Fusion Algebras and Verlinde-Formula in Logarithmic Conformal Field Theories

Fusionalgebren und Verlindeformel in Logarithmisch Konformen Feldtheorien

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Hannover, im Dezember 2006

Zusammenfassung

Die erweiterten minimalen Modelle der $c_{(p,1)}$-Serie sind Beispiele für konforme Feldtheorien mit Korrelationsfunktionen, die logarithmische Divergenzen zeigen. Als eigentlich grundlegende Eigenschaft dieser und anderer logarithmischer CFTs enthalten sie eine Reihe unzerlegbare, aber reduzibele, Darstellungen der Virasoro-Algebra mit zentraler Ladung $c = c_{(p,1)}$ und Ihrer maximal erweiterten Symmetrie-Algebra, der Triplet $\mathcal{W}$-Algebra. Mit ihnen erhält man einen Satz von Darstellungen, der allen Anzeichen zufolge unter Fusion schließt.
Ein komplexer Algorithmus von Gaberdiel und Kausch zur Berechnung dieser Fusionsprodukte wurde von ihnen aufgrund des stark steigenden Rechenaufwands komplett nur für $c = -2$ und $c = -7$ explizit durchgeführt. Praktisch gleichzeitig schlug Flohr eine Methode zur einfacheren Berechnung mit der Verlinde-Formel und einer Wahl für die S-Matrix vor, die auch seine damalige Arbeit an modulinvarianten Partitionsfunktionen nutzte. Seit 3 Jahren gibt es noch einen weiteren Verlinde-ähnlichen Ansatz für die Fusion der irreduziblen Darstellungen in $c_{(p,1)}$ Modellen von Fuchs et al., die allgemein nicht-halbeinfache Fusionsalgebren untersucht haben.
Nach einer Einführung in konforme Feldtheorien im Allgemeinen und die $c_{(p,1)}$ Modelle im Speziellen stelle ich die letzten zwei Ansätze vor. Deren Ergebnisse stimmen mit den vorhandenen Resultaten des Algorithmus überein. Ich zeige den direkten Zusammenhang zwischen beiden Verfahren, der bislang eher im Dunkeln gelegen haben, auf, indem ich die Flohrsche S-Matrix erstmals in geschlossener Form angebe und die Bedeutung einiger Größen des Fuchsschen Ansatzes im Bezug auf die konformen Feldtheorien verdeutliche. Dabei werden diverse Aspekte der beiden Methoden im Detail betrachtet.
In dieser Arbeit wird ausfuehrlich der Weg zu einer Erweitung des Fuchsschen Zugangs auf die unzerlegbaren Darstellungen, die wir gefunden haben, beschrieben und bewiesen, dass deren Ergebnisse auch mit denen der Flohrschen Methode übereinstimmen. Es wird diskutiert, welche Vor- und Nachteile verschiedene Verlinde-ähnliche Formeln für nicht-halbeinfache Fusionsalgebren haben und insbesondere auch der Vergleich zu halbeinfachen Fusionsalgebren gezogen.

Abstract

The extended minimal models of the $c_{(p,1)}$-series are examples for conformal field theories with correlation functions exhibiting logarithmic divergences. As their - and other logarithmic CFTs' - basic feature one finds a number of indecomposable, but reducible, representations of their chiral symmetry algebra, the triplet $\mathcal{W}$-algebra, which is the maximal extended symmetry algebra of the Virasoro algebra at central charge $c = c_{(p,1)}$. These representations together with the irreducible ones form a set, which shows strong evidence of closing under fusion.
An algorithm of Gaberdiel and Kausch to calculate the fusion products has only been completely carried out by them for the cases $c_{(2,1)} = -2$ und $c_{(3,1)} = -7$, because of the strongly increasing complexity towards higher p, also proving the closure under fusion for these two cases.
Virtually at the same time Flohr proposed a simpler method of calculation using the Verlinde formula and a specific choice of the S-matrix, which also took advantage of his work on modular invariant partition sums at that time.
Three years ago Fuchs et al., who studied non-semisimple fusion algebras in general, presented another Verlinde-like proposal for the fusion of only the irreducible representations in $c_{(p,1)}$ models.
After an general introduction to conformal field theories and a one specifically to $c_{(p,1)}$ models I present the last two approaches. Their findings are in correspondence to the known results of the algorithm. I point out the direct connection between both methods, which has been rather befogged until now, as I give for the first time Flohr's S-matrix in closed form and clarify the meaning of some objects in Fuchs' ansatz with respect to the conformal field theories. In the process I examine diverse aspects of both methods in detail.
This thesis will describe at length the path to an extension of Fuchs' approach, which we have found, including the indecomposable representations. It is shown, that the results of this extension are the same as the ones found with Flohr's method. The advantages and disadvantages of the different Verlinde-like formulas for non-semisimple fusion algebras are discussed, in particular also in comparison to semisimple fusion algebras.

Contents

Introduction

Quantum Field Theory and Relationship Problems

The deep relationship between physics and mathematics stems from the unique interaction of these two sciences. On the one hand, mathematics provides fabulous possibilities for the theoretical side of physics to construct theories and proof statements without doubt within the framework of this theory. On the other hand, results from experiments show unprecedented kinds of behaviour between measurable entities, which trigger development in mathematics (like e.g. differential calculus). This has been true for virtually all fields of both sciences.
For centuries of the modern times the needed mathematics could be developed simultaneously with new discoveries in natural sciences and often both was done by the same persons. Only after quantum mechanics, which still used the well-known linear algebra and calculus, had led to a unprecedented change in our picture of nature and its constituents, physics took a direction, in which among other things the language in current research of physics and mathematics diverged far apart form each other.
Nature was not any more thought to be composed out of point-like particles with fixed velocity and momentum. A life or death struggle between wave and particle picture of light – the particle picture seemed already dead as a dodo, until the photo-effect was discovered – made place for peaceful coexistence. Now the world was described with a fuzzy structure of quantum states, which were not localised precisely. A second quantisation added to the states fields, which got the fundamental entities creating also the states. This step to quantum field theory put particles and interaction on equal footing. In the front row quantum electrodynamics appeared with a description of electromagnetism of almost unbelievable accuracy (e.g. [HBH+00]).
Two big drawbacks, connected to each other, exist in quantum field theory. Firstly, one is restricted in the vast majority of interesting cases to perturbation theory. The second one is the general lack of mathematical understanding. For example, there is no graph theory in mathematics, that could deal with Feynman graphs describing the perturbation theory expansion.
Most physicists take little interest in the problems of definitions dealing with distributions, formal series and divergences, which they recipe-like cure with a medicine called regularisation – and subsequent renormalisation. They are ecstatic about the agreement with experiment in theories, where the series seem to converge rather fast, and try to deal with the other cases for the time being by simpler approaches than long investigation of the mathematical structure behind quantum field theories, but also with questionable success.
For decades few mathematician have been able to find a connection to the quantum field theories used by physicist apart from toy models. Simultaneously successes were celebrated on other fields – also new ones –, which have not found the interest of physicists immidiately. As in the last century the size of these sciences literally exploded, communities were fragmented

and communication between them got more difficult.

Conformal Field Theory: Part of a New Common Basis

The communication has drastically improved during the 25 years again. Topics like string theory, operator algebra, non-commutative geometry, knot theory, quantum groups, topological field theory, Hopf algebras in renormalisation and stochastic systems have develeped to a new basis, on which the communities have met again.
Still quantum field theory is hardly understood from the mathematical perspective, but there are exceptions. One of the first reasons for this change has been the invention of conformal field theories (CFT) living on a two dimensional space by Belavin, Polyakov and Zamolodchikov [BPZ84]. They turned out to be a non-trivial quantum field theories – i.e. not equivalent to a free quantum field theories –, which can be treated non-perturbatively and give exact results for measurable quantities. In physics they climbed fast in popularity together with string theory on the search for a theory unifying gravity with the standard model of particle physics. CFTs are an intrinsic part of string theory. Moreover a lot of applications in statistical and condensed matter physics were found (cf. [ISZ88, Aff95, Ber95, Car01]) and thus also a connection to current and near future experiments. Conformal field theories describe systems in these fields of research at so-called critical points, at which the systems have – as one property – no intrinsic length scale.
The algebra governing the symmetry of CFTs, the Virasoro algebra, which is a Lie algebra with central extension, had already found its place in mathematics, before the BPZ paper has appeared. Feigin and Fuks published some work about the representation theory of this algebra in the beginning of the 80ies, which was essential in the development of CFTs [FF82, FF83, FF]. The infinite dimension of the Virasoro algebra is the actual key to the success of conformal field theories. More than restricting the theories so much, that non-perturbative calculations are possible, it also made the development of several mathematically more rigorous approaches possible opposed to other quantum field theories.
R. Longo and K.-H. Rehren found conformal field theories as a prime example for their studies on Haag-Kastler nets of von Neumann algebras, which then contain local observables of the theory [LR95]. This led to an operator algebraic classification of chiral conformal field theories [Kaw03], each of which is the decoupled half of a full conformal field theory without boundary conditions. It has to be seen in the context of axiomatic quantum field theory. A set of axioms, the Wightman axioms, is imposed, among which are e.g. a locality condition and covariance under the Pioncaré group. The study intensively makes use of the Doplicher-Haag-Roberts superselection.
As conformal field theories were put into relation to entities from the study of topological invariants, called topological field theories, a new categorical perspective opened up and many insights have been retrieved for conformal field theory, especially, when they are subject to boundary conditions. In this line of research an exhausting formulation of so-called rational conformal field theories – they are defined by a finiteness condition on the category of irreducible representations of the Virasoro algebra – was developed, in which the full theories are constructed from the chiral halves [FRS02, FRS04a, FRS04b, FRS05, FFRS05].
However, in these two approaches we do not touch the kind of conformal field theories, which this thesis deals with. A promising path was opened in the context of the theory of vertex operator algebras [Hua92, Hua98, Hua00, HK06, HLZ06]. It was crowned by the geometric vertex operator algebras introduced by Yi-Zhi Huang, who found the conformal vertex (oper-

ator) algebras on partial operads over the moduli space of genus-zero Riemann surfaces with analytically parametrised boundaries [Hua91]. This is now also applicable to the theories, we encounter here, which has already made it possible to prove interesting statements about them (e.g. [CF06]).
At last we want to mention the connection of conformal field theories to quantum groups, which is given by a conjecture of Kazdhan and Lusztig. The Kazhdan-Lusztig correspondence proposes the equivalence of integral parts of a CFT on the one side and a corresponding quantum groups on the other side (recent work: [FGST05, FGST06c, FGST06a, FGST06b]).
So we see, how deep the branches of this tree of different research programmes, connected to conformal field theories, reach into the garden of mathematics. Though the physicists and mathematicians speak in different languages about these connected branches, there are some translators at the branching points.

The Verlinde Formula: A Twinkling Diamond Ring

This thesis is certainly in the language of a physicist and awaits translation. Its topic stems from the possibly most remarkable single input, which was translated from the physicists' language for the rest of the tree, and certainly a big step in the development of conformal field theories.
In 1988 Eric Verlinde published a paper on "Fusion Rules and Modular Transformations in 2-D Conformal Field Theory"[Ver88]. He suggested a formula, thereafter associated with his name, which would simplify the calculation of the fusion rules enormously. These rules state, in which representations of the Virasoro algebra the fields are found, that are correlated to a product of two fields of our choice, i.e. one has non-zero (three-point) correlation functions of the former field and the product. It has to be understood on the level of representations of the Virasoro algebra, which contain the fields, and is similar to a tensor product of two representations and its decomposition into irreducible representations for simple Lie algebras. The Verlinde formula uses the fact that the characters of the irreducible representations of the Virasoro algebra – and also the partition function of CFTs – are modular forms, which on its own already leads to great interest among mathematician, as it is hard to find modular forms. The so-called $\mathcal{S}$-transformation, $\tau \rightarrow -1/\tau$, is one of the generators of modular transformations. The associated S-matrix[1] describing the transformation properties of said characters under this $\mathcal{S}$-transformation is all one needs in the Verlinde-formula.
This has been very unexpected. The modular transformation properties of the characters alone give a fusion algebra, which the fusion rules form. The calculation of the fusion algebra is an important step in the understanding of any CFT. Its condition of closure ensures that we know the whole field content of the theory.
The Verlinde formula has soon been recognised to be very interesting for algebraic geometry. The version, known in this field of mathematics as Verlinde formula, calculates the dimension of the space of holomorphic sections of certain line bundles over given moduli spaces. This is connected to the physicists' version for CFTs in a very nice way by Huang's geometric vertex operator algebras.
Over the years many people worked on different proofs for the Verlinde formula in algebraic geometry. A general proof for this formulation has been given by G. Faltings [Fal94]. The

[1] We call any matrix giving the transformation properties for a set of characters under the $\mathcal{S}$-transformation an S-matrix, as a fixed term.

conclusive step that has guarantied the validity for rational conformal field theories, was only made in 2004 by Huang.
We will introduce conformal field theories, the property of rationality and especially the minimal models, a series of rational conformal field theories, in chapter 1. This chapter focusses on all we need for the discussion of fusion for minimal models and the introduction of the Verlinde formula, which in that form is also valid for all rational conformal field theories, in its last section .
In the beginning of the nineties Saleur has argued that conformal field theories with particular scaling properties seem to describe two dimensional systems of polymers or percolation particularly well [Sal92]. The central extension in the Virasoro algebra is zero for these CFT. They allow for logarithmic divergences of the correlation functions. In the following year interesting examples for such logarithmic conformal field theories also with non-vanishing central extension were found. Among these are the models of the $c_{(p,1)}$, with integer $p \geq 2$, series, which are closely related with the minimal models mentioned above. For these models the fusion algebra is non-semisimple. So the Verlinde formula, about which is written above, is not valid for this case. It is our goal here to study conjectures for a different Verlinde formula, which can provide also this non-semisimple fusion algebra without much calculation expense.
But before we get there, the second chapter gives some insights into the $c_{(p,1)}$ models. We see, that these theories are governed by a symmetry algebra, the $\mathcal{W}$-triplet algebra, which is the maximal local extension of the Virasoro algebra here. The definition of fusion can be already given with respect to the Virasoro algebra itself, which then also leads to the fusion rules with respect to the triplet algebra. At the end of chapter 2 the preliminaries for our studies of an adapted Verlinde formula for the $c_{(p,1)}$ series are explained. On the one hand, we discuss the problems, which arise from the different setting in comparison to the case semisimple fusion algebras. On the other hand, we argue for our expectation, that such a formula exists, as we point out the similarities of the $c_{(p,1)}$ models to rational conformal field theories.
In chapter 3 we will learn about two different methods, which have been proposed, to calculate the fusion rules of the $c_{(p,1)}$ models with different adaptations of the Verlinde formula. The first method (cf. [Flo97]) is motivated rather from a physicist's view and uses the characters, as they are given by the relevant representations just counting the elements. We will also present our result for a closed form of the S-matrix used in this approach in section 3.1.
The second approach (cf. [FHST04])has a more mathematical point of view. For the sake of defining a representation of the modular transformations on a certain set of characters, one effectively goes over to a set of linear combinations of these characters, as we show in section 3.2. This new insight improves our understanding of the meaning of the matrices appearing in this approach within the conformal field theory. Also only a smaller fusion algebra is calculated leaving out some relevant representations. However, it is very well founded: The fundamental algebraic statement, that the non-semisimple fusion algebra is the direct sum of its radical and a semisimple algebra, is providing the ansatz for this method.
The task of this thesis has been to connect these two roads to the fusion rules, which have been relatively unconnected, yet. It is intended to lead to better understanding of a Verlinde forumla for $c_{(p,1)}$ models and continue its development. In this respect we have been able to extend the latter approach to incorporate all relevant representations. We describe in section 3.3 in detail, how the two simplest model of the series, $c_{(2,1)} = -2$ and $c_{(3,1)} = -7$, – but mainly the first one – guide the way to this extension. Moreover we see, how it projects back to its smaller archetype.
The last section of the third chapter is devoted to the proof of the equivalence of our extension

and the method in section 3.1. As the starting point of the two methods is the same set of (generalised) characters and the purpose of both is the calculation of the fusion rules, we need to show, that the results coincide for all integer $p \geq 2$. Furthermore we see, how the different matrices appearing in both approaches relate to each other.
Finally, in the conclusion we want to draw attention to some work concerning a Kazhdan-Lusztig-like correspondence for $c_{(p,1)}$ models (among other things). Also the development of the theory of conformal vertex algebra and geometric vertex operator algebra has to be followed. These two subjects are the most promising ones to give a stronger foundation for the generalised Verlinde formula for non-semisimple fusion algebras and the latter might eventually be the connection to a proof for this formula.

Chapter 1

Preliminaries about Conformal Field Theories

In this chapter we introduce conformal field theories starting from the required conformal symmetry. We sketch the derivation of their symmetry algebra, the Virasoro algebra, and give a few details about its representations. We then turn turn to the minimal models and end up with the discussion of fusion in these models. We mainly follow fragments of the book of Di Franscesco [FMS99] and lecture notes of Flohr [Flo03] and Gaberdiel [Gab00].

1.1 Conformal Symmetry and the Virasoro Algebra

Conformal field theories are quantum field theories, which are symmetric under the special conformal group acting on the manifolds, on which the theories live. This group consists of translations, rotations, scaling and so-called special conformal transformations. Talking about CFTs it is often already implied that one only considers the case of two dimensional field theories, because much more work was done and much more results were retrieved for this case. The reason is the larger symmetry these theories have in two dimensions, while they are already nontrivial. The two dimensional space, which is $\mathbb{R}^2$ for now, is complexified, so that we look at the variables $z = x + iy$ and $\bar{z} = x - iy$ as independent variables. The special conformal group in two dimensions $SL(2, \mathbb{C})$ is parametrised by four complex parameters a, b, c and $d \in \mathbb{C}$:

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \qquad \mathbb{C} \to \mathbb{C} \, : \, z \to \frac{az + b}{cz + d} \qquad \text{and } \det A = ad - bc = 1 \, . \tag{1.1}$$

The parameters are grouped in a 2×2 matrix because the composition of two elements of $SL(2, \mathbb{C})$ is just the same as the one given by the product of the two matrices, which contain the parameters of these elements. The group is generated by the transformations we already mentioned:

$$\begin{pmatrix} e^{\frac{1}{2}a} & 0 \\ 0 & e^{-\frac{1}{2}a} \end{pmatrix} \quad : \quad \text{dilation and rotation ,} \tag{1.2}$$

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad : \quad \text{translation ,}$$

$$\tag{1.3}$$

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \quad : \quad \text{special conformal transformation.}$$

The local transformations belonging to the group $SL(2, \mathbb{C})$ are the holomorphic infinitesimal transformations defined by

$$t_\epsilon : \quad z \rightarrow z + \epsilon(z) \quad \epsilon(z) = \sum_{n=-\infty}^{+\infty} \epsilon_n z^{n+1} \ . \tag{1.4}$$

The Laurent expansion of the holomorphic function ϵ makes it obvious, that this group is infinitely generated because of its infinitely many independent degrees of freedom ϵ_n.
We are interested in the change of the expression of a spinless and dimensionless field $\phi(z, \bar{z})$, when the coordinates z and $\bar{z}$ are locally conformally transformed, while the actual field, of course, stays the same.

$$\phi'(z', \bar{z}') = \phi(z, \bar{z}) = \phi(z', \bar{z}') \underbrace{- \epsilon(z') \frac{\partial \phi(z', \bar{z}')}{\partial z'} - \bar{\epsilon}(\bar{z}') \frac{\partial \phi(z', \bar{z}')}{\partial \bar{z}'}}_{\delta\phi} \ . \tag{1.5}$$

We can insert the Laurent expansions for $\epsilon(z')$ and $\bar{\epsilon}(\bar{z}')$ here and introduce $l_n = -z^{n+1} \frac{\partial}{\partial z}$ and $\bar{l}_n = -\bar{z}^{n+1} \frac{\partial}{\partial \bar{z}}$, so that we have:

$$\delta\phi = \sum_{n=-\infty}^{\infty} \left[\epsilon_n l_n \phi(z, \bar{z}) + \bar{\epsilon}_n \bar{l}_n \phi(z, \bar{z}) \right] \ . \tag{1.6}$$

l_n and $\bar{l}_n$then are derivational operators acting on the fields of our theory, which fulfil the commutation relation of the conformal algebra. Their commutators are

$$\begin{aligned} [l_n, l_m] &= (n - m) l_{n+m} \ , \\ \left[\bar{l}_n, \bar{l}_m\right] &= (n - m) l_{n+m} \ , \\ \left[l_n, \bar{l}_m\right] &= 0 \ . \end{aligned} \tag{1.7}$$

The fact, that we get here a direct sum of twice the same algebra acting on the holomorphic and antiholomorphic part of the expressions, makes it possible that we will be able to drop the antiholomorphic part in our notation and only write the holomorphic part, always knowing that the other is also there and looks the same.
Fields of a particularly important kind, called primary fields, have special transformation properties under local conformal transformation. They change only by a given factor depending on the transformation. If we look at an arbitrary local conformal transformation, $z \rightarrow w(z)$ and $\bar{z} \rightarrow \bar{w}(\bar{z})$, a primary field $\phi(z, z)$ transforms as

$$\phi'(w(z), \bar{w}(\bar{z})) = \left(\frac{dw(z)}{dz} \right)^{-h} \left(\frac{d\bar{w}(\bar{z})}{d\bar{z}} \right)^{-\bar{h}} \phi(z, z) \ . \tag{1.8}$$

The negative exponents h and $\bar{h}$ are called the conformal dimensions of the primary field.
In quantum field theory the observable entities are either currents or correlators. So the aim is always to calculate these. The symmetry of the primary fields gives us directly the following relation for correlation functions of n primary fields,

$$\langle \phi_1(w_1, \bar{w}_1) \ldots \phi_n(w_n, \bar{w}_n) \rangle = \prod_{i=1}^{n} \left(\frac{dw}{dz} \right)_{w=w_i}^{-h_i} \left(\frac{d\bar{w}}{d\bar{z}} \right)_{\bar{w}=\bar{w}_i}^{-\bar{h}_i} \langle \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle \ . \tag{1.9}$$

This is a strong statement, which enables one to directly calculate the two- and three-point correlation functions up to a constant. In rational conformal field theories it is closely related to the fusion product of two primary fields, whether any three-point function containing those two is different from zero - so on the constant that is left open. We will come back to this in section 1.2.2.
We have no time coordinate, which would give us a natural ordering, when we quantise the theory, in contrast to theories on normal space-time. There one has a time ordering of the fields and relates it via the Wick theorem to normal ordered products, of which the correlators vanish. The quantisation is realised at the point, where one imposes the commutation relations on the modes of the fields and so lays down the form of the Wick theorem implicitly.
Here we need to use another natural ordering. This is best motivated by the string theory picture. While a closed string propagates through time, it sweeps out a tube isomorphic to a cylinder. Some time axis would go along the cylinder, while a line of constant time would be a circle on the cylinder surface. This can be conformally mapped to the complex plane, where the infinite past would be the origin and the time axis of the world sheet picture would point radially to the future.
So time ordering on the cylinder is radial ordering on the complex plane. The latter is defined for spinless fields as:

$$\mathcal{R}\left(\phi_1(z)\phi_2(w)\right) = \begin{cases} \phi_1(z)\phi_2(w)\text{if} & |z| > |w| \\ \phi_2(z)\phi_1(w)\text{if} & |z| < |w| \quad . \end{cases} \tag{1.10}$$

In a correlation function all fields must be radially ordered, i.e, the fields have to act on the vacuum in the sequence from the origin outwards, in other words from the "past" to the "future". Whenever we want to insert a product of two fields into a correlation function, it is only well-defined in radial ordering. This leads to a commutator $[\cdot,\cdot]$, which is defined as

$$\begin{aligned} \oint_w dza(z)b(w) &= \oint_{C_1} dza(z)b(w) - \oint_{C_2} dzb(w)a(z) =: [A, b(w)] \ , \\ A &:= \oint dza(z) \ , \end{aligned} \tag{1.11}$$

with C_1 and C_2 encircling the origin at distances larger and smaller than $|w|$, respectively, if there are no other fields inserted within the space between these contours in the correlation function. In the general case we can only define the commutator with infinitesimal distance between the two contours. Therefore one looks at the short distance properties of a product of two fields, which are expressed in an operator product expansion (OPE). It shows the singular part of the product of two fields $\phi(z)$ and $\phi(w)$, as their insertion points approach each other expanded in powers of $1/(z-w)$. The OPE of twice the stress energy tensor, for example, is

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{1.12}$$

with a constant c.
Equipped with the commutator (eq. (1.11)), we define the conformal charge

$$Q_\epsilon = \frac{1}{2\pi i}\oint dz\epsilon(z)T(z) \tag{1.13}$$

with the holomorphic part of the stress energy tensor $T(z)$ and can express the conformal Ward identity as

$$\delta_\epsilon\Phi(w) = -\left[Q_\epsilon, \Phi(w)\right] \ . \tag{1.14}$$

This identity describes the variation of a field Φ under a local conformal transformation. Thus the charges Q_ϵ generate the conformal transformations.
We expand the holomorphic stress energy tensor,

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \ . \tag{1.15}$$

The commutation relations of the modes L_n are found to be

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \ . \tag{1.16}$$

This is the Virasoro algebra. It is a infinitely generated Lie algebra with central extension. The central charge c is coming from the OPE of twice the stress energy tensor (eq. (1.12)).
An antiholomorphic pendant belongs to each of the equations (1.13)-(1.16). The modes $\bar{L}_n$ of the antiholomorphic part of the stress energy tensor $\bar{T}(\bar{z})$ also generate the Virasoro algebra and commute with their holomorphic counterparts. So the symmetry algebra of our theory splits into the direct sum of twice the Virasoro algebra for each chiral half. Hence the Virasoro algebra is its chiral symmetry algebra. We now always only deal with the chiral theory, i.e. we look at one chiral half of it.
The Virasoro algebra contains a subalgebra with the modes L_{-1}, L_0 and L_1, which generate the global conformal transformations.
The representation theory of this algebra was greatly brought forward by Feigin and Fuks.
The globally conformally invariant vacuum state is defined as

$$L_n|0\rangle = 0 \quad \forall n \geq -1 \ . \tag{1.17}$$

Other so-called highest weight states $|h\rangle$ are defined by the action of a primary field $\phi(z)$ with conformal dimension h on the vacuum.

$$|h\rangle = \phi(0)|0\rangle \tag{1.18}$$

This gives us a one-to-one correspondence between highest weight states and primary fields. Concerning the action of the Virasoro algebra, one can either look at commutators of the Virasoro generators with primary fields or their action on highest weight states. The highest weight states are eigenstates of the zero mode L_0. The eigenvalue, its weight, is the conformal dimension of the corresponding field. The negative modes, as descendent operators, increase the weight, while the positive modes, as ascendent operators, annihilate the highest weight states. This is the parlance, although the highest weight states have actually the lowest weight in the module generated by the action of the Virsoro algebra on one of them. This module is called Verma module and is given by the following gathered conditions:

$$\text{highest weight state } |h\rangle: \qquad L_0|h\rangle = h|h\rangle \ , \tag{1.19}$$

$$L_n|h\rangle = 0 \quad \forall n > 0 \ , \tag{1.20}$$

$$\text{Verma module consists of the states:} \tag{1.21}$$

$$span(\{L_{-k_1}L_{-k_2}\ldots L_{-k_n}|h\rangle \ \ | \ \ 0 < k_1 \leq k_2 \leq \ldots \leq k_n;\ n \in \mathbb{N}\}) \ . \tag{1.22}$$

The states $L_{-k_1}L_{-k_2}\ldots L_{-k_n}|h\rangle$ spanning this module are also eigenstates of L_0 to the eigenvalue $h+k_1+k_2+\ldots+k_n =: h+N$ and N is called the level of the state. These Verma modules define representations of the Virasoro algebra and are the "building blocks" of all physically relevant representations in CFTs. Their actual properties depend on the central charge.

1.2 Minimal Models

We want to look at a prominent series of models, the minimal models, both as an example and because these models constitute the starting point for the discussion of the $c_{(p,1)}$ series in the next chapter. They are deeply connected with the search for unitary conformal field theories, which contain no states with negative norm, and with the name of V. Kač. One takes the matrix of the inner product of all basis states in a particular level N of a Verma module

$$M_{ij} = \langle i|j\rangle \ , \quad \forall |i\rangle, |j\rangle\text{: basis states of level } N \ . \tag{1.23}$$

Its determinant, the Kač determinant, provides restrictions, for which central charges CFTs can be unitary, for values $0 < c < 1$ and $h > 0$.
This whole parameter strip is virtually excluded this way. Only a discrete series remains. It is parametrised by an integer $m \geq 2$:

$$c = 1 - \frac{6}{m(m+1)} \tag{1.24}$$

$$h_{(r,s)}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)} \quad \forall 1 \leq r < m,\ 1 \leq s \leq r \ . \tag{1.25}$$

This series is part of the bigger one of minimal models. They are constructed in a minimal way looking at the possible conformal dimensions for a given central charge. The defining observation leading to this series is, that for certain values of the central charge the possible conformal weights of primary fields do not form a dense set.
These theories belong to rational conformal field theories (RCFT), which we mentioned already in the introduction. They may be defined as theories, in which the Hilbert space of all fields is a *finite* direct sum of irreducible highest weight representations of the chiral symmetry algebra. If the latter is the direct sum of two copies of the Virasoro algebra, these irreducible highest weight representations are tensor products of two highest weight representations of the two corresponding highest weight states for the holomorphic and antiholomorphic half of the theory. Then the condition follows that there may only be finitely many primary fields. This is only possible with rational central charge. All conformal weights are also rational.
The central charges of the whole series of minimal models are parametrised by a pair of coprime integer numbers, $p > q > 2$:

$$c_{(p,q)} = 1 - 6\frac{(p-q)^2}{pq} \ . \tag{1.26}$$

For $p - q = 1$ the central charge of the unitary minimal models (eq. (1.24)) is recovered, when we set $m = p - 1 = q$. With another pair of integers, $r, s > 0$, we get the possible highest weights,

$$h_{(r,s)} = \frac{(pr - qs)^2 - (p-q)^2}{4pq} \tag{1.27}$$

This equation has the subsequent symmetry

$$h_{(r,s)} = h_{(q-r,p-s)} \ . \tag{1.28}$$

Certain highest weights differ only by integers:

$$h_{(r,s)} + rs = h_{(q+r,p-s)} \tag{1.29}$$

$$h_{(r,s)} + (q-r)(p-s) = h_{(r,2p-s)} \ . \tag{1.30}$$

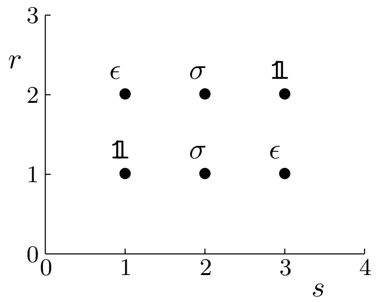

Figure 1.1: *The Kac table of the smallest minimal model $c_{(4,3)}$, the two dimensional critical Ising model. The dots mark the pairs (r, s), which label the conformal dimensions $h_{(r,s)}$ of primary fields in this model. These are the scaling fields, which are quoted next to the dots. Each one appears twice in the Kac table.*

The Verma modules $V_{(r,s)}$ for minimal models with highest weights $h_{(r,s)}$ are reducible. They contain so-called singular – or null – states, each of which is orthogonal to all states in the module except its descendent. The descendents consequently are also orthogonal to the rest of the module. One finds that these singular states are the highest weight states of other Verma modules. For the levels containing a singular state the Kač determinant vanishes.

1.2.1 Irreducible Highest Weight Representations of Minimal Models

The highest weight representations are constructed starting with a particular set of Verma modules, which belong to the highest weights $h_{(r,s)}$ with $0 < r < q$ and $0 < s < p$. This set of highest weights is called the Kač table. As an example the Kač table of the $c_{(4,3)}$ model, which is the two dimensional critical Ising model, is given in figure 1.1. The Verma modules with a highest weight in the Kač table have two singular vectors at levels rs and $(q-r)(p-s)$. They are the ones suggested by equation (1.29), because the weight of a descendent differs by its integer level from the highest weight, and, indeed, the Kač determinant vanishes at these levels.

However, if we now factor out the union of the two Verma modules $V_{(q+r,r-s)}$ and $V_{(r,2p-s)}$, generated from the singular states of the Verma module in the Kač table, we throw away too much. These Verma modules contain again each two singular states. Fortunately it turns out that these two singular states coincides for both modules and we find a structure as in figure 1.2. Every arrow points from a Verma module with the given pair (r, s) to a Verma module, whose highest weight state is a singular state in the former module. The irreducible representations are given by the Verma modules $V_{(r,s)}$ in the Kač table, from which Verma submodules corresponding to the singular states $(q + r, p - s)$ and $(r, 2p - s)$ have been subtracted, from which in turn the Verma submodules on their singular states $(2q + r, s)$ and $(r, 2p + s)$ have been subtracted, from which.... This way we arrive at the following succession of subtractions and additions for an irreducible representation $M_{(r,s)}$.

$$M_{(r,s)} = V_{(r,s)} - (V_{(q+r,r-s)} \cup V_{(r,2p-s)}) + (V_{(2q+r,s)} \cup V_{(r,2p+s)}) - \dots \tag{1.31}$$

with $0 < r < q$ and $0 < s < p$. We can label also the irreducible representations by their place (r, s) in the Kač table.

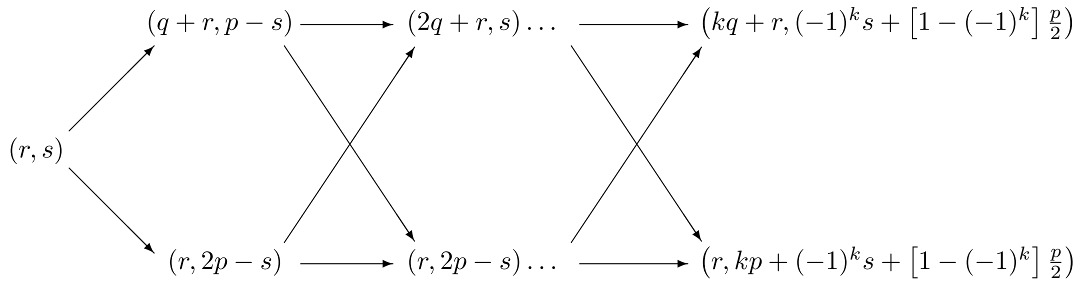

Figure 1.2: *The structure of Verma modules and their singular vectors is symbolically shown. The pairs $(\cdot,\cdot)$ represent the highest weights $h_{(\cdot,\cdot)}$ of Verma modules. The arrows point from one Verma module to its submodules generated from its singular vectors. If (r,s) is in the Kač table, also the sequence of Verma modules is visualised, which need be substracted and added from the one corresponding to (r,s) in order to get an irreducible representation.*

1.2.2 Fusion in Minimal Models

In this section we will introduce the fusion product for minimal models. One wants to associate a representation of the symmetry algebra to a product of two field. This is similar to the case of the tensor product representation of simple Lie algebras, like e.g. the spin algebra $su(2)$, but due to the central extension of the Virasoro algebra can not be the solution here. The tensor product of two Virasoro representations would be a representation of the Virasoro algebra with a different central charge, which is the sum of the two central charges corresponding to the first two the representations.
The problem can be generally solved by the insertion of the stress energy tensor in an arbitrary correlation function with the two fields in our product, which then provides a suitable representation. We look at this in the next chapter. For minimal models it can be shown that the structure constants of the fusion algebra, which is given by the decomposition of the product representation into irreducible representations, follow from the values of three point functions of primary fields.

