Symplectic Fermions – Symmetries of a Vertex Operator Algebra

von

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Diplomarbeit in Physik

angefertigt im

Physikalischen Institut

vorgelegt der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität

Bonn

März 2007

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Abstract

The model of d symplectic fermions constructed by Abe [1] gives an example of a C_2-cofinite vertex operator algebra admitting logarithmic modules. While the case $d = 1$ is a rigorous formulation of the well known triplet algebra, the case $d > 1$ has not yet been analyzed from the perspective of $\mathcal{W}$-algebras. It is shown that in the latter case the $\mathcal{W}(2, 2^{2d^2-d-1}, 3^{2d^2+d})$-algebra is realized. With respect to its zero mode algebra, it is proven that the zero modes corresponding to the vectors of weight 3 form the $2d$-dimensional symplectic Lie algebra. Furthermore, the zero modes corresponding to the vectors of weight 2 form an irreducible representation of this Lie algebra. The use of the zero mode algebra for the classification of irreducible representations of the $\mathcal{W}$-algebra is compared to the approach using Zhu's algebra. The isomorphism between Zhu's algebra and the zero mode algebra as Lie algebras, which is expected from general considerations, is confirmed by explicit calculations.

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Chapter 1

Introduction

Conformal field theories are perhaps the most important example for mathematically rigorously formulated yet nontrivial quantum field theories. Despite many efforts in this direction, the mathematical description of quantum field theory remains unsatisfactory even today. However, in two dimensions and equipped with conformal symmetry as extension of Poincaré symmetry, quantum field theory exhibits a beautiful mathematical structure. This beauty comes along with a great predictive power which originates from the structure of the conformal symmetry algebra. Conformal symmetry may be studied on higher dimensional spaces, but on two dimensional Minkowski space the symmetry algebra has the special feature that it is infinite dimensional. This leads to strong constraints on important quantities. For example, much of the structure of a conformal field theory can be deduced from a special class of fields, which are called primary fields. Computing correlation functions of primary fields can be reduced to solving ordinary differential equations.

Algebraically, the mathematical language of conformal field theory on Riemannian surfaces of genus zero is the theory of vertex operator algebras. While the definition of a vertex operator algebra seems to be very technical at first, all ingredients can be directly interpreted in physical terms, so that the fog of mathematical terminology becomes transparent with the help of physical intuition. From the physical point of view, vertex operator algebras describe the vacuum sector of a conformal field theory. As in general quantum field theory, one is also interested in representations of the field algebra which are inequivalent to the vacuum sector. Mathematically, this means that one would like to classify the representations or modules of the vertex operator algebra.

Irreducible modules are especially important since they serve as building blocks structuring the theory. But it is not always possible to restrict the discussion to irreducible modules. The study of theories where the correlation functions have logarithmic divergencies showed that certain generalized modules which are reducible but indecomposable should also be allowed. Theories admitting such modules have been termed logarithmic and may seem pathological at first. But they actually describe such diverse phenomena as the fractional quantum Hall effect, polymers, abelian sandpiles and disorder. For an introduction to logarithmic conformal field theory the reader is referred to the lecture notes [16], which also contain many references describing the above applications.

Physicists and mathematicians have independently developed tools for the classification of representations of a field algebra. One of the main questions treated in the present work is how these two approaches are connected. In particular, this work contrasts the different perspectives on the symplectic fermion model first formulated by Kausch in [25]. This model realizes the so called triplet algebra at $c = -2$, which has interesting properties with regard to its representations. While having only finitely many irreducible representations, it also admits logarithmic modules. The classification of the representations of the triplet algebra was carried out by Kausch and Gaberdiel in [21] by showing that its zero mode algebra is $\mathfrak{su}(2)$.

In 2005, Abe extended the theory of a pair of symplectic fermions to a model of d pairs in [1] as well as put it within the rigorous framework of vertex operator algebras. Abe also classified the inequivalent modules of the symplectic fermionic vertex operator algebra with help of Zhu's algebra $A(V)$. The usefulness of $A(V)$ stems from Zhu's theorem, which states that there is a one-to-one correspondence of the inequivalent representations of Zhu's algebra to the modules of a vertex operator algebra. As Brungs and Nahm have shown in [7], Zhu's algebra and the zero mode algebra are isomorphic as Lie algebras under mild conditions.

With respect to the work of Gaberdiel and Kausch on one side and Abe's work on the other, some questions naturally arise. Which Lie algebra takes the place of $\mathfrak{su}(2)$ in the case $d > 1$? Is there more structure in terms of Lie algebras or their representations in the zero mode algebra in this case? Can the isomorphism between Zhu's algebra and the zero mode algebra that was described above be shown explicitly? These are the main questions addressed in this work.

Very roughly, the answer to these questions is as follows: The symplectic fermionic vertex operator algebra forms a $\mathcal{W}$-algebra generated by $2d^2 + d$ vectors of weight 2 and $2d^2 - d - 1$ vectors of weight 3. The zero modes corresponding to the vectors of weight 3 form the $2d$-dimensional symplectic Lie algebra. The zero modes of the vectors of weight 2 furnish a representation for this symplectic Lie algebra. Furthermore, it will be shown by explicit calculation of commutators that the zero mode algebra and Zhu's algebra are isomorphic as Lie algebras, answering the last of the three questions above.

This work is organized as follows. Chapter 2 begins with a motivation of the definition of vertex operator algebras, showing how conformal field theory fits into general quantum field theory, which is taken to be characterized by the Wightman axioms. As a prerequisite for the discussion of vertex operator algebras, a short treatment of formal series, focusing on the concept of locality, is given. This is followed by basic definitions from the theory of vertex operator algebras and some of its results, tailored to the needs of Abe's construction. This includes in particular a reconstruction theorem and the notion of twisted modules.

Chapter 3 treats algebras associated to vertex operator algebras which serve as tools for the classification of modules. Concepts emerging from both mathematics and physics are discussed here, among them Zhu's algebra, the zero mode algebra and W-algebras. The analysis of the triplet algebra at $c = -2$ serves as an example of the classification of irreducible representations with the help of the zero mode algebra.

In Chapter 4, Abe's model of d symplectic fermions is presented after a short summary of some aspects of Kausch's original work on symplectic fermions. While the construction is studied preserving mathematical rigor, contact is also made with more physical notions, such as $\mathcal{W}$-algebras. In order to introduce the reader to explicit calculations in the framework of this model, a general commutator formula is derived. This formula is not needed in the following but provides a good example of the calculational reasoning applied in the last section.

The main results are found in Chapter 5, where the $\mathfrak{sp}(2d)$ Lie algebra structure of the zero mode algebra is explored. Additionally, an explicit ismorphism between Zhu's algebra and the zero mode algebra as Lie algebras is exhibited. A brief summary concludes the exposition.

Chapter 2

Vertex Operator Algebras and CFT

This chapter is an introduction to the mathematical description of conformal field theory through vertex operator algebras. It serves both as a motivation for the mathematical formulation and as a basis for the construction of the symplectic fermion theory and the subsequent investigation into its structure in Chapters 4 and 5.

Conformal field theory is a special quantum field theory which has an extension of the Poincaré group as symmetry group. The connection between both theories can be made precise by deriving the axioms of a chiral algebra describing conformal field theory from the Wightman axioms describing a general quantum field theory. This will be done in the first section, but since this serves as a motivation for the axiomatic theory, the treatment proceeds with less rigor than the rest of this work. Especially questions relating to the functional analysis of operators on Hilbert spaces will be ignored.

In order to formulate the axioms of a vertex operator algebra, a minimal introduction to formal series is necessary. Apart from the most basic facts, the section devoted to formal series contains a discussion of the implications of the notion of locality. This is followed by an introduction to vertex operator algebras, containing also the definition of twisted modules, which are needed in the construction of modules for the symplectic fermionic vertex operator algebra.

2.1 From QFT to CFT

The canonical axiom system of quantum field theory was formulated in the 1950s by Gårding and Wightman. This set of axioms, now commonly known as the Wightman axioms, has not been the only attempt at giving quantum field theory a rigorous foundation. A very elegant axiomatic approach that focuses on the algebra of observables, not on the fields, was proposed by Haag and Kastler in the 1960s. Both sets of axioms suffer from the problem that only the most simple theories have been constructed in terms of them.

However, it is possible to derive the axioms of a vertex operator algebra, which encodes the structure of the vacuum sector of a chiral algebra, directly from the Wightman axioms. It is more convenient in this case to start with the Wightman axioms since vertex operators behave very much like quantum fields. Before the Wightman axioms are discussed, a (very short) digression on the symmetry groups of ordinary quantum field theory and conformal field theory will be given. The treatment in this section will follow the book [24] by Kac. A general reference for a mathematical treatment of quantum field theory is the work [3] by Araki, which contains a nice discussion of both the Wightman axioms and the theory of local observables. For a very broad overview of the topics quantum field theory, conformal field theory and vertex operator algebras, see the book [23] by Gannon.

The fundamental symmetry group of quantum field theory is the Poincaré group, the group of isometries of Minkowski space time. Minkowski space time M is a d-dimensional real vector space with the metric $\eta(\cdot,\cdot)$ given by a symmetric non-degenerate bilinear form of signature $(-,+,\dots,+)$. Writing $|a-b|^2$ for $\eta(a,b)$ with $a,b \in M$, two subsets A and B of M are called space-like separated if for any $a \in A$ and $b \in B$ one has $|a-b|^2 < 0$. The forward cone is defined to be the set $\{x \in M \| x|^2 \geq 0, x_0 \geq 0\}$. The causal order on M is given by $a \geq b$ if and only if $a-b$ lies in the forward cone.

The Poincaré group is the semi-direct product of the group of translations and the Lorentz group. When generating the Lorentz group from its Lie algebra, one stays in the connected component of the identity transformation, which is given by the so called proper orthochronous Lorentz group $\mathscr{L}_+^\uparrow$. Its semi-direct product with the group of translations is denoted $\mathscr{P}_+^\uparrow$. The Lorentz group is generated by ordinary spatial rotations and Lorentz boosts. It is well known that Lorentz boosts can be written as a hyperbolic rotation of the coordinates in Minkowski space by a parameter ϕ called rapidity.

Throughout the rest of this section, let x denote a vector in Minkowski space M and μ an index which runs from 0 to $d-1$, where d is the dimension of M. In conformal field theory, one considers a larger symmetry group containing the Poincaré group which only preserves angles, not lengths. The two most simple conformal transformations are the dilations

$$x^\mu \mapsto \lambda x^\mu,$$

with $\lambda \neq 0$ a real number, and the inversion

$$x^\mu \mapsto \frac{x^\mu}{|x|^2}.$$

The latter transformation is of course only defined for x such that $|x|^2 \neq 0$. As opposed to the Euclidean case, the solutions to $|x|^2 = 0$ form a cone in Minkowski space. By composing an inversion with a translation and a further inversion, one obtains the transformation

$$x^\mu \mapsto \frac{x^\mu}{|x|^2} \mapsto \frac{x^\mu}{|x|^2} + a^\mu \equiv \frac{x^\mu + |x|^2 a^\mu}{|x|^2} \mapsto \frac{x^\mu + |x|^2 a^\mu}{1 + 2(x\cdot a) + |x|^2|a|^2},$$

where $a \in \mathbb{R}^4$ and the fact was used that

$$\left(\frac{x^\mu + |x|^2 a^\mu}{|x|^2}\right)^2 = \frac{1}{|x|^4}\left(|x|^2 + 2|x|^2(x\cdot a) + |x|^4|a|^2\right) = \frac{1}{|x|^2}\left(1 + 2(x\cdot a) + |x|^2|a|^2\right).$$

Since the composition of conformal mappings is conformal, the so called special conformal transformation

$$x^\mu \mapsto \frac{x^\mu + |x|^2 a^\mu}{1 + 2(x\cdot a) + |x|^2|a|^2}$$

is a conformal mapping. Like the inversion, it is not defined globally since the denominator may vanish. In two dimensions, which is the case we are interested in, one introduces the light cone coordinates

$$t = x_0 - x_1, \quad \bar{t} = x_0 + x_1.$$

In terms of these coordinates, the translations, Lorentz transformations and dilations become

$$\begin{array}{llll} [\mathrm{T}] & t \mapsto t + a & [\bar{\mathrm{T}}] & \bar{t} \mapsto \bar{t} + \bar{a} \\ [\mathrm{L}] & t \mapsto e^{\alpha} t & [\bar{\mathrm{L}}] & \bar{t} \mapsto e^{-\alpha} \bar{t} \\ [\mathrm{D}] & t \mapsto \lambda t & [\bar{\mathrm{D}}] & \bar{t} \mapsto \lambda \bar{t}, \end{array}$$

where $a \in \mathbb{R}^2$, $\alpha \in \mathbb{R}_{\geq 0}$ and $\lambda \in \mathbb{R}$ nonzero. Keep in mind that there are no spatial rotations in one time and one space dimensions, so Lorentz transformations are given by boosts only. An important feature of the light cone coordinates lies in the decoupling of the special conformal transformations which take the form

$$[\mathrm{S}] \qquad t \mapsto \frac{t + at\bar{t}}{1 + a\bar{t} + \bar{a}t + a\bar{a}t\bar{t}} = \frac{t}{1 + \bar{a}t}, \tag{2.1}$$

$$[\bar{\mathrm{S}}] \qquad \bar{t} \mapsto \frac{\bar{t} + \bar{a}t\bar{t}}{1 + a\bar{t} + \bar{a}t + a\bar{a}t\bar{t}} = \frac{\bar{t}}{1 + a\bar{t}}. \tag{2.2}$$

From the above equations for conformal transformations in light cone coordinates one can infer the general form

$$\gamma(t, \bar{t}) = \left(\frac{at + b}{ct + d}, \frac{\bar{a}\bar{t} + \bar{b}}{\bar{c}\bar{t} + \bar{d}} \right),$$

for all conformal transformations ([T], [L], [D] and [S] and their equivalents for $\bar{t}$), where $a, b, c, d \in \mathbb{R}$. Since the transformations should be locally invertible, we get the constraint $ad - bc \neq 0$. In the unity component of the conformal group one has $ad - bc > 0$. Normalizing to $ad-bc = 1$ does not change the transformation so that each transformation is described by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

These matrices form the group of unimodular 2×2-matrices.

To see how conformal symmetry is implemented in quantum field theory, a formal description of quantum field theory is needed. It was mentioned above that the formulation best suited to the purpose of making contact with the axioms of vertex operator algebras is given by the Wightman axioms. The following is the definition of a quantum field theory on four-dimensional space time containing only scalar fields according to Wightman and Gårding (in the formulation of [3]).

Definition 2.1. *A quantum field theory is the following data: A complex Hilbert space $\mathscr{H}$ containing a distinguished vector $|0\rangle$, called the vacuum, together with a collection of so called fields Φ_a, which are operator valued distributions, mapping functions defined on the Minkowski space to linear operators densely defined on $\mathscr{H}$. These data are required to satisfy the following Wightman axioms*

1. *(quantum field) The operators $\Phi_a(f)$ are given for each C^∞-function with compact support on the Minkowski space $\mathbb{R}^4$. Each $\Phi_a(f)$ and its Hermitian conjugate operator $\Phi_a(f)^*$ are defined at least on a common dense linear subset $\mathfrak{D}$ of the Hilbert space $\mathscr{H}$ and $\mathfrak{D}$ satisfies*

$$\Phi_a(f)\mathfrak{D} \subset \mathfrak{D}, \quad \Phi_a(f)^*\mathfrak{D} \subset \mathfrak{D}$$

for any a and any $f \in C_0^\infty(M)$.

2. *(relativistic symmetry) On $\mathscr{H}$ there exists a unitary representation $U(q,\Lambda)$ of $\mathscr{P}_+^\uparrow$ ($q \in \mathbb{R}^4$, $\Lambda \in \mathscr{L}_+^\uparrow$) satisfying*

$$U(q,\Lambda)\mathfrak{D} = \mathfrak{D},$$
$$U(q,\Lambda)\Phi_a(f)U(q,\Lambda)^{-1} = \Phi((q,\Lambda)f), \tag{2.3}$$

where $(q,\Lambda)f(x) = f(\Lambda^{-1}(x-q))$.

3. *(locality) If the supports of $f \in C^\infty(M)$ and $h \in C^\infty(M)$ are space-like separated, then for any vector $\psi \in \mathfrak{D}$*

$$\Phi_a(f)\Phi_b(h)\psi = (-1)^{p(a)p(b)}\Phi_b(h)\Phi_a(f)\psi,$$

where the parity p associates to each field index a number in $\mathbb{Z}_2$.

4. *(vacuum state) The vacuum $|0\rangle$ satisfies*
   (a) $U(q,\Lambda)|0\rangle = |0\rangle$
   (b) *The set of all vectors obtained by acting with an arbitrary polynomial P of the fields on $|0\rangle$ is dense in $\mathscr{H}$.*
   (c) *The spectrum of the translation group $U(q,1)$ on the orthogonal complement $|0\rangle^\perp$ is contained in*

$$\bar{V}_m = \{p|(p,p) \geq m^2, p^0 > 0\} \quad (m > 0).$$

The usual approach for defining the Poincaré group representation on $\mathscr{H}$ is to first determine the Poincaré Lie algebra and then choose operators on $\mathscr{H}$ by classical correspondence considerations, such that these operators form a representation of the Poincaré Lie algebra. This representation is then carefully (one has to go over to the universal covering group) lifted to a representation of the whole group by exponentiation.

The canonical choice for the representation of P_μ, the generator of translations, is $-i\partial_\mu$. By the covariance law from axiom 2, this implies for the action on fields that

$$i[P_\mu, \Phi_a] = \partial_\mu \Phi_a. \tag{2.4}$$

Applying this equation to the vacuum vector and using its $U(q,1)$-invariance, one obtains

$$\Phi_a(x+q)|0\rangle = \exp\left(i\sum_\mu q_\mu P_\mu\right)\Phi_a(x)|0\rangle, \tag{2.5}$$

where $\Phi_a(x)$ is taken to be a short form for $\Phi_a(f(x))$. With respect to the conformal symmetry, one first fixes the covariance law, which then implies the commutator. A quantum field theory is called conformal if the unitary representation of the Poincaré group in $\mathscr{H}$ extends to a unitary representation of the conformal group parameterized by $(q,\Lambda,b) \mapsto U(q,\Lambda,b)$, such that the vacuum vector $|0\rangle$ is still fixed and such that we have the following covariance law for a scalar field under special conformal transformations

$$U(0,1,b)\Phi_a(x)U(0,1,b)^{-1} = \phi(b,x)^{-\Delta_a}\Phi_a(x^b). \tag{2.6}$$

Here, $\Delta_a \in \mathbb{R}$ is called the conformal weight of the field Φ_a and

$$\phi(b, x) = 1 + 2xb + |x|^2|b|^2.$$

Using Poincaré covariance from Axiom 2, we can extend (2.6) to full conformal covariance:

$$U(q, \Lambda, b)\Phi_a(x)U(q, \Lambda, b)^{-1} = \phi(b, x)^{-\Delta_a}\Phi_a\left((q, \Lambda, b)x\right). \tag{2.7}$$

One can show that this implies that the special conformal generators Q_μ, $\mu = 0, \ldots, 3$ act like

$$i[Q_\mu, \Phi_a(x)] = (|x|^2\partial_{x_\mu} - 2\eta_\mu x_\mu E - 2\Delta_a\eta_\mu x_\mu)\Phi_a(x), \tag{2.8}$$

where $E = \sum_{m=0}^{3} x_m\partial_{x_m}$ is the Euler operator and η_μ are the coefficients of the metric ($\eta_0 = 1$, $\eta_\mu = -1$ for $\mu \geq 1$).

From now on, the discussion will be specialized to the case $d = 2$. In light cone coordinates, equation (2.7) can be written as

$$U(\gamma)\Phi_a(t, \bar{t})U(\gamma)^{-1} = (ct + d)^{-2\Delta_a}(\bar{c}\bar{t} + \bar{d})^{-2\bar{\Delta}_a}\Phi_a(\gamma(t, \bar{t})),$$

where $\Delta_a = \bar{\Delta}_a$. This condition is usually dropped because of the decoupling of the special conformal transformations according to (2.1) and (2.2). One defines the following operators in analogy to the light cone coordinates:

$$P = \frac{1}{2}(P_0 - P_1), \qquad \bar{P} = \frac{1}{2}(P_0 + P_1),$$
$$Q = \frac{1}{2}(Q_0 - Q_1), \qquad \bar{Q} = \frac{1}{2}(Q_0 + Q_1).$$

The action of these operators on fields can be calculated from (2.4) and (2.8) and is given by

$$i[P, \Phi_a(t, \bar{t})] = \partial_t\Phi_a(t, \bar{t}) \tag{2.9a}$$
$$i[\bar{P}, \Phi_a(t, \bar{t})] = \partial_{\bar{t}}\Phi_a(t, \bar{t}) \tag{2.9b}$$
$$i[Q, \Phi_a(t, \bar{t})] = \left(t^2\partial_t + 2\Delta_a t\right)\Phi_a(t, \bar{t}) \tag{2.9c}$$
$$i[\bar{Q}, \Phi_a(t, \bar{t})] = \left(\bar{t}^2\partial_{\bar{t}} + 2\bar{\Delta}_a\bar{t}\right)\Phi_a(t, \bar{t}). \tag{2.9d}$$

In light cone coordinates, equation (2.5) becomes

$$\Phi_a(t + q, \bar{t} + \bar{q})|0\rangle = e^{i(qP + \bar{q}\bar{P})}\Phi_a(t, \bar{t})|0\rangle. \tag{2.10}$$

In the following, we are interested in analytic continuations, permitting complex values for the light cone coordinates t, $\bar{t}$. Since the forward cone is given by the domain $t \geq 0$, $\bar{t} \geq 0$ in light cone coordinates, the joint spectrum of P and $\bar{P}$ lies by the vacuum axiom in this domain. Hence, the operator $\exp(itP + i\bar{t}\bar{P})$ is defined on $\mathcal{D}$ for all values $\mathrm{Im}t \geq 0$, $\mathrm{Im}\bar{t} \geq 0$.

Furthermore, by spectral decomposition, $\exp(i(qP + \bar{q}\bar{P}))$ is, as function of q, the Fourier transform of the operator valued characteristic function of the domain $p \geq 0$, $\bar{p} \geq 0$. Thus, by formula (2.10) the $\mathcal{D}$-valued distribution $\Phi_a|0\rangle$ extends analytically to a function in the domain

$$\{t|\mathrm{Im}t > 0\} \times \{\bar{t}|\mathrm{Im}\bar{t} > 0\} \subset \mathbb{C}^2.$$

It follows, that the value $\Phi_a(t,\bar{t})|0\rangle$ makes sense for Im$t > 0$, Im$\bar{t} > 0$. By translating with equation (2.10), we see that this value is non-zero unless Φ_a vanishes.

As was noted above, conformal transformations are not defined everywhere on the Minkowski plane. In order to remedy this, one considers the compactification of Minkowski space given by

$$z = \frac{1+it}{1-it}, \quad \bar{z} = \frac{1+i\bar{t}}{1-i\bar{t}}.$$

This maps the domain of the above analytic continuation, Im$t > 0$, Im$\bar{t} > 0$, to the domain $|z| < 1$, $|\bar{z}| < 1$. One defines new fields in terms of these variables for $|z| < 1$, $|\bar{z}| < 1$:

$$Y(a,z,\bar{z}) = \frac{1}{(1+z)^{2\Delta_a}(1+\bar{z})^{2\bar{\Delta}_a}}\Phi_a(t,\bar{t}),$$

where $t = i\frac{1-z}{1+z}$, $\bar{t} = i\frac{1-\bar{z}}{1+\bar{z}}$. Note that by the above definition of the coordinates z and $\bar{z}$, $Y(a,z,\bar{z})|0\rangle|_{z=0,\bar{z}=0}$ is a well defined vector in $\mathcal{D}$, denoted by a. Furthermore, $Y(a,z,\bar{z}) \mapsto a$ is a linear injective map since by (2.10) a is zero if and only if Φ_a is zero. In order to recast (2.9a)-(2.9d) in the form usually employed in conformal field theory, one defines

$$\begin{aligned} T &= \frac{1}{2}(P + [P,Q] - Q) \\ H &= \frac{1}{2}(P + Q) \\ T^* &= \frac{1}{2}(P - [P,Q] - Q)\,; \end{aligned}$$

and similarly for $\bar{T}$, $\bar{H}$, $\bar{T}^*$. In terms of these new operators, equations (2.9a)-(2.9d) take the form

$$[T, Y(a,z,\bar{z})] = \partial_z Y(a,z,\bar{z}) \tag{2.11a}$$

$$[H, Y(a,z,\bar{z})] = (z\partial_z + \Delta_a)Y(a,z,\bar{z}) \tag{2.11b}$$

$$[T^*, Y(a,z,\bar{z})] = (z^2\partial_z + 2\Delta_a z)Y(a,z,\bar{z}); \tag{2.11c}$$

and similarly for $\bar{T}$, $\bar{H}$, $\bar{T}^*$. Applying the last two equations to the vacuum vector and setting $z = \bar{z} = 0$ yields

$$Ha = \Delta_a a, \quad T^* a = 0.$$

Since P and $\bar{P}$ are positive semidefinite self-adjoint operators on $\mathscr{H}$ and the same can be proven for Q and $\bar{Q}$, H is also a positive semidefinite self-adjoint operator. As eigenvalue of H, the conformal weight Δ_a consequently has to be non-negative.

The locality axiom, which is central to quantum field theory, has very interesting implications for conformal field theory in two dimensions. Consider the locality axiom in light cone coordinates

$$\Phi_a(t,\bar{t})\Phi_b(t',\bar{t}') = (-1)^{p(a)p(b)}\Phi_b(t',\bar{t}')\Phi_a(t,\bar{t}),$$

if $(t-t')(\bar{t}-\bar{t}') < 0$. For right chiral fields, which are fields with $\partial_{\bar{t}}\Phi_a = 0$, there is no $\bar{t}$ dependence because of (2.9b) and (2.10). Thus, locality specializes to

$$\Phi_a(t)\Phi_b(t') = (-1)^{p(a)p(b)}\Phi_b(t')\Phi_a(t),$$

if $t \neq t'$. Consequently, the support of the commutator is confined to $t = t'$ and one assumes that it has the form

$$[\Phi_a(t), \Phi_b(t')] = \sum_{j\geq 0} \partial_{t'}^j \delta(t - t')\Psi^j(t'),$$

for some fields $\Psi^j(t')$ which are required to satisfy the Wightman axioms but not conformal covariance. Hence, we may add them to the theory and write

$$[Y(a, z), Y(b, w)] = \sum_{j\geq 0} \partial_w^j \delta(z - w)Y(c_j, w). \tag{2.12}$$

By calculating the conformal weight of the fields $Y(c_j, w)$, one concludes that in order for all occurring weights to be positive, the sum on the right hand side must be finite. Interpreted as an equality between formal series (see Theorem 2.7), equation (2.12) can be shown to be equivalent to

$$(z - w)^N [Y(a, z), Y(b, w)] = 0$$

for $N \in \mathbb{N}$ large enough. Assuming that one can expand the chiral field $Y(a, z)$ in a series

$$Y(a, z) = \sum_n a_{(n)} z^{-n-1},$$

the coefficients are operators on $\mathcal{D}$. Denote by V the subspace of $\mathcal{D}$ spanned by all polynomials in the $a_{(n)}$ applied to the vacuum vector $|0\rangle$. Clearly, V is invariant with respect to all $a_{(n)}$ and, by (2.11a), with respect to T. Summarizing all of the above, we arrive at the axioms of a right chiral algebra.

