{\mathbb{1}}(w)\tilde{\mathbb{1}}(w):+\frac{1}{(z-w)^3}:\partial\tilde{\mathbb{1}}(w)\tilde{\mathbb{1}}(w):+\frac{1}{2(z-w)^2}:\partial^2\tilde{\mathbb{1}}(w)\tilde{\mathbb{1}}(w):\right.\\
&\left.+\frac{1}{6(z-w)}:\partial^3\tilde{\mathbb{1}}(w)\tilde{\mathbb{1}}(w):\right)\\
&-2\lambda\log(z-w)\partial\tilde{\mathbb{1}}(w)\frac{\frac{c}{2}}{(z-w)^3}-\lambda\log(z-w)\partial^2\tilde{\mathbb{1}}(w)\frac{\frac{c}{2}}{(z-w)^2}\\
&-\lambda\log(z-w)\partial^3\tilde{\mathbb{1}}(w)\frac{\frac{c}{2}}{3(z-w)}\\
&+\frac{2:T(w)\tilde{\mathbb{1}}(w)\tilde{\mathbb{1}}(w):}{(z-w)^2}+\frac{\partial:T(w)\tilde{\mathbb{1}}(w)\tilde{\mathbb{1}}(w):}{(z-w)}\\
&+\frac{1}{(z-w)^2}\left(-\partial\tilde{\mathbb{1}}(w)\partial\tilde{\mathbb{1}}(w)-2\lambda^2\partial\phi(w)\partial\phi(w)+\left(\frac{\lambda^2}{i\sqrt{2}\alpha_0}+\lambda\mathrm{i}a\sqrt{2}\tilde{\mathbb{1}}(w)\right)\partial^2\phi(w)\right)\\
&+\frac{1}{(z-w)}\left(\partial\tilde{\mathbb{1}}(w)\left(-2\lambda\partial\phi(w)\partial\phi(w)-\left(\frac{\lambda}{i\sqrt{2}\alpha_0}+\mathrm{i}a\sqrt{2}\tilde{\mathbb{1}}(w)\right)\partial^2\phi(w)\right)\right.\\
&\left.+\left(\frac{\lambda^2}{2i\sqrt{2}\alpha_0}+\frac{\lambda\mathrm{i}a\sqrt{2}}{2}\tilde{\mathbb{1}}(w)\right)\partial^3\phi(w)-\lambda^2\partial\phi(w)\partial^2\phi(w)\right).
\end{aligned} \quad (335)$$

A.3.2. The $c = -2$ augmented minimal model

Now we will take a concrete example for LCFT OPEs, namely $c = -2$ which has a fermionic representation with

$$\theta^{\pm} = \theta_0^{\pm} \log(z) + \xi^{\pm} + \sum_{n\neq 0} \theta_n^{\pm} z^{-n} \,, \tag{336}$$

obeying

$$\begin{aligned} \{\theta_n^{\pm}, \theta_m^{\mp}\} &= \frac{1}{n}\delta_{m+n,0} \,, && (337)\\ \{\xi^{\pm}, \theta_0^{\mp}\} &= \pm 1 \,, && (338) \end{aligned}$$

with all other anticommutators vanishing. The contraction rules follow from

$$\overline{\theta^+(z)\theta^-}(w) = -\log(z-w)\,, \tag{339}$$

which implies

$$\begin{aligned} \overline{\theta^-(z)\theta^+}(w) &= -\,\overline{\theta^+(w)\theta^-}(z)\\ &= -(-\log(w-z))\\ &= \log(z-w)\,. && (340) \end{aligned}$$

Since the $\theta^{\pm}$ are fermionic, the OPEs $\theta^+(z)\theta^+(w)$ and $\theta^-(z)\theta^-(w)$ respectively or their descendants are completely regular, i.e. $\theta^{\pm}(z)\theta^{\pm}(w)\tilde{0}$.

