Explicit Formulas for the Scalar Modes in Seiberg–Witten Theory

Nikolas Akerblom

angefertigt am

Institut für Theoretische Physik
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To Heike

These Lectures are about what we can and can't do with machines today, and why. I have attempted to deliver them in a spirit that should be recommended to all students embarking on writing their PhD theses: imagine that you are explaining your ideas to a former smart, but ignorant, self at the beginning of your studies!

Richard P. Feynman
Feynman Lectures on Computation

Say you've got a disease. Werner's granulomatosis or whatever, and you look it up in a medical reference book. You may well find that you then know more about it than your doctor does, although he spent all that time in medical school ... you see? It's much easier to learn about some special, restricted topic than a whole field. The mathematicians are exploring in all directions, and it's quicker for a physicist to catch up on what he needs than to try to keep up with everything that might conceivably be useful.

Richard P. Feynman
The Pleasure of Finding Things Out

So you see, this physics of ours is a lot of fakery—we start out with the phenomena of lodestone and amber, and we end up not understanding either of them very well. But we have *learned a tremendous amount of very exciting and very practical information in the process!*

Richard P. Feynman
The Feynman Lectures on Physics

Abstract

In this thesis, general formulas for the scalar modes a_i and a_D^i in the Seiberg–Witten $SU(N)$ setting are derived in the cases with and without massive hypermultiplets.

These formulas are applied in two cases: (i) the asymptotic behavior of the scalar modes in the asymptotically free region of moduli space is derived in the $SU(2)$ case, (ii) the $SU(3)$ Argyres–Douglas point is studied. In the latter case the approach via the explicit formulas derived raises, for example, the question whether the scalar modes admit an interpretation in terms of BPS mass states everywhere in moduli space.

There is also an appendix giving details on the class of special functions (Lauricella functions of type D) which naturally appear in the formulas for the scalar modes.

Zusammenfassung

In dieser Arbeit werden explizite Formeln für die skalaren Moden a_i und a_D^i im Seiberg–Witten $SU(N)$ Rahmen hergeleitet. Es wird sowohl der Fall mit, als auch der ohne massive Hypermultiplets behandelt.

Für den Gebrauch der Formeln werden zwei Beispiele gebracht: (i) Im $SU(2)$ Fall wird das asymptotische Verhalten der skalaren Moden im asymptotisch freien Bereich des Modulraumes hergeleitet. (ii) Der $SU(3)$ Argyres–Douglas Punkt wird studiert. In der zweiten Anwendung führt der Zugang über die expliziten Formeln z.B. zu der Frage, ob die skalaren Moden überall im Modulraum im Rahmen von BPS Massenzuständen interpretiert werden können.

Ein Anhang bringt Details zu einer bestimmten Klasse von speziellen Funktionen (Lauricella Funktionen vom Typ D), die auf natürliche Weise in den Formeln für die skalaren Moden auftreten.

Contents

1. Introduction and Overwiew

The purpose of this "Introduction and Overview" is to give an idea of where to place this thesis' subject matter, and to roughly explain how the thesis is organized.

In a much celebrated work [33], Seiberg and Witten found an exact solution to $N = 2$ supersymmetric four-dimensional Yang–Mills theory with gauge group $SU(2)$. This paper initiated an avalanche of research leading to a vast set of exactly solvable Yang–Mills theories in several dimensions and with various degrees of supersymmetry. Of particular importance are also string-theoretic derivations of Seiberg–Witten models. See, for example, [36, 35, 25, 18, 3, 27, 19] and references therein.

Of special interest for these solutions is the understanding of the moduli space of vacua, which in many cases turns out to be a hyperelliptic Riemann surface. In particular, simply-laced Lie gauge groups lead to hyperelliptic 'spectral curves.'

The importance of the spectral curves lies in the fact that the physically relevant information of the BPS mass spectrum (of the respective theory) is encoded in a certain meromorphic 1-form defined on the spectral curve (again, of the respective theory). This 1-form is generally known as the *Seiberg–Witten differential*. Masses of BPS states are obtained by integrating the Seiberg–Witten differential along cycles of the homology of the spectral curve.

Any integral along a cycle of the homology of the spectral curve can be expressed as a linear combination (with integer coefficients) of integrals along a basis of the homology. The integrals of the Seiberg–Witten differential along the cycles of a homology basis are called *scalar modes*. From this, one can see that the study of the BPS spectrum of the theory can be reduced to a study of the scalar modes: the scalar modes are important!