Definition 2.2. *A right chiral algebra is a vector space V, the space of states, together with a distinguished non-zero vector $|0\rangle$ called the vacuum, a translation operator $T \in$ End$V$ and a collection of fields*

$$Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$$

for each $a \in V$ with $a_{(n)} \in$ End$V$, such that the following axioms hold:

1. *(translation covariance) $[T, Y(a, z)] = \partial Y(z, a)$;*
2. *(vacuum) $T|0\rangle = 0$ and $Y(a, z)|0\rangle|_{z=0} = a$;*
3. *(completeness) polynomials in the $a_{(n)}$'s with $n < 0$ applied to $|0\rangle$ span V.*
4. *(locality)*

$$(z - w)^N Y(a, z)Y(b, w) = (-1)^{p(a)p(b)}(z - w)^N Y(b, w)Y(a, z)$$

for some $N \in N$ depending on $a, b \in V$.

2.2 Formal Calculus

An important ingredient for the theory of vertex operator algebras is formal calculus. Since we are interested in the application to conformal field theory, we would like to have objects which in some way behave as operator valued distributions. It may be surprising that these objects, the vertex operators, can be defined in a purely algebraic way using formal series. While avoiding any analytical convergence questions, this approach still captures much of the desired behavior of quantum operators. Eventually, one is interested in going over to complex variables, especially when describing measurable objects like correlation functions. While this is possible in a well defined way, the algebraic formulation will be all that is needed for this work. Most general works on vertex operator algebras also discuss formal calculus. For a relatively extensive and pedagogical treatment see the book [26] by Lepowsky and Li. The approach presented here is influenced by the viewpoint taken in the book [24] by Kac.

For a vector space V, the set of formal Laurent series with coefficients in V is defined to be

$$V[[x^{\pm 1}]] = \left\{ \sum_{n\in\mathbb{Z}} v_n x^n \,|v_n \in V \right\}.$$

Using the standard notation, one also defines the following subspaces of $V[[x^{\pm 1}]]$:

$$\begin{aligned} V[x] &= \left\{ \sum_{n\in\mathbb{N}} v_n x^n \,\middle|\, v_n \in V, v_n = 0 \text{ for } n \text{ sufficiently large} \right\} \\ V[x^{\pm 1}] &= \left\{ \sum_{n\in\mathbb{Z}} v_n x^n \,\middle|\, v_n \in V, v_n = 0 \text{ for all but finitely many } n \right\} \\ V[[x]] &= \left\{ \sum_{n\in\mathbb{N}} v_n x^n \,|v_n \in V \right\} \\ V((x)) &= \left\{ \sum_{n\in\mathbb{Z}} v_n x^n \,\middle|\, v_n \in V, v_n = 0 \text{ for } n \text{ sufficiently small} \right\}. \end{aligned}$$

On the space $V[[x^{\pm 1}]]$, two basic operations are defined. Let $f \in V[[x^{\pm 1}]]$ be given by $f(x) = \sum_{n\in\mathbb{Z}} v_n z^n$. Then the formal derivative $\frac{d}{dx}$ and the formal residue Res_x are defined as the operations

$$\begin{aligned} \frac{d}{dx} f(x) &= \sum_{n\in\mathbb{Z}} n v_n z^{n-1}, \\ \mathrm{Res}_x f(x) &= v_{-1}. \end{aligned}$$

With respect to differentiation, the following abbreviation common to analysis will be used:

$$\partial_x^{(n)} f(x) = \frac{1}{n!} \partial_x^n f(x).$$

In the theory of vertex operator algebras, the formal series of interest have coefficients in the space of endomorphisms of a vector space V. In this case, the elements of $\mathrm{End}(V)[[x^{\pm 1}]]$ are conventionally written in the form $\sum_n v_n x^{-1-n}$. Such formal series are called vertex operators if they have certain additional properties which will be described below.

We will frequently need formal sums and formal products of vertex operators. While infinite sums of operators as coefficients will be allowed, they are restricted by the following "universal condition": The coefficient of every monomial of a formal sum or formal product of vertex operators has to act as a finite sum of operators when it is applied to any fixed, but arbitrary, vector in space. For the precise prerequisites for the existence of sums and products as well as limits, see Chapter 2 of [26].

Of course all the preceding definitions can be generalized to the case of several commuting variables in a straightforward way. The following is a definition of the convention for the evaluation of binomials in formal variables, which have to be treated with a bit of care.

Definition 2.3. *The expression $(x+y)^n$ for $n \in \mathbb{Z}$ (in particular, $n < 0$) is defined to be the formal series*

$$(x+y)^n = \sum_{k\in\mathbb{N}} \binom{n}{k} x^{n-k} y^k,$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Note that the order matters: binomial expressions are to be expanded in nonnegative integral powers of the second variable. In particular, while $(x+y)^n = (y+x)^n$ for $n \geq 0$, one cannot exchange the variables for $n < 0$.

Sometimes one also needs a formal exponential notation. Let $S \in x \cdot \mathrm{End}(V)[[x]]$, so that S has no constant term. Then the expression

$$e^S = \sum_{n\in\mathbb{N}} \frac{1}{n!} S^n$$

is well defined by the universal condition and an element of $\mathrm{End}(V)[[x]]$. Consequently, e^S acts as endomorphism of $V[[x]]$. Since one often wants to consider all higher derivatives of a formal expression at once, one would like to know how exponentials of derivatives behave. It is well known that the endomorphisms

$$T_{p(x)} = p(x)\frac{d}{dx}$$

of $\mathbb{C}[x^{\pm1}]$ act as a derivations of $\mathbb{C}[x^{\pm1}]$. By induction, this is also true for the exponential of $T_{p(x)}$. The following Taylor theorem characterizes the action of derivations of this type on formal series.

Theorem 2.4. *Let $v(x) \in V[[x^{\pm1}]]$. Then*

$$e^{y\frac{d}{dx}} v(x) = v(x+y)$$

(and these expressions exist). Also,

$$e^{yx\frac{d}{dx}} v(x) = v(e^y x),$$

where the series $v(e^y x)$ is understood as an element of $V[[x^{\pm1}]][[y]]$. Furthermore,

$$e^{yx^{n+1}\frac{d}{dx}} v(x) = v((x^{-n} - nx)^{-\frac{1}{n}})$$

for $n \in \mathbb{Z}$ and $n \neq 0$.

Proof. See Proposition 8.3.1, p. 182 and Proposition 8.3.10, p. 186 in [19]. □

The most important formal series for the formulation of the theory of vertex operator algebras is

$$\delta(x) = \sum_{m\in\mathbb{Z}} x^m.$$

The δ-series is intimately linked to the so called expansions of zero, which are, informally speaking, the difference between two expansions of the same rational function in opposite directions. In particular, the formal series $\delta(x)$ can be written as an expansion of zero. This fact proves to be helpful in deriving certain identities involving $\delta(x)$ which are relevant to the notion of locality of formal series. Indeed, the equivalence of (2.12) to locality in the sense of Definition 2.6 below can be proven in this way.

Consider the field $\mathbb{C}(x)$ of rational functions in the indeterminate x over $\mathbb{C}$. Define two embeddings

$$\begin{aligned}\iota_+ &: \mathbb{C}(x) \hookrightarrow \mathbb{C}((x))\\ \iota_- &: \mathbb{C}(x) = \mathbb{C}(x^{-1}) \hookrightarrow \mathbb{C}((x^{-1}))\end{aligned}$$

in the following way: For $f \in \mathbb{C}(x)$, $\iota_+ f$ is the expansion of f as a formal Laurent series in x, and $\iota_- f$ is its expansion as a formal Laurent series in x^{-1}. Combining these two maps, one defines the linear map $\Theta : \mathbb{C}(x) \to \mathbb{C}[[x^{\pm1}]]$ by

$$f \mapsto \iota_+ f - \iota_- f.$$

The elements of the image ImΘ are called the expansions of zero. Again, it is possible to generalize this notion to several variables. But only the standard identity (2.14) below will be needed, so that the straightforward generalization will be omitted here. Consider the one-variable example for an expansion of zero,

$$\Theta\left((1-x)^{-1}\right) = (1-x)^{-1} + (x-1)^{-1} = \delta(x). \tag{2.13}$$

It is important to keep in mind that the binomial expansion convention implies that $(1-x)^{-1}$ and $-(x-1)^{-1}$ are not equal. The second equivalence in (2.13) is easily seen to be true by writing out the series expansion. The first equivalence is proven by

$$\iota_+((1-x)^{-1}) = \sum_{n\geq 0} x^n = (1-x)^{-1}$$

and

$$\begin{aligned}\iota_-((1-x)^{-1}) &= \iota_-(-x^{-1}(1-x^{-1})^{-1})\\ &= -\sum_{n<0} x^n = -x^{-1}(1-x^{-1})^{-1} = (-x+1)^{-1}.\end{aligned}$$

In exactly the same way, one can prove the generalization to the case of two variables,

$$x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = \Theta\left((x_1-x_2)^{-1}\right) = (x_1-x_2)^{-1} + (x_2-x_1)^{-1}. \tag{2.14}$$

This expansion of zero can be used to derive a fact related to the concept of locality of vertex operators. First, it is clear that the differentiation operator d/dx commutes with Θ because the ι mappings commute with d/dx. Therefore, one has the equality

$$\frac{1}{n!}\left(\frac{\partial}{\partial x_2}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = \frac{1}{n!}\Theta\left(\left(\frac{\partial}{\partial x_2}\right)^n (x_1 - x_2)^{-1}\right) = \frac{(-1)^n}{n!}\left(\frac{\partial}{\partial x_1}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right).$$

Recalling the fact that $(x + y)^n = (y + x)^n$ only for $n \geq 0$ from the remark in the definition of the binomial expansion, it can be concluded that

$$(x_1 - x_2)^m \left(\frac{\partial}{\partial x_2}\right)^n x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) = n!(x_1 - x_2)^m \left((x_1 - x_2)^{-n-1} - (-x_2 + x_1)^{-n-1}\right) \qquad (2.15)$$

vanishes for $m > n$. All formal series which vanish after multiplication of $(x_1 - x_2)^n$ for some $n > 0$ can be characterized in the following way.

Proposition 2.5. *The null space of the operator of multiplication by* $(x_1 - x_2)^N$, $N \geq 1$ *in* $V[[x_1^{\pm 1}, x_2^{\pm 1}]]$ *is*

$$\sum_{j=0}^{N-1} \partial_{x_1}^{(j)}\delta(x_1 - x_2)V[[x_2^{\pm 1}]].$$

Furthermore, any element $a(x_1, x_2)$ *in the null space is uniquely represented in the form*

$$a(x_1, x_2) = \sum_{j=0}^{N-1} c^j(x_2)\partial_{x_2}^{(j)}\delta(x_1 - x_2),$$

where the $c^j(x_2)$ *are given by* $c^j(x_2) = \mathrm{Res}_{x_1} a(x_1, x_2)(x_1 - x_2)^j$.

Proof. See [24], Corollary 2.2 and its proof. □

2.3 Vertex Operator Algebras

It goes without saying that the present section contains only the most basic notions from the theory of vertex operator algebras. Though vertex operator algebras are a relatively young field, the theory made rapid progress in the last 20 years. The axiomatic description was first given by Borcherds in [6] and has subsequently been developed by numerous authors. Among the texts which may serve as general introduction are the now classic book [19] by Frenkel, Lepowsky and Meurman which summarizes the authors' findings on the connection between vertex operator algebras and the largest sporadic simple group termed the monster, the concise axiomatic exposition [26] by Lepowsky and Li and the work [17] by Frenkel and Ben-Zvi which studies the theory from the perspective of algebraic geometry. A reference stressing the physical viewpoint is the book by Kac, [24]. For an introduction to the theory stressing the super formalism which is introduced below, see the book [31] by Xu.

From section 2.1, we know that fields in a general quantum field theory are taken to be operator valued distributions. In the theory of vertex operator algebras, fields are modeled by formal series with endomorphisms of a vector space as coefficients. In order to implement super commutation rules, it will be assumed that fields are formal series with coefficients in End$V$ where the vector space V has a decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Such a vector space V is called a superspace. Here, $\bar{0}$ and $\bar{1}$ denote the cosets in $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ of 0 and 1. An element is defined to have parity $p(a) \in \mathbb{Z}_2$ if $a \in V_{p(a)}$.

An associative superalgebra $U = U_{\bar{0}} \oplus U_{\bar{1}}$ is a $\mathbb{Z}_2$-graded associative algebra, i.e. $U_\alpha U_\beta \subset U_{\alpha+\beta}$, $\alpha, \beta \in \mathbb{Z}_2$. The endomorphisms of a superspace V carry a natural superalgebra structure, given by the decomposition of End$V$ into the direct sum of the subspaces

$$(\mathrm{End}V)_\alpha = \left\{a \in \mathrm{End}V | aV_\beta \subset V_{\alpha+\beta}\right\}.$$

Define a bracket on this associative superalgebra by setting

$$[a, b] = ab - p(a, b)ba, \tag{2.16}$$

where $a \in (\mathrm{End}V)_\alpha$, $b \in (\mathrm{End}V)_\beta$ and $p(a, b) = (-1)^{\alpha\beta}$. Thus, the bracket of an even element with any other element is the ordinary commutator. In this chapter, the bracket $[\cdot, \cdot]$ is taken to be the general commutator defined by (2.16) while in the following chapters the ordinary commutator will be denoted by $[\cdot, \cdot]$ and the bracket of two odd elements, the anti-commutator, will be written as $\{\cdot, \cdot\}$.

In the following, a field will be assumed to be a formal series $A(x) = \sum_{j\in\mathbb{Z}} A_j x^{-j-1}$ with coefficients in the space of endomorphisms of a superspace V, satisfying the condition that for any $v \in V$, $A_j v = 0$ for j large enough. Suppose V is graded, i.e.

$$V = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} V_n.$$

Then one defines the weight of a homogeneous vector $v \in V_n$ by $\mathrm{wt}v_n = n$. Furthermore, a linear operator $\phi : V \to V$ satisfying $\phi(V_n) \subset V_{n+m}$ for all $n \in \frac{1}{2}\mathbb{Z}$ is said to be homogeneous of weight m.

The locality axiom discussed in section 2.1 is central to quantum field theory. It can also be chosen as the axiom carrying most of the structure of a vertex operator algebra, though it turns out that there exist several equivalent formulations of the locality axiom. As an alternative to locality, one can adopt the Jacobi identity which is preferred in most of the mathematical literature. In this work, however, an effort is made to formulate the axioms of a vertex operator algebra in a way closely related to the physical language of chiral algebras by adopting locality as an axiom.

Definition 2.6. *Two fields $A(x_1)$ and $B(x_2)$ are called mutually local if there exists an $N \in \mathbb{Z}_{>0}$ such that*

$$(x_1 - x_2)^N [A(x_1), B(x_2)] = 0. \tag{2.17}$$

Since the bracket depends on the parity of the coefficients A_j and B_j of $A(x_1)$ and $B(x_2)$, it will be assumed that all coefficients have the same parity, denoted by $p(A)$ and $p(B)$, respectively. Thus, we have $p(A, B) = (-1)^{p(A)p(B)}$ for the parity function in equation (2.16).

When considering products of fields, one notices that for a field $A(x)$, the product $A(x)A(x)$ does in general not exist as a product of formal series because the coefficients violate the universal condition discussed above. This problem is cured by a standard procedure in quantum field theory called normal ordering. Given a field $A(x) = \sum_{n\in\mathbb{Z}} A_{(n)} x^{-n-1}$, one defines

$$A(x)_- = \sum_{n\geq 0} A_{(n)} x^{-n-1} \quad \text{and} \quad A(x)_+ = \sum_{n<0} A_{(n)} x^{-n-1}.$$

This splitting of $A(x)$ into a positive and a negative part is the only one satisfying

$$(\partial A(x))_{\pm} = \partial(A(x)_{\pm}). \tag{2.18}$$

For two fields $A(x_1)$ and $B(x_2)$, the normal ordered product $: A(x_1)B(x_2) :$ is defined by

$$: A(x_1)B(x_2) := A(x_1)_+ B(x_2) + p(A,B) B(x_2) A(x_1)_-. \tag{2.19}$$

Note that one is allowed to set $x_1 = x_2$ in (2.19). When applied to a vector, the first term is a product of two lower-truncated formal Laurent series and the second is a product of an unrestricted formal Laurent series and a Laurent polynomial. Thus, none of the terms violates the universal condition on the coefficients. The following theorem relates the normal ordered product to the expansion of a product of two mutually local fields.

Theorem 2.7. *The following properties are equivalent:*

1. *The fields $A(x_1)$ and $B(x_2)$ are mutually local.*
2. *There exist fields $C^j(x_2) \in \mathrm{End}(V)[[x_2^{\pm 1}]]$, $j \in \{0, \ldots, N-1\}$, such that*

$$[A(x_1), B(x_2)] = \sum_{j=0}^{N-1} \partial_{x_2}^{(j)} \delta(x_1 - x_2) C^j(x_2).$$

3. *There exist fields $C^j(x_2) \in \mathrm{End}(V)[[x_2^{\pm 1}]]$, $j \in \{0, \ldots, N-1\}$, such that*

$$A(x_1)B(x_2) = \sum_{j=0}^{N-1} \left(\iota_{1,2} \frac{1}{(x_1 - x_2)^{j+1}} \right) C^j(x_2) + : A(x_1)B(x_2) :, \tag{2.20}$$

where $\iota_{1,2}(f(x_1, x_2))$ is the formal Laurent series expansion of $f(x_1, x_2)$ involving only finitely many negative powers of x_2.

4. *There exist fields $C^j(x_2) \in \mathrm{End}(V)[[x_2^{\pm 1}]]$, $j \in \{0, \ldots, N-1\}$, such that*

$$[A_{(m)}, B_{(n)}] = \sum_{j=0}^{N-1} \binom{m}{j} C^j_{(m+n-j)}, \tag{2.21}$$

where $m, n \in \mathbb{Z}$.

Proof. See [24], Theorem 2.3, p.25. □

In the physical literature, one encounters equation (2.20) usually in the form

$$A(x_1)B(x_2) = \sum_{j=0}^{N-1} \frac{C^j(x_2)}{(x_1 - x_2)^{j+1}} + : A(x_1)B(x_2) :, \tag{2.22}$$

or, writing out only the singular part

$$A(x_1)B(x_2) \sim \sum_{j=0}^{N-1} \frac{C^j(x_2)}{(x_1 - x_2)^{j+1}}. \tag{2.23}$$

The two equations (2.22) and (2.23) are usually referred to as the operator product expansion, abbreviated OPE. It's a remarkable fact that by (2.21), the singular part of the OPE determines all the brackets between the modes of two mutually local fields, making it an important calculational tool.

With the necessary tools of formal calculus at hand, the definition of a vertex operator algebra can now be given. However, in most of the mathematical literature, vertex operator algebras are defined as vertex algebras with additional properties. As some theorems formulated in terms of vertex algebras will be needed in the construction described in Chapter 4, this approach will be adopted here, too.

Definition 2.8 (vertex superalgebra). *Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a superspace . Suppose we have a linear map $v \mapsto Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$ which associates to each $v \in V$ a field $Y(v,z)$ and we have a distinguished vector $\mathbf{1} \in V$ in $V_{\bar{0}}$. These data are subject to the following properties $\forall v \in V$:*

(VA1) (vacuum) The vertex operator $Y(\mathbf{1}, x)$ is the identity (i.e. $\mathbf{1}_n v = \delta_{n,-1} v$).

(VA2) (state-field correspondence) The vector $Y(v,x)|_{x=0}\, \mathbf{1}$ exists and equals v.

(VA3) (locality) All fields $Y(v,x)$ are mutually local.

(VA4) (translation covariance) There exists a linear operator $T : V \to V$ of even parity, such that for any $v \in V$

$$[T, Y(v,x)] = \partial_x Y(v,x)$$

and $T\,\mathbf{1} = 0$.

A vertex superalgebra is called $\frac{1}{2}\mathbb{Z}$-graded if V is a $\frac{1}{2}\mathbb{Z}$-graded vector space, $\mathbf{1}$ is a vector of weight 0 and T is a linear operator of weight 1. A vertex algebra is a vertex superalgebra V satisfying $V^{\bar{1}} = 0$. Additionally, if there is a grading given on V then $V_n = 0$ for $n \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ for a vertex algebra.

In the following, the word "super" will often be left away when assertions apply to vertex superalgebras as well as vertex algebras. Often, a vertex superalgebra V is denoted by the triple $(V, Y, \mathbf{1})$ in an obvious notation.

As mentioned above, in a large part of the mathematical literature another set of axioms relying on the so called Jacobi identity is preferred. The Jacobi identity has the disadvantage that it has no immediate interpretation in physical terms. However, it allows for a more concise formulation of the axioms, concentrating a large amount of structure in a single identity. In the above axioms of a vertex superalgebra, the locality axiom and the translation covariance axiom may be replaced by the Jacobi identity

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u,x_1)Y(v,x_2) - (-1)^{p(u)p(v)} x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v,x_2)Y(u,x_1) \\ =&x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u,x_0)v, x_2), \end{aligned} \tag{2.24}$$

where $u, v \in V$ arbitrary. Here, the parity $p(v)$ of a vector $v \in V$ is defined by the parity of the modes $v_{(n)}$. Equation (2.24) has several interesting implications. One of them is the commutator formula which, restricting the discussion to vertex algebras where $p(u) = p(v) = 0$, reads

$$[Y(u, x_1), Y(v, x_2)] = \mathrm{Res}_{x_0} x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2). \qquad (2.25)$$

It is derived from (2.24) by taking Res_{x_0}. Note that this describes the failure of two vertex operators to commute. Equation (2.17) characterizing locality (also called weak commutativity in the mathematical literature) can be obtained from (2.25) by multiplying with the polynomial $(x_1 - x_2)^k$ where k large enough. This may be thought of as clearing of a formal pole.

Similarly to the derivation of the commutator formula (2.25), by equating the coefficients of $x_0^{-1} x_1^{-m-1} x_2^{-n-1}$ in the Jacobi identity, one obtains

$$[u_m, v_n] = \sum_{i \geq 0} \binom{m}{i} (u_i v)_{m+n-i}, \qquad (2.26)$$

where $m, n \in \mathbb{Z}$. By the regularity condition of vertex algebras, $(u_i v)_{m+n-i}$ vanishes for i large enough. Thus, the sum on the right hand side of (2.26) is finite and the equation states that the modes form a Lie algebra with the commutator as bracket.

Conformal symmetry. From a physical point of view, vertex (super)algebras are still lacking an important ingredient of two-dimensional conformal field theory, the Virasoro algebra. The Virasoro algebra $\mathfrak{Vir}$ is the one dimensional central extension of the Witt algebra, which is the conformal symmetry algebra of classical theories in two dimensions. The rigorous description of conformal symmetry in two-dimensional quantum theories has some subtle points which are often ignored in the physical literature. For a discussion of these points and proofs of the following assertions, the reader is referred to the lecture on conformal field theory by Schottenloher, [30].

Roughly, one arrives at the Virasoro algebra as follows. First, one considers the conformal group on the Minkowski plane $\mathbb{R}^{1,1}$. As mentioned in the discussion of conformal transformations in section 2.1, one compactifies the Minkowski plane in order for the transformations to be defined everywhere. The compactification of the Minkowski plane is $\mathbb{R}^{1,1} \to \mathbb{S}^1 \times \mathbb{S}^1$. One can show that the conformal group of this compactification is then given by

$$\mathrm{Conf}(\mathbb{R}^{1,1}) \cong \mathrm{Diff}_+(\mathbb{S}^1) \times \mathrm{Diff}_+(\mathbb{S}^1),$$

where $\mathrm{Diff}_+(\mathbb{S}^1)$ is the group of all orientation preserving diffeomorphisms of the circle. The Lie algebra of $\mathrm{Diff}_+(\mathbb{S}^1)$ is the space of smooth vector fields of the circle, $\mathrm{Vect}(\mathbb{S}^1)$. Its complexification $\mathrm{Vect}(\mathbb{S}^1) \times \mathbb{C}$ has a dense subalgebra called the Witt algebra $\mathfrak{W}$. The Witt algebra is the symmetry algebra of a classical theory with conformal symmetry.

After quantization, a symmetry is required to be a unitary representation of the symmetry group on the projective Hilbert space of states. One then tries to lift these representations on the projective Hilbert space to unitary representations on the Hilbert space itself. To achieve this, one has to go over to central extensions of the universal covering of the classical symmetry group. On the level of Lie algebras in the case of conformal symmetry one has to study the central extensions of the Witt algebra. Its unique central extension is given by the Virasoro algebra $\mathfrak{Vir} = \mathfrak{W} \oplus \mathbb{C}C$, which is defined by the bracket relations

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{n,-m}\frac{m(m^2-1)}{12}C, \tag{2.27}$$

$$[L_m, C] = 0. \tag{2.28}$$

The Virasoro algebra is the symmetry algebra of a quantized theory with conformal symmetry in the above sense. The following definition explains how this symmetry is incorporated into the axioms of a vertex algebra in order to obtain a vertex operator algebra describing the vacuum sector of a conformal field theory.

Definition 2.9. *A vertex operator superalgebra is a $\frac{1}{2}\mathbb{Z}$-graded vertex super algebra $(V, Y, \mathbf{1})$ where*

$$V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_{\geq 0}} V_n$$

such that each V_n is finite dimensional, equipped with a distinguished homogeneous weight 2 vector ω called the conformal vector, satisfying the conditions

- *(VOA1) (conformal symmetry) The modes $L_n = \omega_{n+1}$ form a $\mathfrak{Vir}$ module, whose central term C acts as* $c\,\mathrm{id}$ *for some $c \in \mathbb{C}$;*
- *(VOA2) (conformal weight) The mode L_0 acts as $L_0 v = nv$ whenever $v \in V_n$;*
- *(VOA3) (translation generator) The operator T from the definition of a vertex superalgebra is equal to L_{-1}.*

In the context of vertex operator superalgebras, the translation axiom is often termed L_{-1} derivative property. As in the case of vertex algebras, one often denotes vertex operator algebras by the quadruple $(V, Y, \mathbf{1}, \omega)$. A vertex operator algebra V obeying

$$V_0 = \mathbb{C}\mathbf{1} \quad \text{and} \quad V = \bigoplus_{n=0}^{\infty} V_n$$

is said to be of CFT type.

The following proposition yields a quick way of checking the Virasoro algebra in explicit constructions. It also implies that the Virasoro algebra is satisfied on modules (Proposition 2.15).