Ansatz

$$\begin{aligned} T(z) &= {:}\partial\theta^+(z)\partial\theta^-(z){:}\,,\\ \tilde{\mathbb{1}}(z) &= -{:}\theta^+(z)\theta^-(z){:}\,,\\ t(z) &= {:}T(z)\tilde{\mathbb{1}}(z){:}\,. && (341) \end{aligned}$$

The usual OPE - $T(z)T(w)$

$$\begin{aligned} T(z)T(w) &= ({:}\partial\theta^+(z)\partial\theta^-(z){:})({:}\partial\theta^+(w)\partial\theta^-(w){:})\\ &\sim \overline{\partial\theta^+(z)\partial\theta^-}(w)\,\overline{\partial\theta^-(z)\partial\theta^+}(w) + \overline{\partial\theta^+(z)\partial\theta^-}(w)\,{:}\partial\theta^-(z)\partial\theta^+(w){:} + \overline{\partial\theta^-(z)\partial\theta^+}(w)\,{:}\partial\theta^+(z)\partial\theta^-(w){:}\\ &\sim \frac{-1}{(z-w)^2}\frac{1}{(z-w)^2} + \frac{-1}{(z-w)^2}{:}\partial\theta^-(z)\partial\theta^+(w){:}\\ &\quad + \frac{1}{(z-w)^2}{:}\partial\theta^+(z)\partial\theta^-(w){:}\\ &\sim -\frac{1}{(z-w)^4} + \frac{1}{(z-w)^2}\left({:}\partial\theta^+(w)\partial\theta^-(z){:} + {:}\partial\theta^+(z)\partial\theta^-(w){:}\right)\\ &\sim -\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \left({:}\partial\theta^+(w)\partial^3\theta^-(z){:} + {:}\partial^3\theta^+(z)\partial\theta^-(w){:}\right)\,. && (342) \end{aligned}$$

Unfortunately, the last term can not be identified as a linear combination made up of descendants of $T(w)$ or $\tilde{\mathbb{1}}(w)$ only.

The new OPE - $t(z)t(w)$ - auxiliary calculations To calculate the OPE of $t(z)t(w)$ we take the same ansatz as before.

$$\begin{aligned} t(z)t(w) &= {:}\tilde{\mathbb{1}}(z)T(z){::}\tilde{\mathbb{1}}(w)T(w){:}\\ &\sim (\tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w))(T(z)T(w)) + (\tilde{\mathbb{1}}(z)T(w))(T(z)\tilde{\mathbb{1}}(w))\\ &\quad + (\tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w)){:}T(z)T(w){:} + (T(z)T(w)){:}\tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w){:}\\ &\quad + (\tilde{\mathbb{1}}(z)T(w)){:}T(z)\tilde{\mathbb{1}}(w){:} + (T(z)\tilde{\mathbb{1}}(w)){:}\tilde{\mathbb{1}}(z)T(w){:}\,. && (343) \end{aligned}$$

In the following we will calculate again the factors of the six summands for this ansatz.

- $$\begin{aligned}
& \tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w) \\
= \quad & (:\theta^+(z)\theta^-(z):)(:\theta^+(w)\theta^-(w):) \\
\sim \quad & \overline{\theta^+(z)\theta^-}(w)\,\overline{\theta^-(z)\theta^+}(w) + \overline{\theta^+(z)\theta^-}(w)\,:\theta^-(z)\theta^+(w): + \overline{\theta^-(z)\theta^+}(w)\,:\theta^+(z)\theta^-(w): \\
\sim \quad & -\log(z-w)\log(z-w) - \log(z-w):\theta^-(z)\theta^+(w): + \log(z-w):\theta^+(z)\theta^-(w): \\
\sim \quad & -\log^2(z-w) - 2\log(z-w)\tilde{\mathbb{1}}(w) - \log(z-w)\sum_{i=1}^{3}\frac{(z-w)^i}{i!}\partial^i\tilde{\mathbb{1}}(w) \\
& -(z-w)^2\log(z-w)\sum_{i=0}^{1}\frac{(z-w)^i}{(i+1)!}\partial^i T(w) \\
& +\log(z-w)\frac{(z-w)^4}{24}(:\theta^+(w)\partial^4\theta^-(w): + :\partial^4\theta^+(w)\theta^-(w):)\,. \qquad (344)
\end{aligned}$$