The Fusion Algebra

If we look at three primary fields ϕ_i, ϕ_j and ϕ_k, the three point correlation functions with these fields have the general form

$$\langle\phi_k(z_k)\phi_i(z_i)\phi_j(z_j)\rangle = \frac{C_{ijk}}{(z_{ij})^{h_i+h_j-h_k}(z_{ik})^{h_i+h_k-h_j}(z_{jk})^{h_j+h_k-h_i}}\ , \tag{1.32}$$

It can be illustrated by the graph in figure 1.3. This amplitude can only be different from zero, if the representation conjugate to ϕ_k is contained in the product representation of ϕ_i and ϕ_j, which we want to decompose into a direct sum of irreducible representations. For minimal models also the opposite direction of this conclusion is true. In the general case one also has to consider the multiplicity, with which one representation appears in the other, but here all these multiplicities are one.
We define the fusion algebra with the structure constants being the fusion coefficients $(N_{RCFT})_{ij}{}^k$ telling us, which representations are in the decomposition of the fusion product:

$$\phi_i\times\phi_j = \sum_k (N_{RCFT})_{ij}{}^k\phi_k\ , \tag{1.33}$$

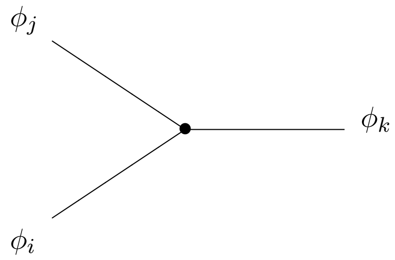

Figure 1.3: *The tree graph corresponding to the three point correlation function.*

where the sum goes over all irreducible representations – or equivalently primary fields. The fusion product closes for the irreducible representations of RCFTs. The fusion coefficients relate to the three point function by

$$(N_{RCFT})_{ij}{}^{k} = 1 \qquad \Leftrightarrow C_{ijk} \neq 0 \tag{1.34}$$

The fusion algebra is associative, commutative and semisimple. The semisimplicity is not generic for fusion algebras, but is required for the Verlinde formula, which we learn about in the next subsection. The unit element of the fusion algebra is the vacuum representation.
If one defines the matrices $N_{RCFT,I}$ with matrix elements

$$(N_{RCFT,I})_{j}{}^{k} = (N_{RCFT})_{Ij}{}^{k} \ , \tag{1.35}$$

these are a representation of the same algebra:

$$N_{RCFT,I}N_{RCFT,J} = \sum_{K} (N_{RCFT})_{IJ}{}^{K} N_{RCFT,K} \ . \tag{1.36}$$

For the vacuum representation one naturally gets the unit matrix.
The decomposition of the fusion products in the case of the minimal models can be written in closed form as the subsequent sum:

$$\phi_{(r,s)} \times \phi_{(r',s')} = \sum_{\substack{k=1+|r-r'| \\ k+r+r'=1 \bmod 2}}^{min(r+r'-1,2q-1-r-r')} \quad \sum_{\substack{l=1+|s-s'| \\ k+s+s'=1 \bmod 2}}^{min(s+s'-1,2p-1-s-s')} \phi_{(k,l)} \tag{1.37}$$

Characters and the Verlinde-Formula

Especially, when different multiplicities become relevant or the direct correspondence between three point functions and fusion coefficients breaks down, the calculation of the fusion rules from the fields is something one rather wants to avoid. Almost twenty years ago a formula has been proposed by E. Verlinde, which leads to the fusion coefficients via the properties of the characters of the irreducible representations.
The characters of highest weight representations are given by the weighted sum of the number $dim(h+n)$ of linearly independent states at level n over all levels from zero to infinity. This is just equal to the following trace over the Hilbert space of the representation:

$$\begin{aligned} \chi_{(c,h)}(\tau) &= tr_{\mathcal{H}_{(c,h)}} q^{L_0 - c/24} \\ &= \sum_{n=0}^{\infty} dim(n+h) q^{n+h-c/24} \end{aligned} \tag{1.38}$$

with $q = e^{2\pi i\tau}$.
One particularly interesting thing about conformal field theories is that the characters and partition functions are modular forms. We find that the characters of the irreducible representations transform into linear combinations of themselves under the modular group $SL(2,\mathbb{Z})$. One can write the characters as the components of a vector $\boldsymbol{\chi}(\tau)$and then describe the linear combinations, which are equal to the transformed characters – the ones evaluated at the point $\gamma\tau$ with $\gamma \in SL(2,\mathbb{Z})$–, by a matrix $G(\gamma)$:

$$\boldsymbol{\chi}(\gamma\tau) = G(\gamma)\boldsymbol{\chi}(\tau) \ . \tag{1.39}$$

The matrices $G(\gamma)$ form a representation of $SL(2,\mathbb{Z})$. It is generated by the two elements for the transformations $\mathcal{T} : \tau \to \tau + 1$ and $\mathcal{S} : \tau \to -1/\tau$.
The latter is particularly important in CFTs because the matrix $S_{RCFT} = G(\mathcal{S})$ – the S-matrix – is the only thing we need for the mentioned Verlinde formula, which directly yields the fusion coefficients:

$$N_{ij}^{k} = \sum_{r} \frac{(S_{RCFT})_j{}^r (S_{RCFT})_i{}^r ({S_{RCFT}}^{-1})_r{}^k}{(S_{RCFT})_{\Omega,r}} \ . \tag{1.40}$$

The inverse is actually superfluous because the square of the S-matrix is the unit matrix, ${S_{RCFT}}^2 = \mathbb{1}$. Furthermore the S-matrix is symmetric. The index Ω refers to the row (or column) of the S-matrix corresponding to the vacuum representation.
The actual statement of the Verlinde formula is, that the S-matrix *simultaneously* diagonalises the fusion coefficient matrices. The diagonal matrix belonging to $N_{RCFT,I}$ is

$$M_{diag,I} = diag\left(\frac{(S_{RCFT})_I{}^1}{(S_{RCFT})_\Omega{}^1}, \frac{(S_{RCFT})_I{}^2}{(S_{RCFT})_\Omega{}^2}, \ldots, \frac{(S_{RCFT})_I{}^n}{(S_{RCFT})_\Omega{}^n} \right) \tag{1.41}$$

We have already mentioned in the introduction that the Verlinde formula has been proven in different versions for semisimple fusion algebras ending with the work of Huang, who connected the pieces. But this is not the end of the road. The assumption needed for the Verlinde formula, as we have discussed it here, can be weakened, without loosing all properties that lead to a similar kind of formula, which we conveniently also call Verlinde formula to induce the right associations. In the next chapters we are going to investigate cases, in which the fusion algebra is not any more semisimple.

Chapter 2

The $\mathbf{c_{(p,1)}}$ Series

In the last chapter we have described conformal field theories in a way, which points directly to such RCFTs, for which the maximal chiral symmetry algebra is the Virasoro algebra. We also have looked at the most prominent example, the minimal models. This might have led to the impression, that, for example, there are only highest weight states associated to ordinary primary fields and all their descendents in every CFT, or that each highest weight state is an eigenstate of the zero mode of the Virasoro algebra. The latter condition says, that L_0 diagonalises on the highest weight states. In this section we take a look at examples for CFTs, the $c_{(p,1)}$ models, where this is not the case. For the case of $p = 2$ we explain, how one is led to highest weight states, which are not eigenstates of L_0, through a logarithmic factor found in the four point correlation function of a primary field in these models. These states form each a Jordan cell in L_0 together with a highest weight eigenstate of L_0. These pairs belong to indecomposable representations of the Virasoro algebra with central charge $c = c_{(p,1)}$. The occurrence of such representations is the actual reason for the Jordan cells and the logarithmic factors. Thus the name "logarithmic conformal field theory", to which the $c_{(p,1)}$ models also belong, does not focus on its basic feature.
The $c_{(p,1)}$ models are border cases in double respect. Firstly, this series is closely related to minimal models. They originate from the idea of an extension of the concept of the minimal models seen in the last chapter. Secondly, we see later on, that they exhibit some kind of rationality reminding of RCFTs or – depending on the exact definition, which we also discuss in the last section of this chapter – even being RCFTs.

2.1 Extension of Minimal Models

To recall, minimal models have been defined basically by their symmetry algebra, the Virasoro algebra, taken at the rational central charge $c_{(p,q)}$ defined in equation (1.26) for p and q coprime and larger than one.
One can now ask – and has asked – the question, if there is a way to define a consistent theory having the Virasoro algebra at $c = c_{(p,1)}$ calculated with the same equation. It is certainly not the kind of minimal model we have already seen because the Kac table in this case is empty. But one can extend this Kac table by its border, which represents $(p-1)$ Verma-modules with highest weight $h_{(1,s)}(p, 1)$ for $0 < s < p$.
From this point these models were developed gradually. On the way many problems were encountered. Among these it was found that this series consists just of the LCFTs sketched in the first lines of this chapter.

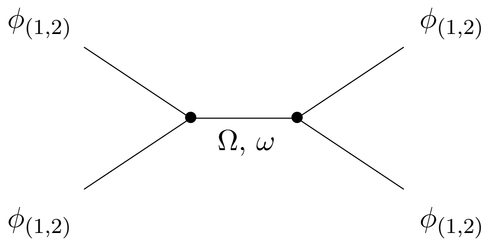

Figure 2.1: *The graph corresponding to the four point correlation function of four times the primary field $\phi_{(1,2)}$ is shown. A logarithmic partner ω of the vacuum Ω has been found asking the question for the internal fields, which are contributing to this amplitude.*

In the beginning Victor Gurarie investigated among others what he called the most simple example of possible four point functions with logarithmic behaviour, which appear in the first model of the series with central charge $c_{(2,1)} = -2$ [Gur93]. In detail he looked at the primary field $\phi_{(1,2)}$ with conformal dimension $h_{(1,2)} = -\frac{1}{8}$ and at the scattering of this field with itself visualised in figure 2.1. It has the form

$$\langle \phi_{(1,2)}(z_1)\, \phi_{(1,2)}(z_2)\, \phi_{(1,2)}(z_3)\, \phi_{(1,2)}(z_4)\rangle = (z_1 - z_3)^{\frac{1}{4}}(z_2 - z_4)^{\frac{1}{4}}\, [x(1-x)]^{\frac{1}{4}}\, F(x) \qquad (2.1)$$

with the anharmonic ratio $x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$. The structure follows from the fact that $\phi_{(1,2)}$ is a primary field and so the correlation functions have to obey eq. (1.9). This also implies a differential equation for $F(x)$.

$$x(1-x)\frac{d^2F(x)}{dx^2} + (1-2x)\frac{dF(x)}{dx} - \frac{1}{4}F(x) = 0\ . \qquad (2.2)$$

If a power series ansatz of not yet fixed lowest power s, with which x appears, is inserted here and the monomials of power $s-1$ are extracted, we get an equation for the power s (cf. [Gab03]).

$$s^2 = 0\ . \qquad (2.3)$$

As the roots of this equation coincide, the second independent solution $F_2(x)$ for $F(x)$ is not a power series, but results from the first solution $F_1(x)$, which is regular at $x = 0$, as

$$F_2(x) = F_1(x)\log(x) + H(x)\ , \qquad (2.4)$$

where $H(x)$ is also regular at $x = 0$. To these two solutions we can associate now the operator product expansion of the field $\phi_{(1,2)}$ with itself.

$$\phi_{(1,2)}(z)\phi_{(1,2)}(w) \sim \omega(w) + \log(z-w)\Omega(z)\ . \qquad (2.5)$$

Here Ω is found to be the unit operator associated to the vacuum state. ω is called its logarithmic partner. This is indeed very matching because it does not only appear here in the same OPE together with Ω, but maps also to Ω under the Virasoro generator L_0. This way those two operators form a Jordan block in the action of L_0:

$$L_0\omega = \Omega\ , \qquad (2.6)$$
$$L_0\Omega = 0\ . \qquad (2.7)$$

These two primary fields correspond to highest weight states. All positive modes L_n, $n > 0$, annihilate them. So in LCFTs L_0 does not diagonalise on highest weight states. Acting on such a pair of states the negative modes L_n, $n < 0$, generate so-called logarithmic highest weight representations.

2.2 Fusion Products of Virasoro Highest Weight Representations

For a systematic analysis of the field content of these theories one needs to understand, how products of fields decompose. It is enough to analyse these for the primary fields or generally on the level of the representations that descend from these fields. Thus we need the fusion products of them. But it already shapes up as a problem to define fusion for the $c_{(p,1)}$ series consistently.
The definition via the three point correlation functions, which we use for the minimal models, is not well defined here. We have already mentioned in our discussion of the minimal models in section 1.2.2 ,that the assumptions, we have imposed there, were rather special. Here we proceed with a widely valid definition consistent with the previous considerations.
Garberdiel and Kausch [GK96a], namely, have found a way to define a tensor-like product, which associates a product representation to two representations of the Virasoro algebra with central charge $c_{(p,1)}$. We will make the difference to the real tensor product clear by an index f , which accompanies the product sign. The product of two primary fields ψ and χ defines a representation with the help of the contour integral of the stress energy tensor and those two fields inserted at points z_1 and z_2, respectively, with the contour going around both insertion points. One always looks at these products in correlation function and so they are only defined in this context. The mentioned countour integral gives us the action of the Virasoro algebra on the product of ψ and χ.

$$\oint_0 dw w^{m+1} \langle \phi(\infty) T(w) \psi(z_1) \chi(z_2) \Omega \rangle \tag{2.8}$$

$$= \sum \langle \phi(\infty) (\Delta^{(1)}_{z_1,z_2}(L_m)\psi)(z_1) (\Delta^{(1)}_{z_1,z_2}(L_m)\chi)(z_2) \Omega \rangle \tag{2.9}$$

with an arbitrary field ϕ and the vacuum Ω. For the action of the Virasoro algebra a comultiplication formula is used, which has been developed before [MS89, Gab94a, Gab94b]. The action of the Virasoro generators in the product representation is given by $\Delta^{(1)}_{z_1,z_2}(L_m)$ and $\Delta^{(2)}_{z_1,z_2}(L_m)$, which combines to the tensor product

$$\Delta_{z_1,z_2}(L_m) = \sum \Delta^{(1)}_{z_1,z_2}(L_m) \otimes \Delta^{(2)}_{z_1,z_2}(L_m) \ , \tag{2.10}$$

which in turn can be explicitly expressed for $n \geq -1$ as

$$\begin{aligned} \Delta_{z_1,z_2}(L_n) &= \tilde{\Delta}_{z_1,z_2}(L_n) \qquad (2.11) \\ &= \sum_{m=-1}^{n} \binom{n+1}{m+1} {z_1}^{n-m} (L_m \otimes \mathbb{1}) + \sum_{l=-1}^{n} \binom{n+1}{l+1} {z_1}^{n-l} (\mathbb{1} \otimes L_l) \end{aligned}$$

and can be expanded in two different ways $\Delta_{z_1,z_2}(L_n)$ and $\tilde{\Delta}_{z_1,z_2}(L_n)$ for $n \leq -2$:

$$\begin{aligned}\Delta_{z_1,z_2}(L_n) &= \sum_{m=-1}^{\infty} \begin{pmatrix} n+m-1 \\ m+1 \end{pmatrix} (-1)^{m+1} {z_1}^{-n-m} (L_m \otimes \mathbb{1}) \\ &\quad + \sum_{l=n}^{\infty} \begin{pmatrix} l-2 \\ n-2 \end{pmatrix} -{z_1}^{l-n} (\mathbb{1} \otimes L_l) \,, \end{aligned} \tag{2.12}$$

$$\begin{aligned}\tilde{\Delta}_{z_1,z_2}(L_n) &= \sum_{l=n}^{\infty} \begin{pmatrix} l-2 \\ n-2 \end{pmatrix} -{z_1}^{l-n} (L_l \otimes \mathbb{1}) \\ &\quad + \sum_{m=-1}^{\infty} \begin{pmatrix} n+m-1 \\ m+1 \end{pmatrix} (-1)^{m+1} {z_1}^{-n-m} (\mathbb{1} \otimes L_m) \,. \end{aligned} \tag{2.13}$$

These two expansions are derived from the same equation (2.8) and must be equal in all correlation functions. However, their action on the product space of the two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of two representations, of which we want to calculate the fusion product, are a priori different. So the fusion product has to be the quotient space of this product space through all states of the form

$$\left[\Delta_{z_1,z_2}(L_n) - \tilde{\Delta}_{z_1,z_2}(L_n)\right] (\psi_1 \otimes \psi_2) \,, \quad m \in \mathbb{Z},\ \psi_i \in \mathcal{H}_i \,, \tag{2.14}$$

as is described in the lecture notes [Gab03] of M. Garberdiel. In these notes the algorithm developed by him and H. Kausch to calculate specific decompositions of fusion products is described and the structure of the contributing representations given [GK96a]. We do not consider these calculations for the Virasoro algebra here, but immediately go on to the extended algebra, which we consider throughout the rest of this thesis.

2.3 The Triplet $\mathcal{W}$-Algebra $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$

It was noticed that the number of relevant irreducible highest weight representations of the Virasoro algebra at $c = c_{(p,1)}$ is infinite, but countable. Furthermore there is a countably infinite set of indecomposable representations, which has to be considered as well, because the fusion algebra, which has been calculated for the cases of $p = 2$, $p = 3$ and partially for higher p (mostly $p = 4$) in [GK96a] using the above-mentioned algorithm, does only close together with these representations. All these irreducible and indecomposable representations have highest weights $h_{(r,s)}$ found in an infinite Kac table with some redundancy – every representation's highest weight appears at two values of (r, s). The indecomposable representations always contain two irreducible subrepresentations. One of these has the same highest weight, the other's highest weight is higher by an integer number.
One of the primary fields with conformal dimension $h_{(3,1)}$ is of particular interest, because $h_{(3,1)} = 2p-1$ is an odd integer. Extensions of the Virasoro algebra with multiplets of fields of half-integer or integer spin has been studied by Horst Kausch in [Kau91]. Especially a series of Virasoro algebras with $c = c_{(p,1)}$ has been found in this work, which can be extended by a triplet of fields $W^{(j)}$ with odd integer spin, which have a structure resembling $SO(3)$, or by a singlet, which is given by the sum of the three triplet fields.

The mentioned triplet is given by the field $\phi_{(3,1)}$ and the action of a screening charge Q on it:

$$W^{(j)} = Q^j \phi_{(3,1)} , \tag{2.15}$$

$$Q = \oint dz V_{\alpha_+}(z) \tag{2.16}$$

with a vertex operator $V_{\alpha_+}(z)$, which can be expressed in terms of oscillators in the frame of a free field construction, just as the whole algebra can be expressed this way (cf. [Kau91, Flo03]). So this triplet $\mathcal{W}$-algebra $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ is generated by the stress energy tensor and the triplet of fields $W^{(j)}$. It is the maximally extended local chiral symmetry algebra of the $c_{(p,1)}$ models. The commutation relations of the modes L_m and $W_m^{(a)}$ of the stress energy tensor and the fields $W^{(a)}$, respectively, for the case $p = 2$, which serves us here as an sufficiently complicated example, then result to

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} - \frac{1}{6}n(n^2-1)\delta_{n+m,0} , \\ \left[L_n, W_m^{(a)}\right] &= (2n-m)W_{n+m}^{(a)} , \\ \left[W_n^{(a)}, W_m^{(b)}\right] &= \delta^{ab}\bigg(2(n-m)\Lambda_{n+m} + \frac{1}{20}(n-m)(2n^2+2m^2-nm-8)L_{n+m} \\ &\quad -\frac{1}{120}n(n^2-1)(n^2-4)\delta_{m+n,0}\bigg) \\ &\quad +i\epsilon^{abc}\left(\frac{5}{14}(2n^2+2m^2-3nm-4)W_{n+m}^{(c)} + \frac{12}{15}V_{n+m}^{(c)}\right) \end{aligned} \tag{2.17}$$

with $\Lambda_m =: L_m^2 : -3/10\,\partial^2 L_m$, $V_m^{(a)} =: L_m W_m^{(a)} : -3/14\,\partial^2 W_m^{(a)}$, $a, b, c \in \{1, 2, 3\}$, $n, m \in \mathbb{Z}$ and expressed in an orthonormal basis. In the first line we repeated the Virasoro algebra with $c = -2$ plugged into equation 1.16. We continue with this example for a while.

2.3.1 Highest Weights and su(2) Structure

The triplet algebra is only associative because of null vectors in the vacuum representation. These lead to constraints expressed in the form that certain combinations of generators of the algebra annihilate any highest weight state ψ. One of these constraints is:

$$\left(W_0^{(a)}W_0^{(b)} - \delta^{ab}\frac{1}{9}{L_0}^2(8L_0+1) - \epsilon^{abc}\frac{1}{5}(6L_0-1)W_0^{(c)}\right)\psi = 0 . \tag{2.18}$$

It follows that the commutator of the zero modes of the fields $W^{(a)}$ becomes:

$$\left[W_0^{(a)}, W_0^{(b)}\right] = \frac{2}{5}(6h-1)\epsilon^{abc}W_0^{(c)} , \tag{2.19}$$

which is a rescaled $su(2)$ algebra. We can now rescale the zero modes

$$W_0'^{(a)} = \frac{5i}{6h-1}W_0^{(a)} \tag{2.20}$$

and find, that the highest weight states of the triplet algebra are eigenvectors of $W_0'^{(3)}$ and the Casimir operator $\sum_a \left(W_0'^{(a)}\right)^2$. We traditionally denote eigenvalues as m and $j(j+1)$,

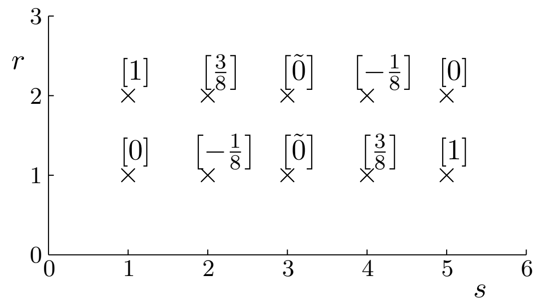

Figure 2.2: *The extended Kač table for the $c = -2$ model. The indecomposable and irreducible representations belonging to the pair (r, s) through their highest weight $h_{(r,s)}$ are denoted next to each cross.*

respectively.
These are the states with highest weights $h_{(r,s)}$ with $0 < r < 3$ and $0 < s < 3p$, which combine to an extended Kač table for the $c_{(p,1)}$ models. After we took the field $\phi_{(3,1)}$, which is on the border of this extended Kac table into the chiral symmetry algebra, all other highest weight states of the Virasoro algebra with $r \geq 3$ and $s \geq 3p$ are now descendants of those in this table with respect to the triplet algebra. This is again something most easy to be seen with the constraints from the null vectors (i.a. eq. (2.18)). They lead to the following equation for an highest weight state ψ:

$$L_0{}^2(8L_0 + 1)(8L_0 - 3)(L_0 - 1)\psi \; . \tag{2.21}$$

The state ψ is per definition an eigenvector of L_0 with the highest weight being the eigenvalue. So this gives a simple equation for possible highest weights with solutions, which are exactly the entries of the extended Kac table for $p = 2$, which figure 2.2 shows. The double root for highest weight zero is due to the indecomposable representation of the triplet algebra $\mathcal{W}(2, 3, 3, 3)$, which contains all the indecomposable Virasoro representations.
Coming back to the $su(2)$ algebra the subsequent values, which we found for j, lead to the classification as singlet and doublet irreducible representations.

- $\mathbf{j = 0}$**:** the singlet representations [0] and $\left[-\frac{1}{8}\right]$,
- $\mathbf{j = \frac{1}{2}}$**:** the doublet representations [1] and $\left[\frac{3}{8}\right]$,

where we introduced the notation $[h]$ for the representation with highest weight h. Note that from equation (2.18) also the relation $W_0^{(a)} W_0^{(a)} = W_0^{(b)} W_0^{(b)}$ follows. So the Casimir operator is actually the same as $3\left(W_0^{(3)}\right)^2$ and we gain m from j and vice versa by

$$j(j + 1) = 3m^2 \; . \tag{2.22}$$

Because m has to be an integer or half-integer number, the possibility of higher values for j is excluded from the outset. No such quantum numbers can be associated to the indecomposable representation because it contains both singlet and doublet subrepresentations, as we are going to see now.

2.3.2 Fusion Products of Triplet Representations

As the next step in the investigation of the $c_{(p,1)}$ series the knowledge about the fusion of Virasoro representations has been used to calculate the fusion rules of the triplet algebra

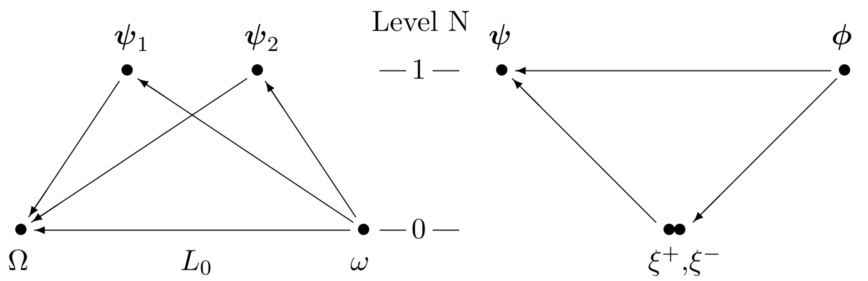

Figure 2.3: *The structure of the indecomposable representations $[\tilde{0}^+]$ (left) and $[\tilde{0}^-]$ (right) of the triplet algebra algebra $\mathcal{W}(2,3,3,3)$. The arrows symbolise the non-vanishing action of zero and ascendent modes on the non-descendent states, which are represented by the dots.*

representations with the same algorithm as before and the action of the extended algebra on the products of states. This was done for $p = 2$ by Garberdiel and Kausch [GK96b], but the rising complexity and sheer amount of calculations for higher p have been too blenching up to date.

Most interesting are the fusion products of the representations $\left[-\frac{1}{8}\right]$ and $\left[\frac{3}{8}\right]$. Their analysis shows that they are always one of two indecomposable representations $[\tilde{0}^+]$ and $[\tilde{0}^-]$, which until now have not and very soon again will not be distinguished because they are isomorphic to each other.

$$\begin{aligned} \left[-\frac{1}{8}\right] \otimes_f \left[-\frac{1}{8}\right] &= [\tilde{0}^+] \ , \\ \left[-\frac{1}{8}\right] \otimes_f \left[\frac{3}{8}\right] &= [\tilde{0}^-] \ , \\ \left[\frac{3}{8}\right] \otimes_f \left[\frac{3}{8}\right] &= [\tilde{0}^+] \ . \end{aligned} \tag{2.23}$$

The structure of $[\tilde{0}^+]$ and $[\tilde{0}^-]$ is given in figure 2.3. The vertices $\bullet$ are states that are no descendants. In both representations two are found on level zero and two on level 1, which is also their conformal weight because the highest weight is zero. The arrows denote the action of ascendent and zero modes of the triplet algebra, which takes us from one to the other. That these are not annihilating the states here makes the representation indecomposable. They are not any more a construction of quotients of Verma modules like the irreducible representations. The representations consist of the respective four non-descendent states and all descendants of these state. We get back to general p now.

We first have a look on the indecomposable representations and their relation to the irreducible ones. The indecomposable representations can be associated with the positions $(r,s) = (1, p+\lambda)$ for $0 < \lambda < p$ in the extended Kac table. We denote these representations by their highest weights in brackets $\left[\widetilde{h_{(1,p+\lambda)}}\right]$ with a tilde pointing out the difference to the irreducible representation $\left[h_{(1,p-\lambda)}\right]$, which has the same highest weight and is a subrepresentation of $\left[\widetilde{h_{(1,p+\lambda)}}\right]$. The other irreducible subrepresentation is $\left[h_{(1,3p-\lambda)}\right]$. The relation between the highest weights is

$$h_{(1,3p-\lambda)} - h_{(1,p-\lambda)} = p - \lambda \ . \tag{2.24}$$

Two further irreducible representations $\left[h_{(1,p)}\right]$ and $[h_{(1,2p)}]$ are not subrepresentations of an indecomposable representation, but are like the indecomposable representations projective modules of the triplet algebra. So we have exhausted the first row of the extended Kac table, but there are no more irreducible or indecomposable representations to come because the second row is redundant to the first one.

2.4 Characters of $\mathcal{W}$-Algebra Representations

The essential thing, we need to know, of the triplet algebra representations for our actual work are their characters. The characters, which need to be calculated using equation (1.38), are those of the Virasoro highest weight representations on the highest weight states $|h_{(2k+1,i)}\rangle$ for $k \in \mathbb{Z}^+$ and $0 < i \leq p$ or $2p \leq i < 3p$. All those, which belong to the same i, are degenerate and have weights that differ by integer numbers. They combine in the light of the triplet algebra to one triplet algebra representation. More precisely the Hilbert space of a triplet algebra representation is the direct sum of the Hilbert spaces of the Virasoro representations each weighted by a multiplicity originating from the $su(2)$ symmetry. The characters of the triplet algebra representations are the sum of the Virasoro characters with the mentioned multiplicities incorporated. Furthermore one can derive from the structure of the indecomposable representations that their characters are the sum of the irreducible subrepresentations with the multiplicity, with which the latter appear in the former.
Here the character of the vacuum representation, $i = 1$, shall exemplify the needed calculation. First the Virasoro character came out of the investigations of the Virasoro algebra of Feigin and Fuks and successors [FF83] in the early 80's:

$$\chi^{Vir}_{2k+1,1} = \frac{1}{\eta(q)}\left(q^{h_{(2k+1,1)}} - q^{h_{(2k+1,-1)}}\right) . \tag{2.25}$$

Here the Dedekind η-function is used:

$$\eta(q) = q^{\frac{1}{24}} \prod_{n\in\mathbb{N}} (1 - q^n) . \tag{2.26}$$

The multiplicities of the Virasoro representations in the triplet algebra representations are $2k+1$ for $h_{(2k+1,1)}$. Especially for $h_{(3,1)}$ the multiplicity is three matching the triplet. So the Hilbert spaces of these representations relate as

$$\mathcal{H}^{\mathcal{W}}_{[0]} = \bigoplus_{k\in\mathbb{Z}_+} (2k+1)\mathcal{H}^{Vir}_{|h_{(2k+1,1)}\rangle} \tag{2.27}$$

with the non-negative integer numbers $\mathbb{Z}_+$. For the characters it follows that

$$\chi^{\mathcal{W}}_{[0]} = \sum_{k\in\mathbb{Z}_+} (2k+1)\chi^{Vir}_{2k+1,1} , \tag{2.28}$$

which evaluates after a few steps to

$$\chi^{\mathcal{W}}_{[0]} = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} (2k+1) q^{(2pk+(p-1))^2} = \frac{1}{p\eta(\tau)} \left(\Theta_{p-1,p}(\tau) + (\partial\Theta)_{p-1,p}(\tau)\right) . \tag{2.29}$$

In the last term the character is expressed in terms of modular forms, namely the Jacobi-Riemann Θ-functions and the affine Θ-functions, which are respectively defined as

$$\Theta_{\lambda,k}(\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}} , \tag{2.30}$$

$$(\partial\Theta)_{\lambda,k}(\tau) = \sum_{n\in\mathbb{Z}} (2kn+\lambda) q^{\frac{(2kn+\lambda)^2}{4k}} . \tag{2.31}$$

All triplet algebra characters of the irreducible and the indecomposable triplet algebra representations can be expressed by these functions. The appearance of modular forms in CFTs has lead to quite some interest among mathematicians working in this field. Recently a conjecture by Werner Nahm ([Nah04]), that constructs a whole family of forms with the help of CFTs is modular, has kindled some work to calculate examples for it [Zag06].
Always the quotient of the Θ-functions and the Dedekind η-function appears in the characters. $\Theta_{\lambda,k}/\eta$ is a modular form of modular weight zero, while $(\partial\Theta)_{\lambda,k}/\eta$ is one of weight one. We choose a notation, where we sort the characters into three groups. Those of irreducible singlet representations $\chi^+_{\lambda,p}$, of irreducible doublet representations $\chi^-_{\lambda,p}$ and of indecomposable representations $\chi^{\mathcal{R}}_{\lambda,p}$ belong to the highest weights $h_{1,p-\lambda}$ for $0 \leq \lambda < p$, $h_{1,3p-\lambda}$ for $0 \leq \lambda < p$ and $h_{1,p+\lambda}$ for $0 < \lambda < p$, respectively. The characters are given by

$$\chi^+_{0,p} = \frac{1}{\eta(\tau)}\Theta_{0,p} , \tag{2.32}$$

$$\chi^-_{p,p} = \frac{1}{\eta(\tau)}\Theta_{p,p} , \tag{2.33}$$

$$\chi^+_{\lambda,p} = \frac{1}{p\eta(\tau)}\left[(p-\lambda)\Theta_{\lambda,p} + (\partial\Theta)_{\lambda,p}\right] , \tag{2.34}$$

$$\chi^-_{\lambda,p} = \frac{1}{p\eta(\tau)}\left[\lambda\Theta_{\lambda,p} - (\partial\Theta)_{\lambda,p}\right] , \tag{2.35}$$

$$\chi^{\mathcal{R}}_{\lambda,p} = \frac{2}{\eta(\tau)}\Theta_{\lambda,p} . \tag{2.36}$$

The vacuum character is $\chi^+_{p-1,p}$ in this notation. We want to sort the characters of the irreducible representations for later use in the vector:

$$\boldsymbol{\chi}^t_{irr,p} = (\chi^+_{0,p}, \chi^-_{p,p}, \chi^+_{p-1,p}, \chi^-_{p-1,p}, \chi^+_{p-2,p}, \chi^-_{p-2,p}, \ldots, \chi^+_{1,p}, \chi^-_{1,p}) . \tag{2.37}$$

This is the same sequence as in the work of Fuchs et al. (e.g. [FHST04]).