Proposition 2.10. *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator (super)algebra. Then the Virasoro algebra relations (2.27) are equivalent to the set of equations*

$$\omega_{(0)}\omega = L(-1)\omega \tag{2.29a}$$

$$\omega_{(1)}\omega = 2\omega \tag{2.29b}$$

$$\omega_{(2)}\omega = 0 \tag{2.29c}$$

$$\omega_{(3)}\omega = \frac{c}{2}\mathbf{1} \tag{2.29d}$$

$$\omega_{(n)}\omega = 0 \text{ for } n \geq 4 \tag{2.29e}$$

Proof. Recall the Lie algebra of modes, (2.26) for $u, v \in V$ and $m, n \in \mathbb{Z}$:

$$[u_m, v_n] = \sum_{i\geq 0} \binom{m}{i} (u_i v)_{m+n-i}.$$

Setting $u = v = \omega$ and using equations (2.29a)-(2.29e), one obtains

$$\begin{aligned}
[\omega_{m+1}, \omega_{n+1}] &= (L_{-1}\omega)_{m+n+2} + 2(m+1)\omega_{m+n+1} + \frac{1}{2}\frac{(m+1)m(m-1)}{6}(c_v\,\mathbf{1})_{m+n-1} \\
&= -(m+n+2)\omega_{m+n+1} + 2(m+1)\omega_{m+n+1} + \frac{1}{12}\left(m^3 - m\right)\delta_{m+n,0}c\,\mathrm{id} \\
&= (m-n)L_{m+n} + \frac{1}{12}\left(m^3 - m\right)\delta_{m+n,0}c\,\mathrm{id}.
\end{aligned}$$

In the second step, the L_{-1} derivative property as well as the vacuum property $Y(\mathbf{1}, x) = \mathrm{id}$ was used. □

Weight formula. It has been stated above that the Jacobi identity implies the locality and the translation axiom. In the case of a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$, (2.25) yields useful commutator formulas involving Virasoro modes and arbitrary vertex operators. Setting $u = \omega$ and applying Res_{x_1} and $\mathrm{Res}_{x_1} x_1$ respectively, one obtains

$$[L_{-1}, Y(v, x)] = Y(L_{-1}v, x) \tag{2.30}$$

and

$$[L_0, Y(v, x)] = xY(L_{-1}v, x) + Y(L_0 v, x). \tag{2.31}$$

Assume that v is homogeneous, i.e. $L_0 v = \mathrm{wt}v\, v$. Using the L_{-1} derivative property, equation (2.31) then implies the weight formula for modes,

$$\mathrm{wt}v_n = \mathrm{wt}v - n - 1. \tag{2.32}$$

Equations (2.30) and (2.31) can be generalized to

$$[L_m, Y(v, x)] = \sum_{n\in\mathbb{N}} \binom{m+1}{n} x^{m+1-n} Y(L_{n-1}v, x) \tag{2.33}$$

by applying the operation $\mathrm{Res}_{x_1} x_1^{m+1}$ to the commutator formula (2.25).

Reconstruction theorem. Verifying all the axioms in explicit constructions of vertex operator algebras is simplified by a general reconstruction theorem that has been obtained by Frenkel, Kac, Radul and Wang in [18] and independently by Meurman and Primc in [28]. It is given here in the slightly less general formulation of [17]. The theorem relies on the existence of a Poincaré-Birkhoff-Witt-like basis of the form $a^{\alpha_1}_{-j_1} \ldots a^{\alpha_m}_{-j_m}\,\mathbf{1}$. This basis makes it possible to extend the vertex algebra axioms from the generating set of vectors $\{a^\alpha\}_{\alpha\in S}$ to arbitrary vectors by induction. The reconstruction theorem will be used in the construction of the symplectic fermionic vertex operator algebra, where one is exactly in the situation described above.

Theorem 2.11. *Let V be a $\mathbb{Z}_{\geq 0}$-graded vector space, $\mathbf{1} \in V_0$ a non-zero vector, and T a degree one endomorphism of V. Let S be a countable ordered set and $\{a^\alpha\}_{\alpha \in S}$ a collection of homogeneous vectors in V. Suppose we are also given fields*

$$a^\alpha(x) = \sum_{n \in \mathbb{Z}} a^\alpha_{(n)} x^{-n-1}$$

such that the following conditions hold:

(R1) For all α, $a^\alpha(x)\,\mathbf{1}|_{x=0} = a^\alpha$.

(R2) The operator T acts according to $T\,\mathbf{1} = 0$ and $[T, a^\alpha(x)] = \partial_x a^\alpha(x)$ for all α.

(R3) All fields a^α are mutually local.

(R4) The vector space V has a basis of vectors

$$a^{\alpha_1}_{-j_1} \dots a^{\alpha_m}_{-j_m}\,\mathbf{1}, \tag{2.34}$$

where $j_1 \geq j_2 \geq \ldots \geq j_m > 0$, and if $j_i = j_{i+1}$, then $\alpha_i \geq \alpha_{i+1}$ with respect to the given order on the set S.

Under these assumptions, the assignment

$$Y(a^{\alpha_1}_{j_1} \dots a^{\alpha_m}_{j_m}\,\mathbf{1}, x) =: \partial^{(j_1-1)} a^{\alpha_1}(x) \dots \partial^{(j_m-1)} a^{\alpha_m}(x) :$$

defines a vertex superalgebra structure on V.

For the proof of locality of arbitrary fields, the following Lemma serves as a basis for a proof by induction.

Lemma 2.12 (Dong's Lemma). *If $A(x)$, $B(x)$, $C(x)$ are three mutually local fields, then the fields $: A(x)B(x) :$ and $C(x)$ are also mutually local.*

Proof. See Lemma 2.3.4 in [17], p.35. □

Proof. (Reconstruction Theorem) The creation property (VA2)

$$\lim_{x \to 0} Y(v, x)\,\mathbf{1} = v \tag{2.35}$$

is satisfied for $Y(a^\alpha, x)$ by assumption. This implies that all nonnegative modes of the a^α have to annihilate the vacuum. Thus, one also has $\partial_x^n Y(a^\alpha, x)\,\mathbf{1} = a^\alpha_{-n+1}\,\mathbf{1}$. The creation property for a general vector $a^{\alpha_1}_{-j_1} \dots a^{\alpha_r}_{-j_r}$ follows by induction over r in the following way. Assume that (2.35) holds for $Y(A, x) = \sum_n A_{(n)} x^{-n-1}$. It has to be shown that it then also holds for

$$Y(a^{\alpha_0}_{-k} A, x) = \frac{1}{(k-1)!} : \partial_x^{k-1} a^{\alpha_0}(x) Y(A, x) :,$$

where $k > 0$ and $i \in S$. By definition (2.19) of the normally ordered product this is equal to

$$\frac{1}{(k-1)!}\sum_{m\in\mathbb{Z}}\sum_{n\leq -k}(-n-1)\cdots(-n-k+1)a^{\alpha_0}_{(n)}A_{(m)}x^{-n-m-k-1}$$
$$+\frac{1}{(k-1)!}\sum_{m\in\mathbb{Z}}\sum_{n>-k}(-n-1)\cdots(-n-k+1)A_{(m)}a^{\alpha_0}_{(n)}x^{-n-m-k-1}.$$

Noting that all terms in the second sum with $n < 0$ have vanishing coefficients and rewriting the sum so that the coefficient of every monomial in x can be read off, the normal ordered product takes the following form:

$$\frac{1}{(k-1)!}\sum_{m\in\mathbb{Z}}\left(\sum_{n\leq -k}(-n-1)\cdots(-n-k+1)a^{\alpha_0}_{(n)}A_{(m-n)}\right.$$
$$\left.\sum_{n\geq 0}(-n-1)\cdots(-n-k+1)A_{(m-n)}a^{\alpha_0}_{(n)}\right)x^{-k-m-1}.$$

The second sum annihilates the vacuum since the creation property is satisfied for the a^α. By the inductive assumption, the positive modes of A also annihilate the vacuum so that in the limit $x \to 0$ the only remaining term is

$$b_{(-k)}A_{(-1)}\mathbf{1} = b_{(-k)}A.$$

The translation axiom (VA4) holds for the a^α by assumption. By differentiation, this implies also $[T, \partial_x^n a^\alpha(x)] = \partial_x^{n+1}a^\alpha(x)$. It can be proven by explicit calculation that the Leibniz rule holds for the normally ordered product, i.e.

$$\partial_x : A(x)B(x) : = : (\partial_x A(x))B(x) : + : A(x)(\partial_x B(x)) : .$$

Using this fact, it can be shown that the translation axiom holds for the normal ordered product of two fields if it holds for the fields themselves. The splitting of a field into its positive and negative part commutes with differentiation according to equation (2.18). This implies

$$[T, A(x)^\pm] = [T, A(x)]^\pm.$$

With the help of this identity the following normal ordered products can be computed:

$$\begin{aligned}
:[T, A(x)]B(x) :=& [T, A(x)]^+B(x) + B(x)[T, A(x)]^-\\
& TA(x)^+B(x) - A(x)^+TB(x) + B(x)TA(x)^- - B(x)A(x)^-T,\\
: A(x)[T, B(x)] :=& A(x)^+[T, B(x)] + [T, B(x)]A(x)^-\\
& A(x)^+TB(x) - A(x)^+B(x)T + TB(x)A(x)^- - B(x)TA(x)^-.
\end{aligned}$$

Adding these two equations, one obtains

$$\begin{aligned}
\partial_x : A(x)B(x) :=& : \partial_x A(x)B(x) : + : A(x)\partial_x B(x) :\\
=& : [T, A(x)]B(x) : + : A(x)[T, B(x)] :\\
=& T(A(x)^+B(x) + B(x)A(x)^-) - (A(x)^+B(x) + B(x)A(x)^-)T\\
=& T : A(x)B(x) : - : A(x)B(x) : T\\
=& [T, : A(x)B(x) :].
\end{aligned}$$

By induction, the translation axiom can then be extended to all vertex operators.

Locality for arbitrary fields is similarly proven by induction, using the mutual locality of all fields of the form $\partial_x^n a^\alpha(x)$ and Dong's Lemma. The locality of derivatives of the $a^\alpha(x)$ can be seen as follows. For any two mutually local fields $A(x_1)$ and $B(x_2)$, one has by definition

$$(x_1 - x_2)^N [A(x_1), B(x_2)] = 0$$

for some $N \in \mathbb{N}$. Differentiating this equation with respect to x_1, one obtains

$$(x_1 - x_2)^{N-1} [A(x_1), B(x_2)] + (x_1 - x_2)^N [\partial_{x_1} A(x_1), B(x_2)] = 0.$$

After multiplication with $(x_1 - x_2)$, the first term vanishes due to mutual locality of $A(x_1)$ and $B(x_2)$. The remaining equation proves mutual locality of $\partial_{x_1} A(x_1)$ and $B(x_2)$. By induction, the locality property can be extended to arbitrary derivatives and then with Dong's Lemma to arbitrary normally ordered products, proving (VA3) and completing the proof. □

Subsets of vertex operator algebras which generate the whole vertex operator algebra according to equation (2.34) are important enough to introduce the following definition.

Definition 2.13. *Let $(V, Y, \mathbf{1})$ be a vertex algebra and S a subset of V consisting of homogeneous vectors. If V is spanned by vectors of the form*

$$\psi^1_{-n_1} \psi^2_{-n_2} \cdots \psi^r_{-n_r} \mathbf{1}$$

for $\psi^i \in S$, $n_i \in \mathbb{Z}$ and $r \in \mathbb{N}$, then S is called a generating set of vectors of V. If the above holds for all $n_i < 0$, then V is called strongly generated by S.

2.4 Modules for VOAs

A vertex operator algebra formalizes the properties of the vacuum sector of a conformal field theory. But as in general quantum field theory, one would also like to study representations of the field algebra which are inequivalent to the vacuum sector. Classifying all inequivalent irreducible representations is an important task and yields a lot of physical information. The mathematical tools for this classification are discussed in Chapter 3. The following is the definition of an ordinary module for a vertex operator algebra V. Generalized modules which are related to the action of automorphisms of V are discussed below.

Definition 2.14. *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator superalgebra. A V-module (W, Y_W) is a superspace W with the grading*

$$W = \bigoplus_{h \in \mathbb{C}} W_h$$

equipped with a linear map $Y_W : V \to \mathrm{End}(W)[[x^{\pm 1}]]$, $v \mapsto Y_W(v, x) = \sum_{m \in \mathbb{Z}} v_m x^{-m-1}$ such that for $u, v \in V$ and $w \in W$, the endomorphisms u_n of W satisfy the condition

(M1) (regularity) $u_n w = 0$ *for* n *sufficiently large;*

(M2) (vacuum) $Y_W(\mathbf{1}, x) = \mathrm{id}_W$*;*

(M3) (Jacobi identity) For Z_2*-homogeneous* $u, v \in V$*, the following Jacobi identity holds:*

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(u, x_1)Y_W(v, x_2) - (-1)^{p(u)p(v)} x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W(v, x_2)Y_W(u, x_1)$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(Y(u, x_0)v, x_2);$$

(M4) (grading restriction) The subspaces W_h *are finite dimensional for all* $h \in \mathbb{C}$*,* $W_h = 0$ *for all* h *whose real part is sufficiently negative and the grading is given by* L_0 *eigenvalues according to*

$$W_h = \{w \in W | L_0 w = hw\}.$$

Inspecting the above definition, the reader might ask why the Jacobi identity was used instead of a locality axiom. The answer is that in the case of modules, the Jacobi identity is equivalent to weak commutativity only in conjunction with another property called weak associativity (c.f. Proposition 4.4.1 in [26], p. 127). Thus, the definition of a module would get unnecessarily complicated if one wants do to without the Jacobi identity.

One might also wonder why the Virasoro algebra is not mentioned in the axioms. The following proposition gives the answer by stating that it can be derived from the other module properties.

Proposition 2.15. *Let* V *be a vertex operator algebra and let* (W, Y_W) *be a module for* V*. Then the following relations hold on* W*:*

$$[L_{-1}, Y_W(v, x)] = Y_W(L_{-1}v, x) = \frac{d}{dx} Y_W(v, x) \quad \text{for } v \in V,$$
$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n}c,$$

for $m, n \in \mathbb{Z}$*.*

Proof. Both identities follow from the Jacobi identity exactly as in the case of vertex operator algebras. See the reasoning leading to equation (2.30) for the L_{-1} commutator formula and the proof of Proposition 2.10 for the derivation of the Virasoro algebra. □

Categorical notions. Studying vertex operator algebras, it is natural to define mappings between them. This leads to the definition of homomorphisms, isomorphisms and so on, which will be given in the following paragraph. These categorical notions for vertex operator algebras and their modules will be needed for the definition of twisted modules.

Given vertex operator algebras V_1, V_2 of the same central charge c, a homomorphism is defined as a grading preserving linear map $f : V_1 \to V_2$ such that

$$f(Y(u,z)v) = Y(f(u),z)f(v) \quad \text{for } u, v \in V_1, \tag{2.36}$$

and such that $f(\mathbf{1}) = \mathbf{1}$ and $f(\omega) = \omega$. An isomorphism is a bijective homomorphism, an endomorphism is a homomorphism with $V_1 = V_2$ and an automorphism is a bijective endomorphism.

In complete analogy, one defines the respective notions for modules. Given two modules W_1, W_2 of the vertex operator algebra V, a homomorphism is defined as a grading preserving linear map $f : W_1 \to W_2$ such that

$$f(Y(v,z)w) = Y(v,z)f(w) \quad \text{for } v \in V, w \in W_1.$$

All the above notions can also be canonically defined for modules.

An important notion with respect to modules is irreducibility. Irreducible or simple modules are modules with no nontrivial submodules. A vertex operator algebra is called simple if it is irreducible as a module for itself. Irreducible modules have a particularly simple grading structure which will be described in the following.

Note that equation (2.31) remains true in the case of modules so that the weight formula for modes, equation (2.32), also holds for the action on modules. Thus, a homogeneous vector $v \in V$ maps W_h to $W_{(h+\mathrm{wt}v-n-1)}$. Since the action of modes of arbitrary homegenous vectors can only shift the degree by an integer, W decomposes into submodules corresponding to congruence classes modulo $\mathbb{Z}$. If we define for $\alpha \in \mathbb{C}/\mathbb{Z}$ the space

$$W_{[\alpha]} = \bigoplus_{h+\mathbb{Z}=\alpha} W_{(h)},$$

then

$$W = \bigoplus_{\alpha\in\mathbb{C}/\mathbb{Z}} W_{[\alpha]}.$$

As an irreducible module, M may have no nontrivial submodules. Consequently,

$$W = W_{[\alpha]} \tag{2.37}$$

for some α. Because of the grading restriction (M4), there is a lowest weight h_0 such that

$$W_{[\alpha]} = \bigoplus_{h=h_0}^{\infty} W_{(h)}, \tag{2.38}$$

The space W_{h_0} is called the lowest weight space or top level of the module W. Since W is irreducible, all vectors in W can be obtained by acting with modes of vectors from V on the lowest weight module W_{h_0} such that

$$W = \mathrm{span}\{v^1_{-n_1} \cdots v^r_{-n_r} w | v^i \in V, w \in W_{h_0}, n_i \in \mathbb{Z}_{>0}\}.$$

Closely related to the top level W_{h_0} is the space of singular vectors $\Omega(W)$, defined by

$$\Omega(W) = \mathrm{span}\{w\,| v_n w = 0 \text{ for all } v \in V_k \text{ where } n > k-1\}.$$

Twisted modules. We will now turn to the definition of twisted modules. These are modules associated to automorphisms of a vertex operator algebra. Vertex operator superalgebras come with a natural automorphism that can be described as follows.

Let $V = V^{\bar{0}} \oplus V^{\bar{1}}$ be a vertex operator superalgebra and let $\theta : V \to V$ be defined by

$$\theta(a+b) = a - b$$

for $a \in V^{\bar{0}}$ and $b \in V^{\bar{1}}$. Since for $a \in V^{\bar{k}}$, $b \in V^{\bar{l}}$ with $k, l \in \{0, 1\}$ the vector $a_{(n)}b$ is in $V^{\bar{k}+\bar{l}}$, we have $\theta(a_n b) = \theta(a)_{(n)}\theta(b)$. Comparing the last equality with (2.36), we see that θ is an automorphism of V.

More generally, let σ be an automorphism of V of order S. Then V is decomposed into a direct sum of eigenspaces V^k of σ

$$V = V^0 \oplus V^1 \oplus \cdots \oplus V^{S-1} \tag{2.39}$$

where V^k is the eigenspace of V for σ with eigenvalue $\exp\left(\frac{2k\pi i}{S}\right)$. Since the vacuum and the conformal vector have to be invariant under an automorphism, V^0 is a vertex operator subsuperalgebra of V. Furthermore, all V^k for $k = 1, \ldots, S-1$ are V^0-modules because the homomorphism condition for σ implies that for $v \in V^0$, $Y(v, z)$ maps vectors from V^k to itself.

Definition 2.16 ([27]). *Let $(V, Y, \mathbf{1})$ be a vertex superalgebra with an automorphism σ of order S. A σ-twisted V-module is a triple (M, d, Y_M) consisting of a superspace M, a $\mathbb{Z}_2$-endomorphism d of M and a linear map $Y_M(\cdot, z)$ from V to $(\mathrm{EndV})[[z^{\pm\frac{1}{S}}]]$ satisfying the following conditions:*

(T1) For any $a \in V$, $u \in M$, $a_n u = 0$ for $n \in \frac{1}{S}\mathbb{Z}$ sufficiently large;

(T2) $Y_M(\mathbf{1}, z) = \mathrm{id}_M$;

(T3) $[d, Y_M(a, z)] = Y_M(D(a), z) = \frac{d}{dz}Y_M(a, z)$ for any $a \in V$, where D is the translation operator of V;

(T4) For any $\mathbb{Z}_2$-homogeneous $a, b \in V$, the following θ-twisted Jacobi identity holds:

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_M(a, z_1)Y_M(b, z_2) - (-1)^{p(a)p(b)}z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)Y_M(b, z_2)Y_M(a, z_1)\\
=&z_2^{-1}\sum_{j=0}^{S-1}\frac{1}{S}\delta\left(\left(\frac{z_1 - z_0}{z_2}\right)^{\frac{1}{S}}\right)Y_M(Y(\sigma^j a, z_0)b, z_2).
\end{aligned}$$

If V is a vertex operator superalgebra, a σ-twisted V-module for V as a vertex superalgebra is called a σ-twisted weak module for V as a vertex operator superalgebra. A σ-twisted weak V-module M is said to be $\frac{1}{2S}\mathbb{Z}$-graded if $M = \bigoplus_{n \in \frac{1}{2S}\mathbb{Z}} M_n$ such that

(T5) $a_n M_r \subseteq M_{r+m-n-1}$ for $a \in V_m$, $m \in \mathbb{Z}$, $n \in \frac{1}{S}\mathbb{Z}, r \in \frac{1}{2S}\mathbb{Z}$.

A σ-twisted module M for V as a vertex superalgebra is called a σ-twisted module for V as a vertex operator superalgebra if $M = \bigoplus_{\alpha \in \mathbb{C}} M_\alpha$ such that

(T6) $L_0 u = \alpha u$ *for* $\alpha \in \mathbb{C}$, $u \in M_\alpha$;

(T7) *For any fixed* α, $M_{\alpha+n} = 0$ *for* $n \in \frac{1}{2S}\mathbb{Z}$ *sufficiently small;*

(T8) $\dim M_\alpha < \infty$ *for any* $\alpha \in \mathbb{C}$.

For the rest of this section, V will be a vertex operator superalgebra and M will be a σ-twisted V-module. Recall that we then have the decomposition given in (2.39). In the case $a \in V^k$ even or odd, the σ-twisted Jacobi identity specializes to

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(a, z_1)Y_M(b, z_2) - (-1)^{p(a)p(b)} z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(b, z_2)Y_M(a, z_1) \\ =& z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\left(\frac{z_1 - z_0}{z_2}\right)^{-\frac{k}{S}} Y_M(Y(a, z_0)b, z_2). \end{aligned}$$

Setting $b = \mathbf{1}$, the terms on the left hand side may be combined:

$$z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(a, z_1) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\left(\frac{z_1 - z_0}{z_2}\right)^{-\frac{k}{S}} Y_M(Y(a, z_0)\,\mathbf{1}, z_2).$$

Taking $\mathrm{Res}_{z_0} z_0^{-1}$, we obtain

$$z_2^{-1}\delta\left(\frac{z_1}{z_2}\right) Y_M(a, z_1) = z_2^{-1}\delta\left(\frac{z_1}{z_2}\right)\left(\frac{z_1}{z_2}\right)^{-\frac{k}{S}} Y_M(a, z_2).$$

Therefore, $z^{\frac{k}{S}} Y_M(a, z)$ is an element of $\mathrm{End}(M)[[z^{\pm 1}]]$ for any $a \in V^k$, where $k \in \{1, \dots, S-1\}$. It follows, that for any homogeneous a in V^k the weight of the operator a_n equals $\mathrm{wt}a - n - 1$ with $n \in \frac{k}{S} + \mathbb{Z}$. In particular, a_n has weight in $\mathbb{Z}$ for $a \in V^0$. Consequently, for $\lambda \in \mathbb{C}$

$$M^0(\lambda) = \bigoplus_{n \in \mathbb{Z}} M_{\lambda+n}$$

is a V^0 submodule of M and $M^0(\lambda) = M^0(\mu)$ if and only if $\mu - \lambda \in \mathbb{Z}$. On the other hand, considering the whole vertex operator algebra V,

$$M(\lambda) = \bigoplus_{n \in \frac{1}{S}\mathbb{Z}} M_{\lambda+n}$$

is a σ-twisted V-submodule of M and $M(\lambda) = M(\mu)$ if and only if $\lambda - \mu \in \frac{1}{S}\mathbb{Z}$. It then follows from the first decomposition, that

$$M(\lambda) = \bigoplus_{k=0}^{S-1} M^0\left(\lambda + \frac{k}{S}\right) \tag{2.40}$$

is a decomposition of $M(\lambda)$ into V^0-modules. If M is a simple module, $M(\lambda) = M$ since we would have a nontrivial submodule of M otherwise. In this case the decomposition will be written in the form

$$M = M^0 \oplus M^1 \oplus \cdots \oplus M^{S-1}. \tag{2.41}$$

This section will be concluded by two theorems that make assertions about the irreducibility of the elements in the decompositions (2.39) and (2.41). The first theorem has been proven by Dong and Mason in [12]. It is given here in the formulation of [11], section 5.1. The second theorem is due to Dong and Lin, c.f. [11]. In their orignal form, both theorems apply only to vertex operator algebras, not superalgebras. But all the arguments carry over to the super case so that we have the following theorems.

Theorem 2.17. *Let V be a simple vertex operator superalgebra and σ an automorphism of V of finite order S. Then the V^0 modules V^k in the decomposition*

$$V = \bigoplus_{k=0}^{S-1} V^k$$

are all irreducible as V^0 modules. In particular, V^0 is a simple vertex operator superalgebra.

Theorem 2.18. *In the setting of Theorem 2.17, let M be a simple σ-twisted V-module. Then the M^k, $k = 1, \ldots, S-1$ in the decomposition*

$$M = \bigoplus_{k=1}^{S-1} M^k$$

are nonzero and non-isomorphic simple V^0 modules.

2.5 A Characterization of Primary Fields

This chapter began with a derivation of the definition of a vertex algebra from a set of physically motivated axioms. This paragraph illustrates the reverse procedure by deriving the transformation behavior of primary fields from the vertex operator algebra structure. Primary vectors are defined by the property $L_m v = 0$ for all $m > 0$ and $L_0 v = hv$ for some $h \in \mathbb{N}$. For a primary vector v, equation (2.33) reads

$$[L_n, Y(v,x)] = \left(x^{n+1} \frac{d}{dx} + h(n+1)x^n \right) Y(v,x). \tag{2.42}$$

This can be rewritten in the form

$$[L_n, Y(v,x)] = x^{-h(n+1)} x^{n+1} \frac{d}{dx} \left(x^{h(n+1)} Y(v,x) \right).$$

Multiplying with x_0 and $x^{h(n+1)}$, one obtains

$$[x_0 L_n, x^{h(n+1)} Y(v,x)] = x_0 x^{n+1} x^{n+1} \frac{d}{dx} \left(x^{h(n+1)} Y(v,x) \right).$$

By an argument common in the theory of formal series, this expression may be exponentiated. First, reinterpret the commutator by rewriting the last equation in the form

$$(L_{x_0 L_n} - R_{x_0 L_n}) x^{h(n+1)} Y(v,x) = x_0 x^{n+1} \frac{d}{dx} \left(x^{h(n+1)} Y(v,x) \right). \tag{2.43}$$

Here, L_X and R_X are taken to be the operators of left and right multiplication with the operator X, respectively. Now, taking the n-th iterate of (2.43), multiplying by $1/n!$ and summing over n yields the exponential form

$$e^{x_0 L_n} x^{h(n+1)} Y(v, x) e^{-x_0 L_n} = e^{x_0 x^{n+1} \frac{d}{dx}} \left(x^{h(n+1)} Y(v, x) \right).$$

By the formal Taylor theorem, Theorem 2.4, the right hand side equals

$$x_1^{h(n+1)} Y(v, x_1),$$

where

$$x_1 = \begin{cases} e^{x_0} x & \text{if } n = 0 \\ (x^{-n} - n x_0)^{-\frac{1}{n}} & \text{if } n \neq 0. \end{cases}$$

Thus, we arrive at the equation

$$e^{x_0 L_n} Y(v, x) e^{-x_0 L_n} = \left(\frac{x_1}{x} \right)^{h(n+1)} Y(v, x_1) = \left(\frac{\partial x_1}{\partial x} \right)^h Y(v, x_1),$$

which is usually taken as a definition of a primary conformal field by physicists.

Chapter 3

Algebras for the Analysis of CFTs

Given a vertex operator algebra V, the most important task often is to classify its modules. In conformal field theory this amounts to finding inequivalent representations of the field algebra. There exist two common approaches to this classification problem. In certain cases, one can infer the commutation relations of the zero mode algebra on the top level of any module. If the zero mode algebra forms a well-known Lie algebra, the representations of the whole field algebra may be found by studying the representations of this Lie algebra.

The second approach is to study the representations of a certain associative algebra first defined by Zhu in [32]. The importance of Zhu's algebra $A(V)$ stems from Zhu's theorem, which states that there is a one to one correspondence between $A(V)$-modules and V-modules. As, for example, Zhu's algebra is finite dimensional in the case of weakly rational vertex operator algebras, this can make the classification of modules significantly easier.

In the following, the definition of Zhu's algebra is given and Zhu's theorem is stated. The classification of vertex operator algebra modules with the help of the Lie algebra of the zero modes is illustrated in the case of the triplet algebra at $c = -2$. This analysis follows the one by Kausch and Gaberdiel in [21].