- $$\begin{aligned}
& T(z)\tilde{\mathbb{1}}(w) \\
= \quad & -(:\partial\theta^+(z)\partial\theta^-(z):)(:\theta^+(w)\theta^-(w):) \\
\sim \quad & -\Big[\overline{\partial\theta^+(z)\theta^-}(w)\,\overline{\partial\theta^-(z)\theta^+}(w) + \overline{\partial\theta^+(z)\theta^-}(w)\,:\partial\theta^-(z)\theta^+(w): + \overline{\partial\theta^-(z)\theta^+}(w)\,:\partial\theta^+(z)\theta^-(w):\Big] \\
\sim \quad & -\frac{-1}{(z-w)}\frac{1}{(z-w)} - \frac{-1}{(z-w)}:\partial\theta^-(z)\theta^+(w): - \frac{1}{(z-w)}:\partial\theta^+(z)\theta^-(w): \\
\sim \quad & \frac{1}{(z-w)^2} + \frac{\partial\tilde{\mathbb{1}}(w)}{(z-w)} + \partial^2\tilde{\mathbb{1}}(w) + 2T(w) + \frac{(z-w)}{2}\partial^3\tilde{\mathbb{1}}(w) + 3\frac{(z-w)}{2}\partial T(w)\,. \qquad (345)
\end{aligned}$$

- $$\begin{aligned}
& \tilde{\mathbb{1}}(z)T(w) \\
= \quad & (:\theta^+(z)\theta^-(z):)(:\partial\theta^+(w)\partial\theta^-(w):) \\
\sim \quad & -\Big[\overline{\theta^+(z)\partial\theta^-}(w)\,\overline{\theta^-(z)\partial\theta^+}(w) + \overline{\theta^+(z)\partial\theta^-}(w)\,:\theta^-(z)\partial\theta^+(w): + \overline{\theta^-(z)\partial\theta^+}(w)\,:\theta^+(z)\partial\theta^-(w):\Big] \\
\sim \quad & -\frac{1}{(z-w)}\frac{-1}{(z-w)} - \frac{1}{(z-w)}:\theta^-(z)\partial\theta^+(w): - \frac{-1}{(z-w)}:\theta^+(z)\partial\theta^-(w): \\
\sim \quad & \frac{1}{(z-w)^2} - \frac{\partial\tilde{\mathbb{1}}(w)}{(z-w)} + 2T(w) + \frac{(z-w)}{2}\partial T(w)\,. \qquad (346)
\end{aligned}$$

- $$T(z)T(w) \quad = \quad \frac{-1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(z)}{(z-w)}\,. \qquad (347)$$

Now we put the factors together yielding the complete OPE.

The new OPE - $t(z)t(w)$ After making use of the anticommutation relations among the $\theta^{\pm}$ which cause some terms to vanish, we can state the final result for the OPE in the case of $c = -2$:

$$
\begin{aligned}
& t(z)t(w) \\
\sim\ & \frac{1}{(z-w)^4}\left(\log^2(z-w) + 2\log(z-w)\tilde{\mathbb{1}}(w) + 1\right) \\
& + \frac{1}{(z-w)^3}\log(z-w)\partial\tilde{\mathbb{1}}(w) \\
& + \frac{1}{(z-w)^2}\left(\left[\log(z-w) - 2\log^2(z-w)\right]T(w) + \left[2 - 4\log(z-w)\right]t(w)\right) \\
& + \frac{1}{(z-w)^2}\left(\frac{\log(z-w)}{2}\partial^2\tilde{\mathbb{1}}(w)\right) \\
& + \frac{1}{2(z-w)}\left(\left[\log(z-w) - 2\log^2(z-w)\right]\partial T(w) + \left[2 - 4\log(z-w)\right]\partial t(w)\right) \\
& + \frac{1}{2(z-w)}\left(\frac{\log(z-w)}{3}\partial^3\tilde{\mathbb{1}}(w) + \partial^3\tilde{\mathbb{1}}(w)\right) \\
& + \left(\log^2(z-w) + 2\log(z-w)\tilde{\mathbb{1}}(w)\right) :T(z)T(w): \\
& + \frac{-\log(z-w)}{24}\left(:\partial^4\theta^+(w)\theta^-(w): + :\theta^+(w)\partial^4\theta^-(w):\right) \\
& - \log^2(z-w)\left(:\partial\theta^+(w)\partial^3\theta^-(w): + :\partial^3\theta^+(w)\partial\theta^-(w):\right) \\
& -2\log(z-w)\left(:\partial\theta^+(w)\partial^3\theta^-(w): + :\partial^3\theta^+(w)\partial\theta^-(w):\right)\tilde{\mathbb{1}}(w)\,. \qquad (348)
\end{aligned}
$$