2.4.1 No Canonical S-matrix

We see, that the characters of the representations, which are the actual projective modules of the triplet algebra, are modular forms of modular weight zero. But we also have to stomach the fact that the other irreducible representations do not transform so nicely under modular transformations due to the second summand, a weight one modular form. The peculiarities of these characters, when modular transformed, gets clear in the example of the vacuum character of $p = 2$, which we take a look at under the transformation $\tau \to -1/\tau$.

$$\begin{aligned}\chi^+_{1,2}\left(-\frac{1}{\tau}\right) &= \frac{1}{2\eta(-1/\tau)}\left[\Theta_{1,2}\left(-\frac{1}{\tau}\right) + (\partial\Theta)_{1,2}\left(-\frac{1}{\tau}\right)\right] \\ &= \frac{1}{2\eta(\tau)}\left[\frac{1}{2}\Theta_{0,2}(\tau) - \frac{1}{2}\Theta_{2,2}(\tau) - i\tau(\partial\Theta)_{1,2}(\tau)\right] \\ &= \frac{1}{4}\chi^+_{0,2} - \frac{1}{4}\chi^-_{0,2} - \frac{i}{2}\tau\chi^+_{1,2} - \frac{i}{2}\tau\chi^-_{1,2} .\end{aligned} \tag{2.38}$$

The factor of τ appearing in the last two lines poses the first big problem. We recall that the goal of the research, which this thesis continues, is to find a Verlinde formula for the $c_{(p,1)}$ series and develop its understanding. Therefore we need to find some kind of S-matrix, which should base on the transformation properties of characters of representations of the chiral symmetry algebra under $\tau \to -1/\tau$, as it is in the proven Verlinde formula for rational conformal field theories. The characters of the irreducible representations, we calculated, are not transforming very wildly here. The transformed characters are almost again a linear combination of these characters themselves. Only the factors of τ spoil at this point the easy walk we had considering RCFTs[1].

But there is another big obstacle. We have seen that some fusion products result in indecomposable representations. So these should be included into our considerations. The S-matrix should have a size corresponding to the number of generators of the fusion algebra, which we want to get at the end. In the case of $p = 2$ it is known because of the calculation of Garberdiel and Kausch – and for other p there are convincing arguments, as we see, for example, in the next section – that the generators of the fusion algebra of the highest weight representations of the triplet $\mathcal{W}$-algebra are the irreducible representations and the indecomposable representations, so $3p - 1$ generators. This leads to the mentioned second complication. For these representations, the characters are linearly dependent. The structure of the indecomposable representations with irreducible ones as subrepresenations causes the following dependence:

$$2\chi^{+}_{\lambda,p} + 2\chi^{-}_{\lambda,p} = \chi^{\mathcal{R}}_{\lambda,p} \,. \tag{2.39}$$

This fits nicely to the image we got of the indecomposable representation for $p = 2$ in figure 2.3 – two singlet and two doublet vertices –, but also lets any $(3p-1) \times (3p-1)$ S-matrix (for the general p) based on the modular transformation of only the characters in eqns. (2.32)-(2.36) be singular.

We even have not considered the question yet, how the two isomorphic versions of every indecomposable representations should be dealt with. If we wanted to distinguish between them the S-matrix would even have to be still bigger. But to prevent toplofty hopes, it should be mentioned right now, that no possibility whatsoever has been seen, how to accomplish this. Thus we will not make a difference between two isomorphic representations and in the next chapter we are going to investigate the possibilities to find a $(3p-1) \times (3p-1)$ S-matrix, which leads us to the fusion rules.

2.5 Quasi-Rationality

For now, we want to take a look at a few properties of the $c_{(p,1)}$ models showing that these theories have much in common with rational conformal field theories and so the expectance of some kind of Verlinde formula is well based.

The definitions of rational and quasi-rational conformal field theories differ slightly in different parts of the literature. The Hilbert space of fields of a conformal field theories can be written in the form ([Flo03]):

$$\mathfrak{H} \otimes \mathfrak{H} = \bigoplus_{\lambda \in \Lambda} \mathfrak{H}^{(\lambda)} \otimes \bar{\mathfrak{H}}^{(\bar{\lambda})} \tag{2.40}$$

$$= \bigoplus_{\lambda \in \Lambda} \left(\bigoplus_{\nu \in N_\lambda} \mathfrak{H}^{(\lambda)}_{\nu} \otimes \bigoplus_{\nu \in N_\lambda} \mathfrak{H}^{(\bar{\lambda})}_{\nu} \right) \tag{2.41}$$

[1]This absolutely only refers to the method, not to the horrendous mathematics that lies in the Verlinde formula in any of its versions.

The Hilbert spaces $\mathfrak{H}^{(\lambda)}$ and $\bar{\mathfrak{H}}^{(\bar{\lambda})}$ belong to irreducible highest weight representations of the chiral symmetry algebra for the two chiral halves of the theory. This algebra may be larger than the Virasoro algebra, as we have seen. These are direct sums of the Hilbert spaces $\mathfrak{H}_\nu^{(\lambda)}$ and $\mathfrak{H}_\nu^{(\bar{\lambda})}$ of highest weight representations of the Virasoro algebra. If the set Λ is finite and the fusion products decompose into finitely many representations, the theory is called rational. If the set Λ is countably infinite and the fusion coefficients $\mathtt{N}_{ij}{}^k$ are zero for almost all k, the theory is called quasi-rational. In contrast to the demand that the fusion coefficients are different from zero for only finitely many k, this leaves us with the possibility, that in certain limits within the index set N_Λ, counting the Virasoro representations, the number of summands in the decomposition of the fusion products may go to infinity. We have introduced here and continue to use the stronger version of rationality. One also finds definition, in which the reference to the fusion products is dropped and only the statement about the index set Λ is kept.
In words a rational conformal field theory is a CFT, in which the irreducible highest weight representations of the Virasoro algebra – possibly infinitly many – are grouped in a finite number of blocks given by an extended chiral symmetry algebra (compared to the Virasoro algebra). With this definition and, if the fusion rules, we propose here, are correct for all p, the $c_{(p,1)}$ models are all rational. For $p = 2$ this was proven by calculating the fusion rules in a paper thus called "A Rational Logarithmic Field Theory" [GK96b]. This is a definition with respect to the extended symmetry algebra.
One often talks in the case of infinitely many irreducible Virasoro representations, though grouped in the described way, already of a quasi-rational CFT with respect to the Virasoro algebra. In this sense we continue here and often compare between rational conformal field theories and quasi-rational $c_{(p,1)}$ models.
There are strong indications from other parts of the $c_{(p,1)}$ series, that build up the parallel between other rational conformal field theories and these models. The search for a partition function of the $c_{(p,1)}$ models has brought many insights [Flo96a]. It ought to be calculated from the characters of the relevant representations and has to be modular invariant. Certain products of characters from the two chiral halves are excluded because of vanishing couplings. The factors of τ in the modular transformation of the characters give a further problem(e.g. eq. (2.38)). These factors are taken care of, if one first introduces further $2(p-1)$ α-dependent forms, which for $\alpha \to 0$ reduce to the characters of the indecomposable representations. In fact the characters of the indecomposable representations are split into a sum:

$$\begin{aligned} \chi^{\mathcal{R}}{}_{\lambda,p} &= \chi^{\mathcal{R}+}_{\lambda,p}(\alpha) + \chi^{\mathcal{R}-}_{\lambda,p}(\alpha) \\ \chi^{\mathcal{R}+}_{\lambda,p}(\alpha) &= \frac{1}{\eta}\left[\Theta_{\lambda,p} + i\alpha\lambda(\nabla\Theta)_{\lambda,p}\right] \\ \chi^{\mathcal{R}-}_{\lambda,p}(\alpha) &= \frac{1}{\eta}\left[\Theta_{\lambda,p} - i\alpha(p-\lambda)(\nabla\Theta)_{\lambda,p}\right] \end{aligned} \tag{2.42}$$

where $(\nabla\Theta)_{\lambda,p}$ is

$$(\nabla\Theta)_{\lambda,p} = i\tau(\partial\Theta)_{\lambda,k} = \frac{1}{2\pi}\log(q)(\partial\Theta)_{\lambda,k} \tag{2.43}$$

With $\chi^{\mathcal{R}+}_{\lambda,p}(\alpha)$, $\chi^{\mathcal{R}-}_{\lambda,p}(\alpha)$ and the characters from equations (2.32)-(2.35) one is let to the following modular invariant partition function consistent with the model and staying modular invariant also for $\alpha \to 0$ (cf. [Flo97]).

$$\begin{aligned} Z_{p,\alpha} &= \left|\chi^+_{o,p}\right|^2 + \left|\chi^-_{o,p}\right|^2 + \sum_{\lambda=1}^{p} \left[\chi^+_{\lambda,p}\chi^{\mathcal{R}+*}_{\lambda,p} + {\chi^+_{\lambda,p}}^*\chi^{\mathcal{R}+}_{\lambda,p} + \chi^-_{\lambda,p}\chi^{\mathcal{R}-*}_{\lambda,p} + {\chi^-_{\lambda,p}}^*\chi^{\mathcal{R}-}_{\lambda,p}\right] \\ &= \frac{1}{\eta\eta^*}\left(|\Theta_{0,p}|^2 + |\Theta_{p,p}|^2 + \sum_{\lambda=1}^{p}\left[2|\Theta_{\lambda,p}|^2 + i\alpha\left((\partial\Theta)_{\lambda,p}(\nabla\Theta)^*_{\lambda,p} - {(\partial\Theta)_{\lambda,p}}^*(\nabla\Theta)_{\lambda,p}\right)\right]\right) \end{aligned} \quad (2.44)$$

For $\alpha = 0$ this is the partition function $Z(\sqrt{p/2})$ of the standard $c = 1$ Gaussian model.
Recently it was also proven for $p = 2$, that $\chi^{\mathcal{R}+}_{\lambda,p}(\alpha)$ and the characters of the irreducible representations also form a basis for the chiral vacuum torus amplitudes (cf. [FG06]). Also strong indications for this statement to generalise to arbitrary p were presented there. In the case of rational conformal field theory the canonical basis for the chiral vacuum torus amplitudes consist just out of the characters of the irreducible representations.
Here obviously we have to take the indecomposable ones into account. This space of amplitudes is also correspondingly larger. Their characters are linearly dependent on the characters of the irreducible representations, but with the new forms $\chi^{\mathcal{R}+}_{\lambda,p}(\alpha)$ we have a representative for the indecomposable representations. This is also particularly interesting, when it now gets to the fusion rules and some kind of Verlinde formula for the $c_{(p,1)}$ series. Of course, these new forms are in no way standing out, as the characters of the irreducible representations do, which leads to the name "canonical basis" in RCFTs, where only they span the space. In the next chapter we are going to see, that we actually have to go over to another form related to $\chi^{\mathcal{R}+}_{\lambda,p}(\alpha)$ in order to calculate the fusion rules.

Chapter 3

Verlinde Formula for $\mathbf{c_{(p,1)}}$ Models

Chapter 2 introduced us to the $c_{(p,1)}$ series and pointed out several properties of these models, which are particularly important in view of the fusion product in these theories. We saw that there are strong arguments for an analogue to the Verlinde formula because of the parallels to rational conformal field theories, the basis of the chiral vacuum torus amplitudes, the modular transformation properties of the characters and the modular invariant partition function. But we also discovered some problems. We have indecomposable representations of the maximally extended chiral symmetry algebra $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$, which have to be considered. Therefore there is no canonical basis of characters of irreducible representations for the vacuum torus amplitudes. The linear dependence of characters of indecomposable and irreducible representations prevents to directly find "the" S-matrix.
In this chapter we are searching for *an* S-matrix. We find various candidates. One of them has been calculated by M. Flohr on the grounds of considerations on characters and partition functions. Another one came from Fuchs and coworkers from the search of an $SL(2,\mathbb{Z})$ representation acting only on the irreducible characters.
We start with the former in section 3.1, which depends on the parameter α, which we have introduced at the very end of the last chapter. This leads to a Verlinde formula, which gives α-dependent (pre-)fusion rules. Thus we are going to call it in this thesis the α-Verlinde formula. The actual fusion rules are recovered after the limit $\alpha \to 0$ is taken and some minor steps thereafter are done, which need some discussion.
Fuchs' S-matrix and the "generelised" (to non-semisimple fusion algebras) Verlinde formula, he gets, are discussed afterwards in section 3.2. In this formula the fusion coefficients are not diagonalised by the S-matrix, but only block-diagonalised. This method is found in a more algebraic approach, but only covers the irreducible representation.
In section 3.3 we find an extended version of the block-diagonalisation method. It reduce to Fuchs' approach in a canonical way and gives the same results for the irreducible representation. In the last section 3.4 we proof that the extended version also results to the same fusion rules as the α-Verlinde formula.

3.1 The α-Verlinde Formula

A decade ago a way to adapt the Verlinde-Formula to $c_{(p,1)}$ models was proposed in [Flo97]. We have already indicated that we can advance on our way to an S-matrix with the help of the forms $\chi^{\mathcal{R}+}_{\lambda,p}(\alpha)$. They are linearly independent from the characters of irreducible representations. What is more, they and the characters of the irreducible representations close under modular

transformations of their argument, i.e. any of these $3p-1$ forms evaluated at $\gamma\tau$, with $\gamma \in SL(2,\mathbb{Z})$, can be written as a linear combination of the same forms evaluated at τ. This is the case for all linear combination of $\chi^{\mathcal{R}+}_{\lambda,p}(\alpha)$ and $\chi^{\mathcal{R}-}_{\lambda,p}(\alpha)$, which are linearly independent from the characters.
With this in hand a $(3p-1)\times(3p-1)$ S-matrix $S_{(p,\alpha)}$ can be uniquely defined for a vector of characters with $2p$ components as in eq. (2.37) and further $p-1$ components being one of the linear combinations, parametrised by $x \in \mathbb{C}$,

$$\tilde{\chi}_{\lambda,p}(\alpha,x) = \frac{2}{p}\left[(p+x-\lambda)\chi^{\mathcal{R}+}_{\lambda,p}(\alpha) + (\lambda-x)\chi^{\mathcal{R}-}_{\lambda,p}(\alpha)\right] . \tag{3.1}$$

The results do not depend on the choice of x. The form $\tilde{\chi}_{\lambda,p}(\alpha,x)$ surely depends on x. However, when we insert the forms $\chi^{\mathcal{R}+}_{\lambda,p}(\alpha)$ and $\chi^{\mathcal{R}-}_{\lambda,p}(\alpha)$ (eq. (2.42)) into equation (3.1), it emerges, that it only depends on the product of x and α:

$$\tilde{\chi}_{\lambda,p}(\alpha,x) = \frac{1}{\eta}\left[2\Theta_{\lambda,p} + 2xi\alpha(\nabla\Theta)_{\lambda,p}\right] \tag{3.2}$$

We can redefine α in a convenient way to incorporate x. Because we take the limit $\alpha \to 0$ at the end, this does not change the results.
For the following $x = -i/2$ is chosen, which corresponds to the choice made in [Flo97]:

$$\tilde{\chi}_{\lambda,p}(\alpha) = \tilde{\chi}_{\lambda,p}(\alpha,-i) = \frac{1}{\eta}\left[2\Theta_{\lambda,p} + \alpha(\nabla\Theta)_{\lambda,p}\right] \tag{3.3}$$

The factor $2/p$ appears in equation (3.1) in contrast to [Flo97] to have a multiplicity - which a priori may be chosen - of 2 in front of the $\Theta_{\lambda,p}/\eta$ term in $\tilde{\chi}_{\lambda,p}(\alpha,-i)$, instead of a multiplicity of p. The result at the end will depend on choice of the multiplicity. Another multiplicity in $\tilde{\chi}_{\lambda,p}(\alpha,x)$ leads qualitatively to the correct fusion rules, but with different multiplicities. Our choice is the one, for which the forms $\tilde{\chi}_{\lambda,p}(\alpha)$ get the character of the indecomposable representations for $\alpha \to 0$.
The following vector of characters and forms, representing the indecomposable representations, is used in the course of this section:

$$\begin{aligned}\boldsymbol{\chi}^t_p(\alpha) = {} & (\chi^+_{0,p}, \chi^-_{p,p}, \chi^+_{p-1,p}, \chi^-_{p-1,p}, \tilde{\chi}_{p-1,p}(\alpha), \chi^+_{p-2,p}, \chi^-_{p-2,p}, \tilde{\chi}_{p-2,p}(\alpha), \ldots, \\ & \chi^+_{1,p}, \chi^-_{1,p}, \tilde{\chi}_{1,p}(\alpha)).\end{aligned} \tag{3.4}$$

The S-transformation ($\tau \to -1/\tau$) of the vector $\boldsymbol{\chi}_p(\alpha)$ is then given by an S-matrix depending on α, as well:

$$\boldsymbol{\chi}_p(\alpha)\left(-\frac{1}{\tau}\right) = \mathtt{S}_{(p,\alpha)}\boldsymbol{\chi}_p(\alpha)(\tau) \tag{3.5}$$

In previous work this α-dependent S-matrix has always been calculated only for particular choices of p. First we express the transformation properties of the Θ-functions under the $\mathcal{S}$-transformation in a matrix $\mathfrak{S}$. One takes Jacobi-Riemann Θ-functions, $\Theta_{\lambda,p}$, affine Θ-functions, $(\partial\Theta)_{\lambda,p}$, and affine Θ-function multiplied with τ, $(\nabla\Theta)_{\lambda,p}$, all divided by the Dedekind η-function in the sequence

$$\frac{1}{\eta} \quad (\Theta_{0,p}, \Theta_{1,p}, \ldots \Theta_{p,p}, (\partial\Theta)_{1,p}, (\partial\Theta)_{2,p}, \ldots (\partial\Theta)_{p-1,p}, \tag{3.6}$$

$$-(\nabla\Theta)_{1,p}, -(\nabla\Theta)_{2,p}, \ldots, -(\nabla\Theta)_{p-1,p}) . \tag{3.7}$$

The matrix $\mathfrak{S}$ describes the transformation $\tau \to -1/\tau$ of this vector analogously to eq. (3.5). It has three blocks different from zero, for which the matrix elements are

$$\begin{aligned}
\mathfrak{S}_{ij} &= \frac{1}{1+\delta_{j,1}+\delta_{j,p+1}}\sqrt{\frac{2}{p}}\cos\left(\frac{\pi(i-1)(j-1)}{p}\right) && \forall\, 0 < i,j \le p+1\,,\\
\mathfrak{S}_{(2p+k)(p+l+1)} &= i\sqrt{\frac{2}{p}}\sin\left(\frac{\pi kl}{p}\right) && \forall\, 0 < k,l < p\,,\\
\mathfrak{S}_{(p+n+1)(2p+m)} &= -i\sqrt{\frac{2}{p}}\sin\left(\frac{\pi nm}{p}\right) && \forall\, 0 < n,m < p\,.
\end{aligned}$$

The δ is the Kronecker symbol.
The matrix B building the characters and forms, which we need, from the Θ-functions has only few non-zero matrix elements, as one can see in equations (2.37) and (3.3). These elements are

$$\begin{aligned}
B_{1,1} &= 1\,, & B_{2p} &= 1\,, \qquad (3.8)\\
B_{(3\lambda)(p-\lambda)} &= \frac{\lambda}{p}\,, & B_{(3\lambda)(2p-\lambda+1)} &= \frac{1}{p}\,,\\
B_{(3\lambda+1)(p-\lambda)} &= \frac{p-\lambda}{p}\,, & B_{(3\lambda+1)(2p-\lambda+1)} &= -\frac{1}{p}\,,\\
B_{(3\lambda+2)(p-\lambda)} &= 2\,, & B_{(3\lambda+2)(2p+\lambda)} &= I\alpha\,,
\end{aligned} \qquad (3.9)$$

The product $B\mathfrak{S}B^{-1}$ finally is equal to $\mathtt{S}_{(p,\alpha)}$.
We have now also calculated this S-matrix for general p and get it in closed form. Its structure has its roots in the fact, that we put the characters of the two irreducible representations, which are not subsets of any indecomposable representations, in the first entries of the vector $\boldsymbol{\chi}_p(\alpha)$, while all other pairs of irreducible representations are accompanied by the indecomposable representation, which contains them.
This leads to a block structure with one 2×2 block $S(p)_{0,0}$ and each $(p-1)$ 2×3 and 3×2 blocks $S(p)_{0,l}$ and $S(p)_{s,0}$, respectively. These blocks do not depend on α. The rest of the matrix is filled with 3×3 blocks $S(p,\alpha)_{s,l}$. Many of the matrices defined throughout this chapter will have this structure, where s and l will always run from 1 to $p-1$ inclusive.
Finally, the matrix $\mathtt{S}_{(p,\alpha)}$ is given for arbitrary p as:

$$\mathtt{S}_{(p,\alpha)} = \begin{pmatrix} S(p)_{0,0} & S(p)_{0,1} & \dots & S(p)_{0,p-1}\\ S(p)_{1,0} & S(p,\alpha)_{1,1} & \dots & S(p,\alpha)_{1,p-1}\\ \vdots & \vdots & \ddots & \vdots\\ S(p)_{p-1,0} & S(p,\alpha)_{p-1,1} & \dots & S(p,\alpha)_{p-1,p-1}\end{pmatrix} \qquad (3.10)$$

with

$$\begin{aligned}
S(p)_{0,0} &= \frac{1}{\sqrt{2p}}\begin{pmatrix}1 & 1\\ 1 & (-1)^p\end{pmatrix}\,,\\
S(p)_{0,l} &= \frac{2}{\sqrt{2p}}\begin{pmatrix}1 & 1 & 0\\ (-1)^{p-l} & (-1)^{p-l} & 0\end{pmatrix}\,,
\end{aligned}$$

$$
\begin{aligned}
S(p)_{s,0} &= \frac{1}{\sqrt{2p}} \begin{pmatrix} \frac{s}{p} & (-1)^{p+s}\frac{s}{p} \\ \frac{p-s}{p} & (-1)^{p+s}\frac{p-s}{p} \\ 2 & 2(-1)^{p+s} \end{pmatrix}, \\
S(p,\alpha)_{s,l} &= \frac{2}{\sqrt{2p}}(-1)^{p+l+s} \times \\
& \begin{pmatrix} \frac{s}{p}\mathfrak{c}_{sl} + \frac{2}{p}\frac{1}{\alpha}\mathfrak{s}_{sl} & \frac{s}{p}\mathfrak{c}_{sl} + \frac{2}{p}\frac{1}{\alpha}\mathfrak{s}_{sl} & -\frac{1}{p\alpha}\mathfrak{s}_{sl} \\ \frac{p-s}{p}\mathfrak{c}_{sl} - \frac{2}{p}\frac{1}{\alpha}\mathfrak{s}_{sl} & \frac{p-s}{p}\mathfrak{c}_{sl} - \frac{2}{p}\frac{1}{\alpha}\mathfrak{s}_{sl} & \frac{1}{p\alpha}\mathfrak{s}_{sl} \\ 2\mathfrak{c}_{sl} - \alpha(p-l)\mathfrak{s}_{sl} & 2\mathfrak{c}_{sl} + \alpha l \mathfrak{s}_{sl} & 0 \end{pmatrix}
\end{aligned}
$$

with the abbreviations $\mathfrak{c}_{sl} = \cos\left(\pi\frac{sl}{p}\right)$ and $\mathfrak{s}_{sl} = \sin\left(\pi\frac{sl}{p}\right)$.
This matrix fulfils ${\mathtt{S}_{(p,\alpha)}}^2 = \mathbb{1}$, but is not symmetric. For $\alpha \to 0$ the forms $\tilde{\chi}_{\lambda,p}(\alpha)$ pass into the characters of the indecomposable representations. So they are linearly dependent with the characters of the irreducible representations in this limit. Consequently some of the entries of $\mathtt{S}_{(p,\alpha)}$ diverge in this case.
For completeness the matrix $\mathtt{T}_{(p,\alpha)}$ for the transformation $\tau \to \tau + 1$ is given here. It is defined as

$$\boldsymbol{\chi}_p(\alpha)(\tau+1) = \mathtt{T}_{(p,\alpha)}\boldsymbol{\chi}_p(\alpha)(\tau) \tag{3.11}$$

and is calculated to be

$$
\begin{aligned}
T(p)_{0,0} &= \begin{pmatrix} e^{-i\frac{\pi}{12}} & 0 \\ 0 & e^{-i\pi\left(\frac{p}{2}-\frac{1}{12}\right)} \end{pmatrix}, \\
T(p,\alpha)_{s,s} &= \begin{pmatrix} \mathfrak{t}_s & 0 & 0 \\ 0 & \mathfrak{t}_s & 0 \\ i\alpha\,(p-s)\,\mathfrak{t}_s & -i\alpha\, s\, \mathfrak{t}_s & \mathfrak{t}_s \end{pmatrix}
\end{aligned}
\tag{3.12}
$$

with

$$\mathfrak{t}_s = e^{-i\pi\left(\frac{(p-s)^2}{2p}-\frac{1}{12}\right)} \tag{3.13}$$

with all other elements of $\mathtt{T}_{(p,\alpha)}$ being zero.
The matrices $\mathtt{S}_{(p,\alpha)}$ and $\mathtt{T}_{(p,\alpha)}$ describe the action of the generators $\mathcal{S}$ and $\mathcal{T}$ of the modular group $SL(2,\mathbb{Z})$ on $\boldsymbol{\chi}_p(\alpha)(\tau)$. So with equations (3.5) and (3.11) any element $\gamma \in SL(2,\mathbb{Z})$ can be represented as a matrix $\mathtt{G}_{(p,\alpha)}(\gamma)$, which is a product only containing copies of $\mathtt{S}_{(p,\alpha)}$ and $\mathtt{T}_{(p,\alpha)}$, such that

$$\boldsymbol{\chi}_p(\alpha)\,(\gamma\tau) = \mathtt{G}_{(p,\alpha)}(\gamma)\boldsymbol{\chi}_p(\alpha)(\tau) \tag{3.14}$$

As the action of $SL(2,\mathbb{Z})$ on functions on $\mathbb{C}$ is linear[1], we directly have for two elements $\gamma, \gamma' \in SL(2,\mathbb{Z})$, that

$$\mathtt{G}_{(p,\alpha)}(\gamma\gamma') = \mathtt{G}_{(p,\alpha)}(\gamma)\mathtt{G}_{(p,\alpha)}(\gamma') \tag{3.15}$$

It follows that $\mathtt{S}_{(p,\alpha)}$ and $\mathtt{T}_{(p,\alpha)}$ generate a representation of $SL(2,\mathbb{Z})$, namely $\mathtt{G}_{(p,\alpha)}(\gamma)$, for a fixed $\alpha \neq 0$. We can also immediately see, that like the generators of $SL(2,\mathbb{Z})$ also $\mathtt{S}_{(p,\alpha)}$ and $\mathtt{T}_{(p,\alpha)}$ have to fulfil the conditions ${\mathtt{S}_{(p,\alpha)}}^2 = \mathbb{1}$ and $\left(\mathtt{S}_{(p,\alpha)}\mathtt{T}_{(p,\alpha)}\right)^3 = \mathbb{1}$.[2]
The matrix $\mathtt{S}_{(p,\alpha)}$ is now plugged into the Verlinde formula as known for rational conformal field theories. This, of course, leads to an object ${\mathtt{N}_{ij}}^k(\alpha)$, which depends on α. But here the

[1] The action of $SL(2,\mathbb{Z})$ on a function $f : \mathbb{C} \to \mathbb{C}$ shall be defined as the composition $f \circ \gamma$ with $\gamma \in SL(2,\mathbb{Z})$.
[2] As an easy check one can calculate these products for any p, which we did for up to $p = 6$.

limit of $\alpha \to 0$ exists. We define the coefficients ${\mathtt{N}_{ij}}^k$ to be exactly this limit and get the α-Verlinde formula:

$$ {\mathtt{N}_{ij}}^k = \lim_{\alpha\to 0} {\mathtt{N}_{ij}}^k(\alpha) = \lim_{\alpha\to 0}\left(\sum_{r=1}^{3p} \frac{(\mathtt{S}_{(p,\alpha)})_{jr}(\mathtt{S}_{(p,\alpha)})_{ir}{(\mathtt{S}_{(p,\alpha)})_r}^k}{(\mathtt{S}_{(p,\alpha)})_{3,r}}\right) \tag{3.16} $$

Note, that the third component of the vector $\boldsymbol{\chi}_p(\alpha)(\tau)$ is the character of the vacuum representation. In contrast to the semisimple case of RCFTs with symmetric S-matrix, the indices of $\mathtt{S}_{(p,\alpha)}$ in the Verlinde formula have to be kept as in this formula. Especially the third line of $\mathtt{S}_{(p,\alpha)}$ – rather than the column – has to be taken for the denominator of the α-Verlinde formula. This is due to a convention of left-multiplication of $\mathtt{S}_{(p,\alpha)}$ with $\boldsymbol{\chi}_p(\alpha)(\tau)$, which we chose quite naturally.

At this point we want to note, that the limit in the α-Verlinde formula has to be taken the way, it is here. One can think of different parameters α in the different copies of the S-matrix $\mathtt{S}_{(p,\alpha)}$, which are not simultaneously put to zero. However, we have found, that only the simultaneous limit results in correct fusion rules. Otherwise one has no unit element, for example. In view of the equivalence of this formula to the block-diagonalisation method – which we show in section 3.4 – this is also, what one would expect. We have detailed our studies on this in appendix A.1.1.

On first sight the results for ${\mathtt{N}_{ij}}^k$ for $p = 2$ and $p = 3$, which are given in the appendices B.1 and C.1, differ quite much from the fusion coefficients ${\mathcal{N}_{ij}}^k$ calculated in [GK96b] and [GK96a]. However, we have to note that any fusion rules we get using eq. (3.16) by itself, can only be taken as true on the level of characters, not representations, because the calculation is based only on the modular transformation properties of the characters. Here we have the problem, that, as soon as we take the limit $\alpha \to 0$, the functions $\tilde{\chi}_{\lambda,}(\alpha)$ degenerate again to a linear combination of characters of irreducible representations given in eq. (2.39). So the method presented here can not distinguish the indecomposable representation from linear combination of irreducible representations in the decomposition of the fusion product.

Indeed, for many fusion products ${\mathtt{N}_{ij}}^k$ has components corresponding to the these linear combinations, while in [GK96b] and [GK96a] the corresponding indecomposable representation was the correct result.

There is another problem that occurs in fusion products of indecomposable representations with some other representation: For one and the same fusion product ${\mathtt{N}_{ij}}^k$ encodes the linear combinations mentioned above together with the corresponding indecomposable representations, which then have a negative integer coefficients. These problems shall be illustrated in the case of $p = 3$.

Example: The matrix $\mathtt{S}(3,\alpha)$ reads

$$ \begin{pmatrix} \frac{1}{2}\hat{r} & \frac{1}{2}\hat{r} & \hat{r} & \hat{r} & 0 & \hat{r} & \hat{r} & 0 \\ \frac{1}{2}\hat{r} & -\frac{1}{2}\hat{r} & \hat{r} & \hat{r} & 0 & -\hat{r} & -\hat{r} & 0 \\ \frac{1}{6}\hat{r} & \frac{1}{6}\hat{r} & -\frac{1}{6}\hat{r}-\hat{s} & -\frac{1}{6}\hat{r}-\hat{s} & \frac{1}{2}\hat{s} & -\frac{1}{6}\hat{r}+\hat{s} & -\frac{1}{6}\hat{r}+\hat{s} & -\frac{1}{2}\hat{s} \\ \frac{1}{3}\hat{r} & \frac{1}{3}\hat{r} & -\frac{1}{3}\hat{r}+\hat{s} & -\frac{1}{3}\hat{r}+\hat{s} & -\frac{1}{2}\hat{s} & -\frac{1}{3}\hat{r}-\hat{s} & -\frac{1}{3}\hat{r}-\hat{s} & \frac{1}{2}\hat{s} \\ \hat{r} & \hat{r} & -\hat{r}+\hat{t} & -\hat{r}-\frac{1}{2}\hat{t} & 0 & -\hat{r}-\frac{1}{2}\hat{t} & -\hat{r}+\hat{t} & 0 \\ \frac{1}{3}\hat{r} & -\frac{1}{3}\hat{r} & -\frac{1}{3}\hat{r}+\hat{s} & -\frac{1}{3}\hat{r}+\hat{s} & -\frac{1}{2}\hat{s} & \frac{1}{3}\hat{r}+\hat{s} & \frac{1}{3}\hat{r}+\hat{s} & -\frac{1}{2}\hat{s} \\ \frac{1}{6}\hat{r} & -\frac{1}{6}\hat{r} & -\frac{1}{6}\hat{r}-\hat{s} & -\frac{1}{6}\hat{r}-\hat{s} & \frac{1}{2}\hat{s} & \frac{1}{6}\hat{r}-\hat{s} & \frac{1}{6}\hat{r}-\hat{s} & \frac{1}{2}\hat{s} \\ \hat{r} & -\hat{r} & -\hat{r}-\hat{t} & -\hat{r}+\frac{1}{2}\hat{t} & 0 & \hat{r}-\frac{1}{2}\hat{t} & \hat{r}+\hat{t} & 0 \end{pmatrix} \tag{3.17} $$

with $\hat{r} = \sqrt{6}/3$, $\hat{s} = \sqrt{2}/(3\alpha)$ and $\hat{t} = \alpha\sqrt{2}$. Eq. (3.16) then gives "pre-fusion-rules" listed in appendix C.1.
It is worth going through some particular fusion products from eq. (C.1) to see the problems arising through the ambiguities in the limit $\alpha \to 0$. In this example there are two indecomposable representations and two corresponding identities of their characters:

$$2\chi^{+}_{i,3} + 2\chi^{-}_{i,3} = \chi^{\mathcal{R}}_{i,3} \quad i = 1,2 \ .$$

These "translate" to identities of representations, which shall symbolise their indistinguishabileness in this calculation:

$$\begin{aligned} 2\left[-\frac{1}{4}\right] + 2\left[\frac{7}{4}\right] &= \widetilde{\left[-\frac{1}{4}\right]} \ , \\ 2\,[0] + 2\,[1] &= [\tilde{0}] \ . \end{aligned} \tag{3.18}$$

Quite typical is the following product:

$$\left[-\frac{1}{4}\right] \otimes^f \left[-\frac{1}{3}\right] = 2\left[-\frac{1}{4}\right] + 2\left[\frac{7}{4}\right] \ . \tag{3.19}$$

Here the first identity in eq. (3.18) is used to get the desired result $\widetilde{\left[-\frac{1}{4}\right]}$. Substitutions of this kind are still quite comprehensable. But there are several results for other fusion products like

$$[\tilde{0}] \otimes^f [1] = 4\,[0] - [\tilde{0}] + 2\left[-\frac{1}{3}\right] + 4\,[1] \ , \tag{3.20}$$

which catch one's eye because of a disturbing minus sign. But it also contains the latter of the linear combinations in eq. (3.18) in a sufficiently high multiplicity, so that we can mend this problem by a calculation on the level of characters. Equation (3.20) then yields

$$2\,[\tilde{0}] - [\tilde{0}] + 2\left[-\frac{1}{3}\right] = [\tilde{0}] + 2\left[-\frac{1}{3}\right] \ . \tag{3.21}$$

This kind of calculation must be done in several fusion products given in eq. (C.1). For those products one gets finally the fusion rules for $\mathcal{W}$-algebra representations, which are listed in eq. (C.3) and are consistent with the fusion rules calculated for the Virasoro modules in [GK96a].

Without clear rules for these substitutions the value of the results would be lost. Fortunately we know, that fusion products of irreducible representations can only decompose into irreducible representations, which have the correct $su(2)$ quantum number j, and any indecomposable representations, because they have no unique $su(2)$ quantum numbers. The quantum number j is additive within the fusion rules. So the fusion product of two singlets gives apart from indecomposable representations only singlets, two doublets also only singlets and one of each gives doublets. As there are no other values for j, these are the only possibilities.
This rules out all the combinations of both singlets and doublets in the decomposition. Now two irreducible representations having the characters on the left hand side of equation (2.39), which gives our "translation" to the correct fusion rules, are exactly a singlet and a doublet and thus forbidden. This justifies the permanent substitution in all fusion product of two irreducible representations, in which the mentioned combinations appear.