3.1 Definition of Zhu's Algebra and Zhu's Theorem

Let V be a vertex operator algebra and $u, v \in V$ with u a homogeneous element. Define a product

$$u \star v = \mathrm{Res}\left(Y(u,z)v\frac{(z+1)^{\mathrm{wt}u}}{z}\right),$$

which, in terms of modes, reads

$$(u \star v)_n = \sum_{m \geq \mathrm{wt}u} u_{-1-m}v_{m+n} + \sum_{m<\mathrm{wt}u} v_{m+n}u_{-1-m}.$$

Extend the product $\star$ linearly to all $u \in V$. The space spanned by elements of the form

$$(L_{-1} + L_0)u \star v$$

with $u, v \in V$ can be shown to be a two-sided ideal for $\star$ and is called $O(V)$. Define Zhu's algebra as the quotient $A(V) = V/O(V)$. It turns out that $(A(V), \star)$ is an associative algebra with the equivalence class of the vacuum [**1**] being the unit and the conformal vector $[\omega]$ being the center. The important relationship between representations of $A(V)$ and V-modules stems from two facts. First, one can calculate that restricted to the top level M_{h_0} of any irreducible V-module M,

$$o(u \star v) = o(u)o(v),$$

where o is the linear map mapping a homogeneous vector $v \in V_k$ to its zero mode $v_{(k-1)}$. Thus, o defines a homomorphism of algebras, $o : A(V) \to Z(V)|_{M_{h_0}}$, where $Z(V) \subset \mathrm{End}(V)$ is the algebra generated by all elements $o(a)$, $a \in V$. The second fact is the identity

$$o(L_{-1}u + L_0 u) = 0$$

for all $u \in V$. From the above two equations, we can infer that $o(u) = 0$ on each top level space M_{h_0} for any $u \in O(V)$. Thus we can assign to each class u in $A(V)$ a zero mode $o(u)$ in a well defined way. Using this mapping, Zhu was able to show the following result.

Theorem 3.1. *Let V be a vertex operator algebra and M a V-module. Then the following holds*

- *The top level M_{h_0} of M is a representation of the associative algebra $A(V)$.*
- *Conversely, if $\pi : A(V) \to \mathrm{End}(W)$ is a representation of $A(V)$, then there exists a V-Module M such that $M_{h_0} = W$.*
- *The isomorphism classes of the irreducible V-modules and the isomorphism classes of irreducible representations of $A(V)$ are in one-to-one correspondence.*

Zhu's algebra can be endowed in a canonical way with a Lie algebra structure using the commutator. Brungs and Nahm showed in [7] that the mapping o gives rise to a Lie algebra isomorphism between Zhu's algebra and the zero mode algebra.

Certain finiteness conditions for vertex operator algebras are closely related to Zhu's algebra. To formulate these conditions, one introduces the notion of a weak V-module, which is a vector space satisfying all V-module axioms except for the ones related to the grading. A weak V-module W admitting an $\mathbb{N}$-grading $W = \bigoplus_{n\in\mathbb{N}} W_n$ such that $v_m W_n \subset W_{\mathrm{wt}v+n-m-1}$ is called admissible.

Following Dong, Li and Mason (c.f. [10]), a vertex operator algebra is called weakly rational if every admissible V-modules can be decomposed into a direct sum of irreducible V-modules. A vertex operator algebra is called rational if every admissible module is a direct sum of irreducible admissible modules. An even stronger statement is regularity, which is satisfied if every weak V-module is a direct sum of irreducible V-modules.

To decide if any of the above conditions are met by a vertex operator algebra, the notion of C_2-cofiniteness has proven useful. A vertex operator algebra V is called C_2-cofinite if the subspace

$$C_2(V) = \mathrm{span}\,\{v_{(-n)}w \,|\, v, w \in V\}$$

is of finite codimension in V, i.e. $\dim(V/C_2(V)) < \infty$. If the vertex operator algebra V is C_2-cofinite it can be proven that Zhu's algebra $A(V)$ is finite. As a finite algebra $A(V)$ then only has finitely many irreducible modules. Thus, by Zhu's theorem, a C_2-cofinite vertex operator algebra V only has finitely many inequivalent irreducible V-modules.

The following result has been proven by Abe, Buhl and Dong in [2].

Theorem 3.2. *A vertex operator algebra V of CFT type is regular if and only if V is C_2-cofinite and rational.*

There existed an important conjecture in the literature, stating that rationality in the above sense and C_2-cofiniteness are equivalent. However, the triplet algebra which will be discussed in the next section serves as a counterexample to this conjecture.

3.2 $\mathcal{W}$-Algebras and the Triplet Algebra

$\mathcal{W}$-algebras are special vertex operator algebras, whose field algebra is an extension of the Virasoro algebra by additional fields. In this sense, $\mathcal{W}$-algebras are also sometimes termed maximally extended symmetry algebras since they form the maximal field algebra which is compatible with the Virasoro algebra for a fixed central charge.

Definition 3.3. *A $\mathcal{W}$-algebra of type $\mathcal{W}(2, h_1, \ldots, h_m)$ is a vertex operator algebra which has a minimal generating set in the sense of Definition 2.13 consisting of the vacuum* **1***, the conformal vector ω of weight two and m additional primary vectors W^i of weight h_i, where $i \in 1, .., m$.*

$\mathcal{W}$-algebras have aroused considerable interest due to their connection with the classification of all rational conformal field theories. For a general introduction and some classification results, see [15] and [5]. Throughout this section, the physical mode convention as opposed to the mathematical mode convention will be used. This means that the field corresponding to a homogeneous vector $v \in V_h$ is expanded in the form

$$Y(v, x) = \sum_{n\in\mathbb{Z}} v_{(n)} x^{-n-h}.$$

From a calculational point of view, $\mathcal{W}$-algebras are characterized by the commutation relations of the modes of their generating vectors or, equivalently, by the OPE of the generating fields. In general, the commutation relations of the modes of the generating fields of a $\mathcal{W}$-algebra cannot be expressed as a Lie algebra in the modes of these fields. However, it was shown by Nahm in [29] that it is always possible to express the commutators in terms of modes of fields which are special normal ordered products of the generating fields.

For the description of $\mathcal{W}$-algebras the notion of a quasi-primary field is needed. This is a vertex operator which satisfies (2.42) only for $n = \pm 1$ and $n = 0$. Denote the family of quasi-primary fields of a given $\mathcal{W}$-algebra by $\{\phi^i\}_{i\in I}$ and their conformal weights by h_i. Then the quasi-primary normal ordered product is defined by

$$\begin{aligned}\mathcal{N}\left(\phi^j, \partial^n\phi^i\right) = &\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{h(ijk)+\sigma(ijk)+2n-1}{r}^{-1}\binom{2h(i)+n-1}{r}\partial^r N(\phi^j, \partial^{n-r}\phi^i)\\ &-(-1)^n\sum_{\{k|h(ikj)\geq 1\}} C^{ij}_k\binom{h(ijk)+n-1}{n}\binom{h(ijk)+\sigma(ijk)+2n-1}{n}^{-1}\\ &\times\binom{2h(i)+n-1}{h(ijk)+n}\binom{\sigma(ijk)-1}{h(ijk)-1}^{-1}\frac{\partial^{h(ijk)+n}\phi^k}{(\sigma(ijk)+n)(h(ijk)-1)!},\end{aligned} \tag{3.1}$$

where $h(ijk) = h_i + h_j - h_k$ and $\sigma(ijk) = h_i + h_j + h_k - 1$. Furthermore, the structure constants C_k^{ij} are defined such that $\sum^l C_l^{ij} d^{lk} = C^{ijk}$, where C^{ijk} and d^{ij} are given by the correlation functions

$$C^{ijk} = \left\langle \mathbf{1}, \phi^k_{(h_k)} \phi^i_{(-h_k+h_j)} \phi^j_{(-h_j)} \mathbf{1} \right\rangle \quad \text{and} \quad d^{ij} = \left\langle \mathbf{1}, \phi^i_{(h_i)} \phi^j_{(-h_j)} \mathbf{1} \right\rangle .$$

An especially well-studied family of $\mathcal{W}$-algebras are the triplet algebras

$$\left\{ \mathcal{W}(2, (2p-1)^3) \right\}_{p \in \mathbb{N}_{\geq 2}} .$$

The triplet algebra at $c = -2$ is the extension of the Virasoro theory by a triplet of fields with $h = 3$, which are commonly denoted by W^i, where the index i takes values from the set $\{+, -, 0\}$.

The triplet algebra has interesting properties related to the finiteness notions introduced in the previous section. While it only has finitely many inequivalent irreducible modules it also admits logarithmic modules. Additionally it serves as a counterexample for the conjecture that C_2-cofiniteness equals rationality in the sense defined in section 3.1. The result that the triplet algebra is C_2-cofinite has independently been obtained by Abe in [1] and Carqueville and Flohr in [9]. It had been known before by Gaberdiel and Neitzke who stated the fact in [22]. In Abe's model, the triplet algebra $\mathcal{W}(2, 3, 3, 3)$ is a special case of a more general $\mathcal{W}$-algebra which will be discussed in Chapter 4, while the result obtained in [9] applies to all triplet algebras $\left\{ \mathcal{W}(2, (2p-1)^3) \right\}_{p \geq 2}$.

The triplet algebra $W(2, 3, 3, 3)$ is characterized by the commutation relations

$$[L_m, L_n] = (m-n) L_{m+n} - \frac{1}{6} m(m^2 - 1) \delta_{m+n}, \tag{3.2a}$$

$$[L_m, W^a_n] = (2m - n) W^a_{m+n}, \tag{3.2b}$$

$$\begin{aligned}[W^a_m, W^b_n] = & g^{ab} \left(2(m-n)\Lambda_{m+n} + \frac{1}{20}(m-n)(2m^2 + 2n^2 - mn - 8) L_{m+n} \right. \\ & \left. - \frac{1}{120} m(m^2-1)(m^2-4)\delta_{m+n} \right) \\ & + f^{ab}_c \left(\frac{5}{14}(2m^2 + 2n^2 - 3mn - 4) W^c_{m+n} + \frac{12}{5} V^c_{m+n} \right),\end{aligned} \tag{3.2c}$$

where Λ and V^a are normal ordered fields in the sense of equation (3.1), given by

$$\Lambda =: L^2 : -\frac{3}{10} \partial^2 L$$

and

$$V^a =: L W^a : -\frac{3}{14} \partial^2 W^a .$$

The non-vanishing components of g^{ab} and f^{ab}_c are defined to be

$$g^{00} = 1, \quad g^{\pm\mp} = 2 \quad \text{and} \quad f^{0\pm}_{\pm} = \pm 1, \quad f^{\pm\mp}_0 = \pm 2.$$

The above commutation relations have been obtained by Kausch, who constructed an explicit realization of $\mathcal{W}(2, 3, 3, 3)$ that will be discussed at the beginning of Chapter 4.

The representations of $\mathcal{W}(2,3,3,3)$ have first been analyzed by Eholzer, Honecker and Hübel in [14] and by Gaberdiel and Kausch in [21]. In the latter work, the generalized highest weight representations of the triplet algebra at $c = -2$ have been classified with the help of certain null vector conditions. Using these conditions, the zero mode algebra on the top level of any highest weight representation is derived. By classifying the representations of the zero mode algebra one then obtains a classification of the representations of the whole field algebra. This is essentially the physicist's version of the classification in terms of Zhu's algebra. In the following, a short summary of the treatment in [21] is given.

Because the fields corresponding to null vectors have to decouple in correlation functions, it has to be required that zero modes of null vectors annihilate all states in a representation. Applying this condition to the explicitly known null vectors N^{ab}, we obtain

$$\left(W_0^a W_0^b - g^{ab}\frac{1}{9}L_0^2(8L_0+1) - f_c^{ab}\frac{1}{5}(6L_0-1)W_0^c\right)\psi = 0, \tag{3.3}$$

where ψ is any state in the top level of a representation. From this equation, one can deduce

$$L_0^2(8L_0+1)(8L_0-3)(L_0-1)\psi = 0. \tag{3.4}$$

We know that any irreducible representation has the structure (2.4), implying that the eigenvalues of the L_0 zero mode have to take one of the values satisfying (3.4) on the top level of each irreducible representation. The possible eigenvalues are called the spectrum of the theory and are given by $h_0 = 0, -\frac{1}{8}, \frac{3}{8}$ and $h_0 = 1$. Writing down (3.3) with a and b exchanged and noting that g^{ab} is symmetric while f_c^{ab} is antisymmetric, one obtains the commutator

$$[W_0^a, W_0^b] = \frac{2}{5}(6h-1)f_c^{ab}W_0^c. \tag{3.5}$$

Thus the zero modes of the weight three fields, restricted to the top level of an arbitrary representation, satisfy the commutation relations of $\mathfrak{su}(2)$. This implies that the top level of every representation of the triplet algebra is a $\mathfrak{su}(2)$-module. According to the $\mathfrak{su}(2)$ representation theory, irreducible representations are labeled by a non-negative half integral number j, where $j(j+1)$ is the eigenvalue of the Casimir operator $\sum_a (W_0^a)^2$. Let m be the eigenvalue of W_0^3. We get one further constraint on j and m, since $W_0^a W_0^a = W_0^b W_0^b$ by (3.3). Thus, $j(j+1) = 3m^2$ has to be satisfied, allowing only $j = 0, \frac{1}{2}$. By examining the concrete realization of the different modules (c.f. Chapter 4), one finds that the one-dimensional singlet representation corresponds to the modules with lowest weight $h = 0, -\frac{1}{8}$ and the two-dimensional doublet representation to the case $h = 1, \frac{3}{8}$. Comparing this to the analysis with the help of Zhu's algebra (c.f. Theorem 4.8), it is important to note that the above classification obviously does not yield a one-to-one correspondence between $\mathfrak{su}(2)$-modules and field algebra modules.

The mathematically rigorous classification of the irreducible representations of the triplet algebra is a special case of Theorem 4.8. This result has been obtained by Abe using Zhu's algebra. In the special case of the symplectic fermion model, the relation between the zero mode algebra and Zhu's algebra will be investigated in Chapter 5.

Chapter 4

Symplectic Fermions

The theory of a pair of symplectic fermions was first described by Kausch in [25]. It furnishes an explicit realization of the triplet algebra $\mathcal{W}(2,3,3,3)$ and serves as an example for a conformal field theory admitting reducible but indecomposable representations. The classification of irreducible representations of $\mathcal{W}(2,3,3,3)$ with the help of the zero mode algebra has been described in Chapter 3.

About ten years after the work of Kausch, Abe generalized the model to d pairs of symplectic fermions by constructing the vertex operator superalgebra SF. Abe was able to show that SF^+, the even part of SF, provides an example of a C_2-cofinite but irrational vertex operator algebra by constructing two reducible but indecomposable SF-modules. He also classified and constructed the four inequivalent irreducible SF^+-modules. Two of these modules are SF^+ itself and the odd part SF^-. The other two modules are realized as even and odd part of the θ-twisted SF-module, where θ is the canonical involution of SF associated to its $\mathbb{Z}_2$ grading.

In this chapter, the construction of SF and of the θ-twisted SF-module $SF(\theta)$ will be given following Abe. These constructions will serve as a basis for Chapter 5, where their properties are investigated. An attempt to make this presentation as self-contained as possible is made, proving the vertex operator and module properties except for the extensive calculations related to identities needed in the twisted module construction. For details concerning these calculations, the reader is referred to the classic book on vertex operator algebras by Frenkel, Lepowsky and Meurman, [19].

Since the construction of SF is first presented with a focus on the mathematical structure, the explicit calculations related to the Virasoro algebra are deferred to 4.2.2. This section also contains some assertions which go beyond the work of Abe, among them the characterization of SF^+ as $\mathcal{W}\left(2, 2^{2d^2-d-1}, 3^{2d^2+d}\right)$-algebra. After the construction of $SF(\theta)$, this chapter is concluded by a derivation of a general commutator formula for modes of the generating vectors of SF^+.

4.1 Kausch's Symplectic Fermions

Kausch's construction of a pair of symplectic fermions described in [25] starts from the mode algebra

$$\{\psi_m^\alpha, \psi_n^\beta\} = mJ^{\alpha\beta}\delta_{m+n}, \tag{4.1}$$

where the indices α, β are equal to + or − and the nonvanishing components of J are defined to be $J^{+-} = -J^{-+} = 1$. In more mathematical terms, this defines a Lie superalgebra with even part $\mathbb{C}\mathrm{id}$ and odd part $\oplus_{n\in\mathbb{Z}}\mathbb{C}\psi_n^+ \oplus \mathbb{C}\psi_n^-$. The mode algebra (4.1) defines an infinite dimensional Clifford algebra C_{SF}. A field representation of C_{SF} is a representation on a vector space V such that $\psi^\pm = \sum_{n\in\mathbb{Z}} \psi_n^\pm x^{-n-1}$ is a field with odd parity.

There exists a field representation of C_{SF} on the Fock space, which, for the moment, will be taken to be the linear space spanned by the formal expressions

$$\psi^\pm_{-n_1} \cdots \psi^\pm_{-n_r}|0\rangle, \tag{4.2}$$

where $n_1, \ldots, n_r \in \mathbb{Z}_{>0}$ (the whole construction will be made more precise in the discussion of Abe's vertex operator algebra SF). This can be interpreted as creation of the states of the theory from the vacuum $|0\rangle$ by the creation operators $\psi^\pm_{-n_1}$. The operators $\psi_n^\pm$ act on (4.2) as an exterior product by $\psi_n^\pm$ for $n < 0$ and as the contraction by $\psi_n^\pm$ for $n \geq 0$.

The representation space can be split into a fermionic (r odd in (4.2)) and a bosonic (r even) part. Kausch shows that the bosonic part is generated by the stress energy tensor

$$T(z) = \frac{1}{2} J_{\alpha\beta} : \psi^\alpha(z)\psi^\beta(z) :$$

and three primary weight 3 fields

$$\begin{aligned}
W^+(z) &=: \partial\psi^+(z)\psi^+(z) : \\
W^0(z) &= \frac{1}{2}\left(: \partial\psi^+(z)\psi^-(z) : + : \partial\psi^-(z)\psi^+(z) :\right) \\
W^-(z) &=: \partial\psi^-(z)\psi^-(z).
\end{aligned}$$

Consequently, the pair of symplectic fermions realizes the triplet algebra at $c = -2$. Kausch has computed the operator product expansion of the generating fields to be

$$\begin{aligned}
T(z)T(w) &\sim \frac{-1}{(z-w)^4} + \frac{2T}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \\
T(z)W^\alpha(w) &\sim \frac{3W^\alpha(w)}{(z-w)^2} + \frac{\partial W^\alpha(w)}{z-w}, \\
W^\alpha(z)W^\beta(w) &\sim g^{\alpha\beta}\left(\frac{1}{(z-w)^6} - 3\frac{T(w)}{(z-w)^4} - \frac{3}{2}\frac{\partial T(w)}{(z-w)^3}\right. \\
&\qquad \left. + \frac{3}{2}\frac{\partial^2 T(w)}{(z-w)^2} - 4\frac{T^2(w)}{(z-w)^2} + \frac{1}{6}\frac{\partial^3 T(w)}{z-w} - 4\frac{\partial T^2(w)}{z-w}\right) \\
&\quad - 5 f^{\alpha\beta}_\gamma \left(\frac{W^\gamma(w)}{(z-w)^3} + \frac{1}{2}\frac{\partial W^\gamma(w)}{(z-w)^2} + \frac{1}{25}\frac{\partial^2 W^\gamma(w)}{z-w} + \frac{1}{25}\frac{(TW^\gamma)(w)}{z-w}\right),
\end{aligned}$$

where $g^{+-} = g^{-+} = 2$, $g^{00} = -1$ and $f^{\alpha\beta}_\gamma$ are the structure constants of $\mathfrak{su}(2)$, normalized to $f_0^{+-} = 2$. These relations are equivalent to the commutators (3.2a)-(3.2c) by Theorem 2.7.

4.2 Abe's Generalized Symplectic Fermion Model

4.2.1 Construction of SF^+

The construction of SF is a generalization of the standard construction of vertex operator algebras through Heisenberg Lie algebras as it is described in, e.g. [26], Section 6.3.

Abe's construction has partly been inspired by a similar treatment of the vertex operator algebra $M(1)^+$ by Dong and Nagatomo, [13]. Roughly, Abe's generalization, which implements super commutation rules, can be described as follows. The construction starts from a d-dimensional vector space $\mathfrak{h}$ to which one associates an affine Lie superalgebra, given by the nontrivial central extension of $\mathfrak{h} \otimes \mathbb{C}[t^{\pm 1}]$ by the one dimensional center $\mathbb{C}K$. The Fock space SF is the space generated from the highest weight vector $\mathbf{1}$ with the property $\mathfrak{h} \otimes \mathbb{C}[t]\mathbf{1} = 0$ and $K\mathbf{1} = \mathbf{1}$. This space naturally carries the structure of a vertex superalgebra with vacuum vector $\mathbf{1}$.

The modes of the vector $\sum_{i=1}^{d}(e^i \otimes t^{-1})(f^i \otimes t^{-1})\,\mathbf{1}$, where $\{e^i, f^i\}_{1\leq i\leq d}$ is the canonical basis of $\mathfrak{h}$, satisfy the Virasoro algebra. Thus, SF can be made into a vertex operator superalgebra. It can be decomposed into an even and an odd part such that the vertex operators corresponding to the vectors of the even and odd part satisfy the locality axiom with a bosonic and a fermionic commutator, respectively. The even part SF^+ is the vertex operator algebra which generalizes the $\mathcal{W}(2,3,3,3)$ triplet algebra at $c = -2$.

As indicated above, the construction of SF^+ starts from a finite dimensional vector space $\mathfrak{h}$ with a skew-symmetric nondegenerate bilinear form $\langle\cdot,\cdot\rangle$. The dimension of $\mathfrak{h}$ has to be even and we can pick a basis $\{e^i, f^i \mid 1 \leq i \leq d\}$ such that

$$\langle e^i, f^j\rangle = -\langle f^j, e^i\rangle = -\delta^{ij}$$

for $1 \leq i, j \leq d$, where $d = \dim\mathfrak{h}/2$ and all other pairings vanish.

One proceeds to construct the Heisenberg superalgebra $\hat{L}(\mathfrak{h}) = \mathfrak{h}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$, such that $\mathbb{C}K$ is the even part and $\mathfrak{h} \otimes \mathbb{C}[t,t^{-1}]$ is the odd part. This is done by equipping $\hat{L}(\mathfrak{h})$ with the bracket relations

$$[\psi \otimes t^m, \psi' \otimes t^n]_+ = m\langle\psi,\psi'\rangle\delta_{m+n}K \tag{4.3}$$

and

$$[K, \hat{L}(\mathfrak{h})] = 0,$$

where $\psi, \psi' \in \mathfrak{h}$, $m, n \in \mathbb{Z}$, making it into a Lie superalgebra. It should be noted that here and in the following, δ_i is taken as an abbreviation for $\delta_{i,0}$.

The vertex operator superalgebra SF is obtained by taking the universal enveloping algebra $U(\hat{L}(\mathfrak{h}))$ and dividing out certain ideals. More precisely, let $\mathcal{A}$ be the quotient algebra of $U(\hat{L}(\mathfrak{h}))$ by the two sided ideal generated by $K - 1$. Furthermore, denoting by $\psi(m)$ the operator of left multiplication by $\psi \otimes t^m$ on $\mathcal{A}$ for $\psi \in \mathfrak{h}$ and $m \in \mathbb{Z}$, we call $\mathcal{A}_{\geq 0}$ the ideal generated by $\psi(m)\,1$ with $\psi \in \mathfrak{h}$, $m \in \mathbb{Z}_{\geq 0}$. Finally, SF, the space which will be given the vertex operator algebra structure, is obtained from $\mathcal{A}$ by dividing out $\mathcal{A}_{\geq 0}$.

The $\mathbb{Z}_2$-grading of $\hat{L}(\mathfrak{h})$ induces a $\mathbb{Z}_2$-grading on $\mathcal{A}$. The even and odd parts are called $\mathcal{A}^{\bar{0}}$ and $\mathcal{A}^{\bar{1}}$, respectively. Thus, SF decomposes as $SF = SF^{\bar{0}} \oplus SF^{\bar{1}}$, where $SF^{\bar{i}} = \mathcal{A}^{\bar{i}}/(\mathcal{A}_{\geq 0} \cap \mathcal{A}^{\bar{i}})$ for $\bar{i} \in \mathbb{Z}_2$.

We have yet to equip SF with the structure of a vertex operator superalgebra. This is done by first defining the modes of a vector and then defining a vertex operator corresponding to this vector as a formal series with the modes as coefficients. It will be shown that this definition is identical to the one given in the reconstruction Theorem 2.11 and that all the assumptions of the theorem are satisfied by SF. Having established in this way that SF is a vertex superalgebra, the $\mathfrak{Vir}$-module structure and grading properties will be verified in a second step, making SF into a vertex operator superalgebra.

The linear map from $SF \to \mathrm{End}(SF)$, $a \mapsto a_{(n)}$ is given by

$$a_{(n)} = \sum_{\substack{i_j \in \mathbb{Z} \\ \sum_{j=1}^r = -\sum_{j=1}^r n_j + n + 1}} \binom{-i_1 - 1}{n_1 - 1} \cdots \binom{-i_r - 1}{n_r - 1} : \psi^1(i_1) \cdots \psi^r(i_r) : \tag{4.4}$$

for $a = \psi^1(-n_1) \cdots \psi^r(-n_r)\, 1$ with $\psi^i \in \mathfrak{h}$ and $n_i \in \mathbb{Z}_{>0}$, where the normal ordered product $: \cdot :$ is the operation on $\mathcal{A}$ defined inductively by $: \psi(n) := \psi(n)$ and

$$: \psi^1(n_1) \cdots \psi^r(n_r) 1 := \begin{cases} \psi^1(n_1) : \psi^2(n_2) \cdots \psi^r(n_r)\, 1 : & \text{if } n_1 < 0, \\ (-1)^{r-1} : \psi^2(n_2) \cdots \psi^r(n_r)\, 1 : \psi^1(n_1) & \text{if } n_1 \geq 0, \end{cases} \tag{4.5}$$

with $r \in \mathbb{Z}_{>0}$, $n, n_i \in \mathbb{Z}$ and $\psi, \psi^i \in \mathfrak{h}$. We obtain the vertex operator as the linear map $Y(\cdot, x) : \; SF \to \mathrm{Hom}(SF, SF((x)))$ by setting

$$Y(a, x) = \sum_{n \in \mathbb{Z}} a_{(n)} x^{-n-1}. \tag{4.6}$$

The vector space $\mathfrak{h}$ may be identified as a subspace of SF by the injective map $\psi \mapsto \psi(-1)\, 1 + \mathcal{A}_{\geq 0}$ and we write $\psi_{(m)}$ for $(\psi(-1)\, 1)_{(m)}$ for $\psi \in \mathfrak{h}$ and $m \in \mathbb{Z}$.

Theorem 4.1. *The above construction endows the space SF with a vertex superalgebra structure, where the vertex operator mapping is given by*

$$Y(a^{\alpha_1}_{j_1} \ldots a^{\alpha_m}_{j_m} \mathbf{1}, x) =: \partial^{(j_1 - 1)} a^{\alpha_1}(x) \ldots \partial^{(j_m - 1)} a^{\alpha_m}(x) : \tag{4.7}$$

Proof. We have to verify the assumptions of the reconstruction theorem. Define the vacuum in SF as the vector $\mathbf{1} = 1 + \mathcal{A}_{\geq 0}$ and complement the definition of Y by setting $Y(\mathbf{1}, x) = \mathrm{id}$. The canonical basis, ordered according to the list $(e^1, \ldots, e^d, f^1, \ldots, f^d)$ is a finite ordered set of homogeneous vectors of SF. The corresponding fields are given by $Y(v^i, x)$, where $v = e, f$.