A.3.3. Calculations for the general tensorized model

We take the ansatz

$$
\begin{aligned}
t^{(0)}(z)t^{(0)}(w) &= t^{(1)}(z)t^{(1)}(w) + \left(\alpha + \beta\tilde{\mathbb{1}}(z)\right)\left(\alpha + \beta\tilde{\mathbb{1}}(w)\right)T^{(2)}(z)T^{(2)}(w) \\
&\sim \left(T^{(1)}(z)T^{(1)}(w) + :T^{(1)}(z)T^{(1)}(w):\right)\left(\tilde{\mathbb{1}}^{(1)}(z)\tilde{\mathbb{1}}^{(1)}(w) + :\tilde{\mathbb{1}}^{(1)}(z)\tilde{\mathbb{1}}^{(1)}(w):\right) \\
&\quad + \left(T^{(1)}(z)\tilde{\mathbb{1}}(w) + :T^{(1)}(z)\tilde{\mathbb{1}}(w):\right)\left(\tilde{\mathbb{1}}^{(1)}(z)T^{(1)}(w) + :\tilde{\mathbb{1}}^{(1)}(z)T^{(1)}(w):\right) \\
&\quad + \left(\alpha^2 + 2\alpha\beta\tilde{\mathbb{1}} + \beta^2\tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w)\right)T^{(2)}(z)T^{(2)}(w)\,. \qquad (349)
\end{aligned}
$$

Note that there are no contractions between the two parts; the tensorized fields factorize into their respective OPEs.

$$
\begin{aligned}
t^{(0)}(z)t^{(0)}(w) \sim\ & \left(\frac{\frac{c_1}{2}}{(z-w)^4} + \frac{2T^{(1)}(w)}{(z-w)^2} + \frac{\partial_w T^{(1)}(w)}{(z-w)} + :T^{(1)}(z)T^{(1)}(w):\right) \\
& \qquad \times\left(\log^2(z-w) - 2\log(z-w)\tilde{\mathbb{1}}(w) + :\tilde{\mathbb{1}}^{(1)}(z)\tilde{\mathbb{1}}^{(1)}(w):\right) \\
& + \left(\frac{\mathbb{1}}{(z-w)^2} + \frac{\partial_w\tilde{\mathbb{1}}(w)}{(z-w)} + :T^{(1)}(z)\tilde{\mathbb{1}}(w):\right) \\
& \qquad \times\left(\frac{\mathbb{1}}{(z-w)^2} - \frac{\partial_w\tilde{\mathbb{1}}(w)}{(z-w)} + :\tilde{\mathbb{1}}^{(1)}(z)T^{(1)}(w):\right) \\
& + \left(\alpha^2 + 2\alpha\beta\tilde{\mathbb{1}}(w) + \beta^2\tilde{\mathbb{1}}(z)\tilde{\mathbb{1}}(w)\right) \\
& \qquad \times\left(\frac{\frac{c_2}{2}}{(z-w)^4} + \frac{2T^{(2)}(w)}{(z-w)^2} + \frac{\partial_w T^{(2)}(w)}{(z-w)} + :T^{(2)}(z)T^{(2)}(w):\right)\,. \qquad (350)
\end{aligned}
$$