The products with the indecomposable representations are a bigger problem because the argument of $su(2)$ quantum numbers can not be applied, when the representation does not have unique quantum numbers. Here a practical argument is given by the negative coefficients. These should be mended, which seems to be possible for all p as well. There are quite a few still left out, in which a substitution should be made, but where we have no argument except the result. For example, for $p = 3$ there are 7 fusion products of this kind left (cf. appendix C.1). But there is also no argument, why now exactly these should be exceptions.
All in all we can surely say, that the following rules are well-founded. There are no indications of deviations whatsoever:

- Replace the left hand side of the following equation by the indecomposable representation on the right hand side, whenever it appears:

$$2\,[h_{1,p-\lambda}(p,1)] + 2\,[h_{1,3p-\lambda}(p,1)] = \left[\widetilde{h_{1,p+\lambda}(p,1)}\right] \quad \lambda = 1 \dots p-1 \quad . \tag{3.22}$$

- If two coefficients appear now for the same indecomposable representation in one fusion rule, add them.

If there is a negative coefficient of an indecomposable representation in the decomposition of the fusion product, it has to be compensated by a higher positive multiplicity from the first rule to make sense. We checked this up to $p = 6$.
Finally – with only one α in the α-Verlinde formula – the following conjecture summarises this method.

Conjecture: *The structure constants $\mathcal{N}_{ij}{}^{k}$ of the fusion algebra of the $c_{(p,1)}$ series are calculated by equation* (3.16) *for all $i = 1 \dots (3p-1)$ and*

- *for all $(j,k) \in \{1,2\} \times \{1,\dots,3p-1\}$ and all $(j,k) \in \{3,\dots,3p-1\} \times \{1,2\}$ as $\mathcal{N}_{ij}{}^{k} = \mathtt{N}_{ij}{}^{k}$*
- *for all $(j,k) \in \{3, 3p-1\} \times \{k \in \{3,\dots,3p-1\} | k \mod 3 = 0\}$ and $\kappa = k, (k+1)$ as*

$$\mathcal{N}_{ij}{}^{\kappa} = \begin{cases} 0 & \textit{if} \quad \mathtt{N}_{ij}{}^{k} = \mathtt{N}_{ij}{}^{(k+1)} \\ \mathtt{N}_{ij}{}^{\kappa} & \textit{else} \end{cases} \quad , \tag{3.23}$$

$$\mathcal{N}_{ij}{}^{(k+2)} = \begin{cases} \mathtt{N}_{ij}{}^{(k+2)} + \mathtt{N}_{ij}{}^{k}/2 & \textit{if} \quad \mathtt{N}_{ij}{}^{k} = \mathtt{N}_{ij}{}^{(k+1)} \\ \mathtt{N}_{ij}{}^{(k+2)} & \textit{else} \end{cases} \quad . \tag{3.24}$$

Here we have stated the proposed connection between the fusion coefficients $\mathcal{N}_{ij}{}^{k}$ and the "pre-fusion" coefficients $\mathtt{N}_{ij}{}^{k}$, which enables us to compute the former for any p with little expenses. However, the limit in this procedure makes it hard to understand the cause, why this leads to the correct result. The situation looks surely a bit better after the work in [FG06] gave us the new perspective on the functions $\tilde{\chi}_{\lambda,p}(\alpha)$ as chiral vacuum torus amplitudes. But still one advantage of a different method, which we will discuss in the next section, is the absence of such a limit.
As mentioned above the ambiguities about the indecomposable representations are generic for methods based on the modular transformation properties of characters. So there is virtually no hope to find a method using some kind of Verlinde formula, which does not exhibit them. But this is something we gladly cope with, as the α-Verlinde formula reduces the amount of needed calculation to get the fusion rules for any particular p enormously.

3.2 Block-Diagonalisation of the Fusion Rules

This section introduces the approach of Fuchs et al., published first in [FHST04]. First we want to mention a few key features already in the beginning. The work of Fuchs et al. yields only the fusion rules of irreducible representations among themselves. A limit like in the last section does not appear. This method is motivated by the theorem that any non-semisimple, finitely generated, associative and commutative algebra, like the fusion algebra we look for here, is the direct sum of its radical and some semisimple algebra. As the key consequence a matrix $P_{irr,p}$ is found, which simultaneously block-diagonalises the matrices $N_{irr,p,I}$ of fusion coefficients for the irreducible representations,[3] in contrast to the case of RCFTs, where the fusion algebra of the Virasoro irreducible modules is semisimple and the S-matrix diagonalises the fusion coefficient matrices N_I simultaneously.
But what are the coefficients $N_{irr,p,I}$ going to be, that are the structure constants of a closed algebra, which is called the "small" fusion algebra in the following? We have seen that the fusion algebras calculated by Gaberdiel and Kausch do not close with just the irreducible representations. But in the last section we have also seen, that on the level of characters we have got (pre-)fusion coefficients $\mathtt{N}_{ij}{}^{k}$, which for the fusion products of two irreducible representations are only different from zero, if they correspond again to irreducible representations. Here the substitution of some of these summands of irreducible representations in the decomposition of the fusion product by the corresponding indecomposable representations is not made. This leads to the closed algebra we have asked for. Now we just have to understand the fusion products in a different way. Namely, we see the irreducible representations in the problematic cases just as sets. The decomposition of the products, which we get with the "small" fusion algebra does not tell us any more the structure of the fusion product, as it was defined for $c_{(p,1)}$ models, e.g. that certain product are the indecomposable representation. They only state, which states belonging to an irreducible representation or a subrepresentation of an indecomposable representation are expected in the decomposition of the product of two fields.
We first find out, how the simultaneous block-diagonalisation comes about and see, that the matrix $P_{irr,p}$ is a matrix consisting of simultaneous eigenvectors of the matrices $N_{irr,p,I}$. Afterwards we find an S-matrix $\mathtt{S}_{irr,p}$ by construction of an $SL(2,\mathcal{Z})$ representation acting on the characters of the irreducible representation, for which a so-called automorphy factor is needed, and put this S-matrix in relation to $P_{irr,p}$. Towards the end of this section we will find a replacement of the automorphy factor, which will also be useful, when it comes to the extension of this method in section 3.3.

3.2.1 Simultaneous Eigen Decomposition of the Fusion Coefficient Matrices for Irreducible Representations

We proceed now with the definition of the matrix $P_{irr,p}$ mentioned above and work out its properties and the simultaneous block-diagonalisation. While the subsequent considerations were presented in [FHST04] in a general setting for non-semisimple fusion algebras, we will restrict ourselves here to the case of the $c_{(p,1)}$ models using the same notation.
In the basis of irreducible representations X we have define our fusion coefficient together with

[3] All the $2p \times 2p$ matrices here carry the index irr to distinguish them from the larger matrices in the section 3.3, where this method is extended beyond the $2p$ irreducible representations to also incorporate the indecomposable representations.

the so-called "small" fusion algebra:

$$X_I X_J = \sum_{K=1}^{2p} (N_{irr,p})_{IJ}{}^K X_K \ . \tag{3.25}$$

One chooses a basis now different from the this one. In view of the direct sum of a semisimple algebra and a radical, which is equal to the "small" fusion algebra, it consists of the union of a set of primitive idempotents, e_A with $A = 1 \dots p+1$, in the semisimple algebra and a basis of the radical, w_A with $A = 3 \dots p+1$. All the primitive idempotents e_A form a partition of the unit element of the semisimple algebra (and also the whole "small" fusion algebra):

$$\sum_{A=1}^{p+1} e_A = \mathbb{1} \ , \tag{3.26}$$

Each w_A corresponds to an e_A with an image of dimension 2. There are two further primitive idempotents in the new basis with a one dimensional image ($A = 1, 2$). The new basis, called Y, is taken in the following order:

$$Y = (e_1, e_2, e_3, w_3, e_4, w_4, \dots, e_{p+1}, w_{p+1}) \ . \tag{3.27}$$

Its elements relate to each other by

$$e_A e_B = \delta_{A,B} e_B \ , \tag{3.28}$$

$$e_A w_C = \delta_{A,C} w_C \ , \tag{3.29}$$

$$w_C w_D = 0 \tag{3.30}$$

with $0 < A, B \leq p+1$, $3 \leq C, D \leq p+1$ and δ being the Kronecker delta.
Because we want to get the fusion coefficient at the end again in the basis X, we want to find the change of basis to the basis Y, by which the fusion coefficients matrices, defined in the usual way (cf. eq. 1.35), are simultaneously block-diagonalised and not only the block-diagonal coefficients in the basis Y. This block structure becomes clear by a few steps of calculation. The change of basis is given by $P_{irr,p}$:

$$X_L = \sum_{J=1}^{2p} (P_{irr,p})_L{}^J Y_J \ . \tag{3.31}$$

X_I is the vector of irreducible representations in the sequence of the characters in eq. (2.37). This defines $P_{irr,p}$.

Proposition: *$P_{irr,p}$ block-daigonalises the matrices $N_{irr,p,I}$ simultaneously, i.e.*

$$N_{irr,p,I} = P_{irr,p} M_{irr,p,I} P_{irr,p}{}^{-1} \ , \tag{3.32}$$

where the matrices $M_{irr,p,I}$, $0 < I \leq 2p$, are block-diagonal and the I-th row of $P_{irr,p}$, π_I, is related to the row corresponding to the vacuum representation, π_Ω, by

$$\pi_I = \pi_\Omega M_{irr,p,I} \tag{3.33}$$

for all $0 < I \leq 2p$.

Remark: We will proof these statements, as we calculate now an explicit expression for $M_{irr,p,I}$ in terms of matrix elements of $P_{irr,p}$.

Proof: We multiply equation (3.25) with $({P_{irr,p}}^{-1})_L{}^J$ and sum over J:

$$\begin{aligned} X_I Y_L &= \sum_{K,J,R,S=1}^{2p} ({P_{irr,p}}^{-1})_L{}^J (N_{irr,p})_{IJ}{}^K (P_{irr,p})_K{}^S ({P_{irr,p}}^{-1})_S{}^R X_R \\ &= \sum_{K,J,S=1}^{2p} \underbrace{({P_{irr,p}}^{-1})_L{}^J (N_{irr,p})_{IJ}{}^K (P_{irr,p})_K{}^S}_{=:(M_{irr,p,I})_L{}^S} Y_S \, . \end{aligned} \tag{3.34}$$

Hence the matrices $M_{irr,p,I}$ give the decompositions of the products of X_I and Y_L into linear combinations of Y_S for $I, L = 1 \ldots 2p$. With the relations between the elements of the basis Y, (eqs. (3.28)-(3.30)) and eq. (3.31), one can calculate the product on the left hand side

$$X_I Y_L = \begin{cases} (P_{irr,p})_{IL} Y_L & \text{for} \quad L = 1, 2 \\ (P_{irr,p})_{IL} Y_L + (P_{irr,p})_{I(L+1)} Y_{L+1} & \text{for} \quad L = 3, 5, 7, \ldots \\ (P_{irr,p})_{I(L-1)} Y_L & \text{for} \quad L = 4, 6, 8, \ldots \quad . \end{cases} \tag{3.35}$$

So the matrices $M_{irr,p,I}$ are block-diagonal with 2×2 blocks and upper-triangular.

$$M_{irr,p,I} = \tag{3.36}$$
$$\begin{pmatrix} (P_{irr,p})_{I1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & (P_{irr,p})_{I2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & (P_{irr,p})_{I3} & (P_{irr,p})_{I4} & \cdots & 0 & 0 \\ 0 & 0 & 0 & (P_{irr,p})_{I3} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & (P_{irr,p})_{I(2p-1)} & (P_{irr,p})_{I(2p)} \\ 0 & 0 & 0 & 0 & \cdots & 0 & (P_{irr,p})_{I(2p-1)} \end{pmatrix} .$$

Now we still need to show the second half of our proposition. The row π_Ω of the matrix $P_{irr,p}$ is determined by the fact that the vacuum representation is the unit element of the fusion algebra. Thus eq. (3.26) tells us, that the sum of all idempotents e_A is just the vacuum representation. Eq. (3.31) for the case of the vacuum, $L = \Omega$, reads

$$X_\Omega = \sum_{K=1}^{2p} (\pi_\Omega)^K Y_K \, . \tag{3.37}$$

A comparison to eq. (3.26), with the order of the basis Y kept in mind, yields

$$\pi_\Omega = (1, 1, 1, 0, 1, 0, \ldots, 1, 0) \, . \tag{3.38}$$

One can plug this into (3.25) with X_J being the vacuum representation:

$$X_I = X_\Omega X_I = \sum_{K=1}^{2p} (\pi_\Omega)^K Y_K X_I \, . \tag{3.39}$$

Because of the commutativity of the algebra we can plug eq. (3.34) into eq. (3.39):

$$X_I = \sum_{K,L=1}^{2p} \underbrace{(\pi_\Omega)^K (M_{irr,p,I})_K{}^L}_{=(\pi_I)^L} Y_L \; . \tag{3.40}$$

Comparing with the definition of $P_{irr,p}$ (eq. (3.31)) equation (3.33) has been shown. □

$\mathbf{P_{irr,p}}$: The Simultaneous Eigen Matrix of the Fusion Coefficient Matrices $\mathbf{N_{irr,I}}$

We have seen now, that the matrices $M_{irr,p,I}$ are the block-diagonalisation of the matrices $N_{irr,I}$ we were looking for. The fusion coefficients are now given by equation (3.32) in terms of the matrix elements of $P_{irr,p}$ only.
Furthermore the columns of the matrix $P_{irr,p}$ are generalised eigenvectors of $N_{irr,p,I}$ for all I, which follows from a small calculation after multiplying eq. (3.32) from the right with a column p_J of $P_{irr,p}$, where e_J is the J-th vector of the canonical orthonormal basis:

$$\begin{aligned} P_{irr,p} M_{irr,p,I} P_{irr,p}{}^{-1} p_J &= P_{irr,p} M_{irr,p,I} e_J \\ &= \begin{cases} P_{irr,p}((P_{irr,p})_{IJ} e_{J-1} + (P_{irr,p})_{I(J-1)} e_J) & \text{for} \quad J = 4,6,8,\dots \\ P_{irr,p}(P_{irr,p})_{IJ} e_J & \text{else} \end{cases} \\ &= \begin{cases} ((P_{irr,p})_{IJ} p_{J-1} + (P_{irr,p})_{I(J-1)} p_J) & \text{for} \quad J = 4,6,8,\dots \\ (P_{irr,p})_{IJ} p_J & \text{else.} \end{cases} \end{aligned} \tag{3.41}$$

The matrix elements of $P_{irr,p}$ coming from $M_{irr,p,I}$ are just scalars and the multiplication of e_J on $P_{irr,p}$ produces again the columns of $P_{irr,p}$. Thus the columns p_J corresponding to the idempotents are eigenvectors to the eigenvalues $(P_{irr,p})_{IJ}$ just as in the semisimple case. Those p_J corresponding to the basis of the radical are generalised eigenvector to the eigenvalue $\lambda_{J-1} := (P_{irr,p})_{I(J-1)}$ – using this abbreviation also for all other $J = 2 \dots n+1$ – spanning the 2×2 blocks each with its partner idempotent:

$$\begin{aligned} & (N_{irr,p,I} - \lambda_{J-1} \mathbb{1})^2 p_J \qquad J = 4,6,8,\dots \\ = \; & N_{irr,p,I}(\lambda_J p_{J-1} + \lambda_{J-1} p_J) - 2\lambda_{J-1}(\lambda_J p_{J-1} + \lambda_{J-1} p_J) + (\lambda_{J-1})^2 p_J \; . \end{aligned} \tag{3.42}$$

The eigenvector p_{J-1} belongs to the same eigenvalue λ_{J-1}, so this is equal to

$$\begin{aligned} & \lambda_J \lambda_{J-1} p_{J-1} + \lambda_{J-1}(\lambda_J p_{J-1} + \lambda_{J-1} p_J) \\ & -2\lambda_{J-1}\lambda_J p_{J-1} - 2(\lambda_{J-1})^2 p_J + (\lambda_{J-1})^2 p_J = 0 \; . \end{aligned} \tag{3.43}$$

To summarise, the matrices $N_{irr,p,I}$ for all I have for each pair of an idempotent e_A and corresponding basis element of the radical w_A an eigenspace or subspace of an eigenspace, which is spanned by the same two generalised eigenvectors. This puts the structure of the matrix $M_{irr,p,I}$ and the meaning of the matrix $P_{irr,p}$ into the context of eigen decomposition. In comparison to the semisimple case, which is detailed in appendix A.1.2, we have here two dimensional eigenspaces instead of one dimensional ones. In the semisimple case one finds a matrix of simultaneous eigenvactors P_{RCFT}, which is related to the S-matrix S_{RCFT} of $\mathcal{S}$-transformation of the characters of irreducible representations by the multiplication of a diagonal matrix K_{RCFT} (cf. eqns. A.9, A.8). Analogously one would expects, that for the non-semisimple case a matrix $K_{irr,p}$ should exist, which connects a suitable S-matrix $\mathtt{S}_{irr,p}$ with the matrix $P_{irr,p}$:

$$P_{irr,p} = \mathtt{S}_{irr,p} K_{irr,p} \ . \tag{3.44}$$

In this case the radical of the fusion algebra entails that the matrix $K_{irr,p}$ need not be diagonal any more, but a 2×2 dimensional taking the two-dimensional eigenspaces into account. Then the S-matrix $\mathtt{S}_{irr,p}$ would also block-diagonalise all the fusion coefficient matrices $N_{irr,p,I}$, because one stays in the same eigenspace, in full analogy to the semisimple case, where the S-matrix S_{RCFT} diagonalises them.
However, because the characters of irreducible representations do not close any more under the transformation $\tau \to -1/\tau$, it is not directly obvious, what this S-matrix $\mathtt{S}_{irr,p}$ has to be. This will be the first point, we have to clarify, before we can afterwards look for the matrix $K_{irr,p}$, which then also fixes $P_{irr,p}$.

3.2.2 An S-Matrix for the Characters of Irreducible Representations

One can also deal in a different way with the fact, that the characters of the irreducible representations do not close under modular transformations of their argument, compared to section 3.1. As we have mentioned in section 2.4.1, there are only some factors of τ, which cause this problem, like e.g. in eq. (2.38). Thus one can define a matrix with entries depending on τ as:

$$\boldsymbol{\chi}_{(irr,p)}\left(-\frac{1}{\tau}\right) = \mathbf{S}_p(\tau)\boldsymbol{\chi}_{(irr,p)}(\tau) \tag{3.45}$$

with the vector of characters of irreducible representations $\boldsymbol{\chi}_{(irr,p)}$ from equation (2.37).
This was the actual starting point in the paper Fuchs et al. [FHST04], which led to this method to calculate fusion rules of the $c_{(p,1)}$-series.
The matrix $\mathbf{S}_p(\tau)$ can be written down in 2×2 blocks $A_{s,j}$ as follows:

$$\mathbf{S}_p(\tau) = \begin{pmatrix} A(p)_{0,0} & A(p)_{0,1} & \dots & A(p)_{0,p-1} \\ A(p)_{1,0} & A(p)_{1,1} & \dots & A(p)_{1,p-1} \\ \vdots & \vdots & \ddots & \vdots \\ A(p)_{p-1,0} & A(p)_{p-1,1} & \dots & A(p)_{p-1,p-1} \end{pmatrix} \tag{3.46}$$

with

$$\begin{aligned} A(p)_{0,0} &= S(p)_{0,0} \\ A(p)_{0,j} &= \frac{2}{\sqrt{2p}}\begin{pmatrix} 1 & 1 \\ (-1)^{p-l} & (-1)^{p-l} \end{pmatrix} \\ A(p)_{s,0} &= S(p)_{s,0} \\ A(p)_{s,j} &= \frac{2}{\sqrt{2p}}(-1)^{p+j+s} \times \\ & \begin{pmatrix} \frac{s}{p}\cos(\pi\frac{sj}{p}) - i\tau\frac{p-j}{p}\sin(\pi\frac{sj}{p}) & \frac{s}{p}\cos(\pi\frac{sj}{p}) + i\tau\frac{j}{p}\sin(\pi\frac{sj}{p}) \\ \frac{p-s}{p}\cos(\pi\frac{sj}{p}) + i\tau\frac{p-j}{p}\sin(\pi\frac{sj}{p}) & \frac{p-s}{p}\cos(\pi\frac{sj}{p}) - i\tau\frac{j}{p}\sin(\pi\frac{sj}{p}) \end{pmatrix} \\ 0 < & \ s,j \ < p \end{aligned}$$

and with the blocks $S(p)_{s,j}$ with $s = 0$ and $0 \le j < p$ from equation (3.10).

Construction of an $SL(2,\mathbb{Z})$ Representation

An analogous definition to (3.45) for the $\mathcal{T}$-Transformation $\tau \to \tau + 1$ gives a $2p \times 2p$ matrix $\mathbf{T}_p$, which does not depend on τ because under this transformation the irreducible characters transform into multiples of themselves, i.e. $\mathbf{T}_p$ is diagonal. All other matrices $\mathbf{J}_p(\gamma,\tau)$, which give the transformation of the irreducible characters under a given $\gamma \in SL(2,\mathbb{Z})$, are products of the matrices $\mathbf{S}_p(\cdot)$ evaluated at different functions of τ and $\mathbf{T}_p$.
The objective now is to find a map $j_p(\gamma,\tau)$ from $SL(2,\mathbb{Z})$ $(\ni \gamma)$ into the $2p \times 2p$ matrices with matrix elements depending on τ –, a so-called automorphy factor – which leads to generators $\mathtt{S}_{irr,p}$ and $\mathtt{T}_{irr,p}$ of an $SL(2,\mathbb{Z})$ representation of constant matrices by multiplication from the left on $\mathbf{S}_p(\tau)$ and $\mathbf{T}_p$, respectively. With the definitions above this gives us a representation ρ_p of $SL(2,\mathbb{Z})$:

$$\rho_p(\gamma) := j_p(\gamma,\tau) J_p(\gamma,\tau)\ . \tag{3.47}$$

This is described in detail in [FHST04]. Obviously $j_p(\gamma,\tau)$ must depend on the element of $SL(2,\mathbb{Z})$ because it has to cancel the τ dependence of $\mathbf{S}_p(\tau)$ and at the same time leave $\mathbf{T}_p$ τ-independent. Furthermore it has to fulfil a cocycle condition in consideration of a similar condition for $J_p(\gamma,\tau)$. It has to preserve the unit element and it commutes "strongly" with ρ:

$$\rho_p(\gamma) j_p(\gamma',\tau) = j_p(\gamma',\tau)\rho_p(\gamma) \quad \forall\, \gamma,\gamma' \in SL(2,\mathbb{Z})\ . \tag{3.48}$$

Moreover $j_p(\gamma,\tau)$ is defined to be block-diagonal:

$$j_p(\gamma,\tau) = \mathbb{1}_{2\times 2} \oplus \bigoplus_{s=1}^{p-1} B_s(\gamma,\tau)\ . \tag{3.49}$$

With all this information one is first led to the matrix $j_p(\mathcal{S},\tau)$ and then through the cocycle condition to $j_p(\mathcal{T},\tau)$ and so gets the generators of the desired representation of $SL(2,\mathbb{Z})$. For our work we only need the former one, which has the blocks

$$B_s(S,\tau) = \begin{pmatrix} \frac{s}{p} + i\frac{p-s}{\tau p} & \frac{s}{p} - i\frac{s}{\tau p} \\ \frac{p-s}{p} - i\frac{p-s}{\tau p} & \frac{p-s}{p} + i\frac{s}{\tau p} \end{pmatrix} \quad s = 1,\dots,p-1\ . \tag{3.50}$$

Plugged into equation (3.47) for $\gamma = \mathcal{S}$ one gets the S-matrix, which we search for and are going to relate to $P_{irr,p}$:

$$\mathtt{S}_{irr,p} := \rho_p(\mathcal{S}) = j_p(\mathcal{S},\tau)J_p(\mathcal{S},\tau) = j_p(\mathcal{S},\tau)\mathbf{S}_p(\tau)\ . \tag{3.51}$$

For the 2×2 blocks of this matrix one finds for $s = 0$ or $j = 0$ the same blocks as in eq. (3.46), $S(irr,p)_{s,j} = A(p)_{s,j}$, and otherwise

$$\begin{aligned} S(irr,p)_{s,j} \quad &= \quad \frac{2}{\sqrt{2p}}(-1)^{p+j+s} \times \\ &\begin{pmatrix} \frac{s}{p}\cos(\pi\frac{sj}{p}) + \frac{p-j}{p}\sin(\pi\frac{sj}{p}) & \frac{s}{p}\cos(\pi\frac{sj}{p}) - \frac{j}{p}\sin(\pi\frac{sj}{p}) \\ \frac{p-s}{p}\cos(\pi\frac{sj}{p}) - \frac{p-j}{p}\sin(\pi\frac{sj}{p}) & \frac{p-s}{p}\cos(\pi\frac{sj}{p}) + \frac{j}{p}\sin(\pi\frac{sj}{p}) \end{pmatrix}\ , \\ 0 < \quad & s,j \quad < p\ . \end{aligned} \tag{3.52}$$

Of course, ${\mathtt{S}_{irr,p}}^2 = \mathbb{1}$. Furthermore it is equal to the τ-dependent S-matrix evaluated at $\tau = i$, $\mathbf{S}_p(i)$. In an analogous way we get to the matrix $\mathtt{T}_{irr,p}$. The relation $(\mathtt{T}_{irr,p}\mathtt{S}_{irr,p})^3 = \mathbb{1}$ holds and the two matrices generate the $SL(2,\mathbb{Z})$ representation ρ.

The Connection between $\mathtt{S}_{\mathbf{irr,p}}$ and $\mathbf{P}_{\mathbf{irr,p}}$

To complete the overview of Fuchs' approach, we only need to state, what the matrix $K_{irr,p}$ is, as defined in equation (3.44). There we already mentioned, that one can have 2×2 blocks around the diagonal in $K_{irr,p}$ with further off-diagonal elements being zero, when one requires $\mathtt{S}_{irr,p}$ to block-diagonalise all the matrices $N_{irr,p,I}$. Because we expect the latter in generalisation of the semisimple case (cf. A.1.2), we take $K_{irr,p}$ now to have this block structure.
Because we know the vacuum row of $P_{irr,p}$, π_Ω, we have some grip on $K_{irr,p}$ through its relation to the vacuum row of $\mathtt{S}_{irr,p}$, σ_Ω. Like in eq. (A.10), we have also here

$$\pi_\Omega = \sigma_\Omega K_{irr,p} \ , \tag{3.53}$$

But to determine it completely the choice of two conditions, that are imposed on each 2×2 block, has to be made. First the elements of the first column of each block must add to zero. Second the determinant of each block is put to one. The following matrix is thus taken for $K_{irr,p}$:

$$\begin{aligned} K_{irr,p} &= (K_{irr,p})_0 \oplus \bigoplus_{s=1}^{p-1} (K_{irr,p})_s \ , \\ (K_{irr,p})_0 &:= \begin{pmatrix} \frac{1}{(\mathtt{S}_{irr,p})_\Omega{}^1} & 0 \\ 0 & \frac{1}{(\mathtt{S}_{irr,p})_\Omega{}^2} \end{pmatrix} , \\ (K_{irr,p})_s &:= \begin{pmatrix} \frac{1}{(\mathtt{S}_{irr,p})_\Omega{}^{2s+1} - (\mathtt{S}_{irr,p})_\Omega{}^{2s+2}} & -(\mathtt{S}_{irr,p})_\Omega{}^{2s+2} \\ \frac{-1}{(\mathtt{S}_{irr,p})_\Omega{}^{2s+1} - (\mathtt{S}_{irr,p})_\Omega{}^{2s+2}} & \frac{1}{(\mathtt{S}_{irr,p})_\Omega{}^{2s+1}} \end{pmatrix} . \end{aligned} \tag{3.54}$$

Said a bit more compactly, it is the unique block-diagonal matrix with the first block being diagonal and the other blocks of the form

$$\begin{aligned} &(K_{irr,p})_s \begin{pmatrix} k_i & \bullet \\ -k_i & \bullet \end{pmatrix} , \\ &det((K_{irr,p})_s) = 1 \ , \end{aligned} \tag{3.55}$$

which relates the vacuum rows as in equation A.10.
Now we can write the "generalised" Verlinde formula, with which this method provides us, in terms of matrix elements of $\mathtt{S}_{irr,p}$:

$$N_{irr,p,I} = \mathtt{S}_{irr,p} K_{irr,p} M_{irr,p,I} (K_{irr,p})^{-1} \mathtt{S}_{irr,p} \ . \tag{3.56}$$

We give the example of $p = 3$ in appendix C.2. There we explicitly state each step of calculation within this method from the τ-dependent S-matrix $\mathtt{S}_{irr,3,1}$ till one of the fusion coefficient matrices $N_{irr,p,I}$ and present all matrices, which are written down in this section only for general p. Fuchs et al. have calculated a closed expression for the fusion rules for irreducible representations also for general p using the trick, that also the matrices $M_{irr,p,I}$ constitute a representation of the "small" fusion algebra (cf. [FHST04]).

3.2.3 A Replacement for the Automorphy Factor

Until now we can only say, that the matrix $\mathtt{S}_{irr,p}$ is the one corresponding to the transformation $\tau \to -\frac{1}{\tau}$ that results from the construction of a closed modular group action on the space generated by the characters of the irreducible representations. To accomplish this an automorphy factor is needed. But an additional interpretation giving a more direct connection to physically relevant quantities or properties would be favourable. This has been the motivation to find a matrix $C_{irr,p}(\tau)$, which almost conjugates[4] – we need a small alteration due to the τ dependence of $C_{irr,p}(\tau)$ – the two matrices $\mathbf{S}_p(\tau)$ and $\mathtt{S}_{irr,p}$ and replaces the automorphy factor. In this way we see $\mathtt{S}_{irr,p}$ as the matrix giving the $\mathcal{S}$-transformation of τ-dependent linear combinations ${\chi'}_{(irr,p)}(\tau)$ of characters $\chi_{(irr,p)}(\tau)$ of irreducible representations given by $C_{irr,p}(\tau)$:

$$
{\chi'}_{(irr,p)}(\tau) = C_{irr,p}(\tau)\chi_{(irr,p)}(\tau) \; . \tag{3.57}
$$

With equation (3.45) one gets the $\mathcal{S}$-transformation of ${\chi'}_{(irr,p)}(\tau)$:

$$
\begin{aligned}
{\chi'}_{(irr,p)}\left(-\frac{1}{\tau}\right) &= C_{irr,p}\left(-\frac{1}{\tau}\right)\chi_{(irr,p)}\left(-\frac{1}{\tau}\right) && (3.58)\\
&= C_{irr,p}\left(-\frac{1}{\tau}\right)\mathbf{S}_p(\tau){C_{irr,p}}^{-1}(\tau)C_{irr,p}(\tau)\chi_{(irr,p)}(\tau) && (3.59)\\
&= \underbrace{C_{irr,p}\left(-\frac{1}{\tau}\right)\mathbf{S}_p(\tau){C_{irr,p}}^{-1}(\tau)}_{=:{\mathbf{S}'}_p(\tau)}{\chi'}_{(irr,p)}(\tau) \; . && (3.60)
\end{aligned}
$$

${\mathbf{S}'}_p(\tau)$ is now set to be equal to $\mathtt{S}_{irr,p}$. So the matrix $C_{irr,p}(\tau)$ we are looking for should relate $\mathtt{S}_{irr,p}$ and $\mathbf{S}_p(\tau)$ through

$$
\mathtt{S}_{irr,p} = C_{irr,p}\left(-\frac{1}{\tau}\right)\mathbf{S}_p(\tau){C_{irr,p}}^{-1}(\tau) \; . \tag{3.61}
$$

The τ-dependence makes the problem to find $C_{irr,p}(\tau)$ a bit more intricate. One cannot just solve the set of equations, which eq. (3.61) represents, for the τ-dependent matrix elements of $C_{irr,p}(\tau)$, as these have to be evaluated once at $\tau = -\frac{1}{\tau}$ instead of τ.
We could calculate $C_{irr,2}$, as we looked at the expansion of equation (3.61) around $\tau = i$. Furthermore we used an argument, which we could derive, for the Laurent coefficients of the determinant of $C_{irr,2}$. These calculations are detailed in appendix A.1.3 and yield the result

$$
C_{irr,2}(\tau) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{4}-\frac{1}{4}i\tau & \frac{1}{4}+\frac{1}{4}i\tau \\ 0 & 0 & \frac{1}{4}+\frac{1}{4}i\tau & \frac{3}{4}-\frac{1}{4}i\tau \; . \end{pmatrix} \tag{3.62}
$$

This provides us with some kind of picture, how a block $(C_{irr,p})_s(\tau)$ of $C_{irr,p}(\tau)$ looks like, where we define

$$
C_{irr,p}(\tau) = \mathbb{1}_{2\times 2} \oplus \bigoplus_{s=1}^{p-1} (C_{irr,p})_s(\tau) \; . \tag{3.63}
$$

[4]Conjugation is always meant in a group theoretical sense – not complex conjugate or suchlike. We say, a matrix M conjugates two (similar) matrices N_1 and N_2, if $N_1 = MN_2M^{-1}$.

To calculate these blocks, we looked at eq. (3.61) block by block

$$
\begin{aligned}
S(irr,p)_{s,0} = A(p)_{s,0} &= (C_{irr,p})_s(-\frac{1}{\tau})A(p)_{s,0} \,, && (3.64)\\
S(irr,p)_{0,j} = A(p)_{0,j} &= A(p)_{0,j}(C_{irr,p})_j{}^{-1}(\tau) \,, && (3.65)\\
S(irr,p)_{s,j} &= (C_{irr,p})_s(-\frac{1}{\tau})A(p)_{s,j}(C_{irr,p})_j{}^{-1}(\tau) \,, && (3.66)\\
& 0 < s, j < p \,. && (3.67)
\end{aligned}
$$

Our ansatz has been

$$
\begin{pmatrix} c_{111} + c_{112}\tau & c_{121} + c_{122}\tau \\ c_{211} + c_{212}\tau & c_{221} + c_{222}\tau \end{pmatrix} \qquad (3.68)
$$

with $c_{ijk} \in \mathbb{C}$ for $ijk \in \{1, 2\}$. We then use the blocks given in (3.46) and (3.52) for different values of s and j, but p left open. To determine one particular block $(C_{irr,p})_\sigma(\tau)$ it is enough to choose a sufficient subset among the three matrix equations above for $s = j = \sigma$ and the condition, that for $\tau = i$ it is the unit matrix. For σ being one or two we have calculated the blocks by solving this system of equations.

$$
\begin{aligned}
(C_{irr,p})_2(\tau) &= \begin{pmatrix} \frac{1+p}{2p} + i\frac{1-p}{2p}\tau & \frac{1}{2p} + i\frac{1}{2p}\tau \\ \frac{p-1}{2p} + i\frac{p-1}{2p}\tau & \frac{2p-1}{2p} + i\frac{-1}{2p}\tau \end{pmatrix} \,, && (3.69)\\
(C_{irr,p})_3(\tau) &= \begin{pmatrix} \frac{2+p}{2p} + i\frac{2-p}{2p}\tau & \frac{2}{2p} + i\frac{2}{2p}\tau \\ \frac{p-2}{2p} + i\frac{p-2}{2p}\tau & \frac{2p-2}{2p} + i\frac{-2}{2p}\tau \end{pmatrix} \,. && (3.70)
\end{aligned}
$$

From these two cases it is not too hard to guess the following matrix, which gives the blocks $(C_{irr,p})_s(\tau)$ for arbitrary s:

$$
(C_{irr,p})_s(\tau) = \begin{pmatrix} \frac{s+p}{2p} - i\frac{p-s}{2p}\tau & \frac{s}{2p} + i\frac{s}{2p}\tau \\ \frac{p-s}{2p} + i\frac{p-s}{2p}\tau & \frac{2p-s}{2p} - i\frac{s}{2p}\tau \end{pmatrix} \,. \qquad (3.71)
$$

We have verified it by calculating $\mathtt{S}_{irr,p}$ block by block using these blocks for $C_{irr,p}(\tau)$ in eq. (3.61).