(R1) Since $\psi_{(i)} \mathbf{1} = 0$ for $\psi \in \mathfrak{h}$ and $i \in \mathbb{Z}_{\geq 0}$, we have $Y(\psi, x)\, \mathbf{1}|_{x=0} = \psi$.

(R2) It will be shown in section 4.2.2 that the mode $(\sum_{i=1}^d e^i_{(-1)} f^i)_{-2}$ satisfies the properties of the weight one operator T.

(R3) Let $\psi, \psi' \in \mathfrak{h}$. Then we have

$$\begin{aligned} \{\psi(x_1), \psi'(x_2)\} &= \sum_{m \in \mathbb{Z}} m \langle \psi, \psi' \rangle \delta_{m+n} x_1^{-m-1} x_2^{-n-1} \\ &= \sum_{m \in \mathbb{Z}} m \langle \psi, \psi' \rangle x_1^{-m-1} x_2^{m-1} \\ &= \langle \psi, \psi' \rangle \frac{\partial}{\partial x_2} x_1^{-1} \delta\left(\frac{x_2}{x_1}\right) \end{aligned}$$

It follows by (2.15) that

$$(x_1 - x_2)^2 \{\psi(x_1, \psi'(x_2)\} = 0,$$

proving (super)locality.

(R4) It follows from the Poincaré-Birkhoff-Witt theorem and the fact that $\psi_n \mathbf{1} = 0$ for $\psi \in \mathfrak{h}$ and $n \in \mathbb{Z}_{\geq 0}$, that SF is isomorphic as a vector space to $\bigwedge(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}])$. This proves (R4).

□

It has yet to be shown that the definition of the vertex operator mapping $Y(\cdot, x)$ in the theorem is compatible with the definition of $Y(\cdot, x)$ in the case of vectors of higher weight than 1 in SF. This means, that (4.6) together with (4.4) should give the same definition of a vertex operator as (4.7) for $r > 1$. This is easy to prove since

$$\begin{aligned}\partial^{n_1 - 1} Y(\psi, x) =& \frac{1}{(n_1 - 1)!} \sum_{i_1 \in \mathbb{Z}} (-1 - i_1)(-1 - i_1 - 1) \cdots (-1 - i_1 - (n_1 - 1)) \psi_{(i_1)} x^{-i_1 - n_1} \\ =& \sum_{i_1 \in \mathbb{Z}} \binom{-i_1 - 1}{n_1 - 1} \psi_{(i_1)} x^{-i_1 - n_1}.\end{aligned}$$

Plugging this into the definition (4.7) and collecting powers of x, we see that the coefficient of x^{-n-1} is given by the mode (4.4), proving the equivalence of the two definitions.

In the following, SF will be made into a vertex *operator* superalgebra. In terms of the canonical basis of $\mathfrak{h}$, the conformal vector is defined by

$$\omega = \sum_{j=1}^{d} e^j_{(-1)} f^j. \tag{4.8}$$

Theorem 4.2. *The $\mathcal{A}$-module SF becomes a simple vertex operator superalgebra of central charge $-2d$ with vacuum vector $\mathbf{1}$ and Virasoro vector ω. Furthermore SF is of CFT type, i.e. the weight 0 space of SF is given by $SF_0 = \mathbb{C}\,1$ and the grading by $SF = \bigoplus_{n=0}^{\infty} SF_n$.*

Proof. The Virasoro algebra relations will be proven below in section 4.2.2. It will furthermore be shown in section 4.3 that for $\psi, \psi^1, \ldots \psi^r \in \mathfrak{h}$

$$[L_n, \psi_{(m)}] = -\, m\psi_{(m+n)}, \tag{4.9}$$

where $m, n \in \mathbb{Z}$. It follows that the action of L_0 on an arbitrary vector of the form $\psi^1_{(-n_1)} \cdots \psi^r_{(-n_r)} \mathbf{1}$ with $n_1, \ldots n_r \in \mathbb{Z}_{\geq 0}$ is given by

$$L_0 \psi^1_{(-n_1)} \cdots \psi^r_{(-n_r)} \mathbf{1} = (n_1 + \cdots + n_r) \psi^1_{(-n_1)} \cdots \psi^r_{(-n_r)} \mathbf{1}.$$

One obtains this result by commuting L_0 with the $\psi^i_{(-n_i)}$, picking up a factor n_i but not altering the modes until L_0 hits the vacuum and annihilates it. Thus $SF_0 = \mathbb{C}\mathbf{1}$ and the grading $SF = \bigoplus_{n=0}^{\infty} SF_n$ is given by L_0 eigenvalues. Because of the reconstruction theorem, Theorem 2.11, it only has to be proven that the L_{-1}-derivative property is satisfied on the fields $Y(\psi, x)$, where $\psi \in \mathfrak{h}$. But by (4.9), we know that $[L_{-1}, \psi_{(m)}] = -m\psi_{(m-1)}$ and since

$$\partial_x Y(\psi, x) = \sum_m (-m - 1)\psi_{(m)} x^{-m-2} = \sum_m (-m)\psi_{(m-1)} x^{-m-1},$$

we have $[L_{-1}, Y(\psi, x)] = \partial_x Y(\psi, x)$ for $\psi \in \mathfrak{h}$.

It was shown in the proof of Theorem 4.1 that SF is isomorphic as a vector space to $\bigwedge(\mathfrak{h}\otimes t^{-1}\mathbb{C}[t^{-1}])$. Thus, elements of the form

$$\psi^1_{(-n_1)}\cdots\psi^r_{(-n_r)}\mathbf{1}, \tag{4.10}$$

where $\psi^1,\dots,\psi^r \in \mathfrak{h}$ and $n_1,\dots,n_r \in \mathbb{Z}_{>0}$ span SF. Assume that there is a nontrivial subspace M of SF which is a submodule of SF. Then there is a vector v in M with minimal weight (not necessarily unique) whose weight is not zero since the vacuum is the only vector with that property. Since v is of the form (4.10), there is a vector $\psi \in SF$ such that $\psi_{(n)}v$ with $n > 0$ is not zero. But this operation lowers the weight which contradicts the assumption that the weight of v is minimal. Consequently, SF has to be irreducible as its own module. □

Recall from Section 2.4 that, as a vertex operator superalgebra, SF is endowed with an automorphism θ of order 2. Let SF^+ be the 1-eigenspace and SF^- the (-1)-eigenspace. Then $SF = SF^+ \oplus SF^-$ and we have the grading

$$SF^+ = \bigoplus_{n=0}^{\infty} \quad \text{and} \quad SF^+_n \quad SF^- = \bigoplus_{n=1}^{\infty} SF^-_n. \tag{4.11}$$

Proposition 4.3. *The space SF^+ is a simple vertex operator algebra and the SF^+-module SF^- is irreducible.*

Proof. Since SF is a simple vertex operator superalgebra and θ is an automorphism of order 2 of SF, we are in the situation of Theorem 2.17 and the assertion follows. □

In [1], Abe proceeds to find a set of generators of the vertex operator algebra SF^+. He proves that the set

$$\operatorname{span}\left\{a^1_{(-n_1)}\cdots a^s_{(-n_s)}\mathbf{1} | a^i \in SF^+_2 \oplus SF^+_3, n_i \in \mathbb{Z}\right\}$$

is indeed the whole space SF^+. In order to perform explicit calculations, the generators

$$e^{i,j} := e^i_{(-1)}e^j \quad f^{i,j} := f^i_{(-1)}f^j \quad h^{i,j} := e^i_{(-1)}f^j$$
$$E^{i,j} := \frac{1}{2}\left(e^i_{(-2)}e^j + e^j_{(-2)}e^i\right) \quad F^{i,j} := \frac{1}{2}\left(f^i_{(-2)}f^j + f^j_{(-2)}f^i\right) \quad H^{i,j} := \frac{1}{2}\left(e^i_{(-2)}f^j + f^j_{(-2)}e^i\right) \tag{4.12}$$

for $1 \le i, j \le d$ are defined. In terms of these generators, SF_2 and SF_3 are given as

$$SF^+_2 = \bigoplus_{i,j=1}^{d} \mathbb{C}h^{i,j} \bigoplus_{1\le i<j\le d} \left(\mathbb{C}e^{i,j} \oplus \mathbb{C}f^{i,j}\right)$$
$$SF^+_3 = \bigoplus_{i,j=1}^{d} \mathbb{C}H^{i,j} \bigoplus_{1\le i\le j\le d} \left(\mathbb{C}E^{i,j} \oplus \mathbb{C}F^{i,j}\right) \oplus L_{-1}SF^+_2.$$

The space $L_{-1}SF^+_2$ is needed to account for the antisymmetric combinations of weight 3, since $L_1\psi_{(-1)}\phi = \psi_{(-2)}\phi - \phi_{(-2)}\psi$. This formula can be verified in a straightforward manner by using the L_{-1}-derivative property and calculating $(\psi_{(-1)}\phi_{(-1)})_{(-2)}\mathbf{1}$. The above can be summarized as follows.

Proposition 4.4. *Let $(e^i, f^i)_{1\le i\le d}$ be a canonical basis. Then SF^+ is strongly generated by the vectors $e^{i,j}$, $h^{i,j}$, $f^{i,j}$, $E^{i,j}$, $H^{i,j}$ and $F^{i,j}$ with $1 \le i, j \le d$.*

4.2.2 Virasoro Algebra Relations in SF

The goal of this section will be to establish the Virasoro algebra relations, which are not explicitly proven in Abe's construction of SF. Proceeding in two steps, first the action of the Virasoro modes on arbitrary generators of SF^+ will be computed and then the Virasoro algebra will be established using Proposition 2.10. This has the advantage that the equations derived in the first step can also be used to show that all the generators of SF^+ are primary.

Looking at the list of generators of SF^+, we can conclude that the most general mode expansion we will need is

$$\begin{aligned}(a_{(-h)}b_{(-1)}\,\mathbf{1})_{(2)} &= \sum_{\substack{i_1,i_2\in\mathbb{Z}\\ i_1+i_2=n-h}} \binom{-i_1-1}{h-1}\binom{-i_2-1}{0} :a_{(i_1)}b_{(i_2)}: \\ &= \sum_{i_1<0}\binom{-i_1-1}{h-1}a_{(i_1)}b_{(n-h-i_1)} - \sum_{i_1\geq 0}\binom{-i_1-1}{h-1}b_{(n-h-i_1)}a_{(i_1)},\end{aligned}$$

where $h = 1, 2$. For the Virasoro modes we obtain with $a = e^j$, $b = f^j$ and $h = 1$

$$L_n = \sum_{j=1}^{d}\left(\sum_{i<0} e^j_{(i)}f^j_{(-i)} - \sum_{i\geq 0} f^j_{(-i)}e^j_{(i)}\right). \tag{4.13}$$

Before verifying the relations given in proposition 2.10, the action of L_n on vectors of the form $x^i_{(-r)}y^j_{(-s)}\,\mathbf{1}$ with x and y being either e or f will be derived. With these relations, one can quickly confirm the Virasoro algebra relations.

The calculations are straightforward with the strategy being to commute the different modes until a nonnegative mode hits the vacuum and annihilates it. Keep in mind that the definition of the anti-commutator specializes to

$$\{e^i_{(m)}, f^j_{(n)}\} = -m\delta^{ij}\delta_{m+n} \quad \text{and} \quad \{e^i_{(m)}, e^j_{(n)}\} = \{f^i_{(m)}, f^j_{(n)}\} = 0.$$

This implies in particular that two negative or two positive modes or two modes of the same "species" e or f can be exchanged, leading only to a change in sign. Beginning with $e^i_{(-r)}f^j_{(-s)}\,\mathbf{1}$, we have

$$\begin{aligned}L_n e^i_{(-r)}f^j_{(-s)}\,\mathbf{1} &= \sum_{l=1}^{d}\sum_{k<0} e^l_{(k)}f^l_{(n-k)}e^i_{(-r)}f^j_{(-s)}\,\mathbf{1} - \sum_{l=1}^{d}\sum_{k\geq 0} f^l_{(n-k)}e^l_{(k)}e^i_{(-r)}f^j_{(-s)}\,\mathbf{1} \\ &= \sum_{l=1}^{d}\sum_{k<0} e^l_{(k)}\left((n-k)\delta^{li}\delta_{n-k-r} - e^i_{(-r)}f^l_{(n-k)}\right)f^j_{(-s)}\,\mathbf{1} \\ &\quad + \sum_{l=1}^{d}\sum_{k\geq 0} f^l_{(n-k)}e^i_{(-r)}\left(-k\delta^{lj}\delta_{k-s} - f^j_{(-s)}e^l_{(k)}\right)\mathbf{1} \\ &= \sum_{l=1}^{d}\sum_{k<0}(n-k)e^l_{(k)}f^j_{(-s)}\delta^{li}\delta_{n-k-r}\,\mathbf{1} - \sum_{l=1}^{d}\sum_{k\geq 0} kf^l_{(n-k)}e^i_{(-r)}\delta^{lj}\delta_{k-s}\,\mathbf{1} \\ &= re^i_{(n-r)}f^j_{(-s)}\,\mathbf{1} - sf^j_{(n-s)}e^i_{(-r)}\,\mathbf{1},\end{aligned}$$

where $n - r < 0$ and $s \geq 0$ have been assumed in the last step. Otherwise, the first and second term would be zero, respectively. The action of L_n on $e^i_{(-r)}e^j_{(-s)}\mathbf{1}$ is given by

$$\begin{aligned}
L_n e^i_{(-r)}e^j_{(-s)} &= \sum_{l=1}^{d}\sum_{k<0} e^l_{(k)} f^l_{(n-k)} e^i_{(-r)} e^j_{(-s)}\mathbf{1} - \sum_{l=1}^{d}\sum_{k\geq 0} f^l_{(n-k)} e^l_{(k)} e^i_{(-r)} e^j_{(-s)}\mathbf{1}\\
&= \sum_{l=1}^{d}\sum_{k<0} e^l_{(k)}\left((n-k)\delta^{li}\delta_{n-k-r} - e^i_{(-r)} f^l_{(n-k)}\right) e^j_{(-s)}\mathbf{1}\\
&= \sum_{l=1}^{d}\sum_{k<0}\left((n-k)e^l_{(k)}e^j_{(-s)}\delta^{li}\delta_{n-k-r}\mathbf{1} - e^l_{(k)}e^i_{(-r)}\left((n-k)\delta^{lj}\delta_{n-k-s} - 0\right)\mathbf{1}\right)\\
&= re^i_{(n-r)}e^j_{(-s)}\mathbf{1} - se^j_{(n-s)}e^i_{(-r)}\mathbf{1}
\end{aligned}$$

The second sum in the first line vanishes because $e^l_{(k)}$ can be commuted through. In the last step, it was assumed that $n - r < 0$ and $n - s < 0$. The action of L_n on the missing two vectors can be calculated in exactly the same way so that we can summarize

$$L_n e^i_{(-r)}e^j_{(-s)}\mathbf{1} = re^i_{(n-r)}e^j_{(-s)}\mathbf{1} - se^j_{(n-s)}e^i_{(-r)}\mathbf{1}, \tag{4.14a}$$

$$L_n f^i_{(-r)}f^j_{(-s)}\mathbf{1} = rf^i_{(n-r)}f^j_{(-s)}\mathbf{1} - sf^j_{(n-s)}f^i_{(-r)}\mathbf{1}, \tag{4.14b}$$

$$L_n e^i_{(-r)}f^j_{(-s)}\mathbf{1} = re^i_{(n-r)}f^j_{(-s)}\mathbf{1} - sf^j_{(n-s)}e^i_{(-r)}\mathbf{1}, \tag{4.14c}$$

$$L_n f^i_{(-r)}e^j_{(-s)}\mathbf{1} = rf^i_{(n-r)}e^j_{(-s)}\mathbf{1} - se^j_{(n-s)}f^i_{(-r)}\mathbf{1}. \tag{4.14d}$$

The first and second term on the right hand of each of these equations vanish unless the conditions

$$n - r < 0, \qquad n - s < 0, \tag{4.15}$$

$$r \geq 0, \qquad s \geq 0, \tag{4.16}$$

$$n - r < 0, \qquad s \geq 0, \tag{4.17}$$

$$n - s < 0, \qquad r \geq 0 \tag{4.18}$$

are satisfied for the first and second term, respectively.

With the help of (4.14c) for $i = j$ and $r = s = 1$ the equations (2.29a)-(2.29e) from Proposition 2.10 can easily be verified. Note that the first of these equations follows directly from the definition $L_n = \omega_{n+1}$. With respect to the other equations, we have

$$\omega_1\omega = L_0\omega = \sum_{i=1}^{d} e^i_{(-1)}f^i_{(-1)}\mathbf{1} - f^i_{(-1)}e^i_{(-1)} = 2\sum_{i=1}^{d} e^i_{(-1)}f^i_{(-1)}\mathbf{1} = 2\omega,$$

$$\omega_2\omega = L_1\omega = -\sum_{i=1}^{d} f^i_{(0)}e^i_{(-1)}\mathbf{1} = \sum_{i=1}^{d} e^i_{(-1)}f^i_{(0)}\mathbf{1} = 0,$$

$$\omega_3\omega = L_2\omega = -\sum_{i=1}^{d} f^i_{(1)}e^i_{(-1)}\mathbf{1} = -\sum_{i=1}^{d}\delta^{ii}\delta_{1-1}\mathbf{1} = -d\,\mathbf{1},$$

$$\omega_n\omega = L_{n-1}\omega = -\sum_{i=1}^{d} f^i_{(n-2)}e^i_{(-1)}\mathbf{1} = 0 \quad \text{for } n \geq 4.$$

It follows from Proposition 2.10 that the modes L_n satisfy the Virasoro algebra relations.

Proposition 4.5. *The spaces of primary vectors $\mathcal{P}$ of weight 2 and 3 in SF^+ are given by*

$$\mathcal{P}(SF_2^+) = \bigoplus_{1\le i<j\le d} \left(\mathbb{C}e^{i,j} \oplus \mathbb{C}f^{i,j} \oplus \mathbb{C}h^{i,j} \oplus \mathbb{C}h^{j,i} \oplus \mathbb{C}(h^{i,i} - h^{j,j})\right),$$

$$\mathcal{P}(SF_3^+) = \bigoplus_{i,j=1}^{d} H^{i,j} \oplus \bigoplus_{1\le i\le j\le d} \left(\mathbb{C}E^{i,j} \oplus \mathbb{C}F^{i,j}\right).$$

It follows that SF^+ can be written as the direct sum

$$SF_2^+ \oplus SF_3^+ = \mathbb{C}\omega \oplus \mathcal{P}(SF_2^+) \oplus L_{-1}SF_2^+ \oplus \mathcal{P}(SF_3^+). \tag{4.19}$$

Therefore, the vertex operator algebra SF^+ forms a $\mathcal{W}(2, 2^{2d^2-d-1}, 3^{2d^2+d})$-algebra.

Proof. We have to establish that the generators of $\mathcal{P}(SF_2^+)$ and $\mathcal{P}(SF_3^+)$ are indeed primary. Recall that a primary vector v is defined by the condition $L_1 v = L_2 v = 0$. Keeping in mind the conditions (4.15)-(4.18) and assuming $i \neq j$, one has

$$\begin{aligned}
L_1 e^{i,j} &= L_2 e^{i,j} = 0,\\
L_1 f^{i,j} &= f^i_{(0)} f^j_{(-1)}\mathbf{1} - f^j_{(0)} f^i_{(-1)}\mathbf{1},\\
L_2 f^{i,j} &= f^i_{(1)} f^j_{(-1)}\mathbf{1} - f^j_{(1)} f^j_{(-1)}\mathbf{1},\\
L_1\left(h^{i,i} - h^{j,j}\right) &= -f^i_{(0)} e^i_{(-1)}\mathbf{1} = 0,\\
L_2\left(h^{i,i} - h^{j,j}\right) &= -f^i_{(1)} e^i_{(-1)}\mathbf{1} + f^j_{(1)} e^j_{(-1)}\mathbf{1} = -\delta^{ii} + \delta^{jj} = 0,\\
L_1 h^{i,j} &= -f^i_{(0)} e^j_{(-1)}\mathbf{1} = 0,\\
L_2 h^{i,j} &= -f^i_{(1)} e^j_{(-1)}\mathbf{1} = -\delta i, j = 0.
\end{aligned}$$

In the case of the generators of weight 3 one has with i and j arbitrary

$$\begin{aligned}
L_1 E^{i,j} &= 2e^i_{(-1)} e^j_{(-1)}\mathbf{1} + 2e^j_{(-1)} e^i_{(-1)}\mathbf{1}, = 2e^i_{(-1)} e^j_{(-1)}\mathbf{1} - 2e^i_{(-1)} e^j_{(-1)}\mathbf{1} = 0,\\
L_2 E^{i,j} &= 0,\\
L_1 F^{i,j} &= 2f^i_{(-1)} f^j_{(-1)}\mathbf{1} - f^i_{(0)} f^j_{(-2)}\mathbf{1} + 2f^j_{(-1)} f^i_{(-1)}\mathbf{1} - f^j_{(0)} f^i_{(-2)}\mathbf{1} = 0,\\
L_2 F^{i,j} &= 2f^i_{(0)} f^j_{(-1)}\mathbf{1} - f^i_{(1)} f^j_{(-2)}\mathbf{1} + 2f^j_{(0)} f^i_{(-1)}\mathbf{1} - f^j_{(1)} f^i_{(-2)}\mathbf{1} = 0,\\
L_1 H^{i,j} &= 2e^i_{(-1)} f^j_{(-1)}\mathbf{1} + 2f^j_{(-1)} e^i_{(-1)}\mathbf{1} - f^j_{(0)} e^i_{(-2)}\mathbf{1} = 0,\\
L_2 H^{i,j} &= 2f^j_{(0)} e^i_{(-1)}\mathbf{1} - f^j_{(1)} e^i_{(-2)}\mathbf{1} = 0.
\end{aligned}$$

Since $h^{i,j}$ with $i, j = 1 \ldots d$ arbitrary can be written as linear combination of ω, $h^{k,l}$, $h^{l,k}$ and $h^{l,l} - h^{k,k}$ with $l < k$, it follows from Corollary 4.4 that SF^+ can be written as a direct sum as in (4.19). Thus, the only nonprimary generating vector of SF^+ is ω, proving the assertion that SF^+ forms a $\mathcal{W}$-algebra. Counting the generators, one gets d^2-1 generators from $h^{i,j}$ without ω and, assuming $i < j$, $2\frac{d}{2}(d-1)$ from $e^{i,j}$ and $f^{i,j}$, adding up to $2d^2-d-1$ primaries of weight 2. For weight 3, one has d^2 from $H^{i,j}$ and, assuming $i \le j$, $\frac{d}{2}(d+1)$ from $E^{i,j}$ and $F^{i,j}$, adding up to $2d^2 + d$. □

4.2.3 Construction of SF^+-Modules and Classification

We already know that SF^+ and SF^- are irreducible SF^+-modules. In the following, two θ-twisted modules $SF(\theta)^\pm$ will be constructed. With the help of Zhu's algebra, Abe has been able to show that $\{SF^\pm, SF(\theta)^\pm\}$ gives a complete list of inequivalent irreducible modules.

The two θ-twisted modules are constructed in close analogy to SF. Starting point is the Lie superalgebra

$$\hat{L}^\theta(\mathfrak{h}) := \mathfrak{h} \otimes t^{\frac{1}{2}}\mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K$$

with bracket relations

$$[\psi \otimes t^m, \psi' \otimes t^n]_+ = m\delta_{m+n,0}\langle \psi, \psi' \rangle K \quad \text{and} \quad [K, \hat{L}^\theta(\mathfrak{h})] = 0 \tag{4.20}$$

for $\psi, \psi' \in \mathfrak{h}$ and $m, n \in \frac{1}{2} + \mathbb{Z}$ such that $\mathbb{C}K$ is the even part and $\mathfrak{h} \otimes t^{\frac{1}{2}}\mathbb{C}[t^{\pm 1}]$ is the odd part. As in the case of SF, one considers the associative Algebra $\mathcal{A}^\theta$ which is the quotient algebra of the universal enveloping algebra $U(\hat{L}^\theta(\mathfrak{h}))$ by the ideal generated by $K - 1$. The algebra $\mathcal{A}^\theta$ inherits the $\mathbb{Z}_2$ grading from $\hat{L}^\theta(\mathfrak{h})$ such that $\mathcal{A}^\theta = \mathcal{A}^\theta_{\bar{0}} \oplus \mathcal{A}^\theta_{\bar{1}}$. With $\mathcal{A}^\theta_{>0}$ defined to be the left ideal of $\mathcal{A}^\theta$ generated by the vectors $\psi \otimes t^m$ for $\psi \in \mathfrak{h}$ and $m \in \frac{1}{2} + \mathbb{Z}$, we set $SF(\theta) = \mathcal{A}/\mathcal{A}^\theta_{>0}$.

Again, the vector on which the Fock space will be built is defined by setting $1_\theta = 1 + \mathcal{A}^\theta_{>0}$. We denote by $\psi(m)$ the operator of left multiplication by $\psi \otimes t^m$ on $SF(\theta)$ for $\psi \in \mathfrak{h}$ and $m \in \frac{1}{2} + \mathbb{Z}$. Since the involution θ preserves $\mathcal{A}^\theta_{>0}$, the $\mathbb{Z}_2$-grading of $\mathcal{A}^\theta$ induces a decomposition $SF(\theta) = SF(\theta)^+ \oplus SF(\theta)^-$, where $SF(\theta)^\pm$ are the (± 1)-eigenspaces of $SF(\theta)$ for θ.

To endow $SF(\theta)$ with an SF-module structure, we have to find a way to define an action of the vertex operator $Y(v, x)$ with $v = \psi^1_{(-n_1)} \cdots \psi^r_{(-n_r)}\mathbf{1}$ on $SF(\theta)$. This is done in two steps. First, we introduce

$$W(\psi, x) = \sum_{i \in \frac{1}{2} + \mathbb{Z}} \psi(i) x^{-i-1}$$

for any $\psi \in \mathfrak{h}$ and define

$$W(v, x) =: \partial^{n_1 - 1} W(\psi^1, x) \cdots \partial^{n_r - 1} W(\psi^r, x) : \tag{4.21}$$

for $v = \psi^1_{(-n_1)} \cdots \psi^r_{(-n_r)}\mathbf{1}$ with $\psi^i \in \mathfrak{h}$ and $n_i \in \mathbb{Z}_{>0}$. Thus we obtain a linear map $W(\cdot, x)$ from SF to $\mathrm{Hom}(SF(\theta), SF(\theta))((x^{\frac{1}{2}}))$. Taking v as above, the vertex operator associated to v is defined with the help of a certain operator $\Delta(x)$ as

$$Y(v, x) = W(e^{\Delta(x)} v, x).$$

The operator $\Delta(x)$ is introduced in Chapter 9.2 of [19]. In the present context, it is defined as

$$\Delta(x) = 2 \sum_{m,n \geq 0} \sum_{i=1}^{d} c_{mn} e^i_{(n)} f^i_{(m)} x^{-m-n},$$

where the coefficients $c_{mn} \in \mathbb{C}$ are determined by the formal expansion

$$\sum_{m,n \geq 0} c_{mn} x^m y^n = -\log\left(\frac{(1+x)^{\frac{1}{2}} + (1+y)^{\frac{1}{2}}}{2}\right).$$

Additionally one sets $Y(\mathbf{1}, x) = \text{id}$. Writing $\psi_{(m)}$ for $(\psi_{(-1)}\mathbf{1})_{(m)}$ for $\psi \in \mathfrak{h}$ and $m \in \frac{1}{2} + \mathbb{Z}$, one can obtain the following identities by direct calculation (c.f. [19]):

$$Y(\psi, x) = W(\psi, x) \tag{4.22}$$

$$Y(\psi_{(-1)}\phi, x) = W(\psi_{(-1)}\phi, x) + \frac{\langle\psi, \phi\rangle}{8}\text{id}x^{-2}. \tag{4.23}$$

Setting $\psi = e^i$ and $\phi = f^i$ we have for the Virasoro vector

$$Y(\omega, x) \equiv \sum_n \omega_n^W x^{-1-n} = W(\omega, x) - \frac{d}{8}\text{id}x^{-2}.$$

Using the definition (4.21), this implies

$$L_n^W = \omega_{n+1}^W = \sum_{i=1}^{d} \sum_{\substack{l<0 \\ l\in\frac{1}{2}+\mathbb{Z}}} e^i_{(l)} f^i_{(n-l)} - \sum_{i=1}^{d} \sum_{\substack{l\geq 0 \\ l\in\frac{1}{2}+\mathbb{Z}}} f^i_{(n-l)} e^i_{(l)} - \frac{d}{8}\text{id}\delta_{n,0}. \tag{4.24}$$

In the following, the W superscript will not be used since the context is clear. We are now in a position to prove the following statement.