After inserting the OPEs (220) and sorting the terms by order of $(z-w)$ we get

$$
\begin{aligned}
\sim\quad & \frac{1}{(z-w)^4}\Big[\Big(1+\alpha^2\frac{c_2}{2}\Big)-\Big(\frac{c_1}{2}+\beta^2\frac{c_2}{2}\Big)\log^2(z-w) \\
& +2\Big(\frac{c_1}{2}+\beta^2\frac{c_2}{2}\Big)\log(z-w)\tilde{\mathbb{1}}(w)+\alpha\beta\tilde{\mathbb{1}}^{(1)}c_2+\Big(\frac{c_1}{2}+\beta^2\frac{c_2}{2}\Big):\tilde{\mathbb{1}}^{(1)}(z)\tilde{\mathbb{1}}^{(1)}(w):\Big] \\
& +\frac{1}{(z-w)^2}\Big[+2t^{(0)}+\alpha\beta\tilde{\mathbb{1}}^{(1)}T^{(2)}(w)+\Big(T^{(0)}(w)-(1-\beta^2)T^{(2)}(w)\Big):\tilde{\mathbb{1}}^{(1)}(w)\tilde{\mathbb{1}}^{(1)}(w): \\
& 2\Big(T^{(0)}(w)-(1-\beta^2)T^{(2)}(w)\Big)\log^2(z-w)-4\Big(t^{(0)}-\alpha\beta\tilde{\mathbb{1}}T^{(2)}(w)\Big)\log(z-w)\Big] \\
& \frac{1}{(z-w)}\Big[\partial t^{(0)}+\frac{1}{2}\alpha\beta\tilde{\mathbb{1}}^{(1)}\partial T^{(2)}(w)+\partial\Big(\Big(T^{(0)}(w)-(1-\beta^2)T^{(2)}(w)\Big):\tilde{\mathbb{1}}^{(1)}(w)\tilde{\mathbb{1}}^{(1)}(w):\Big) \\
& +\Big(\partial T^{(0)}(w)-(1-\beta^2)\partial T^{(2)}\Big)\log^2(z-w)-2\Big(\partial t^{(0)}-\alpha\beta\tilde{\mathbb{1}}\partial T^{(2)}(w)\Big)\log(z-w)\tilde{\mathbb{1}}^{(1)}(w)\Big] \\
& +\Big(\log^2(z-w)+2\log(z-w)\tilde{\mathbb{1}}^{(1)}(w)\Big)\Big(:T^{(0)}(w)T^{(0)}(w):-(1-\beta^2):T^{(2)}(w)T^{(2)}(w):\Big)\ . \qquad (351)
\end{aligned}
$$

A.4. Percolation as a tensor model

A.4.1. The fourfold Ising model

Tensorizing two Ising models with each other yields nine possible fields whose weights are given by

$$((r_1,s_1),(r_2,s_2)) = (0,0),(0,\frac{1}{2}),(\frac{1}{2},0),(\frac{1}{2},\frac{1}{2}),(\frac{1}{16},\frac{1}{16}),(0,\frac{1}{16}),(\frac{1}{16},0),(\frac{1}{2},\frac{1}{16}),(\frac{1}{16},\frac{1}{2}). \qquad (352)$$

However, we can choose the smallest subset which already closes under fusion containing only

$$((r_1,s_1),(r_2,s_2)) = (0,0),(0,\frac{1}{2}),(\frac{1}{2},0),(\frac{1}{2},\frac{1}{2}),(\frac{1}{16},\frac{1}{16}) \qquad (353)$$

For this example, we will state the complete fusion products:

$$
\begin{aligned}
(0,0)\times(X,Y) &= (X,Y) & (354)\\
(0,\tfrac{1}{2})\times(0,\tfrac{1}{2}) &= (0,0) & (355)\\
(0,\tfrac{1}{2})\times(\tfrac{1}{2},0) &= (\tfrac{1}{2},\tfrac{1}{2}) & (356)\\
(0,\tfrac{1}{2})\times(\tfrac{1}{2},\tfrac{1}{2}) &= (\tfrac{1}{2},0) & (357)\\
(0,\tfrac{1}{2})\times(\tfrac{1}{16},\tfrac{1}{16}) &= (\tfrac{1}{16},\tfrac{1}{16}) & (358)\\
(\tfrac{1}{2},0)\times(\tfrac{1}{2},0) &= (0,0) & (359)\\
(\tfrac{1}{2},0)\times(\tfrac{1}{2},\tfrac{1}{2}) &= (0,\tfrac{1}{2}) & (360)\\
(\tfrac{1}{2},0)\times(\tfrac{1}{16},\tfrac{1}{16}) &= (\tfrac{1}{16},\tfrac{1}{16}) & (361)\\
(\tfrac{1}{2},\tfrac{1}{2})\times(\tfrac{1}{2},\tfrac{1}{2}) &= (0,0) & (362)\\
(\tfrac{1}{2},\tfrac{1}{2})\times(\tfrac{1}{16},\tfrac{1}{16}) &= (\tfrac{1}{16},\tfrac{1}{16}) & (363)\\
(\tfrac{1}{16},\tfrac{1}{16})\times(\tfrac{1}{16},\tfrac{1}{16}) &= (0+\tfrac{1}{2},0+\tfrac{1}{2}) & (364)\\
&= (0,0)+(0,\tfrac{1}{2})+(\tfrac{1}{2},0)+(\tfrac{1}{2},\tfrac{1}{2}). & (365)
\end{aligned}
$$