This matrix only replaces the factor $j_p(\gamma,\tau)$ for the case of $\gamma = \mathcal{S}$. Because $j_p(\gamma,\tau)$ depends on γ, the matrix replacing it for other $\gamma \neq \mathcal{S}$ is different from $C_{irr,p}(\tau)$. Hence other elements of the representation $\rho(\gamma)$ are not given by the transformation γ of the same linear combination of characters as given by $C_{irr,p}(\tau)$. The interpretation, it yields for $\mathtt{S}_{irr,p}$, does not hold for the whole representation $\rho(\gamma)$. Consequently the matrix $C_{irr,p}(\tau)$ is of little importance for the original method of Fuchs, which we have discussed in this section.

However, for the extension this method to indecomposable representations this matrix is very helpful, as we will see now, to find the matrix C_p connecting the larger S-matrix, taking the place of $\mathtt{S}_{irr,p}$, with the α dependent S-matrix $\mathtt{S}(p,\alpha)$ from the section 3.1. We have seen in section 3.1, that $\mathtt{S}(p,\alpha)$ belongs to an $SL(2,\mathbb{Z})$ representation $\mathtt{G}_{(p,\alpha)}(\gamma)$ (eq. 3.15). This representation gives the modular transformation properties of a set of forms $\boldsymbol{\chi}_p(\alpha)(\tau)$ without any automorphy factor (eq. 3.14). So we get with the product $C_p\mathtt{G}_{(p,\alpha)}(\gamma){C_p}^{-1}$ another representation of the modular group, which also needs no automorphy factor – or said in another way, its automorphy factor is the unit matrix. Thus we can interpret this new representation there as the one, which gives directly the modular transformation properties of the set of linear combinations of the original forms $\boldsymbol{\chi}_p(\alpha)(\tau)$ given by $C_p\boldsymbol{\chi}_p(\alpha)(\tau)$.

3.2.4 Substitution of τ-Dependent Linear Combinations

We now start to compare the two methods described in this section and section 3.1. Some character identities will help to transfer the τ-dependent matrices $\mathbf{S}_p(\tau)$ and $C_{irr,p}(\tau)$ into α-dependent matrices. This will reveal the connection between $\mathbf{S}_p(\tau)$ and $\mathtt{S}_{(p,\alpha)}$. As a side-effect, we continue our preparations for the next section, because we will use there the α-dependent pendant of $C_{irr,p}(\tau)$, which we call $C'_p(\alpha)$, to find C_p.

Lemma: *The characters given in eq.* (2.34) *and* (2.35) *and the forms from eq.* (3.3) *fulfil the equation*

$$i(s-p)\tau\chi^+_{p-s,p} + is\tau\chi^-_{p-s,p} = -\frac{1}{\alpha}\tilde{\chi}_{p-s,p}(\alpha) + \frac{2}{\alpha}\chi^+_{p-s,p} + \frac{2}{\alpha}\chi^-_{p-s,p}\,. \tag{3.72}$$

Remark: The matrices $\mathbf{S}_p(\tau)$ (eq. (3.46)) and $C_{irr,p}(\tau)$ (eq. (3.71)) are multiplied by the vector of irreducible representations $\boldsymbol{\chi}_{irr,p}$ (eq. (2.37)) in eqns. (3.45) and (3.57). The summands in their matrix elements containing τ always turn up in pairs in a row of one of their blocks. The first of these terms in the pair is multiplied with the character of a singlet representation $\chi^+_{p-s,p}$. The second one is multiplied with the character of the corresponding doublet representation $\chi^-_{p-s,p}$. The only difference in these two terms is a factor of $(s-p)$ in the first and s in the second term. So for $0 < s < p$ the constellation given on the left hand side of equation (3.72) appears in the τ-dependent linear combination of characters all the time. We want to replace this by the right hand side using $2p \times (3p-1)$ matrices, which are multiplied now with the vector $\boldsymbol{\chi}_p(\alpha)$ (eq. (3.4)) instead of $\boldsymbol{\chi}_{irr,p}$, but give the same result.

Proof: We plug in the characters from equations (2.34) and (2.35) and find that the factors match in precisely the way to let the dependence on $\Theta_{p-s,p}$ and on s drop out.

$$i(s-p)\tau\left(\frac{1}{p\eta}\left[s\Theta_{p-s,p} + (\partial\Theta)_{p-s,p}\right]\right) + is\tau\left(\frac{1}{p\eta}\left[(p-s)\Theta_{p-s,p} - (\partial\Theta)_{p-s,p}\right]\right) \tag{3.73}$$

$$= \quad -i\tau\frac{1}{\eta}(\partial\Theta)_{p-s,p} = -\frac{1}{\eta}(\nabla\Theta)_{p-s,p}\,. \tag{3.74}$$

Equation (3.3) guides the way to insert a zero (one of two we need to insert here):

$$-\frac{\alpha}{\alpha\eta}(\nabla\Theta)_{p-s,p} - \frac{1}{\alpha\eta}2\Theta_{p-s,p} + \frac{1}{\alpha\eta}2\Theta_{\lambda,p} \tag{3.75}$$

$$= \quad -\frac{1}{\alpha}\tilde{\chi}_{p-s,p}(\alpha) + \frac{2}{\alpha}\frac{1}{p\eta}\,s\,\Theta_{\lambda,p} + \frac{2}{\alpha}\frac{1}{p\eta}(\partial\Theta)_{p-s,p} + \frac{2}{\alpha}\frac{1}{p\eta}(p-s)\Theta_{\lambda,p} - \frac{2}{\alpha}\frac{1}{p\eta}(\partial\Theta)_{p-s,p}$$

$$= \quad -\frac{1}{\alpha}\tilde{\chi}_{p-s,p}(\alpha) + \frac{2}{\alpha}\chi^+_{p-s,p} + \frac{2}{\alpha}\chi^-_{p-s,p}\,.$$

□

We start with the matrix $\mathbf{S}_p(\tau)$ and write down its partner $2p\times(3p-1)$ matrix . A column must be inserted for each form $\tilde{\chi}_{s,p}(\alpha)$ after the columns multiplied with $\chi^+_{s,p}$ and $\chi^-_{s,p}$ for $0 < s < p$. In the elements in the latter two columns the respective factors $i(s-p)\tau$ and $is\tau$ are both replaced by $2/\alpha$. The added column has to contain $-1/\alpha$. This way we do the following changes for the blocks of $\mathbf{S}_p(\tau)$:

$$\begin{pmatrix} \frac{s}{p}\mathfrak{c}_{sl} - i\tau\frac{p-j}{p}\mathfrak{s}_{sl} & \frac{s}{p}\mathfrak{c}_{sl} + i\tau\frac{j}{p}\mathfrak{s}_{sl} \\ \frac{p-s}{p}\mathfrak{c}_{sl} + i\tau\frac{p-j}{p}\mathfrak{s}_{sl} & \frac{p-s}{p}\mathfrak{c}_{sl} - i\tau\frac{j}{p}\mathfrak{s}_{sl} \end{pmatrix} \to \begin{pmatrix} \frac{s}{p}\mathfrak{c}_{sl} + \frac{2}{p}\frac{1}{\alpha}\mathfrak{s}_{sl} & \frac{s}{p}\mathfrak{c}_{sl} + \frac{2}{p}\frac{1}{\alpha}\mathfrak{s}_{sl} & -\frac{1}{p\alpha}\mathfrak{s}_{sl} \\ \frac{p-s}{p}\mathfrak{c}_{sl} - \frac{2}{p}\frac{1}{\alpha}\mathfrak{s}_{sl} & \frac{p-s}{p}\mathfrak{c}_{sl} - \frac{2}{p}\frac{1}{\alpha}\mathfrak{s}_{sl} & \frac{1}{p\alpha}\mathfrak{s}_{sl}\,. \end{pmatrix} \tag{3.76}$$

The first two rows of the added columns are zero because these rows do not depend on τ. We see, that the matrix we get is just a composition of the α dependent S-matrix $\mathtt{S}_{(p,\alpha)}$ and a subsequent projection onto the components of $\boldsymbol{\chi}_p(\alpha)$ belonging to irreducible representations, as it is expected to be. This is no big deal because we had both matrices already.
The interesting piece is the application of this to the matrix $C_{irr,p}(\tau)$. The diagonal blocks get replaced by 2×3 blocks arranged in a diagonal way, i.e. the whole matrix, called C'_p, is the direct sum of a 2×2 unit matrix and these blocks.

$$\begin{pmatrix} \frac{s+p}{2p} - i\frac{p-s}{2p}\tau & \frac{s}{2p} + i\frac{s}{2p}\tau \\ \frac{p-s}{2p} + i\frac{p-s}{2p}\tau & \frac{2p-s}{2p} - i\frac{s}{2p}\tau \end{pmatrix} \to \underbrace{\begin{pmatrix} \frac{s+p}{2p} + \frac{1}{2p\alpha} & \frac{s}{2p} + \frac{1}{2p\alpha} & -\frac{1}{2p\alpha} \\ \frac{p-s}{2p} - \frac{1}{2p\alpha} & \frac{2p-s}{2p} - \frac{1}{2p\alpha} & \frac{1}{2p\alpha} \end{pmatrix}}_{(C'_p)_s(\alpha)} , \tag{3.77}$$

$$C'_p(\alpha) = 1\!\!1_{2\times 2} \oplus \bigoplus_{s=1}^{p-1} (C'_p)_s(\alpha) .$$

Now the matrix $C'_p(\alpha)$ encodes the linear combinations of characters, for which $\mathtt{S}_p$ gives their transformation under $\tau \to -1/\tau$, as τ-independent linear combinations of these characters and the forms $\tilde{\chi}_{s,p}(\alpha)$.

For example we get for $p = 2$ and $p = 3$

$$C'_2(\alpha) = \begin{pmatrix} 1 & 0 & & 0 & \\ 0 & 1 & & & \\ & & \frac{3\alpha+2}{4\alpha} & \frac{\alpha+2}{4\alpha} & -\frac{1}{4\alpha} \\ & 0 & \frac{\alpha-2}{4\alpha} & \frac{3\alpha-2}{4\alpha} & \frac{1}{4\alpha} \end{pmatrix} \tag{3.78}$$

and

$$C'_3(\alpha) = \begin{pmatrix} 1 & 0 & & 0 & & & 0 & \\ 0 & 1 & & & & & & \\ & & \frac{2\alpha+1}{3\alpha} & \frac{\alpha+2}{6\alpha} & -\frac{1}{6\alpha} & & & \\ 0 & & \frac{\alpha-1}{3\alpha} & \frac{5\alpha-2}{6\alpha} & \frac{1}{6\alpha} & & 0 & \\ & & & & & \frac{5\alpha+2}{6\alpha} & \frac{\alpha+1}{3\alpha} & -\frac{1}{3\alpha} \\ 0 & & & 0 & & \frac{\alpha-2}{6\alpha} & \frac{2\alpha-1}{3\alpha} & \frac{1}{3\alpha} \end{pmatrix} . \tag{3.79}$$

3.3 Extension of the Block-Diagonalisation Ansatz

In this section we will work on an extension for the block-diagonalisation method, which was introduced in the last one. It is absolutely parallel to the version for irreducible representations and the latter will be seen in each step of calculation as a projection of the larger $(3p-1)\times(3p-1)$ matrices, we encounter here, onto a $2p$ dimensional space.
We will start with considerations analogous to the beginning of the last section and derive the form, in which a matrix P_p block-diagonalises the fusion coefficient matrices $N_{p,I}$. It will be very similar to the calculation, we have already seen in section 3.2.1.
Then we need to find the S-matrix S_p,which is connected to P_p by the multiplication of a block-diagonal matrix K_p in complete analogy to (3.44). We have already mentioned at the end of section 3.2.3, that the large block-diagonalising S-matrix, S_p, will be connected to the matrix $\mathtt{S}(p,\alpha)$ by conjugation with the matrix $C_p(\alpha)$. We have also discussed, that S_p is

the representative of the transformation $\tau \to -1/\tau$ in a representation of $SL(2,\mathbb{Z})$ without automorphy factor – given by $C_p(\alpha)\mathtt{G}_{(p,\alpha)}(\gamma){C_p}^{-1}(\alpha)$ – acting on a set of linear combination of generalised characters, $C_p(\alpha)\boldsymbol{\chi}_p(\alpha)(\tau)$. The S-matrix S_p encodes the decompositions of the transformation $\tau \to -1/\tau$ of the linear combinations of generalised characters, given by $C_p(\alpha)$, exactly like in eq. (3.5). But here the linear combination is chosen in such a way, that the S-matrix does not depend on α anymore.

We give a detailed description of the freedom of choice we have in different steps and of the causes for our partially heuristically made descisions. Sometimes the arguments for a particular choice are solely its influence on the actual outcome for $p = 2$. This is, of course, an invalid argument for the cause, why this should be the right scheme. But it is exactly following the intention of our work. In section (3.3), namely, we show the total equality of the results of the extended methods with the approach by Flohr in section 3.1 rather than looking at particular examples. We have described how this latter approach, which uses the α-Verlinde formula, develops from considerations about the partition function and the chiral vacuum torus amplitudes in an unambiguous way. Together with the ubiquitous possibility of projection onto the calculations of the last section we lead the two methods together and state the equivalence of the extended block-diagonalisation method and the α-Verlinde formula.

What we show in the end, is, that it is also possible to get to the proposed true fusion rules in the conjecture in section 3.1 via block-diagonalisation and the subsequent well-defined substitutions of indecomposable representations in the decomposition of the fusion product, which have been described there.

3.3.1 Simultaneous Eigen Decomposition of the Fusion Coefficient Matrices including Indecomposable Representations

We now want to block-diagonalise the matrices of (pre-)fusion coefficients for the full (pre-)fusion algebra including indecomposable representations simultaneously. This fusion algebra is defined in the familiar way:

$$X_I X_J = \sum_{K=1}^{2p} (N_p)_{IJ}{}^K X_K \ . \tag{3.80}$$

The basis X is now larger and also contains the indecomposable representations. Its sequence is the same as the one of the vector $\boldsymbol{\chi}_p(\alpha)$ (eq. (3.4)). There is actually only one difference to the last section.

For the change of basis to the one, in which the matrices block-diagonalise, we have to note, that now the radical of the fusion algebra has the double dimension. We call the extra basis elements of the radical, which take care of the indecomposable representations, w'_A with $A = 3 \dots p+1$. And Y (eq. (3.27)) is then replaced by

$$Y' = (e_1, e_2, e_3, w_3, w'_3, e_4, w_4, w'_4, \dots, e_{p+1}, w_{p+1}, w'_{p+1}) \ . \tag{3.81}$$

The idempotents of the semisimple algebra and the basis of the larger radical relate to each other just as before and we get in addition to the equations (3.28)-(3.30):

$$e_A w'_C = \delta_{A,C} w'_C \ , \tag{3.82}$$

$$w'_C w_D = 0 \ , \tag{3.83}$$

$$w'_C w'_D = 0 \tag{3.84}$$

with $0 < A \leq p+1$ and $3 \leq C, D \leq p+1$. With this basis, Y', instead of Y we can define P_p analogous to eq. (3.31). The proposition in section 3.2.1 translates directly:

Proposition: *P_p block-diagonalises the matrices $N_{p,I}$ simultaneously.*

$$N_{p,I} = P_p M_{p,I} {P_p}^{-1} \tag{3.85}$$

with the block-diagonal matrices $M_{p,I}$, $0 < I \leq 2p$. For the rows of P_p, π_I,

$$\pi_I = \pi_\Omega M_{p,I} \tag{3.86}$$

holds for all $0 < I \leq 2p$.

Proof: The proof is also analogous to the one in section 3.2.1. On the one hand, we get equation (3.34) with Y replaced by Y' without the indices irr from equation (3.80):

$$X_I Y'_L = \sum_{S=1}^{2p} (M_{p,I})_L{}^S Y'_S \,. \tag{3.87}$$

On the other hand this can be calculated from eqns. (3.28)-(3.30) and (3.82)-(3.84):

$$X_I Y_A = \begin{cases} (P_p)_{IA} Y_A & \text{for} \quad A = 1, 2 \\ (P_p)_{IA} Y_A + (P_p)_{I(A+1)} Y_{A+1} + (P_p)_{I(A+2)} Y_{A+2} & \text{for} \quad A = 3, 6, 9, \ldots \\ (P_p)_{I(A-1)} Y_A & \text{for} \quad A = 4, 7, 10, \ldots \\ (P_p)_{I(A-2)} Y_A & \text{for} \quad A = 5, 8, 11, \ldots \end{cases} . \tag{3.88}$$

$M_{p,I}$ is an upper-triangular block-diagonal matrix with all but one 3×3 blocks and reads

$$M_{p,I} = M_{p,I,0} \oplus \bigoplus_{n=1}^{p-1} M_{p,I,n} \,, \tag{3.89}$$

$$M_{p,I,0} = \begin{pmatrix} (P_p)_{I1} & 0 \\ 0 & (P_p)_{I2} \end{pmatrix} \,, \tag{3.90}$$

$$M_{p,I,n} = \begin{pmatrix} (P_p)_{I(3n)} & (P_p)_{I(3n+1)} & (P_p)_{I(3n+2)} \\ 0 & (P_p)_{I(3n)} & 0 \\ 0 & 0 & (P_p)_{I(3n)} \end{pmatrix} \,. \tag{3.91}$$

Because the idempotents are still a partition of the unit element and per definition the vacuum row of P_p gives the unit element of the fusion algebra from the basis Y', it has to be

$$\pi_\Omega = (1, 1, 1, 0, 0, 1, 0, 0, \ldots, 1, 0, 0) \,. \tag{3.92}$$

The exact same steps, which we already had beneath eq. (3.38), show that $M_{p,I}$ relates the I-th row of P_p to its vacuum row. □

The cause, why $M_{p,I}$ and P_p look like this, is again just linear algebra. We have done a simultaneous eigen decomposition for a set of $N_{p,I}$, which was possible, because they happened to be the structure constants of the algebra in equation (3.80) and so are related to each other by the properties of the algebra like commutativity. This has enteredour proof, as we plugged in (3.80) at one point and interchange elements of X.
The eigen decomposition has been nicely encoded in eqns. (3.28)-(3.30) and (3.82)-(3.84) and

we see again, that this way the matrix P_p is a matrix consisting of simultaneous generalised eigenvectors. We can calculate for a column p_J of P_p as in equation (3.41), that

$$N_{p,I}p_J = \begin{cases} \left((P_p)_{IJ}p_{J-1} + (P_p)_{I(J-1)}p_J\right) & \text{for} \quad J = 4, 7, 10, \dots \\ \left((P_p)_{IJ}p_{J-2} + (P_p)_{I(J-2)}p_J\right) & \text{for} \quad J = 5, 8, 11, \dots \\ (P_p)_{IJ}p_J & \text{else} \end{cases} \tag{3.93}$$

and find also (see eq. (3.42))

$$(N_{p,I} - \lambda_{J-1}\mathbb{1})^2 p_J = 0 \quad \text{for } J = 4, 7, 10, \dots \quad , \tag{3.94}$$

$$(N_{p,I} - \lambda_{J-2}\mathbb{1})^2 p_J = 0 \quad \text{for } J = 5, 8, 11, \dots \quad . \tag{3.95}$$

P_p consists of generalised eigenvectors, but here the eigenspaces are three dimensional apart from two one-dimensional ones.

3.3.2 Determination of the Matrix S_2

Again the question for the S-matrix S_p and the matrix K_p arises, which give us P_p,

$$P_p = S_p K_p \ . \tag{3.96}$$

In order to find S_p we ask the question, if find a matrix $C_p(\alpha)$ can be found, which conjugates the α-dependent S-matrix $\mathtt{S}_{(p,\alpha)}$ to a yet unknown α-independent S-matrix, which produces the fusion rules calculated with the α-Verlinde formula without a limit $\alpha \to 0$ to be taken in the way, which we have learned about in the last section.
We have seen in section 3.2.4, that we have already $2p$ forms given by the multiplication of $C'_p(\alpha)$ (eq. (3.78)) with the vector $\boldsymbol{\chi}_p(\alpha)$. It seems, that this is the best point to start by keeping these forms and adding $p-1$ additional ones again. They should also be linear combinations of the elements of $\boldsymbol{\chi}_p(\alpha)$. This means that the rows of the new S-matrix S_p corresponding to irreducible representations are the rows of $\mathtt{S}_{irr,p}$ with zeros in the additional columns.
We start with the simplest case, $p = 2$, and search the matrix $C_2(\alpha)$, for which we will have

$$S_2 = C_2(\alpha)\mathtt{S}_{(2,\alpha)}{C_2}^{-1}(\alpha) \ . \tag{3.97}$$

The matrix $\mathtt{S}_{irr,2}$ appears as a block in S_2. We need to know what the fifth line of S_2 is. The picture we have until now of S_2 is

$$S_2 = \begin{pmatrix} & & & & 0 \\ & & & & 0 \\ & & \mathtt{S}_{irr,2} & & 0 \\ & & & & 0 \\ s_1 & s_2 & s_2 & s_4 & s_5 \end{pmatrix} \tag{3.98}$$

with $s_5 \neq 0$. This is strongly related with the question how we are going to find $C_2(\alpha)$, because the existence of this matrix is certainly giving restrictions on what the fifth line might be. However, S_2 – even if known completely – does not give many restrictions on $C_2(\alpha)$. We will see, that many different matrices can take the place of $C_2(\alpha)$ in equation (3.97) fulfilling all needed conditions, we can think of. For $p = 2$ there will be no difference between them. This is, what we meant with the need of heuristic arguments in the beginning of this section. We will

only be able to single out a specific $C_2(\alpha)$, when we ask, for which we can find a generalisation to arbitrary p. When we have $C_2(\alpha)$, we will also have S_2. We can get possible $C_2(\alpha)$ by looking at the eigenvalues of the matrices S_2 and $\mathtt{S}_{(2,\alpha)}$. The eigenvalues and eigenvectors the different S-matrices for $p = 2$ are listed in table D. S_2 and $\mathtt{S}_{(2,\alpha)}$ are both diagonalisable. The former one has a three dimensional eigenspace for the eigenvalue 1, a two one dimensional eigenspace for the eigenvalue -1 and for the eigenvalue s_5. For the latter one it is not so different. It has eigenvalues 1 and -1 belonging to eigenspaces with dimensions three and two, respectively.
If now S_2 is chosen, so that the fifth eigenvalue – and matrix element – s_5 is also -1, the matrices S_2 and $\mathtt{S}_{(2,\alpha)}$ are diagonalised to the same matrix

$$D_S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} . \tag{3.99}$$

We shortly ignored a small problem with the diagonalisability of S_2. When we set its fifth eigenvalue to -1, the matrix may not be diagonalisable depending on the other matrix elements in the fifth line. We encounter this problem in the list of eigenvectors of S_2 for $s_5 \neq -1$ (cf. D), in which the one belonging to the eigenvalue -1 is in general not defined for $s_5 = -1$, because its fifth component is

$$\frac{-2s_2 - s_3 - s_4 + 2s_1}{s_5 + 1} \tag{3.100}$$

Only when the numerator of this fraction is zero, we get a diagonalisable matrix, i.e in the case of $s_1 = s_2 + s_3/2 + s_4/2$. With this condition and $s_5 = -1$ the eigenvectors have the same first four components as the eigenvectors of the smaller matrix $\mathtt{S}_{irr,2}$ (see eigenvectors of $\mathtt{S}_{irr,2}$ and of S_2 for $s_5 = -1$ in appendix D) and their fifth component depends on the fifth row of S_2.
The next question, we can ask, is now, how S_2 has to look like, so that the matrices $M_{p,I}$ are those given in equation (3.89). For this we take the result, which have been calculated following section 3.1, from the appendix given in equation (B.3). First we argue that S_2 should block-diagonalise the fusion rules. The stated product has the following form

$$S_2 N_{p,1} S_2 = \begin{pmatrix} \bullet & 0 & 0 & 0 & 0 \\ 0 & \bullet & 0 & 0 & 0 \\ 0 & 0 & & & \\ 0 & 0 & & \Box & \\ 0 & 0 & & & \end{pmatrix} . \tag{3.101}$$

Two elements of this matrix provide restrictions for S_2:

$$\left.\begin{array}{rr} (S_2 N_{p,1} S_2)_{51} = & -2 + 2s_2 + s_3 + s_4 = 0 \\ (S_2 N_{p,1} S_2)_{52} = & -2 - 2s_2 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{rr} s_3 = & 4 - s_4 \\ s_2 = & -1 \end{array}\right. . \tag{3.102}$$

The matrix element s_4 is left undetermined by this argument, because with these two conditions also all other matrices $N_{p,I}$ take the form as in eq. (3.101), when they are multiplied with S_2

from both sides. At this point our S-matrix looks like

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & -1 & 0 \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & -1 & 4-s_4 & s_4 & -1 \end{pmatrix} . \tag{3.103}$$

As a consequence the first two columns of the matrices S_2 and $\mathtt{S}_{(p,\alpha)}$ are the same. The first two rows were anyway the same from the beginning. This very much militates in favour of a block-diagonal $C_2(\alpha)$ apart from the good reasons there are anyway because its smaller brother $C_{irr,2}$ is also block-diagonal.

We now look again at equation (3.96) and turn towards K_p. We expect K_p to have the corresponding block-diagonal structure because S_p should block-diagonalises the fusion coefficient matrices. We may impose the same conditions as for $K_{irr,p}$ on it, but the larger blocks, of course, leave us with much more freedom in the choice of K_p.

In the section 3.2.1 we have seen that the matrix elements of the matrix $K_{irr,p}$ are chosen that way because of the two known vacuum rows of $\mathtt{S}_{irr,p}$ and $\mathtt{P}_{irr,p}$, which $K_{irr,p}$ has to connect. Here we also have both vacuum rows. The one of S_p we know from our argument at the very beginning of section 3.3.2.They are the same as before in section 3.2 apart from some zeros in additional columns. Thus any element of the third row of a block of K_p is multiplied with zero and does not contribute. So the same argument applies for the rest and fixes us four matrix elements per block up to normalisation, if we once again demand two of them to be the negative of each other (see eq. (3.55)). As we have done before, we set the additional third column in the first two rows of each block to zero. This gives us the right result for the vacuum row of P_p and also is compatible with our goal to be able to reduce the whole extended method back to the one of the last section by projecting on the $2p$ components of our basis, which represent the irreducible representations. This provides us also with a reason to use the same normalisation for the four matrix elements per block from the last section and to copy them from there. But we also ask the 3×3 blocks to have determinant one, which fixes the third diagonal element of each block to be one. We are left with two undetermined matrix elements per block k_{s1} and k_{s2}.

$$\begin{aligned} K_p &= (K_p)_0 \oplus \bigoplus_{s=1}^{p-1} (K_p)_s \qquad (3.104) \\ (K_p)_0 &:= \begin{pmatrix} \frac{1}{(\mathtt{S}_p)_\Omega{}^1} & 0 \\ 0 & \frac{1}{(\mathtt{S}_p)_\Omega{}^2} \end{pmatrix} \\ (K_{irr,p})_s &:= \begin{pmatrix} \frac{1}{(\mathtt{S}_p)_\Omega{}^{2s+1} - (\mathtt{S}_p)_\Omega{}^{2s+2}} & -(\mathtt{S}_p)_\Omega{}^{2s+2} & 0 \\ \frac{-1}{(\mathtt{S}_p)_\Omega{}^{2s+1} - (\mathtt{S}_p)_\Omega{}^{2s+2}} & \frac{1}{(\mathtt{S}_p)_\Omega{}^{2s+1}} & 0 \\ k_{s1} & k_{s2} & 1 \end{pmatrix} \end{aligned}$$

We come back to the case $p = 2$ and have

$$K_2 = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & k_1 & k_2 & 1 \end{pmatrix} . \tag{3.105}$$

With the S-matrix in equation (3.103) and this matrix K_2 we can go through the calculations, which are needed to get to the coefficient matrices. The equations (3.96), (3.89) and (3.85) lead us via the matrices P_2 and $M_{2,I}$ to the matrices $N_{2,I}$. Here we are left with the argument that the result should agree to the result of M. Flohr, which in turn after some substitutions agree with the result of M. Gaberdiel and H. Kausch.
We compare the following results for $N_{p,1}$ from the calculations just described and from equation (B.3):

$$\begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 4-s_4-\frac{1}{2}k_1 & d+\frac{1}{2}k_1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \end{pmatrix} . \tag{3.106}$$

This leads only to a single condition, which then fixes all five matrices $N_{p,I}$.

$$k_1 = 4 - 2s_4 . \tag{3.107}$$

For $p = 2$ we have found now the conditions, which allow the correct results.
Two matrix elements are left open: k_2 and s_4. Here the good old philosophically deeply discussed[5] argument of simplicity will be the guide. We set the two not yet fixed matrix elements of K_2 to zero:

$$K_2 = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.108}$$

It follows, that $s_4 = 2$. This also gives some more "symmetry" to the S-matrix. The elements of the third and fourth column are now the same modulo minus signs. We make the corresponding choice for all blocks of K_p (eq. 3.104), so that the third row of each block is $(0, 0, 1)$. Now we have all matrices, which we need for the extended blockdiagonalisation method for $p = 2$. But $C_2(\alpha)$ is still missing, which should tell us, how the matrix we have found now is related to the matrix $\mathtt{S}_{(2,\alpha)}$. Furthermore we also long for a generalisation to arbitrary p.

3.3.3 Observations about Similar S-Matrices and the Matrix $\mathbf{C_p(\boldsymbol{\alpha})}$

Because our original intention was to find a matrix $C_p(\alpha)$, which relates them to $\mathtt{S}_{(p,\alpha)}$ to a matrix S_p, which contains $S_{irr,p}$, our way to the S-matrix S_p for arbitrary p will go this way and we will first find $C_p(\alpha)$. More precisely we will take a few pages to find $C_2(\alpha)$ and then have a comparably easy task to guess the general $C_p(\alpha)$.
We find a possible matrix C_2 fulfilling (3.97) from the matrices, which diagonalise S_2 and $\mathtt{S}_{(2,\alpha)}$, to the same diagonal matrix D_S (see eq. 3.99) If these two diagonalising matrices are U_1 and $U_2(\alpha)$, respectively, we have

$$U_1^{-1} S_2 U_1 = D_S = U_2(\alpha)^{-1} \mathtt{S}_{(2,\alpha)} U_2(\alpha) . \tag{3.109}$$

This directly gives us a lot of possible matrices $C_2(\alpha)$ by rearrangement of equation (3.97).

$$C_2(\alpha) = U_1 {U_2}^{-1}(\alpha) , \tag{3.110}$$

[5]What is simple?

As the eigenspaces, we are looking at, are two or three dimensional, there are quite a lot of matrices U_1 and $U_2(\alpha)$ that meet our needs. Every possible basis of eigenvectors spanning a particular eigenspace may be taken as the columns of these matrices. In other words the columns may be any linear combination of the eigenvectors of one eigenspace listed in appendix D for $\mathtt{S}_{(2,\alpha)}$ and S_2 (with the matrix elements inserted, which we have found now), as long as they are linearly independent.
First we just state the one possible $C_2(\alpha)$ here, which computes plainly using the eigenvectors in appendix appendix D as columns of U_1 and $U_2(\alpha)$:

$$C_{2,1}(\alpha) = \begin{pmatrix} \frac{9}{8} & \frac{7}{8} & -\frac{3}{2}+\frac{1}{2\alpha} & \frac{1}{2\alpha} & -\frac{1}{4\alpha} \\ -\frac{1}{8} & \frac{1}{8} & \frac{3}{2}-\frac{1}{2\alpha} & -\frac{1}{2\alpha} & \frac{1}{4\alpha} \\ 0 & 0 & \frac{1}{2}-\frac{1}{\alpha} & \frac{1}{2}-\frac{1}{\alpha} & \frac{1}{2\alpha} \\ \frac{1}{8} & \frac{7}{8} & -\frac{1}{2\alpha} & -\frac{1}{2}-\frac{1}{2\alpha} & \frac{1}{4\alpha} \\ \frac{5}{8} & \frac{11}{8} & -1+\frac{1}{2\alpha} & \frac{1}{2}-\frac{1}{2\alpha} & \frac{1}{4\alpha} \end{pmatrix} \tag{3.111}$$

This does not fit our expectations. This matrix does not have the block structure, which our thoughts about the triples of irreducible and indecomposable representations would suggest. We also recall, that we would like to have a matrix with the first four rows equal to the matrix from equation (3.78). Fortunately this is not the matrix, we will deal with. We only need it to find other ones.
But how much choice do we actually have for $C_2(\alpha)$? Or even better, what is the most general $C_2(\alpha)$, which we get from equation (3.110), and are there others – not in the form of eq. (3.110) –, that fulfil equation (3.97)? The answers are given by the following linear algebraic statement and during its proof.

Lemma: *Let S, $\tilde{S} \in M_{n\times n}(\mathbb{C})$ be two diagonalisable matrices, which are diagonalised to the same matrix. Then they are similar to each other and all matrices $C \in M_{n\times n}(\mathbb{C})$ fulfilling the equation*

$$CSC^{-1} = \tilde{S}\ , \tag{3.112}$$

are given by the product of a particular $C = C_1$ times a matrix $A \in M_{n\times n}(\mathbb{C})$, which commutes with S or $\tilde{S}$, and conversely any such product fulfils equation (3.112).

Remark: Equation (3.112) can also be defined with the matrices S and $\tilde{S}$ interchanged. But this does not make a difference, when we go over from $C = C_1A$ to $C^{-1} = A^{-1}C_1{}^{-1}$. Note that the inverse of A commutes with the same matrices as A itself.
It is not needed here, but it is one line to see, that any two matrices S and $\tilde{S}$, which are conjugate through a matrix C are diagonalised to the same diagonal matrix. If S is diagonalised by P,

$$PDP^{-1} = S = C\tilde{S}C^{-1}\ . \tag{3.113}$$

$\tilde{S}$ is diagonalised by $C^{-1}P$ to the same diagonal matrix D.