Theorem 4.6. *The space $SF(\theta)$ carries the structure of a θ-twisted SF module.*

Proof. All axioms follow easily from the definitions except for the translation axiom, the Jacobi identity and the L_0-grading related axioms. For the Jacobi identity, the reader is referred to Theorem 9.5.3 in [19] and its proof therein.

With the help of identity (4.24), one concludes that $[L_m, \psi_{(n)}] = -n\psi_{(m+n)}$ for $m \in \mathbb{Z}$ and $n \in \frac{1}{2} + \mathbb{Z}$ as in the untwisted case since the term proportional to the identity does not contribute to the commutator. This implies the translation axiom (c.f. Section 4.2.1). Moreover, it follows as in the untwisted case that

$$L_0 1_\theta = -\frac{d}{8} 1_\theta$$

and

$$L_0 \psi^1_{(-n_1)} \cdots \psi^r_{(-n_r)} 1_\theta = \left(-\frac{d}{8} + \sum n_i\right) \psi^1_{(-n1)} \cdots \psi^r_{(-n_r)} 1_\theta \tag{4.25}$$

for any $\psi^i \in \mathfrak{h}$ and $n_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$. We know that $SF(\theta) \cong \bigwedge(\mathfrak{h} \otimes t^{-\frac{1}{2}}\mathbb{C}[t^{-1}])$ by the same arguments as in the construction of SF. Thus, elements of the form (4.25) form a basis of $SF(\theta)$ as a vector space. Consequently the grading of $SF(\theta)$ is given by

$$SF(\theta) = \bigoplus_{i=0}^{\infty} SF(\theta)_{-\frac{d}{8}+\frac{i}{2}}. \tag{4.26}$$

□

Let $\Omega(SF(\theta))$ be the space of singular vectors of $SF(\theta)$. Since $SF(\theta)$ has a Poincaré-Birkhoff-Witt basis of the form (4.25) and the action of positive modes is given by the contraction according to (4.20), 1_θ spans $\Omega(SF(\theta))$. But that means that there can be no vector w which is linearly independent from 1_θ such that one can generate a submodule by acting on w with modes $\psi_{(-n)}$, where $n \in -\frac{1}{2} + \mathbb{Z}_{>0}$ and $\psi \in \mathfrak{h}$. Consequently, $SF(\theta)$ has to be irreducible.

Recall the decomposition (2.41) of an irreducible twisted module. Since θ is an automorphism of SF of order 2, we have the decomposition $SF(\theta) = SF(\theta)^+ \oplus SF(\theta)^-$. According to the discussion in Section 2.4, $SF(\theta)^\pm$ are SF^+-modules and by (2.40), they have the grading

$$SF(\theta)^+ = \bigoplus_{i=0}^{\infty} SF(\theta)_{-\frac{d}{8}+i} \quad \text{and} \quad SF(\theta)^- = \bigoplus_{i=0}^{\infty} SF(\theta)_{-\frac{d+4}{8}+i}. \tag{4.27}$$

Proposition 4.7. *The SF^+-modules $SF(\theta)^\pm$ are irreducible as SF^+-modules.*

Proof. Since $SF(\theta)$ is an irreducible θ-twisted SF module, the assertion follows from Theorem 2.18. □

So far, the construction of SF and of $SF(\theta)$ has yielded four SF^+-modules: $SF^\pm$ and $SF(\theta)^\pm$. The classification of all irreducible modules is given by the following result obtained by Abe.

Theorem 4.8. *For the vertex operator algebra $SF+$, any irreducible $A(SF^+)$-module is isomorphic to one in the list $\{\Omega(SF^\pm), \Omega(SF(\theta)^\pm)\}$.*

Proof. The proof will only be outlined here since it requires extensive calculations. For a detailed proof see the proofs of Theorems 4.8 and 4.15 in [1]. The cases $d = 1$ and $d > 1$ are treated separately. Using Zhu's theorem, the problem is reduced to a classification of irreducible $A(SF^+)$-modules. In the case $d = 1$, Abe has shown that $A(SF^+)$ decomposes into a direct sum of four ideals on which $[\omega]$ acts as scalar multiple of the identity. These ideals are either commutative or homorphic images of simple algebras. Therefore there exist at most four irreducible $A(SF^+)$-modules. But since the existence of at least four irreducible modules has been proven by the above construction, there are exactly four irreducible $A(SF^+)$-modules.

The case $d > 1$ is more intricate. First, one has to prove that any irreducible $A(SF^+)$-module W is a direct sum of eigenspaces for the action of all $[h^{i,i}]$ for $i = 1, \ldots, d$. By explicit calculations in $A(SF^+)$, Abe derived the following equation constraining the eigenvalues:

$$[h^{i,i}]^2 \star \left([h^{i,i}] - 1\right) \star \left(8[h^{i,i}] + 1\right) \star \left(8[h^{i,i}] - 3\right) = 0.$$

This equation is the exact analogue of equation (3.4) in the case $d > 1$. It follows that the eigenvalues $\lambda_1, \ldots, \lambda_d$ take values in the set $\{0, -\frac{1}{8}, 1, \frac{3}{8}\}$.

Abe then shows that the following implication holds: if there exists a nonzero simultaneous eigenspace for all $[h^{i,i}]$ such that λ_i is equal to one of the elements of $\{1, \frac{3}{8}, 0, -\frac{1}{8}\}$, then W is isomorphic to the respective element of the list

$$\{\Omega(SF^-), \Omega(SF(\theta)^-), \Omega(SF^+), \Omega(SF(\theta)^+)\}.$$

□

The spectrum of SF^+ can be read off from the gradings (4.11) and (4.27). It is given by

$$h = 0, 1, -\frac{d}{8}, \frac{-d+4}{8}. \tag{4.28}$$

4.3 Explicit Calculations in SF^+

In this section a general commutator formula for modes of vectors which have the form $a_{(-n_1)}b_{(-n_2)}$ will be derived. Since all generators are of this form, all relevant commutators in SF^+ may be calculated with this formula. This result will not be needed in the calculations of Chapter 5 since it suffices to study certain special cases of modes applied to a fixed SF^+-module. However, it would in principle be possible to calculate all commutators of the $\mathcal{W}(2, 2^{2d^2-d-1}, 3^{2d^2+d})$ algebra, so that the result may be of interest. Besides, the commutator formula allows for an explicit calculation of the Virasoro algebra relations and formula (4.9).

In the case considered here, the mode definition (4.4) specializes to

$$(\psi^1_{(-n_1)}\psi^2_{(-n_2)}\,\mathbf{1})_{(n)} = \sum_{\substack{i_1,i_2\\ i_1+i_2=-n_1-n_2+n+1}} \binom{-i_1-1}{n_1-1}\binom{-i_2-1}{n_2-1} :\psi^1_{(i_1)}\psi^2_{(i_2)}: .$$

Observing that one summation breaks down and using the definition (4.5) of normal ordering, we obtain

$$\begin{aligned}(\psi_{(-n_1)}\phi_{(-n_2)}\,\mathbf{1})_{(n)} &= \sum_{\nu<0}\binom{-\nu-1}{n_1-1}\binom{-(-n_1-n_2-\nu+n+1)-1}{n_2-1}\psi_{(\nu)}\phi_{(n-n_1-n_2-\nu+1)} &\text{(4.29a)}\\ &-\sum_{\nu\geq 0}\binom{-\nu-1}{n_1-1}\binom{-(-n_1-n_2-\nu+n+1)-1}{n_2-1}\phi_{(n-n_1-n_2-\nu+1)}\psi_{(\nu)} &\text{(4.29b)}\end{aligned}$$

and

$$\begin{aligned}(\rho_{(-m_1)}\sigma_{(-m_2)}\,\mathbf{1})_{(m)} &= \sum_{\mu<0}\binom{-\mu-1}{m_1-1}\binom{-(-m_1-m_2-\mu+m+1)-1}{m_2-1}\rho_{(\mu)}\sigma_{(m-m_1-m_2-\mu+1)} &\text{(4.29c)}\\ &-\sum_{\mu\geq 0}\binom{-\mu-1}{m_1-1}\binom{-(-m_1-m_2-\mu+m+1)-1}{m_2-1}\sigma_{(m-m_1-m_2-\mu+1)}\rho_{(\mu)}. &\text{(4.29d)}\end{aligned}$$

The calculation of the commutator of the above two modes is straightforward but we will need some notation to keep the presentation clear. In the following, (4.29a)-(4.29d) will be referred to as $A-D$, where the right hand side is meant in the case of (4.29a) and (4.29c). With this convention, the commutator becomes

$$\left[(\psi_{(-n_1)}\phi_{(-n_2)}\,\mathbf{1})_{(n)}, (\rho_{(-m_1)}\sigma_{(-m_2)}\,\mathbf{1})_{(m)}\right] = [A,C]-[A,D]-[B,C]+[B,D]. \tag{4.30}$$

The following notation also helps to keep the formulas compact:

$$\mu' = m - m_1 - m_2 - \mu + 1$$
$$\nu' = n - n_1 - n_2 - \nu + 1$$
$$C_1 = \binom{-\nu - 1}{n_1 - 1}\binom{-(-n_1 - n_2 - \nu + n + 1) - 1}{n_2 - 1}\binom{-\mu - 1}{m_1 - 1}\binom{-(-m_1 - m_2 - \mu + m + 1) - 1}{m_2 - 1}.$$

Consider the commutator $[A, C]$ as an example. Using the above notation, it is given by

$$[A, C] = \sum_{\substack{\nu<0\\ \mu<0}} C_1\psi_{(\nu)}\phi_{(\nu')}\rho_{(\mu)}\sigma_{(\mu')} - \sum_{\substack{\nu<0\\ \mu<0}} C_1\rho_{(\mu)}\sigma_{(\mu')}\psi_{(\nu)}\phi_{(\nu')}. \tag{4.31}$$

The strategy for the computation of the commutator is to bring the second term in the order of the first term. This is done with the help of the anticommutation rule

$$\left\{\psi_{(m)}, \psi'_{(n)}\right\} = m\langle\psi, \psi'\rangle\delta_{n+m,0}.$$

Thus, in the process of rearranging, applying the anticommutator will yield a term of four modes which has the desired order and terms involving only two modes because the other two have been contracted. Since an even number of transpositions is needed to arrange the second term of (4.31) in the order of the first term, its sign will not be altered. Thus, the terms involving four modes will cancel each other, leaving only terms involving two modes. For the following calculation, keep in mind that δ_x should always be read as $\delta_{x,0}$.

$$\begin{aligned}
\rho_{(\mu)}\sigma_{(\mu')}\psi_{(\nu)}\phi_{(\nu')} =& \rho_{(\mu)}\left(\mu'\langle\sigma,\psi\rangle\delta_{\mu'+\nu} - \psi_{(\nu)}\sigma_{(\mu')}\right)\phi_{(\nu')}\\
=& \mu'\langle\sigma,\psi\rangle\delta_{\mu'+\nu}\rho_{(\mu)}\phi_{(\nu')} - \left(\mu\langle\rho,\psi\rangle\delta_{\mu+\nu} - \psi_{(\nu)}\rho_{(\mu)}\right)\sigma_{(\mu')}\phi_{(\nu')}\\
=& \mu'\langle\sigma,\psi\rangle\delta_{\mu'+\nu}\rho_{(\mu)}\phi_{(\nu')} - \mu\langle\rho,\psi\rangle\delta_{\mu+\nu}\sigma_{(\mu')}\phi_{(\nu')}\\
&+ \psi_{(\nu)}\rho_{(\mu)}\left(\mu'\langle\sigma,\phi\rangle\delta_{\mu'+\nu'} - \phi_{(\nu')}\sigma_{(\mu')}\right)\\
=& \mu'\langle\sigma,\psi\rangle\delta_{\mu'+\nu}\rho_{(\mu)}\phi_{(\nu')} - \mu\langle\rho,\psi\rangle\delta_{\mu+\nu}\sigma_{(\mu')}\phi_{(\nu')} + \mu'\langle\sigma,\phi\rangle\delta_{\mu'+\nu'}\psi_{(\nu)}\rho_{(\mu)}\\
&- \mu\langle\rho,\phi\rangle\delta_{\mu+\nu'}\psi_{(\nu)}\sigma_{(\mu')} + \psi_{(\nu)}\phi_{(\nu')}\rho_{(\mu)}\sigma_{(\mu')}
\end{aligned}$$

Performing the same calculations for all the other commutators, we obtain the following terms for the commutators $[A, C]$, $[A, D]$, $[B, C]$ and $[B, D]$, respectively.

$$\begin{aligned}
E = \rho_{(\mu)}\sigma_{(\mu')}\psi_{(\nu)}\phi_{(\nu')} =& \mu'\langle\sigma,\psi\rangle\delta_{\mu'+\nu}\rho_{(\mu)}\phi_{(\nu')} - \mu\langle\rho,\psi\rangle\delta_{\mu+\nu}\sigma_{(\mu')}\phi_{(\nu')} + \mu'\langle\sigma,\phi\rangle\delta_{\mu'+\nu'}\psi_{(\nu)}\rho_{(\mu)}\\
&- \mu\langle\rho,\phi\rangle\delta_{\mu+\nu'}\psi_{(\nu)}\sigma_{(\mu')} + \psi_{(\nu)}\phi_{(\nu')}\rho_{(\mu)}\sigma_{(\mu')}
\end{aligned} \tag{4.32a}$$

$$\begin{aligned}
F = \sigma_{(\mu')}\rho_{(\mu)}\psi_{(\nu)}\phi_{(\nu')} =& -\mu'\langle\sigma,\psi\rangle\delta_{\mu'+\nu}\rho_{(\mu)}\phi_{(\nu')} + \mu\langle\rho,\psi\rangle\delta_{\mu+\nu}\sigma_{(\mu')}\phi_{(\nu')} - \mu'\langle\sigma,\phi\rangle\delta_{\mu'+\nu'}\psi_{(\nu)}\rho_{(\mu)}\\
&+ \mu\langle\rho,\phi\rangle\delta_{\mu+\nu'}\psi_{(\nu)}\sigma_{(\mu')} + \psi_{(\nu)}\phi_{(\nu')}\rho_{(\mu)}\sigma_{(\mu')}
\end{aligned} \tag{4.32b}$$

$$\begin{aligned}
G = \rho_{(\mu)}\sigma_{(\mu')}\phi_{(\nu')}\psi_{(\nu)} =& \mu'\langle\sigma,\psi\rangle\delta_{\mu'+\nu}\phi_{(\nu')}\rho_{(\mu)} - \mu\langle\rho,\psi\rangle\delta_{\mu+\nu}\phi_{(\nu')}\sigma_{(\mu')} + \mu'\langle\sigma,\phi\rangle\delta_{\mu'+\nu'}\rho_{(\mu)}\psi_{(\nu)}\\
&- \mu\langle\rho,\phi\rangle\delta_{\mu+\nu'}\sigma_{(\mu')}\psi_{(\nu)} + \psi_{(\nu)}\phi_{(\nu')}\rho_{(\mu)}\sigma_{(\mu')}
\end{aligned} \tag{4.32c}$$

$$\begin{aligned}
H = \sigma_{(\mu')}\rho_{(\mu)}\phi_{(\nu')}\psi_{(\nu)} =& -\mu'\langle\sigma,\psi\rangle\delta_{\mu'+\nu}\phi_{(\nu')}\rho_{(\mu)} + \mu\langle\rho,\psi\rangle\delta_{\mu+\nu}\phi_{(\nu')}\sigma_{(\mu')} - \mu'\langle\sigma,\phi\rangle\delta_{\mu'+\nu'}\rho_{(\mu)}\psi_{(\nu)}\\
&+ \mu\langle\rho,\phi\rangle\delta_{\mu+\nu'}\sigma_{(\mu')}\psi_{(\nu)} + \psi_{(\nu)}\phi_{(\nu')}\rho_{(\mu)}\sigma_{(\mu')}
\end{aligned} \tag{4.32d}$$

Except for the coefficients which have to be calculated by evaluating the Kronecker deltas, the computation of the commutators is already finished. In the following, E', F', G', H' will be taken to denote the respective terms E, F, G, H after subtraction of $\psi_{(\nu)}\phi_{(\nu')}\rho_{(\mu)}\sigma_{(\mu')}$. Note that for $[A, C]$ and $[A, D]$, we have the summation regions $\mu < 0, \nu < 0$ and $\mu < 0, \nu \geq 0$, respectively. Furthermore, comparing (4.32a) and (4.32b), we see that $E' = -F'$ so that the sums corresponding to $[A, C]$ and $[A, D]$ in (4.30) add such that the sum over ν gets unrestricted. The same is true for the commutators $[B, C]$ and $[B, D]$. This implies that the whole commutator takes the form

$$\begin{aligned} K = \left[(\psi_{(-n_1)}\phi_{(-n_2)}\mathbf{1})_{(n)}, (\rho_{(-m_1)}\sigma_{(-m_2)}\mathbf{1})_{(m)}\right] &= [A, C] - [A, D] - [B, C] + [B, D] \\ &= -\sum_{\substack{\nu<0\\ \mu}} C_1 E' + \sum_{\substack{\nu\geq 0\\ \mu}} C_1 G'. \end{aligned} \tag{4.33}$$

Hence, the Kronecker deltas can be evaluated without taking into account additional conditions coming from restricted summation indices. Evaluating these Kronecker deltas, one obtains the conditions

$$\delta_{\mu+\nu} \Rightarrow \mu = -\nu \tag{4.34a}$$
$$\delta_{\mu'+\nu} \Rightarrow \mu = m - m_1 - m_2 + \nu + 1 \tag{4.34b}$$
$$\delta_{\mu+\nu'} \Rightarrow \mu = -n + n_1 + n_2 + \nu - 1 \tag{4.34c}$$
$$\delta_{\mu'+\nu'} \Rightarrow \mu = m + n - m_1 - m_2 - n_1 - n_2 - \nu + 2. \tag{4.34d}$$

Additional notation is needed to denote the coefficients with the above conditions applied,

$$\begin{aligned} C_{11} &= \mu C_1 \quad \text{where (4.34a) is satisfied,} \\ C_{12} &= \mu' C_1 \quad \text{where (4.34b) is satisfied,} \\ C_{13} &= \mu C_1 \quad \text{where (4.34c) is satisfied,} \\ C_{14} &= \mu' C_1 \quad \text{where (4.34d) is satisfied.} \end{aligned}$$

It is straightforward to compute the coefficients under the respective conditions, yielding

$$C_{11} = -\nu\binom{-\nu-1}{n_1-1}\binom{-m+m_1+m_2-\nu-2}{m_2-1}\binom{\nu-1}{m_1-1}\binom{-n+n_1+n_2+\nu-2}{n_2-1}, \tag{4.35a}$$

$$C_{12} = -\nu\binom{-\nu-1}{n_1-1}\binom{-m+m_1+m_2-\nu-2}{m_1-1}\binom{\nu-1}{m_2-1}\binom{-n+n_1+n_2+\nu-2}{n_2-1}, \tag{4.35b}$$

$$\begin{aligned} C_{13} = &(-n+n_1+n_2+\nu-1)\binom{-\nu-1}{n_1-1}\binom{n-n_1-n_2-\nu}{m_1-1}\binom{-n+n_1+n_2+\nu-2}{n_2-1} \\ &\times\binom{-m+m_1+m_2-n+n_1+n_2+\nu-3}{m_2-1}, \end{aligned} \tag{4.35c}$$

$$\begin{aligned} C_{14} = &(-n+n_1+n_2+\nu-1)\binom{-\nu-1}{n_1-1}\binom{n-n_1-n_2-\nu}{m_2-1}\binom{-n+n_1+n_2+\nu-2}{n_2-1} \\ &\times\binom{-m+m_1+m_2-n+n_1+n_2+\nu-3}{m_1-1}. \end{aligned} \tag{4.35d}$$

We can now put the pieces together by plugging (4.32a) and (4.32c) into (4.33).

$$\begin{aligned}
K = &- \sum_{\nu<0,\mu} C_1 \Big[\mu' \langle\sigma,\psi\rangle \delta_{\mu'+\nu} \rho_{(\mu)}\phi_{(\nu')} - \mu\langle\rho,\psi\rangle\delta_{\mu+\nu}\sigma_{(\mu')}\phi_{(\nu')} \\
&\qquad + \mu'\langle\sigma,\phi\rangle\delta_{\mu'+\nu'}\psi_{(\nu)}\rho_{(\mu)} - \mu\langle\rho,\phi\rangle\delta_{\mu+\nu'}\psi_{(\nu)}\sigma_{(\mu')}\Big] \\
&+ \sum_{\nu\geq 0,\mu} C_1 \Big[\mu'\langle\sigma,\psi\rangle\delta_{\mu'+\nu}\phi_{(\nu')}\rho_{(\mu)} - \mu\langle\rho,\psi\rangle\delta_{\mu+\nu}\phi_{(\nu')}\sigma_{(\mu')} \\
&\qquad + \mu'\langle\sigma,\phi\rangle\delta_{\mu'+\nu'}\rho_{(\mu)}\psi_{(\nu)} - \mu\langle\rho,\phi\rangle\delta_{\mu+\nu'}\sigma_{(\mu')}\psi_{(\nu)}\Big] \\
= &- \sum_{\nu<0}\Big[C_{12}\langle\sigma,\psi\rangle\rho_{(\mu)}\phi_{(\nu')} - C_{11}\langle\rho,\psi\rangle\sigma_{(\mu')}\phi_{(\nu')} + C_{14}\langle\sigma,\phi\rangle\psi_{(\nu)}\rho_{(\mu)} - C_{13}\langle\rho,\phi\rangle\psi_{(\nu)}\sigma_{(\mu')}\Big] \\
&+ \sum_{\nu\geq 0}\Big[C_{12}\langle\sigma,\psi\rangle\phi_{(\nu')}\rho_{(\mu)} - C_{11}\langle\rho,\psi\rangle\phi_{(\nu')}\sigma_{(\mu')} + C_{14}\langle\sigma,\phi\rangle\rho_{(\mu)}\psi_{(\nu)} - C_{13}\langle\rho,\phi\rangle\sigma_{(\mu')}\psi_{(\nu)}\Big].
\end{aligned} \tag{4.36}$$

In the second step the commutator has been rewritten using the conventions (4.35a)-(4.35d). Equation (4.36) is the general commutator formula advertised in the beginning of this section.

In the following, the commutator will be used to calculate directly the Virasoro algebra relations in SF^+, which have already been established using Proposition 2.10. According to the definition (4.8) of the conformal vector of SF^+, let $\psi = \rho = e^i$, $\phi = \sigma = f^i$ and $n_1 = n_2 = m_1 = m_2 = 1$. The general commutator formula then specializes to (a minus sign is introduced to obtain the usual ordering)

$$\begin{aligned}
-[L_{n-1}, L_{m-1}] &= \sum_{i=1}^{d}\Bigg(\sum_{\nu<0}\Big((-n+\nu+1)e^i_{(\nu)}f^i_{(m+n-\nu-2)} - \nu e^i_{(m+\nu-1)}f^i_{(n-\nu-1)}\Big) \\
&\quad + \sum_{\nu\geq 0}\Big(\nu f^i_{(n-\nu-1)}e^i_{(m+\nu-1)} - (-n+\nu+1)f^i_{(m+n-\nu-2)}e^i_{(\nu)}\Big)\Bigg) \\
&= \sum_{i=1}^{d}\Bigg(\sum_{\nu<0}(-n+\nu+1)e^i_{(\nu)}f^i_{(m+n-\nu-2)} - \sum_{\nu<-1+m}(\nu-m+1)e^i_{(\nu)}f^i_{(m+n-\nu-2)} \\
&\quad + \sum_{\nu\geq m-1}(\nu-m+1)f^i_{(m+n-\nu-2)}e^i_{(m+\nu-1)} - \sum_{\nu\geq 0}(-n+\nu+1)f^i_{(m+n-\nu-2)}e^i_{(\nu)}\Bigg).
\end{aligned}$$

In the last step, the indices of two sums have been shifted so that the mode combinations can later be interpreted as modes of the conformal vector. In the following, the two cases $m < 1$ and $m \geq 1$ have to be treated separately. In these respective cases, the sums need to be split according to

$$\begin{aligned}
\sum_{\nu<0} &= \sum_{\nu<m-1} + \sum_{m-1\leq\nu<0} \\
\sum_{\nu\geq 0} &= \sum_{\nu\geq m-1} - \sum_{m-1\leq\nu<0}
\end{aligned} \tag{4.37}$$

in the case $m < 1$ and according to

$$\begin{aligned}\sum_{\nu<0} &= \sum_{\nu<m-1} - \sum_{0\le\nu<m-1} \\ \sum_{\nu\ge 0} &= \sum_{\nu\ge m-1} + \sum_{0\le\nu<m-1}\end{aligned} \tag{4.38}$$

in the case $m \geq 1$. The inequalities for $m < 1$ and $m \geq 1$,

$$\begin{aligned} m-1 \le \nu < 0 \\ 0 \le \nu < m-1, \end{aligned}$$

will be denoted by I_1 and I_2, respectively. The notation

$$\pm\sum_{I_1,I_2} = \begin{cases} \sum\limits_{m-1\le\nu<0} & \text{for } m<1 \\ -\sum\limits_{0\le\nu<m-1} & \text{for } m\ge 1 \end{cases}$$

serves to unify the following manipulations of the commutator for the two cases. Using the sum splitting (4.37) and (4.38) the commutator $-[L_{n-1}, L_{m-1}]$ can be written as

$$\sum_{i=1}^{d}\Bigg(\sum_{\nu<m-1}(m-n)e^i_{(\nu)}f^i_{(m+n-\nu-2)} \pm \sum_{I_1,I_2}(-n+\nu+1)e^i_{(\nu)}f^i_{(m+n-\nu-2)} \\ + \sum_{\nu\ge m-1}(n-m)f^i_{(m+n-\nu-2)}e^i_{(\nu)} \pm \sum_{I_1,I_2}(-n+\nu+1)f^i_{(m+n-\nu-2)}e^i_{(\nu)}\Bigg).$$