Following the same procedure, we try to construct a $c=2$ model out the tensor product of two dublicated smallest Ising models. The smallest possible setup can be constructed by symmetrizing over all fields of the same weight, i.e.

$$\begin{aligned}
\mathbb{1} &= (0,0,0,0) && (366)\\
E_1 &= \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4 && (367)\\
E_2 &= \varepsilon_{12}+\varepsilon_{23}+\varepsilon_{34}+\varepsilon_{14}+\varepsilon_{24}+\varepsilon_{23} && (368)\\
E_3 &= \varepsilon_{123}+\varepsilon_{234}+\varepsilon_{134}+\varepsilon_{124} && (369)\\
E &= \varepsilon_{1234} && (370)\\
S &= \sigma_{1234} && (371)
\end{aligned}$$

in an obvious notation for the tensor product. Its closure under fusion can be shown by explicit calculation. We will give only half of the table since the fusion product is symmetric:

$\times$	$\mathbb{1}$	E_1	E_2	E_3	E	S
$\mathbb{1}$	$\mathbb{1}$	E_1	E_2	E_3	E	S
E_1	–	$E_2+\mathbb{1}$	$2E_3+2E_1$	$E+3E_2$	E_3	$4S$
E_2	–	–	$E+4E_2+\mathbb{1}$	$3E_2+3E_1$	E_2	$6S$
E_3	–	–	–	E_1	$E_2+\mathbb{1}$	$4S$
E	–	–	–	–	$\mathbb{1}$	S
S	–	–	–	–	–	$\mathbb{1}+E_1+E_2+E_3+E$

(372)

A.4.2. Percolation as a tensor model

Starting with the assumption that percolation should be a $c=0$ theory with critical scaling dimensions $h=\frac{1}{8}$ and $h=\frac{5}{8}$ we will try a tensor model as described above:

$$c_{2,1}+4\cdot c_{4,3}=-2+4\cdot\frac{1}{2}=0. \quad (373)$$

Taking all fields of the smallest fourfold Ising model and the $c=-2$ theory, we see that we can again omit some fields and thus the smallest ansatz for $c=0$ contains

- two $h=0$ fields: $(\mathcal{R}_{\mathbb{1}},\mathbb{1})$,
- ten fields of which any two partners having a weight of $h=\frac{1}{2},1,\frac{3}{2},2,\frac{1}{4}$,respectively: $(\mathcal{R}_{\mathbb{1}},E_1)$, $(\mathcal{R}_{\mathbb{1}},E_2)$, $(\mathcal{R}_{\mathbb{1}},E_3)$, $(\mathcal{R}_{\mathbb{1}},E)$, $(\mathcal{R}_{\mathbb{1}},S)$,
- one field with $h=\frac{1}{8}$ - (μ,S),
- one field with $h=\frac{5}{8}$ - (ν,S).

Remark: the last two fields exhibit exactly the two critical scaling exponents as weights that are believed to come up in a percolation model.

It is obvious that the above fields close under fusion since $\mathcal{R}_{\mathbb{1}}\times\mathcal{R}_{\mathbb{1}}\propto\mathcal{R}_{\mathbb{1}}$ and $E_X\times E_Y\propto E_Y$ oder $\mathcal{R}_{\mathbb{1}}$, μ oder $\nu\times\mathcal{R}_{\mathbb{1}}\propto\mu+\nu$ and $S\times E_X\propto S$ as well as $\mu\times\nu\propto\mathcal{R}_{\mathbb{1}}$ and $S\times S\propto\mathcal{R}_{\mathbb{1}}+E+E_1+E_2+E_3$.