Proof: We have already shown the existence, because a particular solution for C can be retrieved via the eigenvectors of S and $\tilde{S}$ from equation (3.110), as described above.
Let C and C' be two matrices, which conjugate S and $\tilde{S}$ as in equation (3.112), so that we have

$$C\,S = \tilde{S}\,C\ , \tag{3.114}$$

$$C'\,S = \tilde{S}\,C'\ . \tag{3.115}$$

A is defined by $A = C^{-1}C'$ and $C' = CA$ is plugged into the last equation:

$$C\,A\,S \;=\; \tilde{S}\,C\,A\,. \tag{3.116}$$

Now use equation (3.114) on the right hand side to get

$$C\,A\,S \;=\; C\,S\,A\,. \tag{3.117}$$

Multiplying the inverse of C shows, that A commutes with S. A commutes also with $\tilde{S}$ because its inverse does. This is directly seen, when one goes through the analogous steps starting with $C = C'A^{-1}$ plugged into (3.114) and uses eq. (3.115).
For the backwards direction we need only to multiply equation (3.114) by an arbitrary matrix A, which commutes with S, from the right side. We can interchange those two matrices on the left hand side and find that CA conjugates S with $\tilde{S}$. □

We now need to find all matrices, which commute with $\mathtt{S}_{(2,\alpha)}$. We continue to call them A and multiply the commutation relation of those two matrices with the matrix $U_2(\alpha)$ and its inverse from opposite sides. We get

$$U_2{}^{-1}(\alpha)\,A\,U_2(\alpha)\,U_2{}^{-1}(\alpha)\,\mathtt{S}_{(2,\alpha)}\,U_2(\alpha) = U_2{}^{-1}(\alpha)\mathtt{S}_{(2,\alpha)}\,U_2(\alpha)\,U_2{}^{-1}(\alpha)\,A\,U_2(\alpha)\,. \tag{3.118}$$

We simplify this with the help of equation (3.109).

$$U_2{}^{-1}(\alpha)\,A\,U_2(\alpha)\,D_S \;=\; D_S\,U_2{}^{-1}(\alpha)\,A\,U_2(\alpha)\,. \tag{3.119}$$

Hence we see, that $A' := U_2{}^{-1}(\alpha)\,A\,U_2(\alpha)$ has to commute with the diagonal matrix D_S.
Which matrices commute with a diagonal matrix? Any matrix element $(A')_{ij}$ is multiplied on the right hand side of the last equation with $(D_S)_{ii}$, while on the left hand side there is the product $(A')_{ij}(D_S)_{jj}$. This is only the same on the diagonal and for off-diagonal elements, for which $(D_S)_{ii} = (D_S)_{jj}$. So the other off-diagonal elements $(A')_{ij}$ have to be zero. This tells us that all matrices A', which commute with D_S (eq. (3.99)), are given by

$$A' = \begin{pmatrix} (A')_{11} & (A')_{12} & (A')_{13} & 0 & 0 \\ (A')_{21} & (A')_{22} & (A')_{23} & 0 & 0 \\ (A')_{31} & (A')_{32} & (A')_{33} & 0 & 0 \\ 0 & 0 & 0 & (A')_{44} & (A')_{45} \\ 0 & 0 & 0 & (A')_{54} & (A')_{55} \end{pmatrix} \tag{3.120}$$

with arbitrary $(A')_{ij}$ for $1 \le i,j \le 3$ or $4 \le i,j \le 5$, so that the matrix has full rank.
Now we take this together with the definition of A' beneath equation (3.119) and the lemma to get via A all possible $C_{2,gen}(\alpha)$ (eq. (3.110)) from the one particular $C_{2,1}(\alpha)$ (eq. (3.111)):

$$C_{2,gen}(\alpha) \;=\; C_{2,1}(\alpha)\,A \;=\; C_{2,1}(\alpha)\,U_2(\alpha)\,A'\,U_2{}^{-1}(\alpha)\,. \tag{3.121}$$

Of course, with so many unknowns the matrix $C_{2,gen}(\alpha)$ gets very lengthy. Now we simply require that the first four rows of this matrix are equal to the matrix $C_2'(\alpha)$ from equation (3.78). We recall, that this was justified by the correspondence of τ-dependent and α-dependent matrices. We want to get an extension of Fuchs' approach, which goes over to the latter one, when one projects to the irreducible representations. In this case the matrix $C_2(\alpha)$ should project to $C_2'(\alpha)$, which corresponds to $C_{irr,2}(\tau)$, because the projection of $\mathtt{S}_{(2,\alpha)}$ corresponds

to $\mathtt{S}_{irr,2}$.
The fifth row then is the transpose of the following vector.

$$\begin{pmatrix} \frac{1}{2} - \frac{1}{8}(A')_{54} - \frac{1}{8}(A')_{55} \\ -\frac{1}{2} + \frac{1}{8}(A')_{54} + \frac{1}{8}(A')_{55} \\ 1 + \left(\frac{1}{2} - \frac{1}{2\alpha}\right)(A')_{54} + \frac{1}{2\alpha}(A')_{55} \\ 1 - \frac{1}{2\alpha}(A')_{54} + \left(\frac{1}{2} + \frac{1}{2\alpha}\right)(A')_{55} \\ \frac{1}{4\alpha}(A')_{54} - \frac{1}{4\alpha}(A')_{55} \end{pmatrix} . \tag{3.122}$$

In analogy to $C_{irr,p}$ this matrix should be block-diagonal This gives twice the same condition, which solves to

$$(A')_{54} = 4 - (A')_{55} . \tag{3.123}$$

3.3.4 Generalisation to Arbitrary Values of p

At last $(A')_{55}$ has been preliminary set to three because of more esthetic reasons. This way the matrix $C_2(\alpha)$ simplifies to

$$C_2(\alpha) = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} \frac{3\alpha+2}{4\alpha} & \frac{\alpha+2}{4\alpha} & -\frac{1}{4\alpha} \\ \frac{\alpha-2}{4\alpha} & \frac{3\alpha-2}{4\alpha} & \frac{1}{4\alpha} \\ \frac{3\alpha+2}{2\alpha} & \frac{5\alpha+2}{2\alpha} & -\frac{1}{2\alpha} \end{matrix} \end{pmatrix} . \tag{3.124}$$

It seems natural to have $(C_2)_{55} = 1/(2\alpha)$. Firstly, it fits to the grouping of terms, we have seen in section 3.2.4. The factors of the $1/\alpha$-terms in the last row are twice as large than in the first and second row. This is expected because of the double multiplicities in the indecomposable representation. Also the inverse of this matrix is quite simple

$$C_2^{-1}(\alpha) = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} 2 & 1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \alpha+2 & 3\alpha+2 & -\alpha \end{matrix} \end{pmatrix} . \tag{3.125}$$

This is very much in our favour, because we can now guess the inverse of $C_3(\alpha)$ with not much effort. The last row in every block is fixed looking at the result for $C_3(\alpha)$, which it would lead to. We require once more, that the first two rows of both blocks are the blocks of the matrix $C'_3(\alpha)$. We get

$$C_3^{-1}(\alpha) = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} 2 & 1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ \alpha+2 & 4\alpha+2 & -\frac{3}{2}\alpha \end{matrix} & 0 \\ 0 & 0 & \begin{matrix} 2 & 1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ 2\alpha+2 & 5\alpha+2 & -\frac{3}{2}\alpha \end{matrix} \end{pmatrix} . \tag{3.126}$$

Its inverse is

$$C_3(\alpha) = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} \frac{2}{3}+\frac{1}{3\alpha} & \frac{1}{6}+\frac{1}{3\alpha} & -\frac{1}{6\alpha} \\ \frac{1}{3}-\frac{1}{3\alpha} & \frac{5}{6}-\frac{1}{3\alpha} & \frac{1}{6\alpha} \\ \frac{4}{3}+\frac{2}{3\alpha} & \frac{7}{3}+\frac{2}{3\alpha} & -\frac{1}{3\alpha} \end{matrix} & 0 \\ 0 & 0 & \begin{matrix} \frac{5}{6}+\frac{1}{3\alpha} & \frac{1}{3}+\frac{1}{3\alpha} & -\frac{1}{6\alpha} \\ \frac{1}{6}-\frac{1}{3\alpha} & \frac{2}{3}-\frac{1}{3\alpha} & \frac{1}{6\alpha} \\ \frac{5}{3}+\frac{2}{3\alpha} & \frac{8}{3}+\frac{2}{3\alpha} & -\frac{1}{3\alpha} \end{matrix} \end{pmatrix} . \tag{3.127}$$

This result, $C_3(\alpha)$, also determines the matrix S_3. This matrix S_3 gives the correct fusion rules through our extended block-diagonalisation method. The caculations are given in appendix C.3.

Another choice of $(A')_{55}$, which we considered, is $(A')_{55} = 1$. This gives a very similar inverse of $C_2(\alpha)$ and we can also guess the inverse of a potential $C_3(\alpha)$, but this though similar has not the required same elements as $C_3'(\alpha)$. Actually the rows equal to $C_3'(\alpha)$ are interchanged and in these rows the first two columns are exchanged in each block, while in the additional rows there are also differences of one or the other minus sign. Remarkably, this version gives an S-matrix and fusion rules that are qualitatively correct. Only the multiplicities are wrong and turn out to be fractional.

Here we can directly do the step to arbitrary p. Once more we find the inverse of $C_p(\alpha)$ first and thus have also $C_p(\alpha)$ itself:

$$C_p{}^{-1}(\alpha) = \mathbb{1}_{2\times 2} \oplus \bigoplus_{s=1}^{p-1} C_{p,s}{}^{-1}(\alpha) , \tag{3.128}$$

$$C_{p,s}{}^{-1}(\alpha) = \begin{pmatrix} 2 & 1 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2} \\ s\alpha+2 & (p+s)\alpha+2 & -\frac{p}{2}\alpha \end{pmatrix} ,$$

$$C_p(\alpha) = \mathbb{1}_{2\times 2} \oplus \bigoplus_{s=1}^{p-1} C_{p,s}(\alpha) , \tag{3.129}$$

$$C_{p,s}(\alpha) = \begin{pmatrix} \frac{p+s}{2p}+\frac{1}{p\alpha} & \frac{s}{2p}+\frac{1}{p\alpha} & -\frac{1}{2p\alpha} \\ \frac{p-s}{2p}-\frac{1}{p\alpha} & \frac{2p-s}{2p}-\frac{1}{p\alpha} & \frac{1}{2p\alpha} \\ \frac{p+s}{p}+\frac{2}{p\alpha} & \frac{2p+s}{p}+\frac{2}{p\alpha} & -\frac{1}{p\alpha} \end{pmatrix} .$$

A block-wise calculation shows that this is exactly the matrix we looked for. With the blocks of $\mathtt{S}_{(p,\alpha)}$ (eq. (3.10)) we have to calculate the following expressions:

$$S(p)_{0,l} C_{p,l}{}^{-1}(\alpha) \tag{3.130}$$

$$C_{p,s}(\alpha) S(p)_{s,0} \tag{3.131}$$

$$C_{p,s}(\alpha) S(p)_{s,l} C_{p,l}{}^{-1}(\alpha) \tag{3.132}$$

The block $S(p)_{0,0}$ is not touched at all. $S(p)_{0,l}$ and $S(p)_{s,0}$, for $0 < s, l < p$, are also not changed by the multiplication. And the last product gives

$$S(p)_{s,j} = \frac{2}{\sqrt{2p}}(-1)^{p+j+s} \times \tag{3.133}$$
$$\begin{pmatrix} \frac{s}{p}\cos(\pi\frac{sj}{p}) + \frac{p-j}{p}\sin(\pi\frac{sj}{p}) & \frac{s}{p}\cos(\pi\frac{sj}{p}) - \frac{j}{p}\sin(\pi\frac{sj}{p}) & 0 \\ \frac{p-s}{p}\cos(\pi\frac{sj}{p}) - \frac{p-j}{p}\sin(\pi\frac{sj}{p}) & \frac{p-s}{p}\cos(\pi\frac{sj}{p}) + \frac{j}{p}\sin(\pi\frac{sj}{p}) & 0 \\ 2\cos(\pi\frac{sj}{p}) + 2\sin(\pi\frac{sj}{p}) & 2\cos(\pi\frac{sj}{p}) + 2\sin(\pi\frac{sj}{p}) & -\sin(\pi\frac{sj}{p}) \end{pmatrix} .$$

This is now the last S-matrix $\mathtt{S}_p$ of this thesis. The (pre-)fusion coefficients are given in terms of its matrix elements by the "generalised" Verlinde formula again (cf. (3.56)):

$$N_{p,I} = S_p K_p M_{p,I} (K_p)^{-1} S_p \ . \tag{3.134}$$

3.3.5 Projection of the Extended Block-Diagonalisation Method on Irreducible Representations

We have now all ingredients to carry through calculations for any value of p in our extension of the method of Fuchs et. al.. P_p is also in the general case invertible because with the invertible $\mathtt{S}_{(p,\alpha)}$ also S_p has to be invertible and K_p was constructed as a full rank matrix.

There are two open tasks for the rest of this chapter. Firstly, we show in this section, that the calculations in the extended block-diagonalisation method, presented in this section, lead to the same results for the irreducible representations as the original version for all $p \geq 2$. Secondly, the next section deals with the proof, that the extended block-diagonalisation method moreover gives the same fusion coefficients for all irreducible and indecomposable representations as the method of M. Flohr in section 3.1.

For the former task we change the sequence of the representations from the groups of three – two irreducible and one indecomposable representations – to the following one:

$$[h_{1,p}]\,,\ [h_{1,2p}]\,,\ [h_{1,1}]\,,\ [h_{1,2p+1}]\,,\ [h_{1,2}]\,,\ [h_{1,2p+2}]\,,\ldots,\ [h_{1,p-1}]\,,\ [h_{1,3p-1}]\,, \tag{3.135}$$
$$\left[\widetilde{h_{1,p+1}}\right]\,,\ \left[\widetilde{h_{1,p+2}}\right]\,,\ldots,\ \left[\widetilde{h_{1,2p-1}}\right]$$

with the indecomposable representations all put to the end. This leads to the permutation of both, rows and columns, in the matrices S_p, K_p, P_p, $M_{p,I}$ and finally $N_{p,I}$. Also the sequence of the latter two groups of matrices is changed, as the index I is affected by the same permutation. The reason is the form all these matrices take after the permutation. All the zeros, which we inserted in some matrices and consequently appeared in other matrices are grouped together with the indecomposable representations in the last columns.

We introduce the following notation, which tells us that a matrix has some form without specifying all matrix elements or the size of the matrix. The matrix S_p has now the form (cf. eq. (3.133)):

$$S_p \triangleq \begin{pmatrix} \boxed{S_{irr,p}} & 0 \\ \multicolumn{2}{|c|}{\qquad} \end{pmatrix} . \tag{3.136}$$

This states that the box on the upper left contains exactly the matrix $S_{irr,p}$ (eq. (3.52)), the box at the bottom contains the other a priori non-zero elements of S_p and on the upper right all matrix elements are zero. With this notation we give the statement, which we want to proof.

Proposition: *The fusion coefficients matrices $N_{p,I}$ each contain the coefficients of the "small" fusion algebra $N_{irr,p,I}$ for $0 < I \leq 2p$ in the subsequent form:*

$$N_{p,I} \triangleq \begin{pmatrix} \boxed{N_{irr,p,I}} & 0 \\ \multicolumn{2}{|c|}{\quad} \end{pmatrix} . \tag{3.137}$$

Proof: The only coefficients of the matrix K_p, which are different from zero and do not come from the matrix $K_{irr,p}$, are the additional diagonal matrix elements. The permutation of rows and columns leaves them on the diagonal and assembles them in a block, which is equal to the unit matrix in $p - 1$ dimensions (cf. eq. (3.104)):

$$K_p \triangleq \begin{pmatrix} \boxed{K_{irr,p}} & 0 \\ 0 & \boxed{\mathbb{1}} \end{pmatrix} . \tag{3.138}$$

The matrix P consequently looks like

$$P_p = S_p K_p \triangleq \begin{pmatrix} \boxed{P_{irr,p}} & 0 \\ \multicolumn{2}{|c|}{\quad} \end{pmatrix} . \tag{3.139}$$

We also note that the $(p-1) \times (p-1)$ elements in the very lower right corner are unchanged by K_p and so equal to those elements of S_p.

We construct the matrices $M_{p,I}$ for $0 < I \leq 2p$ in the new sequence. Each block defined in equation (3.89) has (in the sequence of representations we used there) the element $(M_{p,I,n})_{13} = (P_p)_I^{3n+2}$. These elements are of interest because the permutation to the new sequence of representations bring them from the 5th, 8th, 11th etc. column, where they are not on the diagonal to a new position in the last $p - 1$ columns and the first $2p$ rows, which need to be zero, as we will see next. Fortunately all of them are equal to elements of P_p, for which exactly the same argument tells us, that they actually are zero: In the old sequence they are the 5th, 8th, 11th etc. column and off-diagonal. So after permutation they lie in the part of P_p marked in equation (3.139) to have only zero elements. Hence $M_{p,I}$ appears in the form

$$M_{p,I} \triangleq \begin{pmatrix} M_{irr,p,I} & 0 \\ 0 & \Box \end{pmatrix} . \tag{3.140}$$

We also need to know the form of the inverse of P_p. We look at the definition of the inverse on the level of matrix elements:

$$\sum_{k=1}^{3p-1} (P_p)_{ik} (P_p)^{-1}_{kj} = \delta_{ij} \; . \tag{3.141}$$

For $0 < i \leq 2p$ we can split the sum into two parts:

$$\begin{aligned} \delta_{ij} &= \sum_{k=1}^{2p} (P_p)_{ik} (P_p)^{-1}_{kj} + \sum_{k=2p+1}^{3p-1} (P_p)_{ik} (P_p)^{-1}_{kj} \\ &= \sum_{k=1}^{2p} (P_{irr,p})_{ik} (P_p)^{-1}_{kj} + 0 \; , \end{aligned} \tag{3.142}$$

where we used eq. (3.139) in the last line: The first $2p$ rows of P_p are the same as those of $P_{irr,p}$ filled up with zeros. From these equations for $0 < j \leq 2p$ it follows, that $(P_p)^{-1}_{mn} = (P_{irr,p})^{-1}_{mn}$ for $0 < m, n \leq 2p$ because of the uniqueness of the inverse, which eq. (3.142) defines – or in other words because the system of equations, which is given there, with $2p$ unknowns and $2p$ equations is determined due to the full rank of $P_{irr,p}$. So $(P_p)^{-1}$ has the form

$$(P_p)^{-1} \triangleq \begin{pmatrix} (P_{irr,p})^{-1} & 0 \\ \multicolumn{2}{c}{\Box} \end{pmatrix} \tag{3.143}$$

and we end up with the product (see eqns. (3.139) and (3.140)) for $0 < I \leq 2p$

$$\begin{aligned} N_{p,I} &= P_p M_{p,I} (P_p)^{-1} \\ &\triangleq \begin{pmatrix} P_{irr,p} & 0 \\ \multicolumn{2}{c}{\Box} \end{pmatrix} \begin{pmatrix} M_{irr,p,I} & 0 \\ 0 & \Box \end{pmatrix} \begin{pmatrix} (P_{irr,p})^{-1} & 0 \\ \multicolumn{2}{c}{\Box} \end{pmatrix} , \end{aligned} \tag{3.144}$$

which has the form eq. (3.137). □

We also see, that a projection to the first four components in the sequence of representations, we have temporarily used just now, is the one we already mentioned in the introduction and also afterwards. Indeed, it transfers all these matrices S_p, K_p, P_p, $M_{p,I}$ and $N_{p,I}$, which appear in the extended block-diagonalisation method, to the matrices of its archetype for irreducible representations, $S_{irr,p}$, $K_{irr,p}$, $P_{irr,p}$, $M_{irr,p,I}$ and $N_{irr,p,I}$. We can now again forget about the sequence of eq. (3.135) and go back to the sequence of the vector $\boldsymbol{\chi}_p(\alpha)$ (eq. (3.4)), which is more convenient for the other calculations, because the matrices fragment into 3×3 blocks. Furthermore there we actually have got the name-giving block-diagonalisation.

3.4 Equivalence of both Approaches

The α-Verlinde formula, which we have learned about in section 3.1, expresses the possibility to simultaneously diagonalise the set of matrices $\mathtt{N}_{p,I}(\alpha)$. Unfortunately these are not the matrices of fusion coefficients as in the case of rational conformal field theories. They rather only become matrices of fusion coefficients after the limit $\alpha \to 0$ has been taken – to be precise we can map these coefficients then to the proposed true fusion coefficients in an unambiguous way. But still it gives us the possibility to write the equation for the matrix elements of $\mathtt{N}_{p,I}(\alpha)$ (3.16) as

$$\mathtt{N}_{p,I}(\alpha) = \mathtt{S}_{(p,\alpha)} M_{diag,\alpha,I} {\mathtt{S}_{(p,\alpha)}}^{-1} , \tag{3.145}$$

with $M_{diag,\alpha,I}$ given by

$$M_{diag,\alpha,I} = diag\left(\frac{{\mathtt{S}_{(p,\alpha)}}_I^1}{{\mathtt{S}_{(p,\alpha)}}_3^1}, \frac{{\mathtt{S}_{(p,\alpha)}}_I^2}{{\mathtt{S}_{(p,\alpha)}}_3^2}, \dots, \frac{{\mathtt{S}_{(p,\alpha)}}_I^{3p-1}}{{\mathtt{S}_{(p,\alpha)}}_3^{3p-1}} \right) , \tag{3.146}$$

One can also introduce the matrix $K_{diag,\alpha}$ defined as the diagonal matrix with the vacuum row of $\mathtt{S}_{(p,\alpha)}$ on the diagonal,

$$K_{diag,\alpha} = diag\left(\frac{1}{{\mathtt{S}_{(p,\alpha)}}_3^1}, \frac{1}{{\mathtt{S}_{(p,\alpha)}}_3^2}, \dots, \frac{1}{{\mathtt{S}_{(p,\alpha)}}_3^{3p-1}} \right) , \tag{3.147}$$

which of course commutes in equation (3.145) with the matrices $M_{diag,\alpha,I}$, because these are also diagonal. In this way we are able to see it parallel to our earlier notation. $M_{diag,\alpha,I}$ is given by the I-th line of the product $\mathtt{S}_{(p,\alpha)} K_{diag,\alpha}$ and

$$\mathtt{N}_{p,I}(\alpha) = \mathtt{S}_{(p,\alpha)} K_{diag,\alpha} M_{diag,\alpha,I} {K_{diag,\alpha}}^{-1} {\mathtt{S}_{(p,\alpha)}}^{-1} . \tag{3.148}$$

But this is only to give a more rounded picture. We now get to the second central proposition in this thesis.

Proposition: *The fusion coefficients calculated with the α-Verlinde formula are the same as the ones calculated with the extended block-diagonalisation method:*

$$\lim_{\alpha \to 0} {\mathtt{N}_{ij}}^k(\alpha) = (N_{p,I})_j^k . \tag{3.149}$$

Proof: We plug equations (3.145)and (3.134) into eq. (3.149) and have

$$\Leftrightarrow \lim_{\alpha\to 0}\left(\mathtt{S}_{(p,\alpha)} M_{diag,\alpha,I}\mathtt{S}_{(p,\alpha)}\right) = S_p K_p M_{p,I}(K_p)^{-1} S_p \ . \tag{3.150}$$

We insert two unit matrices into the left hand side of this equation:

$$\mathtt{S}_{(p,\alpha)} M_{diag,\alpha,I}\mathtt{S}_{(p,\alpha)} = \mathtt{S}_{(p,\alpha)} E_{p,\alpha} {E_{p,\alpha}}^{-1} M_{diag,\alpha,I} E_{p,\alpha} {E_{p,\alpha}}^{-1}\mathtt{S}_{(p,\alpha)}$$

with $E_{p,\alpha}$ defined as

$$E_{p,\alpha} := {\mathtt{S}_{(p,\alpha)}}^{-1} S_p = \mathtt{S}_{(p,\alpha)} S_p \ , \tag{3.151}$$

in order to have

$$\mathtt{S}_{(p,\alpha)} M_{diag,\alpha,I}\mathtt{S}_{(p,\alpha)} == S_p {E_{p,\alpha}}^{-1} M_{diag,\alpha,I} E_{p,\alpha} S_p \ , \tag{3.152}$$

With a blockdiagonal ansatz $E_{p,\alpha}$, one can directly calculate the blocks as in the equation (3.151):

$$\begin{aligned} E_{p,\alpha} &= \mathbb{1}_{2\times 2} \oplus \bigoplus_{s=1}^{p-1} (E_{p,\alpha})_s & (3.153) \\ (E_{p,\alpha})_s &= S(p,\alpha)_{s,l} S(p)_{s,j} = \begin{pmatrix} \frac{s}{p} - \frac{2}{p\alpha} & \frac{s}{p} - \frac{2}{p\alpha} & \frac{1}{p\alpha} \\ \frac{p-s}{p} + \frac{2}{p\alpha} & \frac{p-s}{p} + \frac{2}{p\alpha} & -\frac{1}{p\alpha} \\ 2-(p-s)\alpha & 2+s\alpha & 0 \end{pmatrix} . & (3.154) \end{aligned}$$

where we used the blocks from eqns. (3.10) and (3.133).
We are going to show that the product ${E_{p,\alpha}}^{-1} M_{diag,\alpha,I} E_{p,\alpha}$ has a well defined limit for $\alpha \to 0$. This is not clear. For the whole term at the end of equation (3.152) this limit is well defined. They are the fusion coefficients ${\mathtt{N}_{ij}}^k(\alpha)$. But still singular terms in the mentioned product could drop out through the multiplication of S_p from both sides.
We simply calculate first the matrices $M_{diag,\alpha,I}$. We need to consider the following cases. For $I = 1,2$ the matrices $M_{diag,\alpha,I}$ differ by two minus signs. There are three more groups to be distinguished, which belong each to one row of the blocks of $\mathtt{S}_{(p,\alpha)}$. We use again the same abbreviations as for $\mathtt{S}_{(p,\alpha)}$ in eq. (3.10).

$$M_{diag,\alpha,I} = (M_{diag,\alpha,I})_0 \oplus \bigoplus_{l=1}^{p-1} (M_{diag,\alpha,I})_l \tag{3.155}$$

$$I = 1,2: \quad (M_{diag,\alpha,I})_0 = \begin{pmatrix} p & 0 \\ 0 & (-1)^I p \end{pmatrix}$$

$$(M_{diag,\alpha,I})_l = (-1)^{I(p-l)} \begin{pmatrix} \frac{-p\alpha}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & 0 & 0 \\ 0 & \frac{-p\alpha}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$I = 3,6,\ldots: \quad (M_{diag,\alpha,I})_0 = \begin{pmatrix} I & 0 \\ 0 & (-1)^I I \end{pmatrix}$$

$$(M_{diag,\alpha,I})_l = \begin{pmatrix} (-1)^{I+1}\frac{I\alpha\mathfrak{c}_{Il}+2\mathfrak{s}_{Il}}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & 0 & 0 \\ 0 & (-1)^{I+1}\frac{I\alpha\mathfrak{c}_{Il}+2\mathfrak{s}_{Il}}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & 0 \\ 0 & 0 & (-1)^{I+1}\frac{\mathfrak{s}_{Il}}{\mathfrak{s}_{1l}} \end{pmatrix}$$

$$I = 4,7,\dots: \qquad (M_{diag,\alpha,I})_0 = \begin{pmatrix} p-I & 0 \\ 0 & (-1)^I(p-I) \end{pmatrix}$$

$$(M_{diag,\alpha,I})_l = \begin{pmatrix} (-1)^{I+1}\frac{(p-I)\alpha\mathfrak{c}_{Il}-2\mathfrak{s}_{Il}}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & 0 & 0 \\ 0 & (-1)^{I+1}\frac{(p-I)\alpha\mathfrak{c}_{Il}-2\mathfrak{s}_{Il}}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & 0 \\ 0 & 0 & (-1)^I\frac{\mathfrak{s}_{Il}}{\mathfrak{s}_{1l}} \end{pmatrix}$$

$$I = 5,8,\dots: \qquad (M_{diag,\alpha,I})_0 = \begin{pmatrix} 2p & 0 \\ 0 & (-1)^I 2p \end{pmatrix}$$

$$(M_{diag,\alpha,I})_l = \begin{pmatrix} (-1)^I p\alpha\frac{(p-l)\alpha\mathfrak{s}_{Il}-2\mathfrak{c}_{Il}}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & 0 & 0 \\ 0 & (-1)^{I+1}p\alpha\frac{l\alpha\mathfrak{s}_{sl}+2\mathfrak{c}_{Il}}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

For these four cases we can now calculate the product

$$\tilde{M}_{I,\alpha} = E_{p,\alpha}{}^{-1} M_{diag,\alpha,I} E_{p,\alpha} = (\tilde{M}_{I,\alpha})_0 \oplus \bigoplus_{l=1}^{p-1} \left[(-1)^I (\tilde{M}_{I,\alpha})_l\right] \tag{3.156}$$

$$(\tilde{M}_{I,\alpha})_0 = (M_{diag,\alpha,I})_0$$

$$I \;=\; 1,2:$$

$$(\tilde{M}_{I,\alpha})_l = (-1)^{(p+l)} \begin{pmatrix} -\frac{l_1}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & -\frac{l_1}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & 0 \\ \frac{l_2}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & \frac{l_1}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & 0 \\ 0 & 0 & -\frac{\alpha p}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} \end{pmatrix}$$

$$I \;=\; 3,6,9,\dots:$$

$$(\tilde{M}_{I,\alpha})_l = \begin{pmatrix} -\frac{(Il_1\mathfrak{c}_{Il}+2p\mathfrak{s}_{Il})\mathfrak{s}_{1l}-l_2\mathfrak{c}_{1l}\mathfrak{s}_{Il}}{p\mathfrak{s}_{1l}(\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l})} & -\frac{l_1(I\mathfrak{s}_{1l}\mathfrak{c}_{Il}-\mathfrak{c}_{1l}\mathfrak{s}_{Il})}{p\mathfrak{s}_{1l}(\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l})} & 0 \\ \frac{l_1(I\mathfrak{s}_{1l}\mathfrak{c}_{Il}-\mathfrak{c}_{1l}\mathfrak{s}_{Il})}{p\mathfrak{s}_{1l}(\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l})} & \frac{(Il_2\mathfrak{c}_{Il}-2p\mathfrak{s}_{Il})\mathfrak{s}_{1l}-l_1\mathfrak{c}_{1l}\mathfrak{s}_{Il}}{p\mathfrak{s}_{1l}(\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l})} & 0 \\ 0 & 0 & -\frac{I\alpha\mathfrak{c}_{Il}+2\mathfrak{s}_{Il}}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} \end{pmatrix}$$

$$I \;=\; 4,7,10,\dots:$$

$$(\tilde{M}_{I,\alpha})_l = \begin{pmatrix} -\frac{((p-I)l_1\mathfrak{c}_{Il}-2p\mathfrak{s}_{Il})\mathfrak{s}_{1l}+l_2\mathfrak{c}_{1l}\mathfrak{s}_{Il}}{p\mathfrak{s}_{1l}(\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l})} & -\frac{l_1((p-I)\mathfrak{s}_{1l}\mathfrak{c}_{Il}+\mathfrak{c}_{1l}\mathfrak{s}_{Il})}{p\mathfrak{s}_{1l}(\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l})} & 0 \\ \frac{l_1((p-I)\mathfrak{s}_{1l}\mathfrak{c}_{Il}+\mathfrak{c}_{1l}\mathfrak{s}_{Il})}{p\mathfrak{s}_{1l}(\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l})} & \frac{((p-I)l_2\mathfrak{c}_{Il}+2p\mathfrak{s}_{Il})\mathfrak{s}_{1l}+l_1\mathfrak{c}_{1l}\mathfrak{s}_{Il}}{p\mathfrak{s}_{1l}(\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l})} & 0 \\ 0 & 0 & -\frac{(p-I)\alpha\mathfrak{c}_{Il}-2\mathfrak{s}_{Il}}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} \end{pmatrix}$$

$$I \;=\; 5,8,11,\dots:$$

$$(\tilde{M}_{I,\alpha})_l = \begin{pmatrix} -\frac{2l_1(\mathfrak{c}_{Il}+\mathfrak{s}_{Il})}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & -\frac{2l_1(\mathfrak{c}_{Il}+\mathfrak{s}_{Il})}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & \frac{l_1\mathfrak{s}_{Il}}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} \\ \frac{2l_2(\mathfrak{c}_{Il}+\mathfrak{s}_{Il})}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & \frac{2l_2(\mathfrak{c}_{Il}+\mathfrak{s}_{Il})}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & -\frac{l_2\mathfrak{s}_{Il}}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} \\ \frac{l_3(2-l)\alpha p\mathfrak{s}_{Il}}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & \frac{l_3(2-l)\alpha p\mathfrak{s}_{Il}}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} & -\frac{p\alpha(2\mathfrak{c}_{Il}-(l_3-\alpha l)\mathfrak{s}_{Il})}{\alpha\mathfrak{c}_{1l}+2\mathfrak{s}_{1l}} \end{pmatrix}$$

with $l_1 = 2 + l\alpha$, $l_2 = 2-(p-l)\alpha$ and $l_3 = 2+(p-l)\alpha$. Hence these matrices are well-defined in the limit of $\alpha \to 0$ and we can take the limit of $\tilde{M}_I$ rather than of the whole product in equation (3.152):

$$(N_{p,I})_j{}^k = S_p \lim_{\alpha\to 0}\left(E_{p,\alpha}{}^{-1} M_{diag,\alpha,I} E_{p,\alpha}\right) S_p \;. \tag{3.157}$$

We now continue with the right hand side of equation (3.150) and see that we need to show that

$$K_p M_{p,I} (K_p)^{-1} = \lim_{\alpha\to 0}\left(E_{p,\alpha}{}^{-1} M_{diag,\alpha,I} E_{p,\alpha}\right) \;. \tag{3.158}$$