By adding and subtracting certain sums, the argument of the outer sum over i can be rewritten as

$$\begin{aligned} &\sum_{\nu<m-1}(m-n)e^i_{(\nu)}f^i_{(m+n-\nu-2)} \pm \sum_{I_1,I_2}(m-n)e^i_{(\nu)}f^i_{(m+n-\nu-2)} \\ &+ \sum_{\nu\ge m-1}(n-m)f^i_{(m+n-\nu-2)}e^i_{(\nu)} \mp \sum_{I_1,I_2}(n-m)f^i_{(m+n-\nu-2)}e^i_{(\nu)} \\ &\mp \sum_{I_1,I_2}(m-n)e^i_{(\nu)}f^i_{(m+n-\nu-2)} \pm \sum_{I_1,I_2}(-n+\nu+1)e^i_{(\nu)}f^i_{(m+n-\nu-2)} \\ &\pm \sum_{I_1,I_2}(n-m)f^i_{(m+n-\nu-2)}e^i_{(\nu)} \pm \sum_{I_1,I_2}(-n+\nu+1)f^i_{(m+n-\nu-2)}e^i_{(\nu)}. \end{aligned}$$

Collecting terms with the same ordering of the modes, we obtain

$$\begin{aligned} &\sum_{\nu<0}(m-n)e^i_{(\nu)}f^i_{(m+n-\nu-2)} + \sum_{\nu\ge 0}(n-m)f^i_{(m+n-\nu-2)}e^i_{(\nu)} \\ \pm &\sum_{I_1,I_2}(-m+\nu+1)e^i_{(\nu)}f^i_{(m+n-\nu-2)} \pm \sum_{I_1,I_2}(-m+\nu+1)f^i_{(m+n-\nu-2)}e^i_{(\nu)}. \end{aligned}$$

With the help of the Virasoro modes (4.13), we can identify the sum over i of the first two sums as $(m-n)L_{m+n-2}$. The second two sums are just the anticommutator $\left\{e^i_{(\nu)}, f^i_{m+n-\nu-2}\right\} =$

$\nu\delta_{m+n-2,0}$. Thus, the full commutator becomes

$$\begin{aligned}-[L_{n-1}, L_{m-1}] &= (m-n)L_{m+n-2} \mp \sum_{i=1}^{d}\sum_{I_1,I_2}(-m+\nu+1)\nu\delta_{m+n-2,0} \\ &= (m-n)L_{m+n-2} + \frac{-2d}{12}\left((m-1)^3 - m + 1\right)\delta_{m+n-2,0}.\end{aligned}$$

This is of course equivalent to the usual notation for the Virasoro algebra with the central charge $c = -2d$,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n,0}.$$

For the proof of Theorem 4.1, formula (4.9) for the commutator of the Virasoro modes with the modes of a primary vector $\psi_{(-1)}\mathbf{1}$ with $\psi \in \mathfrak{h}$ is needed. In the rest of this section, formula (4.9) will be derived. The reasoning is very similar to the one in the derivation of the general commutator formula. Consider the commutator

$$\left[h^{i,j}_{n+1}, \psi_{(m)}\right] = \sum_{k\in\mathbb{Z}} :e^i_{(k)}f^j_{(n-k)}: \psi_{(m)} - \sum_{k\in\mathbb{Z}} \psi_{(m)} :e^i_{(k)}f^j_{(n-k)}: . \tag{4.39}$$

Writing out the normal ordering, the second term becomes

$$\sum_{k\in\mathbb{Z}} \psi_{(m)} :e^i_{(k)}f^j_{(n-k)}: = \sum_{k<0} \psi_{(m)}e^i_{(k)}f^j_{(n-k)} - \sum_{k\geq 0} \psi_{(m)}f^j_{(n-k)}e^i_{(k)}.$$

Using the same reordering strategy as in the derivation of the general commutator formula, the right hand side can be rewritten as

$$\begin{aligned}&\sum_{k<0}\left(m\langle\psi, e^i\rangle\delta_{m+k}f^j_{(n-k)} - m\langle\psi, f^j\rangle\delta_{m+n-k}e^i_{(k)} + e^i_{(k)}f^i_{(n-k)}\psi_{(m)}\right) \\ &-\sum_{k\geq 0}\left(m\langle\psi, f^j\rangle\delta_{m+n-k}e^i_{(k)} - m\langle\psi, e^i\rangle\delta_{m+k}f^j_{(n-k)} + f^i_{(n-k)}e^j_{(k)}\psi_{(m)}\right).\end{aligned} \tag{4.40}$$

In the commutator (4.39), the terms with three modes cancel each other, so that only terms with one mode remain. Evaluating the Kronecker deltas in (4.40), we obtain conditions on m and n. For the terms in the first row these are $m > 0$ and $m+n < 0$, while for the terms in the second row these are $m \leq 0$ and $m+n \geq 0$. Thus, for each of the modes $e^i_{(n+m)}$ and $f^j_{(n+m)}$ exactly one term does not vanish so that we arrive at

$$\left[h^{i,j}_{n+1}, \psi_{(m)}\right] = -m\langle\psi, f^j\rangle e^i_{(m+n)} + m\langle\psi, e^i\rangle f^j_{(m+n)}.$$

Since the expansion of ψ in basis elements of the symplectic vector space $\mathfrak{h}$ can be written as

$$\psi = \sum_{i=1}^{d}\left(\langle\psi, e^i\rangle f^j - \langle\psi, f^i\rangle e^i\right),$$

we finally obtain

$$\sum_{i=1}^{d}\left[h^{i,i}_{n+1}, \psi_{(m)}\right] = [L_n, \psi_{(m)}] = -m\psi_{(m+n)}.$$

Chapter 5

Lie Algebra Structure of *d* Symplectic Fermions

The investigation into the structure of the zero mode algebra of SF^+ presented in this chapter starts from the question if there is a generalization to the Lie algebra $\mathfrak{su}(2)$ which is found in the case $d = 1$. As mentioned in section 3.2, Gaberdiel and Kausch have used certain explicitly known null vectors to establish the zero mode Lie algebra.

Without these known null vectors, using the general commutator formula derived in section 4.3 may be thought to be an alternative. This approach has the disadvantage that generally the result of the commutator of two zero modes is not in a form that can easily be interpreted in terms of zero modes of a generating vector. This interpretation gets much easier if the zero mode algebra is restricted to a certain module. For this reason, the above problem is circumvented here by computing the Lie algebra of zero modes on the top level of the SF^+-module SF^-. It will be explained below how this restriction to one module is lifted afterwards.

Even by confining the analysis to the top level of SF^-, the resulting zero mode Lie algebra is still relatively large and complicated, so that no immediate interpretation is possible. A first hint comes from the heuristics of simple counting arguments. It was shown in section 4.2.2, that there are $2d^2 + d$ generating vectors of weight 3. Incidentally, the dimension of both $\mathfrak{su}(2d)$ and $\mathfrak{sp}(2d)$ equals this number of generators. Though the $d = 1$ case points to the special unitary Lie algebra, the construction of SF with the help of a symplectic vector space makes the symplectic Lie algebra more likely; and this is indeed the structure that is found. Furthermore it will be shown that the zero modes corresponding to the vectors of weight 2 form an irreducible representation of $\mathfrak{sp}(2d)$.

So far, the whole analysis into the structure of the zero mode algebra applies only to the top level of SF^-. Thus, the question is how to extend the assertions about the structure on all SF^+-modules. The answer will be given with the help of Zhu's algebra. We know from Zhu's theorem that the top level of every SF-module is a representation of Zhu's algebra. All that is needed then is a relation of the zero mode algebra to Zhu's algebra. It will be shown that this relation is given by an isomorphism of both algebras. More precisely, the zero mode Lie algebra restricted to the top level of SF^- is isomorphic to Zhu's algebra with the canonical Lie algebra structure induced by the commutator.

5.1 Commutation Relations of the Zero Mode Algebra

The first step in the approach outlined above is to compute the action of the zero modes of the generators on the top level of SF^-. The top level of the SF^+-module SF^- has the special property that it is given by $\mathfrak{h} = \{\psi_{(-1)}\mathbf{1} \,|\psi \in SF^+\}$. The dimension of $\mathfrak{h}$ is taken to be fixed and equal to $2d$ for the rest of this chapter. The generators of SF^+ of weight 2 and 3 have the form $a_{(-1)}b_{(-1)}\mathbf{1}$ and $a_{(-2)}b_{(-1)}\mathbf{1}$ with $a, b \in SF^+$, respectively. Therefore, it is prudent to calculate the action of zero modes of such general vectors. We have the mode expansion

$$(a_{(-2)}b_{(-1)}\mathbf{1})_{(2)} = \sum_{i<0}(-i-1)a_{(i)}b_{(-i)} - \sum_{i\geq 0}(-i-1)b_{(-i)}a_{(i)}.$$

Applying this zero mode on a vector $\psi_{(-1)}\mathbf{1} \in SF_1^-$ and using $\{\psi_{(m)}, \psi'_{(n)}\} = m\langle\psi, \psi'\rangle\delta_{m+n}$ we obtain

$$\begin{aligned}(a_{(-2)}b_{(-1)})_{(2)}\,\psi_{(-1)}\mathbf{1} &= \sum_{i<0}(-i-1)a_{(i)}(-i)\delta_{-i-1}\langle b, \psi\rangle\,\mathbf{1}\\ &\quad - \sum_{i\geq 0}(-i-1)b_{(-i)}i\delta_{i-1}\langle a, \psi\rangle\,\mathbf{1}\\ &= 0 + 2b_{(-1)}\langle a, \psi\rangle\,\mathbf{1}\\ &= 2\langle a, \psi\rangle b.\end{aligned}$$

The calculation for $a_{(-1)}b_{(-1)}\mathbf{1}$ is very similar. Starting from the mode expansion

$$(a_{(-1)}b_{(-1)}\mathbf{1})_{(1)} = \sum_{i<0}a_{(i)}b_{(-i)} - \sum_{i\geq 0}b_{(-i)}a_{(i)},$$

one obtains

$$\begin{aligned}(a_{(-1)}b_{(-1)}\mathbf{1})_{(1)}\,\psi_{(-1)}\mathbf{1} &= \sum_{i<0}a_{(i)}(-i)\delta_{-i-1}\langle b, \psi\rangle\,\mathbf{1} - \sum_{i\geq 0}b_{(-i)}i\,\delta_{i-1}\langle a, \psi\rangle\,\mathbf{1}\\ &= \langle b, \psi\rangle a - \langle a, \psi\rangle b.\end{aligned}$$

Using the definitions (4.12) for the generators of SF^+, we obtain the following list for the action of the zero modes on the top level of SF^- (the parantheses around mode indices will be omitted for the modes of the generators of SF^+):

$$(E^{i,j})_2\psi = \langle e^i, \psi\rangle e^j + \langle e^j, \psi\rangle e^i \tag{5.1a}$$

$$(F^{i,j})_2\psi = \langle f^i, \psi\rangle f^j + \langle f^j, \psi\rangle f^i \tag{5.1b}$$

$$(H^{i,j})_2\psi = \langle e^i, \psi\rangle f^j + \langle f^j, \psi\rangle e^i \tag{5.1c}$$

$$(e^{i,j})_1\psi = \langle e^j, \psi\rangle e^i - \langle e^i, \psi\rangle e^j \tag{5.1d}$$

$$(f^{i,j})_1\psi = \langle f^j, \psi\rangle f^i - \langle f^i, \psi\rangle f^j \tag{5.1e}$$

$$(h^{i,j})_1\psi = \langle f^j, \psi\rangle e^i - \langle e^i, \psi\rangle f^j. \tag{5.1f}$$

To establish that these endomorphisms of $\mathfrak{h}$ are linearly independent, the explicit matrix form is helpful. It will not be needed for the subsequent discussion, though. The description of the matrix entries will be given in the canonical basis $e^1, \ldots, e^d, f^1, \ldots, f^d$ of $\mathfrak{h}$ and with respect to the block diagonal form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are assumed to be $d \times d$-matrices. With this convention the entries are given according to the following table, as one can easily confirm by calculating the action of the various zero modes on basis elements.

	A_{kl}	B_{kl}	C_{kl}	D_{kl}
$E^{i,j}$	0	$-\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$	0	0
$F^{i,j}$	0	0	$\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$	0
$H^{i,j}$	$\delta_{ik}\delta_{jl}$	0	0	$-\delta_{il}\delta_{jk}$
$e^{i,j}$	0	$\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}$	0	0
$f^{i,j}$	0	0	$-\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}$	0
$h^{i,j}$	$\delta_{ik}\delta_{jl}$	0	0	$\delta_{il}\delta_{jk}$

From this table it is immediately obvious that all zero modes are linearly independent endomorphisms and that the matrices span the entire space of $2d \times 2d$ matrices. Thus the zero modes span the entire space of endomorphisms of SF_1^-.

In order to analyze the Lie algebra structure of this matrix algebra, one has to calculate the commutators between all the zero modes. Starting with the commutator $[E_2^{i,j}, F_2^{k,l}]$, one has to calculate the products

$$\begin{aligned}
E_2^{i,j}F_2^{k,l}\psi =& E_2^{i,j}\left(\langle f^k,\psi\rangle f^l + \langle f^l,\psi\rangle\psi^k\right)\\
=&\langle e^i, f^l\rangle\langle f^k,\psi\rangle e^j + \langle e^i, f^k\rangle\langle f^l,\psi\rangle e^j + \langle e^j, f^l\rangle\langle f^k,\psi\rangle e^i + \langle e^j, f^k\rangle\langle f^l,\psi\rangle e^i\\
=& -\delta^{il}\langle f^k,\psi\rangle e^j - \delta^{ik}\langle f^l,\psi\rangle e^j - \delta^{jl}\langle f^k,\psi\rangle e^i - \delta^{jk}\langle f^l,\psi\rangle e^i
\end{aligned}$$

and

$$\begin{aligned}
F_2^{k,l}E_2^{i,j}\psi =& F_2^{k,l}\left(\langle e^i,\psi\rangle e^j + \langle e^j,\psi\rangle e^i\right)\\
=&\langle e^i,\psi\rangle\langle f^k, e^j\rangle f^l + \langle e^j,\psi\rangle\langle f^k, e^i\rangle f^l + \langle e^i,\psi\rangle\langle f^l, e^j\rangle f^k + \langle e^j,\psi\rangle\langle f^l, e^i\rangle f^k\\
=&\delta^{kj}\langle e^i,\psi\rangle f^l + \delta^{ki}\langle e^j,\psi\rangle f^l + \delta^{lj}\langle e^i,\psi\rangle f^k + \delta^{li}\langle e^j,\psi\rangle f^k.
\end{aligned}$$

The commutator is of course given as the difference of the above products:

$$\begin{aligned}
\left(E_2^{i,j}F_2^{k,l} - F_2^{k,l}E_2^{i,j}\right)\psi =& -\delta^{kj}\left(\langle f^l,\psi\rangle e^i + \langle e^i,\psi\rangle f^l\right) - \delta^{ki}\left(\langle f^l,\psi\rangle e^j + \langle e^j,\psi\rangle f^l\right)\\
&-\delta^{lj}\left(\langle f^k,\psi\rangle e^i + \langle e^i,\psi\rangle f^k\right) - \delta^{li}\left(\langle f^k,\psi\rangle e^j + \langle e^j,\psi\rangle f^k\right)\\
=&\left(-\delta^{kj}H_2^{i,l} - \delta^{ki}H_2^{j,l} - \delta^{lj}H_2^{ik} - \delta^{li}H_2^{j,k}\right)\psi.
\end{aligned}$$

As another example, consider the commutator $[h_1^{i,j}, e_1^{k,l}]$, which is calculated as above by first considering the products

$$\begin{aligned}
h_1^{i,j}e_1^{k,l}\psi =& h_1^{i,j}\left(\langle e^l,\psi\rangle e^k - \langle e^k,\psi\rangle e^l\right)\\
=&\langle e^l,\psi\rangle\langle f^j, e^k\rangle e^i - \langle e^k,\psi\rangle\langle f^j, e^l\rangle e^i\\
=&\delta^{jk}\langle e^l,\psi\rangle e^i - \delta^{jl}\langle e^k,\psi\rangle e^i
\end{aligned}$$

and

$$\begin{aligned} e_1^{k,l} h_1^{i,j} \psi =& e_1^{i,j} \left(\langle f^j, \psi \rangle e^i - \langle e^i, \psi \rangle f^j \right) \\ =& - \langle e^i, \psi \rangle \langle e^l, f^j \rangle e^k + \langle e^i, \psi \rangle \langle e^k, f^j \rangle e^l \\ =& \delta^{jl} \langle e^i, \psi \rangle e^k - \delta^{jk} \langle e^i, \psi \rangle e^l . \end{aligned}$$

Thus, we have the commutator

$$\begin{aligned} [h_1^{i,j}, e_1^{k,l}] \psi =& \delta^{jk} \left(\langle e^i, \psi \rangle e^l + \langle e^l, \psi \rangle e^i \right) - \delta^{jl} \left(\langle e^i, \psi \rangle e^k + \langle e^k, \psi \rangle e^i \right) \\ =& \left(\delta^{jk} E_2^{i,l} - \delta^{jl} E_2^{k,i} \right) \psi . \end{aligned}$$

The commutator $[H_2^{i,j}, h_1^{k,l}]$ shall serve as final example. Proceed as above by calculating

$$\begin{aligned} H_2^{i,j} h_1^{k,l} \psi =& H_2^{i,j} \left(\langle f^l, \psi \rangle e^k - \langle e^k, \psi \rangle f^l \right) \\ =& \langle f^l, \psi \rangle \langle f^j, e^k \rangle e^i - \langle e^k, \psi \rangle \langle e^i, f^l \rangle f^j \\ =& \delta^{jk} \langle f^l, \psi \rangle e^i - \delta^{il} \langle e^k, \psi \rangle f^j \end{aligned}$$

and

$$\begin{aligned} h_1^{k,l} H_2^{i,j} \psi =& h_1^{k,l} \left(\langle e^i, \psi \rangle f^j + \langle f^j, \psi \rangle e^i \right) \\ =& - \langle e^i, \psi \rangle \langle e^k, f^j \rangle f^l + \langle f^j, \psi \rangle \langle f^l, e^i \rangle e^k \\ =& \delta^{jk} \langle e^i, \psi \rangle f^l + \delta^{il} \langle f^j, \psi \rangle e^k , \end{aligned}$$

leading to the commutator

$$\begin{aligned} [H_2^{i,j}, h_1^{k,l}] \psi =& \delta^{jk} \left(\langle f^l, \psi \rangle e^i - \langle e^i, \psi \rangle f^l \right) + \delta^{il} \left(\langle e^k, \psi \rangle f^j - \langle f^j, \psi \rangle e^k \right) \\ =& \left(\delta^{jk} h_1^{i,l} - \delta^{i,l} h_1^{k,j} \right) \psi . \end{aligned}$$

Calculating all the missing commutators, one obtains the following list of all non-vanishing commutators

$$\left[h_1^{i,j}, e_1^{k,l} \right] = \delta^{jk} E_2^{i,l} - \delta^{jl} E_2^{i,k} \tag{5.2a}$$
$$\left[h_1^{i,j}, f_1^{k,l} \right] = \delta^{ik} F_2^{j,l} - \delta^{li} F_2^{j,k} \tag{5.2b}$$
$$\left[h_1^{i,j}, h_1^{k,l} \right] = \delta^{jk} H_2^{i,l} - \delta^{li} H_2^{j,k} \tag{5.2c}$$
$$\left[e_1^{i,j}, f_1^{k,l} \right] = -\delta^{jk} H_2^{i,l} + \delta^{ik} H_2^{j,l} + \delta^{jl} H_2^{i,k} - \delta^{il} H_2^{j,k} \tag{5.2d}$$

$$\left[H_2^{i,j}, E_2^{k,l} \right] = \delta^{jl} E_2^{i,k} + \delta^{kj} E_2^{i,l} \tag{5.3a}$$
$$\left[H_2^{i,j}, F_2^{k,l} \right] = -\delta^{ik} F_2^{j,l} - \delta^{il} F_2^{j,k} \tag{5.3b}$$
$$\left[H_2^{i,j}, H_2^{k,l} \right] = \delta^{jk} H_2^{i,l} - \delta^{il} H_2^{j,k} \tag{5.3c}$$
$$\left[E_2^{i,j}, F_2^{k,l} \right] = -\delta^{jk} H_2^{i,l} - \delta^{ik} H_2^{j,l} - \delta^{jl} H_2^{i,k} - \delta^{li} H_2^{j,k} \tag{5.3d}$$

$$\left[H_2^{i,j}, e_1^{k,l}\right] = \delta^{jk} e_1^{i,l} + \delta^{jl} e_1^{k,i} \tag{5.4a}$$
$$\left[h_1^{i,j}, E_2^{k,l}\right] = \delta^{jk} e_1^{i,l} + \delta^{jl} e_1^{i,k} \tag{5.4b}$$
$$\left[H_2^{i,j}, f_1^{k,l}\right] = \delta^{ik} f_1^{l,j} + \delta^{il} f_1^{k,j} \tag{5.4c}$$
$$\left[h_1^{i,j}, F_2^{k,l}\right] = \delta^{ik} f_1^{j,l} + \delta^{il} f_1^{j,k} \tag{5.4d}$$
$$\left[H_2^{i,j}, h_1^{k,l}\right] = \delta^{jk} h_1^{i,l} - \delta^{il} h_1^{k,j} \tag{5.4e}$$
$$\left[E_2^{i,j}, f_1^{k,l}\right] = -\delta^{jk} h_1^{i,l} - \delta^{ik} h_1^{j,l} + \delta^{jl} h_1^{i,k} + \delta^{il} h_1^{j,k} \tag{5.4f}$$
$$\left[e_1^{i,j}, F_2^{k,l}\right] = -\delta^{jk} h_1^{i,l} + \delta^{ik} h_1^{j,l} - \delta^{jl} h_1^{i,k} + \delta^{il} h_1^{j,k}. \tag{5.4g}$$

For the discussion of the above commutators, it is useful to introduce some notation.

Definition 5.1. *Let Z_2 and Z_3 be the following subspaces of* $\mathrm{End} S\, F_1^-$,

$$Z_2 = \bigoplus_{1 \le i < j \le d} \left(\mathbb{C}e_1^{i,j} \oplus \mathbb{C}f_1^{i,j} \oplus \mathbb{C}h_1^{i,j} \oplus \mathbb{C}h_1^{j,i} \oplus \mathbb{C}(h^{i,i} - h^{j,j})_1\right) \oplus \mathbb{C}\omega$$

$$Z_3 = \bigoplus_{i,j=1}^{d} H_2^{i,j} \oplus \bigoplus_{1 \le i \le j \le d} \left(\mathbb{C}E_2^{i,j} \oplus \mathbb{C}F_2^{i,j}\right).$$

Denote by Z_2' the space Z_2 without the span of the conformal vector.

Note that the reason for choosing a special basis for Z_2 that distinguishes the conformal vector ω will become clear in Section 5.3. Some conclusions can be drawn simply by looking at the commutator list. First of all, it can be concluded from (5.3a)-(5.3d) that the zero modes of the vectors of weight 3 form a Lie algebra with the commutator as Lie bracket. Rephrased in the notation introduced above, this means that $(Z_3, [\cdot,\cdot])$ is a Lie algebra which has yet to be determined.

Since the commutators of the zero modes of the vectors of weight 2, (5.2a)-(5.2d), do not close among the weight 2 vectors, they cannot form a Lie algebra. This may be surprising since there seems to be no reason why the zero mode algebra of the vectors of weight 2 should be fundamentally different from that of the vectors of weight 3. Since the Lie algebra of all zero modes is isomorphic to the Lie algebra of all $2d \times 2d$ matrices, the problem could be cured by going over to another basis. But we are interested in an interpretation in terms of the generating fields of the $\mathcal{W}$-algebra, so this is not an option.

However, it is still possible to find interesting structure in the space Z_2. Observe that the last nine commutators (5.4a)-(5.4g) indicate that the space Z_2 is a representation space for the representation $\rho : Z_3 \to \mathrm{End}(Z_2)$, defined by $\rho(X_2^{i,j})x_1^{k,l} = [X_2^{i,j}, x_1^{k,l}]$, for $X = E, F, H$ and $x = e, f, h$. This means that the zero modes corresponding to the vectors of weight 3 are represented on the zero modes corresponding to the vectors of weight 2 by the adjoint action. The question of irreducibility of this representation will be examined after the Lie algebra of the space Z_3 has been determined.

As one would expect, the commutators corresponding to weight 3 reproduce the commutator found by Gaberdiel and Kausch in the case $d = 1$. In this case, equations (5.3a)-(5.3d) become

$$[H_2, E_2] = 2E_2, \quad [H_2, F_2] = -2F_2 \quad \text{and} \quad [E_2, F_2] = -4H_2.$$

These equations can be combined in the notation of Section 3.2 into

$$\left[W^a, W^b\right] = 2f^{ab}_c W^c, \tag{5.5}$$

where $W^+ = F$, $W^- = E$ and $W^0 = H$. Equation (5.5) is equal to (3.5) with $h = 1$. This was to be expected since $h = 1$ is the weight of the top level SF^- on which the zero mode algebra has been calculated.

5.2 The Symplectic Lie Algebra Structure

It has been mentioned in the introduction to this chapter that the dimension of the space of generating vectors of SF^+ of weight 3 equals $\dim \mathfrak{su}(2d) = \dim \mathfrak{sp}(2d) = 2d^2 + d$. Thus, from dimensional considerations alone, both $\mathfrak{so}(2d)$ and $\mathfrak{sp}(2d)$ are candidates for the Lie algebra $(Z_3, [\cdot, \cdot])$. It was also argued above that the symplectic Lie algebra seems more natural in this context. To determine if it is indeed realized, one could rewrite the commutators in some standard basis and compare them with the expressions given in the literature (see e.g. [4], section V). However, there is a much better way of establishing the $\mathfrak{sp}(2d)$ structure directly from its definition.[1]

The symplectic group $Sp(n)$ is defined to be the group of automorphisms of an n-dimensional vector space V preserving a nondegenerate, skew-symmetric bilinear form Q. Given a one parameter subgroup A_t of $Sp(n)$ with $A_0 = \mathrm{id}$ and $\frac{d}{dt}A_t|_{t=0} = X$ this means that

$$Q(A_t u, A_t v) = Q(u, v)$$

for all $u, v \in V$. Taking derivatives, this translates to

$$Q(Xu, v) + Q(u, Xv) = 0 \tag{5.6}$$

for the elements X of the symplectic Lie algebra $\mathfrak{sp}(n)$. If (5.6) can be verified, the zero modes of the vectors of weight 3 have to form a subalgebra of the symplectic Lie algebra. But since the dimension of $\mathfrak{sp}(2d)$ equals the number of these zero modes and all zero modes are linearly independent, this subalgebra has indeed to be the full algebra.

The verification of (5.6) is straightforward and uses only $\langle u, v\rangle = -\langle v, u\rangle$:

$$\begin{aligned}
\langle E_2^{i,j} u, v\rangle =& \langle \langle e^i, u\rangle e^j + \langle e^j, u\rangle e^i, v\rangle \\
=& \langle e^i, u\rangle\langle e^j, v\rangle + \langle e^j, u\rangle\langle e^i, v\rangle \\
\langle u, E_2^{i,j} v\rangle =& \langle u, \langle e^i, v\rangle e^j + \langle e^j, v\rangle e^i\rangle \\
=& -\langle e^i, u\rangle\langle e^j, v\rangle - \langle e^j, u\rangle\langle e^i, v\rangle,
\end{aligned}$$

$$\begin{aligned}
\langle F_2^{i,j} u, v\rangle =& \langle \langle f^i, u\rangle f^j + \langle f^j, u\rangle f^i, v\rangle \\
=& \langle f^i, u\rangle\langle f^j, v\rangle + \langle f^j, u\rangle\langle f^i, v\rangle \\
\langle u, F_2^{i,j} v\rangle =& \langle u, \langle f^i, v\rangle f^j + \langle f^j, v\rangle f^i\rangle \\
=& -\langle f^i, u\rangle\langle f^j, v\rangle - \langle f^j, u\rangle\langle f^i, v\rangle,
\end{aligned}$$

[1] I am grateful to Nils Carqueville for proposing this approach.

$$\begin{aligned}\langle H_2^{i,j}u, v\rangle =&\langle\langle e^i, u\rangle f^j + \langle f^j, u\rangle e^i, v\rangle\\ =&\langle e^i, u\rangle\langle f^j, v\rangle + \langle f^j, u\rangle\langle e^i, v\rangle\\ \langle u, H_2^{i,j}v\rangle =&\langle u, \langle e^i, v\rangle f^j + \langle f^j, v\rangle e^i\rangle\\ =&-\langle e^i, u\rangle\langle f^j, v\rangle - \langle f^j, u\rangle\langle e^i, v\rangle.\end{aligned}$$

Thus, we have $Q(X^{i,j}u, v) + Q(u, X^{i,j}v) = 0$ for $X = E, F, H$ and $i, j = 1, \dots, d$. The result of this section can be summarized as follows.