A.5. Nullvectors in $c=0$

Since we do not know anything about the commutators of the l_m-modes alone, we will explain only the necessary conditions for μ; whether the null states on level three really exist or not remains to be shown. For the level two null state this question is trivial, since the action of l_1 already annihilates the state as mentioned before, and thus we have $l_1^2|\chi^{(2)}_{(h,c)}\rangle=0$, anyway.

A.5.1. The level two null state in $c = 0$

As mentioned before, the level two null state is given by

$$|\chi^{(2)}_{(h,c)}\rangle = \left(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2\right)|h\rangle \,. \tag{374}$$

Letting l_2 act on this state, we have to compute two commutators - $[l_2, L_{-2}]$ and $[l_2, L_{-1}^2]$. Since the action of positive modes on $|h\rangle$ vanishes, we know that in this case any expression equals zero in which a negative mode stands on the right. This way, the commutators can easily be simplified.

$$\begin{aligned} l_2 L_{-2}|h\rangle &= [l_2, L_{-2}]|h\rangle \\ &= (4l_0 + h + \mu)\,|h\rangle \qquad (375) \\ l_2 L_{-1}^2 &= [l_2, L_{-1}^2]|h\rangle \\ &= [\,[l_2, L_{-1}], L_{-1}]\,|h\rangle \\ &= 6l_0|h\rangle \qquad (376) \end{aligned}$$

The null state condition now translates into

$$\left(4l_0 + \mu + h - \frac{9}{2h+1} l_0\right)|h\rangle = 0 \tag{377}$$

which means that for $l_0|h\rangle = h|h\rangle$ we have $\mu = \frac{9h}{2h+1} - 5h$ which is $\mu_{h=0} = 0$ and $\mu_{h=5/8} = -5/8$ and for $l_0|h\rangle = 0$ we have $\mu = -h$ which is $\mu_{h=0} = 0$ and $\mu_{h=5/8} = -5/8$. Thus, for this computation the choice of the action of the l_0 modes does not make any difference.

A.5.2. The level three null state in $c = 0$

Now we will compute the conditions for level three analogously. The level two null state is given by

$$|\chi^{(3)}_{(h,c)}\rangle = \left(L_{-3} - 2(h+1)L_{-2}L_{-1} + h(h+1)L_{-1}^3\right)|h\rangle \,. \tag{378}$$

With the same argumentation as above, we can now simplify the action of l_3 on this state.

$$\begin{aligned} l_3 L_{-3}|h\rangle &= [l_3, L_{-3}]|h\rangle \\ &= (6l_0 + 2h + 4\mu)\,|h\rangle \qquad (379) \\ l_2 L_{-2}L_{-1} &= [l_2, L_{-2}L_{-1}]|h\rangle \\ &= [\,[l_3, L_{-2}], L_{-1}]\,|h\rangle \\ &= (10l_0 + 2h)\,|h\rangle \qquad (380) \\ l_3 L_{-1}^3 &= [l_3, L_{-1}^3]|h\rangle \\ &= [\,[\,[l_3, L_{-1}], L_{-1}, L_{-1}]]\,|h\rangle \\ &= 24l_0|h\rangle \qquad (381) \end{aligned}$$

The null state condition now translates into

$$(24l_0 - 2(h+1)(10l_0 + 2h) + h(h+1)(6l_0 + 2h + 4\mu))\,|h\rangle = 0 \tag{382}$$

which means that for $l_0|h\rangle = h|h\rangle$ we have $\mu = \frac{6h}{h+1} - 2h$ which is $\mu_{h=2} = 0$ and $\mu_{h=1/3} = 5/6$ and for $l_0|h\rangle = 0$ we have $\mu = 1 - h/2$ which is $\mu_{h=2} = 0$ and $\mu_{h=1/3} = 5/6$. Thus, again, the results are the same for both choices of l_0.

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