First we need to calculate the matrix K_p and P_p in order to get the matrices afterwards with eqns. (3.104) (we have set $k_{s1} = k_{s2} = 0$) and (3.96), respectively:

$$\begin{aligned} K_p &= K_{p,I} \oplus \bigoplus_{l=1}^{p-1} K_{p,l} \qquad (3.159) \\ K_{p,0} &= \begin{pmatrix} \sqrt{2p^3} & 0 \\ 0 & (-1)^{p+1}\sqrt{2p^3} \end{pmatrix} \\ K_{p,l} &= \begin{pmatrix} (-1)^{p+l+1}\sqrt{\frac{p}{2}}\,\mathfrak{s}_{1l} & (-1)^{p+l}\sqrt{\frac{2}{p^3}}\,(\mathfrak{c}_{1l} - l\mathfrak{s}_{1l}) & 0 \\ (-1)^{p+l}\sqrt{\frac{p}{2}}\,\mathfrak{s}_{1l} & (-1)^{p+l+1}\sqrt{\frac{2}{p^3}}\,(\mathfrak{c}_{1l} + (p-l)\mathfrak{s}_{1l}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

$$P_p = \begin{pmatrix} P(p)_{0,0} & P(p)_{0,1} & \dots & P(p)_{0,p-1} \\ P(p)_{1,0} & P(p)_{1,1} & \dots & P(p)_{1,p-1} \\ \vdots & \vdots & \ddots & \vdots \\ P(p)_{p-1,0} & P(p)_{p-1,1} & \dots & P(p)_{p-1,p-1} \end{pmatrix} \qquad (3.160)$$

$$\begin{aligned} P(p)_{0,0} &= \frac{1}{\sqrt{2p}} \begin{pmatrix} p & (-1)^{p+1}p \\ p & -p \end{pmatrix} \\ P(p)_{0,l} &= \frac{2}{\sqrt{2p}} \begin{pmatrix} 0 & (-1)^{p+l+1}\frac{2}{p}\mathfrak{s}_{1l} & 0 \\ 0 & -\frac{2}{p}\mathfrak{s}_{1l} & 0 \end{pmatrix} \\ P(p)_{s,0} &= \frac{1}{\sqrt{2p}} \begin{pmatrix} s & (-1)^{s+1}s \\ p-s & (-1)^{s+1}(p-s) \\ 2p & 2(-1)^{s+1}p \end{pmatrix} \end{aligned}$$

$$\begin{aligned} P(p,\alpha)_{s,l} &= \times \\ &\begin{pmatrix} (-1)^{s+1}\frac{\mathfrak{s}_{sl}}{\mathfrak{s}_{1l}} & (-1)^{s+1}\frac{2}{p^2}\,(s\mathfrak{c}_{sl}\mathfrak{s}_{1l} - \mathfrak{s}_{sl}\mathfrak{c}_{1l}) & 0 \\ (-1)^{s}\frac{\mathfrak{s}_{sl}}{\mathfrak{s}_{1l}} & (-1)^{s+1}\frac{2}{p^2}\,((p-s)\mathfrak{c}_{sl}\mathfrak{s}_{1l} + \mathfrak{s}_{sl}\mathfrak{c}_{1l}) & 0 \\ 0 & (-1)^{s+1}\frac{4}{p}\,(\mathfrak{c}_{sl} + \mathfrak{s}_{sl})\,\mathfrak{s}_{1l} & (-1)^{p+s+l+1}\sqrt{\frac{2}{p}}\mathfrak{s}_{sl} \end{pmatrix}. \end{aligned}$$

One can simply read off the matrices $M_{p,I}$ from the rows of this matrix (see eq. (3.89)). We plug these matrices into the left hand side of equation (3.158):

$$\begin{aligned} \tilde{M}_I &= K_p M_{p,I} (K_p)^{-1} = (\tilde{M}_I)_0 \oplus \bigoplus_{l=1}^{p-1} (\tilde{M}_I)_l \qquad (3.161) \\ &\quad (\tilde{M}_I)_0 = (M_{diag,\alpha,I})_0 \\ I &= 1,2: \\ &\quad (\tilde{M}_{I,\alpha})_l = (-1)^{I(p+l)} \begin{pmatrix} -\frac{1}{\mathfrak{s}_{1l}} & -\frac{1}{\mathfrak{s}_{1l}} & 0 \\ \frac{1}{\mathfrak{s}_{1l}} & \frac{1}{\mathfrak{s}_{1l}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
I &= 3,6,9,\dots : \\
&(\tilde{M}_I)_l = (-1)^I \begin{pmatrix} -\frac{(I\mathfrak{c}_{Il}+p\mathfrak{s}_{Il})\mathfrak{s}_{1l}-\mathfrak{c}_{1l}\mathfrak{s}_{Il}}{p{\mathfrak{s}_{1l}}^2} & -\frac{I\mathfrak{s}_{1l}\mathfrak{c}_{Il}-\mathfrak{c}_{1l}\mathfrak{s}_{Il}}{p{\mathfrak{s}_{1l}}^2} & 0 \\ \frac{I\mathfrak{s}_{1l}\mathfrak{c}_{Il}-\mathfrak{c}_{1l}\mathfrak{s}_{Il}}{p{\mathfrak{s}_{1l}}^2} & \frac{(I\mathfrak{c}_{Il}-p\mathfrak{s}_{Il})\mathfrak{s}_{1l}-\mathfrak{c}_{1l}\mathfrak{s}_{Il}}{p{\mathfrak{s}_{1l}}^2} & 0 \\ 0 & 0 & -\frac{\mathfrak{s}_{Il}}{\mathfrak{s}_{1l}} \end{pmatrix} \\
I &= 4,7,10,\dots : \\
&(\tilde{M}_I)_l = (-1)^I \begin{pmatrix} -\frac{((p-I)\mathfrak{c}_{Il}-p\mathfrak{s}_{Il})\mathfrak{s}_{1l}+\mathfrak{c}_{1l}\mathfrak{s}_{Il}}{p\mathfrak{s}_{1l}\mathfrak{s}_{1l}} & -\frac{(p-I)\mathfrak{s}_{1l}\mathfrak{c}_{Il}+\mathfrak{c}_{1l}\mathfrak{s}_{Il}}{p{\mathfrak{s}_{1l}}^2} & 0 \\ \frac{(p-I)\mathfrak{s}_{1l}\mathfrak{c}_{Il}+\mathfrak{c}_{1l}\mathfrak{s}_{Il}}{p\mathfrak{s}_{1l}\mathfrak{s}_{1l}} & \frac{((p-I)\mathfrak{c}_{Il}+p\mathfrak{s}_{Il})\mathfrak{s}_{1l}+\mathfrak{c}_{1l}\mathfrak{s}_{Il}}{p{\mathfrak{s}_{1l}}^2} & 0 \\ 0 & 0 & -\frac{\mathfrak{s}_{Il}}{\mathfrak{s}_{1l}} \end{pmatrix} \\
I &= 5,8,11,\dots : \\
&(\tilde{M}_I)_l = (-1)^I \begin{pmatrix} -\frac{2(\mathfrak{c}_{Il}+\mathfrak{s}_{Il})}{\mathfrak{s}_{1l}} & -\frac{2(\mathfrak{c}_{Il}+\mathfrak{s}_{Il})}{\mathfrak{s}_{1l}} & \frac{\mathfrak{s}_{Il}}{\mathfrak{s}_{1l}} \\ \frac{2(\mathfrak{c}_{Il}+\mathfrak{s}_{Il})}{\mathfrak{s}_{1l}} & \frac{2(\mathfrak{c}_{Il}+\mathfrak{s}_{Il})}{\mathfrak{s}_{1l}} & -\frac{\mathfrak{s}_{Il}}{\mathfrak{s}_{1l}} \\ 0 & 0 & 0 \end{pmatrix} .
\end{aligned}
$$

Finally we compare the matrices $\tilde{M}_I$ with the respective matrices $\tilde{M}_{I,\alpha}$, which constitute the right hand side of said equation (3.158), and notice that the limit of the latter matrices for $\alpha \to 0$ yields the former ones. □

We have promised already in the introduction, that we are going to see for any matrix in one of the presented methods the corresponding matrices in the other ones during this thesis and the direct connection between them. For the latter, the direct connection between $M_{diag,\alpha,I}$ and $M_{p,I}$ and between $K_{diag,\alpha}$ and K_p can be clarified a bit more. With a small step we can do this. We take in eq. (3.158) the matrix K_p and its inverse to the other side. As they do not depend on α, we can take them into the limit.

$$M_{p,I} = \lim_{\alpha\to 0} \left((K_p)^{-1} {E_{p,\alpha}}^{-1} M_{diag,\alpha,I} E_{p,\alpha} K_p\right) . \tag{3.162}$$

This gives us already the relation between $M_{diag,\alpha,I}$ and $M_{p,I}$, but we want to have the other one simultaneously, as we look at the Verlinde formula.
$K_{diag,\alpha,I}$ commutes with $M_{diag,\alpha,I}$. So if we insert once the unit matrix, we get

$$M_{p,I} = \lim_{\alpha\to 0} \left((K_p)^{-1} {E_{p,\alpha}}^{-1} K_{diag,\alpha,I} M_{diag,\alpha,I} K^{-1}_{diag,\alpha,I} E_{p,\alpha} K_p\right) . \tag{3.163}$$

We then define the matrix

$$F_{p,\alpha} := K^{-1}_{diag,\alpha,I} E_{p,\alpha} K_p . \tag{3.164}$$

This can be easily calculated with equations (3.147), (3.151) and (3.104):

$$F_{p,\alpha} = \mathbb{1}_{2\times 2} \oplus \bigoplus_{j=1}^{p-1} (F_{p,\alpha})_j \tag{3.165}$$

$$(F_{p,\alpha})_j = \frac{1}{p^3\alpha^2} \begin{pmatrix} 0 & 2(j\alpha-2)\mathfrak{s}_{1j}\,(\mathfrak{c}_{1j}\alpha+2\mathfrak{s}_{1j}) & (-1)^{j+p+1}\sqrt{2p}\,(\mathfrak{c}_{1j}\alpha+2\mathfrak{s}_{1j}) \\ 0 & 2((p-j)\alpha+2)\mathfrak{s}_{1j}\,(\mathfrak{c}_{1j}\alpha+2\mathfrak{s}_{1j}) & (-1)^{j+p}\sqrt{2p}\,(\mathfrak{c}_{1j}\alpha+2\mathfrak{s}_{1j}) \\ 1 & -2p\alpha\mathfrak{s}_{1j}\,(\mathfrak{c}_{1j}\alpha+2\mathfrak{s}_{1j}) & 0 \end{pmatrix} .$$

With these matrices we have the following, derived from the α-Verlinde formula ((3.145)):

$$\mathtt{N}_{p,I}(\alpha) = \tag{3.166}$$
$$\underbrace{\mathtt{S}_{(p,\alpha)} E_{p,\alpha}}_{=S_p} \underbrace{{E_{p,\alpha}}^{-1} K_{diag,\alpha,I} F_{p,\alpha}}_{=K_p} {F_{p,\alpha}}^{-1} M_{diag,\alpha,I} F_{p,\alpha} \underbrace{{F_{p,\alpha}}^{-1} {K_{diag,\alpha,I}}^{-1} 1 E_{p,\alpha}}_{{K_p}^{-1}} \underbrace{{E_{p,\alpha}}^{-1} {\mathtt{S}_{(p,\alpha)}}^{-1}}_{=S_p} .$$

One find that the three matrices in the middle have a regular limit

$$\lim_{\alpha \to 0} \left(F_{p,\alpha}{}^{-1} M_{diag,\alpha,I} F_{p,\alpha} \right) = M_{p,I} \tag{3.167}$$

and has the "generalised" Verlinde formula for the extended block-diagonalisation method. We have shown in this section, that both approaches including the indecomposable representations,which we learned about in the sections 3.1 and 3.3, give the same fusion rules. Moreover it becomes also clear at this point, that Fuchs et al. found in their recent work a way to calculate the fusion rules for irreducible representations in a perhaps mathematically more appealing and certainly algebraicly better motivated way, which are the same as the ones given by the α-Verlinde formula. On the other hand the connection to the work of M. Flohr provides its CFT-side motivation, needs less many different matrices and is also easier to calculate. Moreover the limit in the α-Verlinde formula has now found its justification through its equality to the "generalised" Verlinde formula in our extension of Fuchs' approach.

Conclusions

In this thesis we have given a detailed description of two different possibilities to calculate the fusion rules of the $c_{(p,1)}$ models.
One of them has been until now only based on the modular transformation properties of the characters of irreducible representations of the $\mathcal{W}$ triplet algebra and consequently only has given the fusion rules of these representations. For this a "generalised" Verlinde formula has been suggested by Fuchs et. al., in which the fusion coefficient matrices are simulaneously block-diagonalised in contrast to the diagonalisation in the case of semisimple fusion algebras. We have shown, that the automorphy factor needed in the definition of the $SL(2,\mathbb{Z})$ representation, from which the S-matrix for this "generalised" Verlinde formula has been taken, is effectively giving the modular transformations for a linear combination of characters of the irreducible representations. This way we have found a meaning of this matrix in the actual conformal field theory, which supplements the one as one of the generators of the $SL(2,\mathbb{Z})$ representation. The latter meaning determines, of course, through the construction of this representation the specific linear combinations of characters, which need to be taken.
However, for the physical picture it is important to take the indecomposable representations, which exist in $c_{(p,1)}$ models, into account, as they are the key feature of these models. They lead to the appearance of logarithmic divergences of correlation functions. The latter in turn has caused the interest for these models to describe, for example, two dimensional polymers or turbulent systems. We have been able now to extend the block-diagonalisation method to also incorporate the indecomposable representations and so give the complete fusion algebra. For the extension a set of forms, which originate from the study of the partition function of the $c_{(p,1)}$ models and reappeared in the basis of the chiral vacuum torus amplitudes, has been used as representatives of the indecomposable representations. The modular transformation properties of linear combination of this set and the characters of irreducible representations provides the S-matrix S_p, which, as the final one, is supposed to give the simultaneous eigen matrix P_p of all fusion coefficient matrices of the whole fusion algebra of irreducible and indecomposable representations, which also has been calculated by Gaberdiel and Kausch for $p = 2$.
For both the original and the extended blockdiagonalisation method one can also argue in the opposite direction compared to the sequence, with which we have passed through the calculations in sections 3.2 and 3.3. We name here the matrices of the extended method, but the same is true for the other one. The theory gives us S_p from some modular transformation properties and, with the definition of a blockdiagonal K_p, we get a matrix P_p. With P_p one can construct, via the block-diagonal matrices $M_{p,I}$, which consist of P_p's matrix elements, other matrices $N_{p,I}$. Then these naturally have all the conditions in relation to P_p, which we have found (simultaneous block-diagonalisation, matrix of generalised eigenvectors, etc.). And these matrices define – as structure constants – some algebra. Then the actual statement is, that this algebra is the fusion algebra of the $c_{(p,1)}$ models modulo some substitutions. It forms

a well-founded proposition, how the fusion rules are calculated, justified by many indications, and gives the – seemingly for higher p – right result. A proof keeps owing and the question, why $\mathtt{S}_p$ block-diagonalises the matrices of fusion coefficients $N_{p,I}$ simultaneously in this way, stands central in it.
However the additional forms, used in our extension, were first successfully used for the calculation of fusion rules by the second one of the two method we mentioned in the first line of this conclusions. We have detailed the path leading to this method and the exact calculations needed. We have given a closed form of the parameter-dependent S-matrix $\mathtt{S}_{\alpha,p}$, which has been used for the fitting adaption of the Verlinde formula there, which we call α-Verlinde formula. Although $\mathtt{S}_{\alpha,p}$ does not diagonalise the fusion coefficient matrices, it simultaniously diagonalises a set of matrices depending on α as well, which in the limit $\alpha \to 0$ are in accord with the known fusion rules with respect to both the triplet algebra and the Virasoro algebra. We have seen, that the fusion rules, we get from any Verlinde formula for $c_{(p,1)}$ models, can not distinguish between indecomposable representations and certain combinations of irreducible representations. As we discussed at first appearance of this indistinguishability in this thesis, in context of the results of the α-Verlinde formula, this is intrinsic to the whole calculation on grounds of modular transformations of characters. At that point it was particularly clear, because the limit $\alpha \to 0$ made the used forms linearly dependent. But also for our extended block-diagonalisation method it is expected, when one goes over to linear combinations of the same forms, which though linearly independent are taken in just the way to make the resulting S-matrix α-independent. Obviously this is achieved only, if terms drop out, that come from the α-dependent summand of the forms $\tilde{\chi}_{\lambda,p}(\alpha)$, which distinguish them from the characters of the indecomposable representations linearly dependent on those of the irreducible ones.
We have shown, that both approaches, which take the indecomposable representations into account, provide the same results. Moreover for any matrix in any of all three methods – the α-Verlinde formula, the original and the extended block-diagonalisation method – the corresponding matrices in the other two approach were given. Especially for the two "large" versions this is helpful for further investigations, as one has a better understanding with respect to standard entities on the conformal field theory side like characters, vacuum amplitudes and the partition sum, while the other's home is the algebraic side around the fusion algebra corresponding to the Virasoro algebra at central charge $c = c_{(p,1)}$.
Also other work connected both sides. The best support for Fuchs' approach until now came from work of Feigin et al. on a Kazhdan-Lusztig-like correspondence. They found, that the closed from for the "small" fusion algebra of $c_{(p,1)}$-models, found by Fuchs et al., is the same as the Grothendieck ring in the quantum groups, which are conjectured to correspond to this series, namely, a reduced quantum group $\bar{\mathcal{U}}_{\mathfrak{q}}sl(2)$ at root of unity $\mathfrak{q} = e^{\frac{i\pi}{p}}$. This is exactly one statement of the Kazhdan-Lusztig correspondence. One other claim is, that a modular group representation associated with the conformal blocks on a torus is equivalent to a modular group representation on the center of the quantum group. It has been shown in [FGST06c], that an $SL(2,\mathbb{Z})$ representation of the $(3p-1)$-dimensional space of conformal blocks on a torus is equivalent to one given on the center of the quantum group. Also there the automorphy factor and the S-matrix of Fuchs are found as a projection on a part of this space up to simple multipliers. This now triggers the question, if perhaps the large S-matrix and other matrices of our extension can be found in connection to the $SL(2,\mathbb{Z})$ representation found by Feigin et al. on the conformal blocks. Is it perhaps equivalent to this representation and consequently also to the one on the quantum group center? Is there a ring in this quantum group, which coincides with the entire fusion algebra of the $c_{(p,1)}$ models? The last question seems only

to make sense, if one asks for the fusion algebra before the ominous substitutions are made, because the Grothendieck ring already found in the quantum group corresponds to the "small" fusion algebra, which is a part of that algebra. So this apparently has to be seen on the level of characters. If these questions can be answered positive, this would certainly speak for our extension.

In the end the actual proof of the extended "generalised" Verlinde formula is perhaps again a question of translation to algebraic geometry and a subsequent proof there. The corresponding version of the formula should be connected to or should even give the dimension of the space of holomorphic section of line bundles on the moduli space of principal G-bundles over a Riemann surface, now for a non-semisimple algebraic group G. It could well also be, that this time the entire proof comes first from the theory of vertex operator algebras, where work is progressing steadily. Recently the first part of a extensive review of conformal vertex algebra applying to LCFT was published ([HLZ06]). The geometric interpretation would then give the mentioned algebraic geometric side.

But with all the indications for our results, like the parallels to the case of rational conformal field theory or the comparison to the results of Gaberdiel and Kausch, one can be virtually sure, that we have here the correct fusion rules for all $c_{(p,1)}$ models in hand.

And the advantage of the Verlinde formula in any of its version is enormous. With this one need not give up at the thought of calculating the fusion rules for interesting models discussed in the literature in the context of e.g. two dimensional magnetohydrodynamics ($c_{(6,1)}$; [ST98] or two dimensional turbulence ($c_{(8,1)}$; [RTR96]). One just calculates it.

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Appendix A

Some Calculations in even more Detail

A.1.1 Different limits in the α-Verlinde formula

The matrix $\mathtt{S}_{(p,\alpha)}$ appears in equation (3.16) four times. One could now think of plugging in this matrix with four different parameters α_i. Then one can take the limit differently fast for each of the four α_i, instead of taking the simultaneous limit, i.e. we have imposed different power-law behaviour, with which the $\alpha_i = x^{a_i}$ is sent to zero:

$$\mathtt{N}_{ij}{}^{k}(x) = \sum_{r=1}^{3p} \frac{(\mathtt{S}_{(p,x^{a_3})})_{jr}(\mathtt{S}_{(p,x^{a_2})})_{ir}(\mathtt{S}_{(p,x^{a_4})})_{r}{}^{k}}{(\mathtt{S}_{(p,x^{a_1})})_{3,r}} \tag{A.1}$$

Our aim has been to get the correct fusion rules for $p = 2$ in the cases, where no indecomposable representations at all are appearing and no substitution would be needed, when we then take the limit $x \to 0$. This is, for example, the case for fusion product with the vacuum representation. For example, one can take the subsequent three fusion products, which are given by (A.1):

$$[1] \otimes^f [0] = \frac{1}{2}[1] + \frac{1}{2}[1]\, x^{a_1+a_4-a_2-a_3} + \frac{1}{2}[0] - \frac{1}{2}[0]\, x^{a_1+a_4-a_2-a_3} \tag{A.2}$$

$$\left[-\frac{1}{8}\right] \otimes^f [0] = \frac{1}{2}\left[-\frac{1}{8}\right] + \frac{1}{2}\left[-\frac{1}{8}\right] x^{a_1-a_3} + \frac{1}{2}[0] - \frac{1}{2}[0]\, x^{a_1-a_3} \tag{A.3}$$

$$\left[-\frac{1}{8}\right] \otimes^f [1] = \frac{1}{2}\left[-\frac{1}{8}\right] - \frac{1}{2}\left[-\frac{1}{8}\right] x^{a_1-a_2} + \frac{1}{2}\left[\frac{3}{8}\right] + \frac{1}{2}\left[\frac{3}{8}\right] x^{a_1-a_2}\ . \tag{A.4}$$

In comparison with section B.1 all the exponents of x appearing in these decompositions have to be zero. The fusion products must be independent of x already, before the limit is taken, so that the vacuum representation is the unit element of the fusion algebra and the representation [1] (for $p = 2$) just "switches" between the corresponding singlet and doublet representations, i.e.

$$a_1 + a_4 = a_2 + a_3 \quad a_1 = a_3 \quad a_1 = a_2 \tag{A.5}$$

So all four must be the same. We also looked at the option of taking the limits of the different α_i manually one after the other, but only the parameter of the S-matrix in the denominator of the equation can be taken first to zero – meaning that the others would cause divergence. If one does so, the other parameters drop out immediately and one is left with no unit element and other problems.

So in summary it turns out that the simultaneous limit is the *only* one leading to the correct fusion rules. Of course, there are good reasons, why these parameters should be the same anyway. One can, for example, only talk of diagonalisation of the fusion coefficient matrices, if the same S-matrix is multiplied from both sides. But still it is a piece of information worth mentioning that the rather implausible possibilities to take this limit consequently also do not give the correct result.

A.1.2 Simultaneous Eigen Decomposition of the Semisimple-Fusion Coefficient Matrices

We have seen in section 1.2.2 that the S-matrix S_{RCFT} diagonalises the matrices of fusion coefficients. But it is not the counterpart of the matrix $P_{irr,p}$ for this case – and $P_{irr,p}$ is not $\mathtt{S}_{irr,p}$ – , as the diagonalised matrices $M_{diag,I}$ (eq. (1.41)) have not matrix elements coming from the I-th line of the S-matrix. But such a counterpart P_{RCFT} does exist, because arguments analogous to those just presented hold. The difference is now, that we already deal with a semisimple algebra and so do not have a direct sum of such an algebra and a radical. Consequently the basis Y_{RCFT} only contains idempotents, which again form a partition of the unit as in equation (3.26). We are left with relation (3.28) for the idempotents. P_{RCFT} gives the canonical basis X_{RCFT} of irreducible representations in terms of elements of the basis Y_{RCFT} (cf. eq. (3.31)). Finally with equations (3.34)-(3.40) we end up with

$$(X_{RCFT})_L \, (Y_{RCFT})_A = (P_{RCFT})_{LA} \, (Y_{RCFT})_A \tag{A.6}$$

instead of (3.35), which gives $M_{diag,I}$.
The fusion coefficients are then given by

$$N_{RCFT,I} = P_{RCFT} M_{diag,I} {P_{RCFT}}^{-1} \; . \tag{A.7}$$

The diagonalising matrix P_{RCFT} consists of simultaneous eigenvectors of the matrices $N_{RCFT,I}$ for all I labeling the n irreducible representations. For any specific I one has lots of other matrices, which diagonalise $N_{RCFT,I}$, by taking multiples – or in the same eigenspace even linear combinations – of the columns of P_{RCFT} for the columns of them. Considering the simultaneous diagonalisation the eigenspaces are a priori different for different I and we are left with the possibility of multiplying each column by a non-zero constant. Extraordinarily, one of the matrices related to P_{RCFT} in this way is just the S-matrix S_{RCFT}:

$$P_{RCFT} = S_{RCFT} K_{diag} \; . \tag{A.8}$$

The diagonal matrix K_{diag} just divides each column of S_{RCFT} through the element in this column and the row corresponding to the vacuum representation in this column:

$$K_{diag} = diag\left(\frac{1}{{(S_{RCFT})_\Omega}^1}, \frac{1}{{(S_{RCFT})_\Omega}^2}, \ldots, \frac{1}{{(S_{RCFT})_\Omega}^n} \right) \; . \tag{A.9}$$

Note that this is already fixed by the vacuum line π_Ω of P_{RCFT}. With σ_Ω denoting the vacuum line of the S-matrix, K_{diag} results from

$$\pi_\Omega = \sigma_\Omega K_{diag} \; , \tag{A.10}$$

which fortunately holds also for all other lines. This fact incorporates the central point of the theorem which the Verlinde formula makes up. As reviewed in [Fuc06] the actual insight of the Verlinde formula is, that the not uniquely defined diagonalising matrix can be chosen as a matrix coming from the braiding of the representation category of the chiral symmetry algebra of the RCFT (e.g. the Virasoro algebra) and – more importantly – the latter actually is the same as our character S-matrix S_{RCFT}.

A.1.3 Calculations leading to $\mathbf{C_{irr,2}}$

We want to solve the following equation for $C_{irr,p}$ in the case of $p = 2$:

$$\mathtt{S}_{irr,p} = C_{irr,p}\left(-\frac{1}{\tau}\right)\mathbf{S}_p(\tau){C_{irr,p}}^{-1}(\tau)\ . \tag{A.11}$$

We start our calculations without fixing p and derive general statement for $C_{irr,p}$. Afterwards apply to $p = 2$, what we have found.
First of all one can use the fact, that at $\tau = i$ the two matrices $\mathtt{S}_{irr,p}$ and $\mathbf{S}_p(\tau)$ are the same and so $C_{irr,p}(i) = \mathbb{1}$. After multiplying eq. (A.11) from the right with $C_{irr,p}(\tau)$ one can expand the matrix elements of $C_{irr,p}(\tau)$, $C_{irr,p}(-1/\tau)$ and $\mathbf{S}_p(\tau)$ around $\tau = i$. We write this here as a matrix equation for simplicity. The derivatives are acting on each matrix element.

$$\begin{aligned}\mathtt{S}_{irr,p}\Bigg(C_{irr,p}(i) \;&+\; \left.\frac{\partial C_{irr,p}(\tau)}{\partial\tau}\right|_{\tau=i}(\tau-i) + \mathcal{O}((\tau-i)^2)\Bigg) \\ =\;& \left(C_{irr,p}\left(-\frac{1}{i}\right) + \left.\frac{\partial C_{irr,p}\left(-\frac{1}{\tau}\right)}{\partial\tau}\right|_{\tau=i}(\tau-i) + \mathcal{O}((\tau-i)^2)\right) \\ & \left(\mathbf{S}_p(i) + \left.\frac{\partial \mathbf{S}_p(\tau)}{\partial\tau}\right|_{\tau=i}(\tau-i) + \mathcal{O}((\tau-i)^2)\right)\ .\end{aligned} \tag{A.12}$$

The constant parts of $C_{irr,p}$ and $\mathbf{S}_p$ are the unit matrix and $\mathtt{S}_{irr,p}$, respectively. The chain rule on the linear part of the right hand side just gives a minus sign from the inner derivative. We get

$$\begin{aligned}\mathtt{S}_{irr,p} \;&+\; \mathtt{S}_{irr,p}\left.\frac{\partial C_{irr,p}(\tau)}{\partial\tau}\right|_{\tau=i}(\tau-i) + \mathcal{O}((\tau-i)^2) \\ =\;& \mathtt{S}_{irr,p} - \left.\frac{\partial C_{irr,p}(\tau)}{\partial\tau}\right|_{\tau=i}\mathtt{S}_{irr,p}(\tau-i) + \left.\frac{\partial \mathbf{S}_p(\tau)}{\partial\tau}\right|_{\tau=i}(\tau-i) + \mathcal{O}((\tau-i)^2)\ .\end{aligned} \tag{A.13}$$

Here we can extract the linear part, which leads to the following equation

$$\mathtt{S}_{irr,p}\left.\frac{\partial C_{irr,p}(\tau)}{\partial\tau}\right|_{\tau=i} = \left.\frac{\partial C_{irr,p}(\tau)}{\partial\tau}\right|_{\tau=i}\mathtt{S}_{irr,p} + \left.\frac{\partial \mathbf{S}_p(\tau)}{\partial\tau}\right|_{\tau=i}\ . \tag{A.14}$$

Now we looked at the case $p = 2$. If one makes the assumption that this matrix should look similar to $j_2(\mathcal{S},\tau)$, i.e it is block-diagonal and the first 2×2 block is the unit matrix, one can easily calculate the derivative of $C_{irr,2}(\tau)$ with the subsequent ansatz,

$$\left.\frac{\partial C_{irr,2}(\tau)}{\partial\tau}\right|_{\tau=i} = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & (\partial c)_{33} & (\partial c)_{34}\\ 0 & 0 & (\partial c)_{43} & (\partial c)_{44}\end{pmatrix}\ . \tag{A.15}$$

The derivative of $\mathbf{S}_2(\tau)$ is a constant matrix looking just alike. The system of equations, which is given by eq. (A.14), can be solved now easily. The solution reads

$$\left.\frac{\partial C_{irr,2}(\tau)}{\partial\tau}\right|_{\tau=i} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4}i & \frac{1}{4}i \\ 0 & 0 & \frac{1}{4}i & -\frac{1}{4}i \end{pmatrix} . \tag{A.16}$$

Now there is no reason, why $C_{irr,2}(\tau)$ should depend particularly complicated on τ. We have seen that $\mathbf{S}_2(\tau)$ is linear in τ. Consequently $j_2(\mathcal{S},\tau)$ is antilinear in τ. If one takes the determinant of equation (A.11) for this case, we get another hint,

$$\begin{aligned} \det\left(\mathbf{S}_{irr,p}\right)\det\left(C_{irr,2}(\tau)\right) &= \det\left(C_{irr,2}\left(-\frac{1}{\tau}\right)\right)\det\left(\mathbf{S}_2(\tau)\right) \qquad \text{(A.17)} \\ \Rightarrow \quad -\det\left(C_{irr,2}\right)(\tau) &= \left(\det\left(C_{irr,2}\right)\left(-\frac{1}{\tau}\right)\right) i\tau \, . \end{aligned}$$

Inserting the Laurent series of $\det\left(C_{irr,2}\right)$ leads to a relation between its modes.

$$\sum_{-\infty}^{+\infty} -c_n\,\tau^n = \sum_{-\infty}^{+\infty} i\,c_n\left(-\frac{1}{\tau}\right)^n\tau \tag{A.18}$$

$$\Rightarrow \quad c_n = (-1)^n i\,c_{-n+1} \, . \tag{A.19}$$

So starting from the simplest possibility, we could have here only a constant and linear contribution or we would have – the next to simplest case – already an antilinear and quadratic contribution in the determinant.
If one looks at the result for the derivative at $\tau = i$, eq. (A.16), it looks very simple. One can make a first guess that it is not a sum of several terms with different power of τ evaluated at that particular point, but that the derivative $\frac{\partial C_{irr,2}(\tau)}{\partial\tau}$ itself actually is constant and equal to the matrix in eq. (A.16). As a consequence the following ansatz seems reasonable

$$C_{irr,2}(\tau) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_{33}-\frac{1}{4}i\tau & c_{34}+\frac{1}{4}i\tau \\ 0 & 0 & c_{43}+\frac{1}{4}i\tau & c_{44}-\frac{1}{4}i\tau \end{pmatrix} . \tag{A.20}$$

The determinant has no term quadratic in τ, which is a first sign for this ansatz, because otherwise it could not have been correct without a term antilinear in τ. Finally this leads us to a system of equations determined by eq. (A.11) that can be solved for the four constant variables in our ansatz. We find

$$C_{irr,2}(\tau) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{4}-\frac{1}{4}i\tau & \frac{1}{4}+\frac{1}{4}i\tau \\ 0 & 0 & \frac{1}{4}+\frac{1}{4}i\tau & \frac{3}{4}-\frac{1}{4}i\tau \, . \end{pmatrix} \tag{A.21}$$

Appendix B

Fusion rules for p = 2

B.1 Results of the α-Verlinde Formula

The S-matrix plugged into the eq. 3.16 is for $p = 2$:

$$\mathtt{S}_{(p,\alpha)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & -1 & 0 \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{\alpha} & \frac{1}{\alpha} & -\frac{1}{2\alpha} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{\alpha} & -\frac{1}{\alpha} & \frac{1}{2\alpha} \\ 1 & -1 & -\alpha & \alpha & 0 \end{pmatrix} . \tag{B.1}$$

The "pre-fusion-rules" calculated the α-Verlinde Formula then are:

$$\begin{aligned}
[0] \quad &\otimes^f \quad [h] = [h] \quad \forall h \in \left\{0, -\frac{1}{8}, \frac{3}{8}, 1, \tilde{0}, \widetilde{-\frac{1}{4}}\right\} , \\
\left[-\frac{1}{8}\right] \quad &\otimes^f \quad \left[-\frac{1}{8}\right] = 2\,[0] + 2\,[1] \ , \\
\left[-\frac{1}{8}\right] \quad &\otimes^f \quad [\tilde{0}] = 2\left[-\frac{1}{8}\right] + 2\left[\frac{3}{8}\right] \ , \\
\left[-\frac{1}{8}\right] \quad &\otimes^f \quad \left[\frac{3}{8}\right] = 2\,[0] + 2\,[1] \ , \\
\left[-\frac{1}{8}\right] \quad &\otimes^f \quad [1] = \left[\frac{3}{8}\right] \ , \\
[\tilde{0}] \quad &\otimes^f \quad [\tilde{0}] = 8\,[0] + 8\,[1] \ , \\
[\tilde{0}] \quad &\otimes^f \quad \left[\frac{3}{8}\right] = 2\left[-\frac{1}{8}\right] + 2\left[\frac{3}{8}\right] \ , \\
[\tilde{0}] \quad &\otimes^f \quad [1] = 4\,[0] + 4\,[1] - [\tilde{0}] \ , \\
\left[\frac{3}{8}\right] \quad &\otimes^f \quad \left[\frac{3}{8}\right] = 2\,[0] + 2\,[1] \ , \\
\left[\frac{3}{8}\right] \quad &\otimes^f \quad [1] = \left[-\frac{1}{8}\right] \ , \\
[1] \quad &\otimes^f \quad [1] = [0] \ .
\end{aligned} \tag{B.2}$$

The fusion coefficients appearing in these fusion rules are found in the following matrices:

$$\mathtt{N}_{2,1} = \begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \end{pmatrix} \tag{B.3}$$

$$\mathtt{N}_{2,2} = \begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathtt{N}_{2,4} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 4 & -1 \end{pmatrix}$$

$$\mathtt{N}_{2,5} = \begin{pmatrix} 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 4 & -1 \\ 0 & 0 & 8 & 8 & 0 \end{pmatrix}$$