Theorem 5.2. *The Lie algebra of the zero modes corresponding to the vectors of weight three,* $(Z_3, [\cdot, \cdot])$*, is isomorphic to the symplectic Lie algebra* $\mathfrak{sp}(2d)$*.*

5.3 How do the Weight 2 Zero Modes Fit In?

As was noted in the discussion following the list of commutators, the zero modes of the vectors of weight 2 form a representation for the Lie algebra Z_3 which has now been established to be $\mathfrak{sp}(2d)$. This immediately brings up the question if this representation is irreducible. To answer this question, some facts (cf. [20], chapter 17 and [8], chapter VI.5) from the representation theory of $\mathfrak{sp}$ will be needed. The defining representation of $\mathfrak{sp}(2d)$ on a $2d$-dimensional vector space V is of course irreducible. All other irreducible representations ρ^k are subspaces of the k-th exterior power $\Lambda^k V$. The irreducible representations arising in this way have dimension

$$\dim\rho^k = \binom{2n}{k} - \binom{2n}{k-2}$$

for $k \geq 2$. This result can be obtained with help of the Weyl character formula, see [8], chapter VI.5. As we know, $\dim(\mathcal{P}(SF_2^+)) = 2d^2 - d - 1$ which equals $\dim\rho^2 = 2d^2 - d - 1$. This means that the space of the zero modes of weight 2, without the span of the conformal vector, is a candidate for an irreducible representation. Of course the fact that this space has the right dimension is not yet a proof for the irreducibility of the representations. The representation could still have trivial subrepresentations or it could decompose into a direct sum of irreducible representations.

The latter case can be excluded from dimensional considerations alone. First of all, one observes that $\dim\rho^k > \dim\rho^2$ for $k = 3, \dots, n$ (note that $\dim\rho^k = 0$ for $k > n$). Consequently, the only irreducible representation a representation of dimension $d^2 - d - 1$ can decompose into is the defining representation of dimension $2d$. But this is impossible since

$$\frac{2d^2 - d - 1}{2d} = \frac{2d-1}{2} - \frac{1}{2d},$$

which cannot be an integer. Thus, one only has to exclude the possibility of trivial subrepresentations. This will be done by showing that the representation acts nontrivially on at least $d(d-1)$ elements. Since $d(d-1) > 2d$ for $d > 3$, the representation cannot decompose into the defining representation and a trivial representation for $d > 3$ and has to be irreducible in this case.

Consider the $d(d-1)$ elements $P = \{e_1^{i,j}, f_1^{i,j}\}_{i<j}$. If it can be shown that there exists no linear combination of these elements on which the representation acts trivially, then

the condition above is met and it is shown that the representation is irreducible for $d > 3$. This will be achieved by first acting with a certain element of $\mathfrak{sp}(2d)$ and then projecting out a single basis vector[2] of the representation space by acting with a second element of $\mathfrak{sp}(2d)$.

In the rest of this section, the mode index will be left away. All elements $x^{i,j}$ with $x = e, f, h$ should be read as $x_1^{i,j}$. Then an arbitrary linear combination of the $d(d-1)$ zero modes $e^{i,j}$ and $f^{i,j}$ can be written as

$$p = \sum_{i<j} a_{ij} e^{i,j} + \sum_{i<j} b_{ij} f^{i,j}.$$

Acting on p with $F^{k,k}$, we can ignore the $f^{i,j}$. Since $[F^{k,k}, e^{i,j}] = 2\delta^{jk} h^{i,k} - 2\delta^{ik} h^{j,k}$, one obtains

$$\begin{aligned}
\left[F^{k,k}, \sum_{i<j} a_{ij} e^{i,j}\right] =& 2\sum_{i<j} a_{ij}\delta^{jk} h^{i,k} - 2\sum_{i<j} a_{ij}\delta^{ik} h^{j,k} \\
=& 2\sum_{j=1}^{d} \delta^{jk} \sum_{i=1}^{j} a_{ij} h^{i,k} - 2\sum_{j=1}^{d}\sum_{i=1}^{j} a_{ij}\delta^{ik} h^{j,k} \\
=& 2\sum_{i=1}^{k} a_{ik} h^{i,k} - 2\sum_{j=1}^{k-1}\sum_{i=1}^{j} a_{i,j}\delta^{ik} h^{j,k} - 2\sum_{j=k}^{d}\sum_{i=1}^{j} a_{ij}\delta^{ik} h^{j,k} \\
=& 2\sum_{i=1}^{k} a_{ik} h^{i,k} - 2\sum_{j=k}^{d} a_{kj} h^{j,k},
\end{aligned}$$

where the second sum was split from the second to the third line. Acting now with $H^{l,l}$, where $l \neq k$ and assuming without loss of generality $l > k$, we obtain

$$\left[H^{l,l}, \left[F^{k,k}, \sum_{i<j} a_{ij} e^{i,j}\right]\right] = 2\sum_{i=1}^{k} \left(a_{ik} h^{l,k}\delta^{il} - a_{ik} h^{i,l}\delta^{kl}\right) - 2\sum_{j=k}^{d} \left(a_{kj} h^{l,k}\delta^{jl} - a_{kj} h^{j,l}\delta^{kl}\right).$$

Most of the above terms are zero. The second terms of both sums are zero since $k \neq l$. The first term of the first sum vanishes because the summation index i never reaches l. Thus the result is

$$\left[H^{l,l}, \left[F^{k,k}, \sum_{i<j} a_{ij} e^{i,j}\right]\right] = -2a_{kl} h^{l,k},$$

yielding the desired projection on a single basis vector of the representation space. If one acts first with $E^{k,k}$, one obtains by a similar calculation

$$\left[E^{k,k}, \sum_{i<j} b_{ij} f^{ij}\right] = 2\sum_{i=1}^{k} b_{ik} h^{k,i} - 2\sum_{j=k}^{d} b_{kj} h^{k,j},$$

[2] Note that these vectors are endomorphisms themselves, i.e. zero modes of vectors of weight 2.

since $[E^{k,k}, f^{i,j}] = -2\delta^{ki}h^{k,j} + 2\delta^{kj}h^{k,i}$. We can project out with $H^{l,l}$ in the same way as above,

$$\left[H^{l,l}, \left[E^{k,k}, \sum_{i<j} b_{ij}f^{i,j}\right]\right] = -2b_{kl}h^{k,l}.$$

Therefore, given any linear combination of vectors from P, we now know that for both $E^{k,k}$ and $F^{k,k}$ to act trivially on it, a_{kl} and b_{kl} have to be zero. Since k and l can be chosen arbitrarily, it can be concluded that the representation acts trivially only on the trivial linear combination with $a_{kl} = b_{kl} = 0$ for all $k, l = 1, \dots d$. This yields the desired assertion that there are at least $d(d-1)$ vectors on which the representation acts nontrivially.

This leaves us with the cases $d = 2$ and $d = 3$, where $d(d-1)$ elements are not enough for proving that there is no irreducible $2d$-dimensional subrepresentation. For these two cases, the basis $\{e^{i,j}, f^{i,j}, h^{i,j}, h^{j,i}, h^{i,i} - h^{j,j}\}_{i<j} \cup \omega$ is chosen. In the case $d = 2$, it has to be shown that the representation acts nontrivially on $h^{1,2}$, $h^{2,1}$ and $h^{1,1} - h^{2,2}$, since we already know that the representation acts nontrivially on the $e^{i,j}$ and $f^{i,j}$. This is shown by the following list of commutators,

$$[E^{2,2}, h^{1,2}] = -2e^{1,2}$$
$$[E^{1,1}, h^{2,1}] = 2e^{1,2}$$
$$[E^{1,2}, h^{1,1} - h^{2,2}] = -2e^{1,2}.$$

In the case $d = 3$, the first commutator from the above three suffices to prove that we have $7 > 6 = 2d$ vectors on which the representation acts nontrivially.

Theorem 5.3. *The zero modes of the vectors of weight 3, which form the $\mathfrak{sp}(2d)$ Lie algebra, are irreducibly represented on the space of the zero modes of the vectors of weight 2 without the conformal vector. The representation $\rho : Z_3 \to Z_2'$ is given by*

$$\rho(X_2^{i,j})x_1^{k,l} = [X_2^{i,j}, x_2^{k,l}],$$

where $X = E, F, H$, $x = e, f, h$ and $i, j, k, l \in \{1, \dots, d\}$. Note that the statement of the theorem is trivial for $d = 1$ since then the representation space is the null space.

5.4 An Isomorphism Between $A(SF^+)$ and the Zero Mode Algebra

The results obtained on the zero mode algebra so far are restricted to the SF^+-module SF^-. In order to extend the Lie algebra structure which has been found above to all other irreducible modules, the following assertion will be proven in this section: For $d > 1$, Zhu's algebra $A(SF^+)$ is isomorphic as Lie algebra to the zero mode algebra EndSF_0^-.

The proof will be given in two steps. First, it will be shown that the commutators of the zero mode algebra for the weight 2 fields, (5.2a)-(5.2d), are reproduced in the Zhu algebra. Then it will be proven that the fact that $[e^{i,j}], [f^{i,j}]$ and $[h^{i,j}]$ already generate Zhu's algebra carries over to the zero mode algebra. With the commutators for the weight

2 fields being the same in the two algebras and the commutators of the weight 3 fields fixed by those of the weight 2 fields, the isomorphism follows.

Before establishing the commutators in Zhu's algebra, some notation has to be introduced. Setting

$$\Theta_m(\psi, \phi) = \frac{1}{m-1}[\psi_{(-m+1)}\phi],$$

where the bracket denotes equivalence classes in Zhu's algebra, the fields can be expressed in the following way:

$$\begin{aligned}
&[e^{i,j}] = \Theta_2(e^i, e^j) \qquad [h^{i,j}] = \Theta_2(h^i, f^j) \qquad [e^{i,j}] = \Theta_2(f^i, f^j)\\
&[E^{i,j}] = \Theta_3(e^i, e^j) + \Theta_3(e^j, e^i) = 2\Theta_3(e^i, e^j) + \Theta_2(e^i, e^j)\\
&[H^{i,j}] = \Theta_3(e^i, f^j) + \Theta_3(f^j, e^i) = 2\Theta_3(e^i, f^j) + \Theta_2(e^i, f^j)\\
&[F^{i,j}] = \Theta_3(f^i, f^j) + \Theta_3(f^j, f^i) = 2\Theta_3(f^i, f^j) + \Theta_2(f^i, f^j).
\end{aligned}$$

Calculations in Zhu's algebra require identities for simplifying products and permuting factors. The following Lemma consists of assertions which are proven in section 4.2 of [1].

Lemma 5.4. *For any $\psi, \phi \in \mathfrak{h}$ and $m \geq 2$, arguments of Θ can be exchanged according to the equation*

$$\Theta_m(\phi, \psi) = (-1)^{m-1} \sum_{i=0}^{m-2} \binom{m-2}{i} \Theta_{m-i}(\psi, \phi). \tag{5.7}$$

For $\psi, \phi, \xi, \eta \in \mathfrak{h}$ and $m \geq 2$ the following Θ-product identity holds:

$$\begin{aligned}
\Theta_2(\psi, \phi) \star \Theta_m(\xi, \eta) = &\frac{1}{m-1}[\psi_{(-1)}\phi_{(-1)}\xi_{(-m+1)}\eta]\\
&+ \langle \phi, \xi \rangle \left((m+1)\Theta_{m+2}(\psi, \eta) + 2m\Theta_{m+1}(\psi, \eta) + (m-1)\Theta_m(\psi, \eta)\right)\\
&- \langle \psi, \xi \rangle \left((m+1)\Theta_{m+2}(\phi, \eta) + 2m\Theta_{m+1}(\phi, \eta) + (m-1)\Theta_m(\phi, \eta)\right)\\
&+ \langle \phi, \eta \rangle \left(\binom{m+1}{2}\Theta_{m+2}(\xi, \psi) + 2\binom{m}{2}\Theta_{m+1}(\xi, \psi) + \binom{m-1}{2}\Theta_m(\xi, \psi)\right)\\
&- \langle \psi, \eta \rangle \left(\binom{m+1}{2}\Theta_{m+2}(\xi, \phi) + 2\binom{m}{2}\Theta_{m+1}(\xi, \phi) + \binom{m-1}{2}\Theta_m(\xi, \phi)\right).
\end{aligned} \tag{5.8}$$

The commutator $[h^{i,j}, h^{k,l}]$ is the easiest to calculate. Using (5.8), one obtains

$$\begin{aligned}
[[h^{i,j}], [h^{k,l}]] &= \Theta_2(e^i, f^j) \star \Theta_2(e^k, f^l) - \Theta_2(e^k, f^l) \star \Theta_2(e^i, f^j)\\
&= [e^i_{(-1)} f^j_{(-1)} e^k_{(-1)} f^l_{(-1)} 1] + \delta^{jk}(\Theta_2(e^i, f^l) + 4\Theta_3(e^i, f^l) + 3\Theta_4(e^i, f^l))\\
&+ \delta^{il}(2\Theta_3(e^k, f^j) + 3\Theta_3(e^k, f^j))\\
&- [e^k_{(-1)} f^l_{(-1)} e^i_{(-1)} f^j_{(-1)} 1] - \delta^{jk}(2\Theta_3(e^i, f^l) + 3\Theta_4(e^i, f^l))\\
&- \delta^{il}(\Theta_2(e^k, f^j) + 4\Theta_3(e^k, f^j) + 3\Theta_4(e^k, f^j))\\
&= \delta^{jk}(\Theta_2(e^i, f^l) + 2\Theta_3(e^i, f^l)) - \delta^{il}(\Theta_2(e^k, f^j) + 2\Theta_3(e^k, f^j))\\
&= \delta^{jk}[H^{i,l}] - \delta^{il}[H^{j,k}],
\end{aligned}$$

where the definition of $[H^{i,j}]$ has been used in the last step. Note also that the two terms with four modes cancel each other since permutations of the modes only lead to sign changes. Calculating the other commutators requires two identities which follow from equation (5.7) for exchanging Θ-arguments. First, the following equations are obtained from (5.7) by setting $m = 2, 3, 4$:

$$\begin{aligned}
\Theta_4(a,b) &= -\Theta_4(b,a) - 2\Theta_3(b,a) - \Theta_2(b,a)\\
\Theta_3(a,b) &= \Theta_3(b,a) + \Theta(b,a)\\
\Theta_2(a,b) &= -\Theta_2(b,a),
\end{aligned}$$

Using these three equations, a straightforward calculation shows that the identities

$$\begin{aligned}
&2(\Theta_3(a,b) + \Theta_3(b,a)) + 3(\Theta_4(a,b) + \Theta_4(b,a))\\
=&-\Theta_2(a,b) - 2\Theta_3(a,b),
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
&\Theta_2(a,b) + \Theta_2(b,a) + 4(\Theta_3(a,b) + \Theta_3(b,a)) + 3(\Theta_4(a,b) + \Theta_4(b,a))\\
=&\Theta_2(a,b) + 2\Theta_3(a,b)
\end{aligned} \tag{5.10}$$

hold for $a, b \in \mathfrak{h}$. With the help of these two identities the rest of the commutators can be computed. Starting with $[[h^{i,j}], [e^{k,l}]]$, we obtain

$$\begin{aligned}
[[h^{i,j}], [e^{k,l}]] =&\Theta_2(e^i, f^j) \star \Theta_2(e^k, e^l) - \Theta_2(e^k, e^l) \star \Theta_2(e^i, f^j)\\
=&[e^i_{(-1)} f^j_{(-1)} e^k_{(-1)} e^l_{(-1)} 1] - [e^k_{(-1)} e^l_{(-1)} e^i_{(-1)} f^j_{(-1)} 1]\\
=&\delta^{jk}((\Theta_3(e^i, e^l) + \Theta_2(e^i, e^l))\\
&+ \delta^{jl}(2(\Theta_3(e^i, e^k) + \Theta_3(e^k, e^i)) + 3(\Theta_4(e^i, e^k) + \Theta_4(e^k, e^i))\\
=&\delta^{jk}[E^{i,l}] - \delta^{jl}[E^{i,k}],
\end{aligned}$$

where (5.9) has been used in the last step. Using (5.10), we can calculate

$$\begin{aligned}
[[h^{i,j}], [f^{k,l}]] =& -\delta^{il}(2\Theta_3(f^k, f^j) + \Theta_2(f^k, f^j))\\
&+ \delta^{ik}\Big((\Theta_2(f^j, f^l) + \Theta_2(f^l, f^j) + 4(\Theta_3(f^j, f^l) + \Theta_3(f^l, f^j))\\
&+3(\Theta_4(f^j, f^l) + \Theta_4(f^l, f^j))\Big)\\
=&\delta^{ik}[F^{j,l}] - \delta^{il}[F^{k,j}].
\end{aligned}$$

This leaves us only with the commutator

$$\begin{aligned}
[[e^{i,j}], [f^{k,l}]] =&[e^i_{(-1)} e^j_{(-1)} f^k_{(-1)} f^l_{(-1)} 1] - [f^k_{(-1)} f^l_{(-1)} e^i_{(-1)} e^j_{(-1)} 1]\\
=&\delta^{jk}\left(\Theta_2(e^i, f^l) + 2\Theta_3(e^i, f^l)\right) - \delta^{il}\left(\Theta_2(f^k, e^j) + 2\Theta_3(f^k, e^j)\right)\\
&- \delta^{jl}\left(2\Theta_3(e^i, f^k) + 2\Theta_3(f^k, e^i) + 3\left(\Theta_4(e^i, f^k) + \Theta_4(f^k, e^i)\right)\right)\\
&+ \delta^{ik}\Big(\Theta_2(e^j, f^l) + \Theta_2(f^l, e^j) + 4\left(\Theta_3(e^j, f^l) + \Theta_3(f^l, e^j)\right)\\
&+3\left(\Theta_4(e^j, f^l) + \Theta_4(f^l, e^j)\right)\Big)\\
=&\delta^{ik}[H^{j,l}] + \delta^{jl}[H^{i,k}] - \delta^{jk}[H^{i,l}] - \delta^{il}[H^{j,k}].
\end{aligned}$$

Obviously the commutators in Zhu's algebra for the fields of weight two are exactly the same as (5.2a)-(5.2d). Thus, the first step of the proof is completed. In the second step, calculating the commutators for the fields of weight 3 will be avoided by exploiting the fact that $[e^{i,j}]$, $[f^{i,j}]$ and $[h^{i,j}]$ generate Zhu's algebra for $d > 1$. It now has to be shown that this applies to the zero mode algebra as well. From [1], we have the identities

$$[E^{i,j}] = ([h^{i,i}] - [h^{j,j}]) \star [e^{i,j}] \tag{5.11a}$$
$$[H^{i,j}] = ([h^{i,i}] - [h^{j,j}]) \star [h^{i,j}] \tag{5.11b}$$
$$[F^{i,j}] = ([h^{i,i}] - [h^{j,j}]) \star [f^{i,j}] \tag{5.11c}$$
$$[E^{i,i}] = -2[h^{i,j}] \star [e^{i,j}] \tag{5.11d}$$
$$[F^{i,i}] = 2[h^{i,j}] \star [f^{i,j}] \tag{5.11e}$$
$$[H^{j,j}] = [e^{i,j}] \star [f^{i,j}] - [h^{i,j}] \star [h^{j,i}]. \tag{5.11f}$$

Simply inserting action of the zero modes (5.1a-5.1f) on the right hand side of the equations in (5.11) yields

$$\begin{aligned}(h_1^{i,i} - h_1^{j,j})e_1^{i,j}\psi =& \langle e^j, \psi\rangle e^i - \delta^{ij}\langle e^i, \psi\rangle e^i - \delta^{ij}\langle e^j, \psi\rangle e^j + \langle e^i, \psi\rangle e^j \\ =& \langle e^j, \psi\rangle e^i + \langle e^i, \psi\rangle e^j \\ =& E_2^{i,j}\psi,\end{aligned}$$

$$\begin{aligned}(h_1^{i,i} - h_1^{j,j})h_1^{i,j}\psi =& \langle f^j, \psi\rangle e^i - \delta^{ij}\langle e^i, \psi\rangle f^i - \delta^{ij}\langle f^j, \psi\rangle e^j + \langle e^i, \psi\rangle f^j \\ =& H_2^{i,j}\psi,\end{aligned}$$

$$\begin{aligned}(h_1^{i,i} - h_1^{j,j})f_1^{i,j}\psi =& \langle f^j, \psi\rangle f^i - \delta^{ij}\langle f^i, \psi\rangle f^i - \delta^{ij}\langle f^j, \psi\rangle f^j + \langle f^i, \psi\rangle f^j \\ =& F_2^{i,j}\psi,\end{aligned}$$

$$\begin{aligned}2h_1^{i,j}e_1^{i,j} =& -2\delta^{ij}\langle e^j, \psi\rangle e^i + 2\delta^{jj}\langle e^i, \psi\rangle e^i \\ =& 2\langle e^i, \psi\rangle e^i \\ =& E_2^{i,i},\end{aligned}$$

$$\begin{aligned}2h_1^{i,j}f_1^{i,j} =& 2\delta^{jj}\langle f^i, \psi\rangle f^i - 2\delta^{ij}\langle f^j, \psi\rangle f^j \\ =& 2\langle f^i, \psi\rangle f^i \\ =& F_2^{i,i},\end{aligned}$$

$$\begin{aligned}e_1^{i,j}f_1^{i,j} - h_1^{i,j}h_1^{j,i} =& \langle f^j, \psi\rangle e^j + \langle f^i, \psi\rangle e^i - \langle f^i, \psi\rangle e^i + \langle e^j, \psi\rangle f^j \\ =& \langle e^j, \psi\rangle f^j + \langle f^j, \psi\rangle e^j \\ =& H_2^{j,j}.\end{aligned}$$

Thus, the respective elements corresponding to the fields of weight two generate the zero mode algebra in exactly the same way as Zhu's algebra. With the commutators for the fields of weight 2 fixed and equal in both algebras, the commutators for the fields of weight 3 also have to be the same. Consequently, the two Lie algebras are isomorphic.

This isomorphism extends the results on the zero mode algebra on SF_0^- in the following way to all modules.

Theorem 5.5. *For all $d \geq 1$, the top level space M_{h_0} of an arbitrary SF^+ module M is a representation of the symplectic Lie algebra $\mathfrak{sp}(2d)$. Furthermore, the statement of Theorem 5.3 extends to the zero mode algebra on arbitrary SF^+-modules.*

Proof. For $d > 1$, the zero mode algebra on SF_0^- contains the symplectic Lie algebra $\mathfrak{sp}(2d)$ as the Lie subalgebra which is spanned by the zero modes of the vectors of weight 3. As was shown above, Zhu's algebra is isomorphic in this case as a Lie algebra to the zero mode algebra on SF_0^-. By Theorem 3.1, the top level M_{h_0} of every SF^+-module M is a representation of Zhu's algebra $A(SF^+)$ and the assertion follows. For $d = 1$ the assertion follows from the fact (c.f. [1], Remark 4.5) that any module for $A(SF^+)$ is a module for $\mathfrak{sl}(2)$, which is isomorphic to $\mathfrak{sp}(2)$. The second statement of the theorem follows directly from the isomorphism. □

Chapter 6

Conclusion

The aim of this work has been to investigate the generalized symplectic fermion model constructed by Abe. This model, which is actually a family of theories parameterized by a natural number d, has a rich structure. For $d = 1$ it reproduces the well known triplet algebra at $c = -2$, while the case $d > 1$ has not yet been studied from a physical point of view. In an attempt to explore the relation of the $d > 1$ case to the $d = 1$ case, the first step consisted in establishing the $\mathcal{W}$-algebra structure of SF^+. This algebra is considerably more complicated than in the $d = 1$ case; in particular, one needs certain weight 2 fields to generate the whole algebra which are not present in the $d = 1$ case.

The zero mode algebra has been chosen as the most effective tool for exploring the structure of this $\mathcal{W}$-algebra. This has technical as well as conceptual reasons. The technical reason is that it is easier to work with the zero mode algebra than to calculate the commutators of all modes of the generating fields. On the conceptual level, it has been the aim to make contact with the analysis of the triplet algebra in terms of zero modes by Kausch and Gaberdiel.

By exploiting the special structure of the top level of the fermionic part of SF, it could be shown that the zero modes of the weight 3 vectors behave very similar to the $d = 1$ case. The symplectic Lie algebra formed by them is a generalization of the Lie algebra found in the $d = 1$ case. For the zero modes corresponding to vectors of weight 2, there is no information which can be extrapolated from the $d = 1$ to the $d > 1$ case since these vectors do not arise in the $d = 1$ case. One might guess that they form a Lie algebra of their own, but it was shown that this is not the case since their commutators do not close. However, it has been proven that they still have an interesting structure because they form an irreducible representation of the zero mode Lie algebra corresponding to the weight 3 vectors.

One of the questions posed in the introduction pertains to the relation of the physical and the mathematical approach to the problem of classifying the modules of a given field algebra. A Mathematician might think that this question is not important since the relevance of Zhu's algebra is given by Zhu's theorem while there is no such theorem for the zero mode algebra. However, physicists have used the zero mode algebra successfully to investigate the representation structure of, e.g. the triplet algebra. Also, it has been proven by Brungs and Nahm that the zero mode algebra and Zhu's algebra are isomorphic as Lie algebras. This isomorphism has been explicitly exhibited in the present work. While such an explicit isomorphism is useful in itself, it also serves to extend the results of the previous paragraph from one representation of the $\mathcal{W}$-algebra to all irreducible representations.

However, it should be noted that this does not make the zero mode *Lie* algebra and the Zhu algebra approach equivalent. As one can see in the analysis by Gaberdiel and Kausch, the classification with the zero mode Lie algebra gives no one-to-one correspondence between Lie algebra modules and field algebra modules. One can therefore make the general statement that by going over from Zhu's algebra to its Lie algebra, one loses information: the Lie algebra is not able to discern modules as finely as Zhu's algebra itself.

Acknowledgments

I would like to thank Michael Flohr for supervising me under circumstances which were sometimes difficult and for giving me the opportunity to study a beautiful field of physics. Every time we met, he took great interest in my progress and motivated me through his genuine fascination with physics and his warm encouragement.

I am grateful to Toshiyuki Abe who patiently answered my questions concerning the symplectic fermionic vertex operator superalgebra. Studying his article on symplectic fermions has been very rewarding. A very special thanks goes to Nils Carqueville, whose help has been invaluable. Not only did he take great interest in this work, but he also constantly encouraged me, furthered my understanding of the theory of vertex operator algebras in various discussions and provided several fruitful ideas. He also read a preliminary version of this work and suggested many improvements.

I thank Melanie Büchel and Daniel Seibert for carefully proof-reading my thesis and providing competent advice with respect to the English language. I am also grateful to Kenny Doberenz for sacrificing a night's sleep to perfect the layout. Finally, I would like to thank Sandra. Although I took the last meters by myself, I would not have overcome the mountain alone.

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