The matrix belonging to the vacuum representation, $\mathtt{N}_{2,3}$, is always the unit matrix.
After the substitutions explained towards the end of section 3.1 have been carried out in equation (B.2), we get the following altered fusions rules are the same as in [GK96b]. We state only the fusion rules here, in which a substitution is done:

$$\begin{aligned} \left[-\frac{1}{8}\right] &\otimes^f \left[-\frac{1}{8}\right] = [\tilde{0}] \ , \\ \left[-\frac{1}{8}\right] &\otimes^f \left[\frac{3}{8}\right] = [\tilde{0}] \ , \\ [\tilde{0}] &\otimes^f [\tilde{0}] = 4\,[\tilde{0}] \ , \\ [\tilde{0}] &\otimes^f [1] = [\tilde{0}] \ , \\ \left[\frac{3}{8}\right] &\otimes^f \left[\frac{3}{8}\right] = [\tilde{0}] \ . \end{aligned} \tag{B.4}$$

Appendix C

Fusion rules for p = 3

C.1 Results of the α-Verlinde Formula

The S-matrix $\mathtt{S}(3,\alpha)$ is given in equation (3.17). The "pre-fusion-rules" calculated with eq. 3.16 are:

$$
\begin{aligned}
[0] &\otimes^f \quad [h] = [h] \quad \forall h \in \left\{0, -\frac{1}{4}, -\frac{1}{3}, \frac{5}{12}, 1, \frac{7}{4}, \tilde{0}, \widetilde{-\frac{1}{4}}\right\} \ , \qquad \text{(C.1)} \\
\left[-\frac{1}{4}\right] &\otimes^f \left[-\frac{1}{4}\right] = [0] + \left[-\frac{1}{3}\right] \ , \\
\left[-\frac{1}{4}\right] &\otimes^f \left[-\frac{1}{3}\right] = 2\left[-\frac{1}{4}\right] + 2\left[\frac{7}{4}\right] \ , \\
\left[-\frac{1}{4}\right] &\otimes^f \left[\widetilde{-\frac{1}{4}}\right] = 2\left[-\frac{1}{3}\right] + [\tilde{0}] \ , \\
\left[-\frac{1}{4}\right] &\otimes^f [\tilde{0}] = \left[\widetilde{-\frac{1}{4}}\right] + 2\left[\frac{5}{12}\right] \ , \\
\left[-\frac{1}{4}\right] &\otimes^f \left[\frac{5}{12}\right] = 2\,[0] + 2\,[1] \ , \\
\left[-\frac{1}{4}\right] &\otimes^f [1] = \left[\frac{5}{12}\right] + \left[\frac{7}{4}\right] \ , \\
\left[-\frac{1}{4}\right] &\otimes^f \left[\frac{7}{4}\right] = [1] \ , \\
\left[-\frac{1}{3}\right] &\otimes^f \left[-\frac{1}{3}\right] = 2\,[0] + \left[-\frac{1}{3}\right] + 2\,[1] \ , \\
\left[-\frac{1}{3}\right] &\otimes^f \left[\widetilde{-\frac{1}{4}}\right] = 4\left[-\frac{1}{4}\right] + 4\left[\frac{5}{12}\right] + 4\left[\frac{7}{4}\right] \ , \\
\left[-\frac{1}{3}\right] &\otimes^f [\tilde{0}] = 4\,[0] + 2\left[-\frac{1}{3}\right] + 4\,[1] \ , \\
\left[-\frac{1}{3}\right] &\otimes^f \left[\frac{5}{12}\right] = 2\left[-\frac{1}{4}\right] + \left[\frac{5}{12}\right] + 2\left[\frac{7}{4}\right] \ , \\
\left[-\frac{1}{3}\right] &\otimes^f [1] = 2\,[0] + 2\,[1] \ ,
\end{aligned}
$$

$$\begin{aligned}
\left[-\frac{1}{3}\right] &\otimes^f \left[\frac{7}{4}\right] = \left[\frac{5}{12}\right] \ , \\
\widetilde{\left[-\frac{1}{4}\right]} &\otimes^f \widetilde{\left[-\frac{1}{4}\right]} = 8\,[0] + 4\left[-\frac{1}{3}\right] + 8\,[1] \ , \\
\widetilde{\left[-\frac{1}{4}\right]} &\otimes^f [\tilde{0}] = 8\left[-\frac{1}{4}\right] + 4\left[\frac{5}{12}\right] + 8\left[\frac{7}{4}\right] \ , \\
\widetilde{\left[-\frac{1}{4}\right]} &\otimes^f \left[\frac{5}{12}\right] = 4\,[0] + 2\left[-\frac{1}{3}\right] + 4\,[1] \ , \\
\widetilde{\left[-\frac{1}{4}\right]} &\otimes^f [1] = 4\left[-\frac{1}{4}\right] - \widetilde{\left[-\frac{1}{4}\right]} + 2\left[\frac{5}{12}\right] + 4\left[\frac{7}{4}\right] \ , \\
\widetilde{\left[-\frac{1}{4}\right]} &\otimes^f \left[\frac{7}{4}\right] = 4\,[0] - [\tilde{0}] + 4\,[1] \ , \\
[\tilde{0}] &\otimes^f [\tilde{0}] = 8\,[0] + 4\left[-\frac{1}{3}\right] + 8\,[1] \ , \\
[\tilde{0}] &\otimes^f \left[\frac{5}{12}\right] = 4\left[-\frac{1}{4}\right] + 2\left[\frac{5}{12}\right] + 4\left[\frac{7}{4}\right] \ , \\
[\tilde{0}] &\otimes^f [1] = 4\,[0] + 2\left[-\frac{1}{3}\right] - [\tilde{0}] + 4\,[1] \ , \\
[\tilde{0}] &\otimes^f \left[\frac{7}{4}\right] = 4\left[-\frac{1}{4}\right] - \widetilde{\left[-\frac{1}{4}\right]} + 4\left[\frac{7}{4}\right] \ , \\
\left[\frac{5}{12}\right] &\otimes^f \left[\frac{5}{12}\right] = 2\,[0] + \left[-\frac{1}{3}\right] + 2\,[1] \ , \\
\left[\frac{5}{12}\right] &\otimes^f [1] = 2\left[-\frac{1}{4}\right] + 2\left[\frac{7}{4}\right] \ , \\
\left[\frac{5}{12}\right] &\otimes^f \left[\frac{7}{4}\right] = \left[-\frac{1}{3}\right] \ , \\
[1] &\otimes^f [1] = [0] + \left[-\frac{1}{3}\right] \ , \\
[1] &\otimes^f \left[\frac{7}{4}\right] = \left[-\frac{1}{4}\right] \ , \\
\left[\frac{7}{4}\right] &\otimes^f \left[\frac{7}{4}\right] = [0] \ .
\end{aligned}$$

The fusion coefficients appearing in these fusion rules are found in the following matrices:

$$\mathtt{N}_{3,1} = \begin{pmatrix} 1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 4 & 4 & 0 \end{pmatrix} \tag{C.2}$$

$$\mathtt{N}_{3,2} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 2 & 2 & 0 \\ 1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathtt{N}_{3,4} = \begin{pmatrix} 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & 4 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 4 & 4 & 0 \end{pmatrix}$$

$$\mathtt{N}_{3,5} = \begin{pmatrix} 2 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 4 & 4 & -1 & 0 & 0 & 0 \\ 4 & 0 & 8 & 8 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 & 4 & -1 \\ 0 & 4 & 0 & 0 & 0 & 8 & 8 & 0 \end{pmatrix}$$

$$\mathtt{N}_{3,6} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 8 & 8 & 0 \end{pmatrix}$$

$$\mathtt{N}_{3,7} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 4 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 4 & 4 & -1 \\ 0 & 0 & 4 & 4 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathtt{N}_{3,8} = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 4 & 4 & 0 \\ 0 & 2 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 4 & 4 & -1 \\ 0 & 4 & 0 & 0 & 0 & 8 & 8 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & -1 & 0 & 0 & 0 \\ 4 & 0 & 8 & 8 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix belonging to the vacuum representation, $\mathtt{N}_{3,3}$, is always the unit matrix.
After the substitutions explained towards the end of section 3.1 have been carried out in equation (C.1), we get the following altered fusions rules in correspondence to the results of [GK96a].

$$\begin{aligned}
\left[-\tfrac{1}{4}\right] &\otimes^f \left[-\tfrac{1}{3}\right] = \widetilde{\left[-\tfrac{1}{4}\right]} \;, \\
\left[-\tfrac{1}{4}\right] &\otimes^f \left[\tfrac{5}{12}\right] = [\tilde{0}] \;, \\
\left[-\tfrac{1}{3}\right] &\otimes^f \left[-\tfrac{1}{3}\right] = [\tilde{0}] + \left[-\tfrac{1}{3}\right] \;, \\
\left[-\tfrac{1}{3}\right] &\otimes^f \widetilde{\left[-\tfrac{1}{4}\right]} = 2\,\widetilde{\left[-\tfrac{1}{4}\right]} + 4\left[\tfrac{5}{12}\right] \;, \\
\left[-\tfrac{1}{3}\right] &\otimes^f [\tilde{0}] = 2\,[\tilde{0}] + 2\,[\tilde{0}] + 2\left[-\tfrac{1}{3}\right] \;, \\
\left[-\tfrac{1}{3}\right] &\otimes^f \left[\tfrac{5}{12}\right] = \widetilde{\left[-\tfrac{1}{4}\right]} + \left[\tfrac{5}{12}\right] \;, \\
\left[-\tfrac{1}{3}\right] &\otimes^f [1] = [\tilde{0}] \;, \\
\widetilde{\left[-\tfrac{1}{4}\right]} &\otimes^f \widetilde{\left[-\tfrac{1}{4}\right]} = 4\,[\tilde{0}] + 4\left[-\tfrac{1}{3}\right] \;, \\
\widetilde{\left[-\tfrac{1}{4}\right]} &\otimes^f [\tilde{0}] = 4\,\widetilde{\left[-\tfrac{1}{4}\right]} + 4\left[\tfrac{5}{12}\right] \;, \\
\widetilde{\left[-\tfrac{1}{4}\right]} &\otimes^f \left[\tfrac{5}{12}\right] = 2\,[\tilde{0}] + 2\left[-\tfrac{1}{3}\right] \;, \\
\widetilde{\left[-\tfrac{1}{4}\right]} &\otimes^f [1] = \widetilde{\left[-\tfrac{1}{4}\right]} + 2\left[\tfrac{5}{12}\right] \;, \\
\widetilde{\left[-\tfrac{1}{4}\right]} &\otimes^f \left[\tfrac{7}{4}\right] = [\tilde{0}] \;, \\
[\tilde{0}] &\otimes^f [\tilde{0}] = 4\,[\tilde{0}] + 4\left[-\tfrac{1}{3}\right] \;, \\
[\tilde{0}] &\otimes^f \left[\tfrac{5}{12}\right] = 2\,\widetilde{\left[-\tfrac{1}{4}\right]} + 2\left[\tfrac{5}{12}\right] \;,
\end{aligned}$$

$$\begin{aligned}
[\tilde{0}] &\otimes^f & [1] &= 2\left[-\frac{1}{3}\right] + [\tilde{0}] \ , \\
[\tilde{0}] &\otimes^f & \left[\frac{7}{4}\right] &= \left[\widetilde{-\frac{1}{4}}\right] \ , \\
\left[\frac{5}{12}\right] &\otimes^f & \left[\frac{5}{12}\right] &= [\tilde{0}] + \left[-\frac{1}{3}\right] \ , \\
\left[\frac{5}{12}\right] &\otimes^f & [1] &= \left[\widetilde{-\frac{1}{4}}\right] \ .
\end{aligned} \tag{C.3}$$

All other products stay the same as in eq. C.1.

C.2 Matrices in the Block-Diagonalisation Method for irreducible representations

We start with the matrix $\mathbf{S}_3(\tau)$

$$\mathbf{S}_3(\tau) = \begin{pmatrix}
\frac{1}{2}\hat{r} & \frac{1}{2}\hat{r} & \hat{r} & \hat{r} & \hat{r} & \hat{r} \\
\frac{1}{2}\hat{r} & -\frac{1}{2}\hat{r} & \hat{r} & \hat{r} & -\hat{r} & -\hat{r} \\
\frac{1}{6}\hat{r} & \frac{1}{6}\hat{r} & -\frac{1}{6}\hat{r} + \hat{\tau} & -\frac{1}{6}\hat{r} - \frac{1}{2}\hat{\tau} & -\frac{1}{6}\hat{r} - \frac{1}{2}\hat{\tau} & -\frac{1}{6}\hat{r} + \hat{\tau} \\
\frac{1}{3}\hat{r} & \frac{1}{3}\hat{r} & -\frac{1}{3}\hat{r} - \hat{\tau} & -\frac{1}{3}\hat{r} + \frac{1}{2}\hat{\tau} & -\frac{1}{3}\hat{r} + \frac{1}{2}\hat{\tau} & -\frac{1}{3}\hat{r} - \hat{\tau} \\
\frac{1}{3}\hat{r} & -\frac{1}{3}\hat{r} & -\frac{1}{3}\hat{r} - \hat{\tau} & -\frac{1}{3}\hat{r} + \frac{1}{2}\hat{\tau} & \frac{1}{3}\hat{r} - \frac{1}{2}\hat{\tau} & \frac{1}{3}\hat{r} + \hat{\tau} \\
\frac{1}{6}\hat{r} & -\frac{1}{6}\hat{r} & -\frac{1}{6}\hat{r} + \hat{\tau} & -\frac{1}{6}\hat{r} - \frac{1}{2}\hat{\tau} & \frac{1}{6}\hat{r} + \frac{1}{2}\hat{\tau} & \frac{1}{6}\hat{r} - \hat{\tau}
\end{pmatrix} \tag{C.4}$$

with $\hat{r} = \sqrt{6}/3$ and $\hat{\tau} = i\tau\sqrt{2}/3$. The automorphy factor reads

$$j_p(\mathcal{S}, \tau) = \begin{pmatrix}
\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 & 0 \\
0 & \begin{matrix} \frac{\tau+2i}{3\tau} & \frac{\tau-i}{3\tau} \\ \frac{2(\tau-i)}{3\tau} & \frac{2\tau+i}{3\tau} \end{matrix} & 0 \\
0 & 0 & \begin{matrix} \frac{2\tau+i}{3\tau} & \frac{2(\tau-i)}{3\tau} \\ \frac{\tau-i}{3\tau} & \frac{\tau+2i}{3\tau} \end{matrix}
\end{pmatrix} . \tag{C.5}$$

The product of the last two matrices is the S-matrix $\mathtt{S}_{irr,3}$.

$$\mathtt{S}_{irr,3} = \begin{pmatrix}
\frac{1}{2}\hat{r} & \frac{1}{2}\hat{r} & \hat{r} & \hat{r} & \hat{r} & \hat{r} \\
\frac{1}{2}\hat{r} & -\frac{1}{2}\hat{r} & \hat{r} & \hat{r} & -\hat{r} & -\hat{r} \\
\frac{1}{6}\hat{r} & \frac{1}{6}\hat{r} & -\frac{1}{6}\hat{r} - \hat{u} & -\frac{1}{6}\hat{r} + \frac{1}{2}\hat{u} & -\frac{1}{6}\hat{r} + \frac{1}{2}\hat{u} & -\frac{1}{6}\hat{r} - \hat{u} \\
\frac{1}{3}\hat{r} & \frac{1}{3}\hat{r} & -\frac{1}{3}\hat{r} + \hat{u} & -\frac{1}{3}\hat{r} - \frac{1}{2}\hat{u} & -\frac{1}{3}\hat{r} - \frac{1}{2}\hat{u} & -\frac{1}{3}\hat{r} + \hat{u} \\
\frac{1}{3}\hat{r} & -\frac{1}{3}\hat{r} & -\frac{1}{3}\hat{r} + \hat{u} & -\frac{1}{3}\hat{r} - \frac{1}{2}\hat{u} & \frac{1}{3}\hat{r} + \frac{1}{2}\hat{u} & \frac{1}{3}\hat{r} - \hat{u} \\
\frac{1}{6}\hat{r} & -\frac{1}{6}\hat{r} & -\frac{1}{6}\hat{r} - \hat{u} & -\frac{1}{6}\hat{r} + \frac{1}{2}\hat{u} & \frac{1}{6}\hat{r} - \frac{1}{2}\hat{u} & \frac{1}{6}\hat{r} + \hat{u}
\end{pmatrix} \tag{C.6}$$

with $\hat{r} = \sqrt{6}/3$ as defined before and $\hat{u} = \sqrt{2}/3$. This matrix is also the result of equation (A.11) with the matrix

$$C_{irr,3}(\tau) = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} \frac{2-i\tau}{3} & \frac{1+i\tau}{6} \\ \frac{1+i\tau}{3} & \frac{5-i\tau}{6} \end{matrix} & 0 \\ 0 & 0 & \begin{matrix} \frac{5-i\tau}{6} & \frac{1+i\tau}{3} \\ \frac{1+i\tau}{6} & \frac{2-i\tau}{3} \end{matrix} \end{pmatrix}. \tag{C.7}$$

With the vacuum row of $\mathtt{S}_{irr,3}$, which is for all p the third row, we get the matrix $K_{irr,3}$,

$$K_{irr,3} = \begin{pmatrix} \begin{matrix} 3\sqrt{6} & 0 \\ 0 & 3\sqrt{6} \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} -\sqrt{2} & \frac{\sqrt{6}-3\sqrt{2}}{18} \\ \sqrt{2} & -\frac{\sqrt{6}+6\sqrt{2}}{18} \end{matrix} & 0 \\ 0 & 0 & \begin{matrix} \sqrt{2} & \frac{\sqrt{6}+6\sqrt{2}}{18} \\ -\sqrt{2} & -\frac{\sqrt{6}-3\sqrt{2}}{18} \end{matrix} \end{pmatrix}. \tag{C.8}$$

The product of this matrix with $\mathtt{S}_{irr,3}$ is the eigenmatrix $P_{irr,3}$,

$$P_{irr,3} = \begin{pmatrix} 3 & 3 & 0 & -\frac{1}{3}\sqrt{3} & 0 & \frac{1}{3}\sqrt{3} \\ 3 & -3 & 0 & -\frac{1}{3}\sqrt{3} & 0 & -\frac{1}{3}\sqrt{3} \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & -1 & \frac{1}{6}\sqrt{3} & -1 & -\frac{1}{6}\sqrt{3} \\ 2 & -2 & -1 & \frac{1}{6}\sqrt{3} & 1 & \frac{1}{3}\sqrt{3} \\ 1 & -1 & 1 & 0 & -1 & 0 \end{pmatrix}. \tag{C.9}$$

With this matrix we can first write down the matrices $M_{irr,3,I}$ and by conjugation with $P_{irr,3}$ find the fusion coefficients. Here we just want to look at the first matrix, $M_{irr,3,1}$, corresponding to the irreducible representation with highest weight $h = -1/3$,

$$M_{irr,3,1} = \begin{pmatrix} \begin{matrix} 3 & 0 \\ 0 & 3 \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} 0 & -\frac{1}{3}\sqrt{3} \\ 0 & 0 \end{matrix} & 0 \\ 0 & 0 & \begin{matrix} 0 & \frac{1}{3}\sqrt{3} \\ 0 & 0 \end{matrix} \end{pmatrix}. \tag{C.10}$$

The decompositions of the fusion rules are encoded in the matrices

$$N_{irr,3,1} = \begin{pmatrix} 1 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.11}$$

We see some pairs of the number two here. The 5th line is the fusion product

$$\left[-\frac{1}{3}\right] \otimes_f \left[-\frac{1}{4}\right] = 2\left[-\frac{1}{4}\right] + 2\left[\frac{7}{4}\right] , \tag{C.12}$$

which has been also an example in the section 3.1.

C.3 Matrices in the extended Block-Diagonalisation Method

The product of the last two matrices is the S-matrix S_3.

$$S_3 = \begin{pmatrix} \frac{1}{2}\hat{r} & \frac{1}{2}\hat{r} & \hat{r} & \hat{r} & 0 & \hat{r} & \hat{r} & 0 \\ \frac{1}{2}\hat{r} & -\frac{1}{2}\hat{r} & \hat{r} & \hat{r} & 0 & -\hat{r} & -\hat{r} & 0 \\ \frac{1}{6}\hat{r} & \frac{1}{6}\hat{r} & -\frac{1}{6}\hat{r} - \hat{u} & -\frac{1}{6}\hat{r} + \frac{1}{2}\hat{u} & 0 & -\frac{1}{6}\hat{r} + \frac{1}{2}\hat{u} & -\frac{1}{6}\hat{r} - \hat{u} & 0 \\ \frac{1}{3}\hat{r} & \frac{1}{3}\hat{r} & -\frac{1}{3}\hat{r} + \hat{u} & -\frac{1}{3}\hat{r} - \frac{1}{2}\hat{u} & 0 & -\frac{1}{3}\hat{r} - \frac{1}{2}\hat{u} & -\frac{1}{3}\hat{r} + \hat{u} & 0 \\ \hat{r} & \hat{r} & -\hat{r} - 3\hat{u} & \hat{r} - 3\hat{u} & \sqrt{\frac{1}{2}} & -\hat{r} + 3\hat{u} & -\hat{r} + 3\hat{u} & -\sqrt{\frac{1}{2}} \\ \frac{1}{3}\hat{r} & -\frac{1}{3}\hat{r} & -\frac{1}{3}\hat{r} + \hat{u} & -\frac{1}{3}\hat{r} - \frac{1}{2}\hat{u} & 0 & \frac{1}{3}\hat{r} + \frac{1}{2}\hat{u} & \frac{1}{3}\hat{r} - \hat{u} & 0 \\ \frac{1}{6}\hat{r} & -\frac{1}{6}\hat{r} & -\frac{1}{6}\hat{r} - \hat{u} & -\frac{1}{6}\hat{r} + \frac{1}{2}\hat{u} & 0 & \frac{1}{6}\hat{r} - \frac{1}{2}\hat{u} & \frac{1}{6}\hat{r} + \hat{u} & 0 \\ \hat{r} & \hat{r} & -\hat{r} + 3\hat{u} & \hat{r} + 3\hat{u} & -\sqrt{\frac{1}{2}} & \hat{r} + 3\hat{u} & \hat{r} + 3\hat{u} & -\sqrt{\frac{1}{2}} \end{pmatrix} \tag{C.13}$$

with $\hat{r} = \sqrt{6}/3$ as defined before and $\hat{u} = \sqrt{2}/3$. With the vacuum row of S_3, which is for all p the third row, we get the matrix K_3,

$$K_3 = \begin{pmatrix} 3\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & \frac{\sqrt{6}-3\sqrt{2}}{18} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\frac{\sqrt{6}+6\sqrt{2}}{18} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \frac{\sqrt{6}+6\sqrt{2}}{18} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & -\frac{\sqrt{6}-3\sqrt{2}}{18} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{C.14}$$

The product of this matrix with S_3 is the eigenmatrix P_3,

$$P_{irr,3} = \begin{pmatrix} 3 & 3 & 0 & -\frac{1}{3}\sqrt{3} & 0 & 0 & \frac{1}{3}\sqrt{3} & 0 \\ 3 & -3 & 0 & -\frac{1}{3}\sqrt{3} & 0 & 0 & -\frac{1}{3}\sqrt{3} & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & \frac{1}{6}\sqrt{3} & 0 & -1 & -\frac{1}{6}\sqrt{3} & 0 \\ 6 & 6 & 0 & \frac{1}{3}\sqrt{3}+1 & \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{3}\sqrt{3}+1 & -\frac{1}{2}\sqrt{2} \\ 2 & -2 & -1 & \frac{1}{6}\sqrt{3} & 0 & 1 & \frac{1}{3}\sqrt{3} & 0 \\ 1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 3 & -3 & 0 & -\frac{1}{3}\sqrt{3} & 0 & 0 & -\frac{1}{3}\sqrt{3} & 0 \end{pmatrix} . \tag{C.15}$$

With this matrix we can first write down the matrices $M_{3,I}$ and by conjugation with P_3 find the fusion coefficients. Here we just want to look at the first matrix, $M_{3,5}$, corresponding to the indecomposable representation with highest weight $h = 0$,

$$M_{irr,3,1} = \begin{pmatrix} \begin{matrix} 6 & 0 \\ 0 & 6 \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} 0 & \frac{1}{3}\sqrt{3}+1 & \frac{1}{2}\sqrt{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & 0 \\ 0 & 0 & \begin{matrix} 0 & -\frac{1}{3}\sqrt{3} & -\frac{1}{2}\sqrt{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{pmatrix}. \tag{C.16}$$

This matrix directly leads through equation (3.85) to the matrix $\mathtt{N}_{3,5}$ in eq. (C.2), where it was calculated with the α-Verlinde formula, as it is also the case for the other matrices $\mathtt{N}_{3,I}$, $0 < I < 9$.

Appendix D

Eigenvalues and eigenvectors of Different S-Matrices

The eigenvalues are always written in the first line of each tables, which is entitled by the matrix they belong to. The corresponding eigenvectors are found beneath them.

$\mathtt{S}_{irr,2}$:

1			-1
$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}2\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}2\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}-2\\2\\1\\1\end{pmatrix}$

$\mathtt{S}_{(2,\alpha)}$:

1			-1	
$\begin{pmatrix}2\\0\\1\\0\\2-\alpha\end{pmatrix}$	$\begin{pmatrix}2\\0\\0\\1\\2+\alpha\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-1\\1\\1\\0\\2+\alpha\end{pmatrix}$	$\begin{pmatrix}-1\\1\\0\\1\\2-\alpha\end{pmatrix}$

S_2 **for** $s_5 \neq -1$:

1			-1	s_5
$\begin{pmatrix}\frac{2s_2-s_3}{s_1+s_2}\\-\frac{2s_1+s_3}{s_1+s_2}\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}\frac{2s_2-s_4}{s_1+s_2}\\-\frac{2s_1+s_3}{s_1+s_2}\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}\frac{1-s_5}{s_1+s_2}\\\frac{1-s_5}{s_1+s_2}\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}-2\\2\\1\\1\\\frac{-2s_2-s_3-s_4+2s_1}{s_5+1}\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$

S_2 **for** $s_5 = -1$ **and** $s_1 = \frac{2s_2+s_3+s_4}{2}$:

1			−1	
$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \frac{s_4}{4} + \frac{s_3}{4} + s_2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ \frac{s_4}{2} + s_3 + s_2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \\ s_4 + \frac{s_3}{2} + s_2 \end{pmatrix}$	$\begin{pmatrix} -2 \\ 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

List of Notation used

Symbol	Description	Equation
B	matrix Matrix composing the characters from the Jacobi-Riemann Θ-function and the affine Θ-function.	(3.8)
$c_{(p,1)}$	Central charges of the Virasoro algebra of the $c_{(p,1)}$ models with $p \geq 2$	via (1.26)
$c_{(p,q)}$	Central charges of the minimal series parametrised by coprime integer $p, q > 1$	(1.26)
$\chi^{+}_{\lambda,p}$	Characters of irreducible singlet representations of the triplet algebra.	(2.32),(2.34)
$\chi^{-}_{\lambda,p}$	Characters of irreducible doublet representations of the triplet algebra.	(2.33),(2.35)
$\chi^{\mathcal{R}}_{\lambda,p}$	Characters of indecomposable representations of the triplet algebra.	(2.36)
$\boldsymbol{\chi}_{irr,p}$	Vector of all irreducible representations of the triplet algebra.	(2.37)
$\chi^{\mathcal{R}+}(\alpha)$	Basis element of the basis of chiral vacuum torus amplidutes.	(2.42)
$\tilde{\chi}_{\lambda,p}(\alpha)$	A specified linear combination of $\chi^{\mathcal{R}+}(\alpha)$ and $\chi^{\mathcal{R}+}(\alpha)$.	(3.3)
$\boldsymbol{\chi}_p(\alpha)$	Vector of character of irreducible representations of the triplet algebra and further forms $\tilde{\chi}_{\lambda,p}(\alpha)$.	(3.4)
$C_{irr,p}(\tau)$	Replacement of the automorphy factor; it gives τ-dependent combinations of characters, for which $\mathtt{S}_{irr,p}$ describes the $\mathcal{S}$-transformation directly.	(3.57)
$C'_3(\alpha)$	Alternative $(3p-1) \times 2p$ matrix for τ-dependent $2p \times 2p$ matrix $C_{irr,p}(\tau)$. Now multiplied on $\boldsymbol{\chi}_p(\alpha)$ instead of $\boldsymbol{\chi}_{irr,p}$, it yields the same result.	(3.78)
$C_p(\alpha)$	Conjugates S_p and $\mathtt{S}_{(p,\alpha)}$; it gives an α-dependent combinations of elements of the vector $\boldsymbol{\chi}_p(\alpha)$, for which S_p gives the $\mathcal{S}$-transformation directly.	(3.57)
$E_{p,\alpha}$	Matrix connecting $\mathtt{S}_{(p,\alpha)}$ and S_p.	(3.151)
$F_{p,\alpha}$	A matrix giving the connection between "generelised" Verlinde formula and α-Verlinde formula in a nice way	(3.164)
$\mathtt{G}_{(p,\alpha)}$	Representation of $SL(2,\mathbb{Z})$ generated by $\mathtt{S}_{(p,\alpha)}$ and $\mathtt{T}_{(p,\alpha)}$	(3.14)
$\vert h\rangle$	Highest weight state	(1.18)
$h_{(r,s)}$	Highest weights of the Verma-modules of the Virasoro algebra with central charge $c_{(p,q)}$	(1.27)

Symbol	Description	Equation
$[h]$	Irreducible representation with highest weight h.	
$\left[\tilde{h}\right]$	Indecomposable representation with highest weigh h.	
j, m	Quantum numbers for the rescaled $su(2)$ algebra given by the zero modes of $W^{(a)}$.	
$K_{diag,p}$	Diagonal matrix connecting P_{RCFT} and S_{RCFT}.	(A.8)
$K_{irr,p}$	2×2 Block-diagonal matrix connecting $P_{irr,p}$ and $\mathtt{S}_{irr,p}$.	(3.44)
K_p	3×3 Block-diagonal matrix connecting P_p and S_p.	(3.96)
L_n	Modes of the holomorphic stress energy tensor and generators of the Virasoro algebra.	(1.15),(1.16)
$M_{diag,I}$	Diagonal matrices of $N_{RCFT,I}$ diagonalised by S_{RCFT}.	
$M_{irr,p,I}$	Simultaneous block-diagonalisation of $N_{irr,p,I}$, for all $0 < I \leq 2p$, through $P_{irr,p}$, consisting of matrix elements of $P_{irr,p}$.	(3.36)
$M_{p,I}$	Simultaneous block-diagonalisation of $N_{p,I}$, for all $0 < I \leq 3p-1$, through P_p, consisting of matrix elements of P_p.	(3.89)
${(N_{RCFT})_{ij}}^k$	Fusion coefficients for a semisimple fusion algebra.	(1.33)
$N_{RCFT,I}$	Fusion coefficient matrices for a semisimple fusion algebra.	(1.35)
${\mathcal{N}_{ij}}^k$	Fusion rules of the $c_{(p,1)}$ models, which are directly calculated only directly, as Gaberdiel and Kausch did for $p = 2$.	
${\mathtt{N}_{ij}}^k(\alpha)$	α-dependent coefficients from the α-Verlinde formula.	(3.16)
${\mathtt{N}_{ij}}^k$	(Pre-)Fusion coefficients calculated with the α-Verlinde formula.	(3.16)
$N_{irr,p,I}$	Fusion coefficient matrices for the "small" fusion algebra of irreducible representations of the triplet algebra.	(3.32)
$N_{p,I}$	Fusion coefficient matrices for the (pre-)fusion algebra of irreducible and indecomposable representations of the triplet algebra.	(3.85)
$P_{irr,p}$	Matrix of simultaneous generalised eigenvectors of $N_{irr,p,I}$ for all $0 < I \leq 2p$.	(3.41)
P_p	Matrix of simultaneous generalised eigenvectors of $N_{p,I}$ for all $0 < I \leq 3p-1$.	(3.93)
$\mathcal{S}$	Transformation $\tau \to -1/\tau$; a generator of modular transformations	
S_{RCFT}	S-matrix for rational conformal field theories w.r.t. the Virasoro algebra.	(1.39)$\gamma = \mathcal{S}$
$\mathtt{S}_{(p,\alpha)}$	S-matrix, which gives the $\mathcal{S}$-transformation of the vector $\boldsymbol{\chi}_p^t(\alpha)$ with blocks $S(p)_{s,l}(\alpha)$.	(3.5)
$S(p)_{s,l}$	Blocks of the matrix $\mathtt{S}_{(p,\alpha)}$, $0 \leq s, l > p$.	(3.10)
$\mathfrak{S}$	Matrix, which gives the $\mathcal{S}$-transformation of the Jacobi-Riemann Θ-function and the affine Θ-function.	(3.8)

Symbol	Description	Equation
$\mathbf{S}_p(\tau)$	τ-dependent S-matrix, which gives the $\mathcal{S}$-transformation of the vector $\boldsymbol{\chi}_{irr,p}$ with blocks $A(p)_{s,l}$.	(3.45)
$A(p)_{s,l}$	Blocks of $\mathbf{S}_p(\tau)$.	(3.46)
$\rho_p(\gamma)$	Representation of $SL(2,\mathbb{Z})$ generated by $\mathtt{S}_{irr,p}$ and $\mathtt{T}_{irr,p}$	(3.47)
$j_p(\gamma,\tau)$	Automorphy factor of the $SL(2,\mathbb{Z})$ representation $\rho_p(\gamma)$, which calculates $\mathtt{S}_{irr,p}$ from $\mathbf{S}_p(\tau)$ for $\gamma = \mathcal{S}$.	(3.49)
$\mathtt{S}_{irr,p}$	Generator of $SL(2,\mathbb{Z})$ representation $\rho_p(\gamma)$ and S-matrix simultaneously block-diagonalising the fusion coefficient matrices of the "small" fusion algebra; blocks: $S(irr,p)_{s,j}$.	(3.51)
S_p	S-matrix simultaneously block-diagonalising the fusion coefficient matrices of the (pre-)fusion algebra; blocks: $S(p)_{s,j}$.	(3.133)
$\mathcal{T}$	Transformation $\tau \to \tau + 1$; a generator of modular transformations	
$\mathtt{T}_{(p,\alpha)}$	Matrix, which gives the $\mathcal{T}$-transformation of the vector $\boldsymbol{\chi}_p^t(\alpha)$.	(3.11)
$T(p,\alpha)_{s,s}$	Blocks of the matrix $\mathtt{T}_{(p,\alpha)}$, $0 \leq s > p$.	
$T(z)$	Holomorphic stress energy tensor.	(1.12)
$\Theta_{\lambda,k}(\tau)$	Jacobi-Riemann Θ-function.	(2.30)
$(\partial\Theta)_{\lambda,k}(\tau)$	Affine Θ-function.	(2.31)
$V_{(r,s)}$	Verma module with highest weight $h_{(r,s)}$	(1.19)
$W^{(a)}$	Triplet of fields for $a = 1,2,3$, which extends the Virasoro algebra to the $\mathcal{W}$ triplet algebra, the maximally extended local chiral symmetry algebra of the $c_{(p,1)}$ models.	
$Z_{p,\alpha}$	Partition function of the $c_{(p,1)}$ models.	(2.44)

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