It is a (well) known fact that the scalar modes are solutions to differential equations in the complex plane—these equations are called Picard–Fuchs equations. It is, in general, difficult to set up Picard–Fuchs equations and even more difficult to solve them. Finding and solving Picard–Fuchs equations is a general method insofar one can presumably always do so. But it does not yield general results. For instance, suppose one has solved the $SU(2)$ case. Then, for the $SU(3)$ case one has to set up and solve Picard–Fuchs equations all over again.

In this thesis, I derive general formulas for the scalar modes for $N = 2$ supersymmetric $SU(N)$ Yang–Mills theory, in the cases with and without massive hypermultiplets. The 'dimension' of the gauge group enters as a parameter. For the reasons just mentioned, I do not discuss Picard–Fuchs equations in the general case. However, I do discuss the Picard–Fuchs equations for the $SU(2)$ case.

One might ask what is won by the formulas. To give some idea of an answer to that question I discuss two applications of them. The first application (in the $SU(2)$ case) is intended to convince the reader that the formulas are indeed correct. The second application is to a particularly puzzling (only partially understood) phenomenon in the $SU(3)$ case, generally referred to by the name 'Argyres–Douglas point.' As can be seen from Chapter 5 the ease of use of the formulas is nice. In the application to the Argyres–Douglas point one can also see that they allow to address some questions very sharply.

Let me give an overview of how the thesis is organized. The first chapter is introductory and intended to serve as a review of the work of Seiberg and Witten on $N = 2$ supersymmetric $SU(2)$ Yang–Mills theory. I also try to give an overview of the general concept of *duality*.

The second chapter deals with the generalization of the work of Seiberg and Witten to gauge group $SU(N)$ and also to the case where there are massive hypermultiplets present.

In the fourth chapter I finally derive the above-mentioned general formulas for the scalar modes and apply them in Chapter 5 to the above-mentioned cases. To get an impression of what is done in Chapter 5 it is perhaps best to have a look at it. Also, at the end of chapter five I discuss some general questions which arise in connection with the Argyres–Douglas point. Really, these questions point to possible future research and so are not fully answered.

There follows a discussion of the thesis (a sort of looking back what has been done) and an appendix on the special function Lauricella $F_D^{(n)}$. The Appendix should be considered an integral part of the thesis since it supplies the reader with information on this special function, needed to understand the discussion in Chaper 5.

Large parts of this thesis have appeared as a preprint, together with M. Flohr, see [1]. It has been submitted to JHEP for publication.

2. Duality in $N = 2$ SUSY $SU(2)$ Yang–Mills Theory

This chapter consists of two sections: In the first section I aim at introducing the reader to the general concept of duality. The second section is a short introduction to the seminal work [33] of Seiberg and Witten on $N = 2$ supersymmetric $SU(2)$ Yang–Mills theory.

My purpose is to pave the way for extensions of this to other gauge groups, to be discussed in later chapters, and my own discoveries (together with M. Flohr), which are also discussed in [1].

2.1 What is Duality?

To explain what goes under the general heading of *duality* I will first consider an example in a familiar setting. Afterwards I will try to explain the *general* features of duality and also how it relates to the topic of the present thesis.

The following discussion originated with Dirac [11] but some details of it (the *local* vector potentials) are due to Wu and Yang [38].

Consider the source-free Maxwell equations

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t}$$

Obviously, they are invariant under the mapping $(\vec{E}, \vec{B}) \mapsto (\vec{B}, -\vec{E})$. Loosely speaking, this mapping amounts to 'exchanging the electric and magnetic fields.'

The Maxwell equations in the presence of sources

$$\nabla \cdot \vec{E} = \rho_e \qquad \nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \vec{\jmath}_e + \frac{\partial \vec{E}}{\partial t}$$

are *not* invariant under the transformation just considered. The reason for this appears to be the absence of magnetic charges and currents.

Then, to maintain the symmetry, let us suppose that there exist non-vanishing magnetic charges and currents, so that the Maxwell equations read

$$\nabla \cdot \vec{E} = \rho_e \qquad \nabla \cdot \vec{B} = \rho_m$$

$$\nabla \times \vec{E} = \vec{\jmath}_m - \frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \vec{\jmath}_e + \frac{\partial \vec{E}}{\partial t}$$

This set of equations is invariant under $(\vec{E}, \vec{B}) \mapsto (\vec{B}, -\vec{E})$, provided that also $(\rho_e, \rho_m) \mapsto (\rho_m, -\rho_e)$ and $(\vec{\jmath}_e, \vec{\jmath}_m) \mapsto (\vec{\jmath}_m, -\vec{\jmath}_e)$.[1]

We now wish to argue, using quantum mechanics, that the Maxwell equations with magnetic charges and currents imply that magnetic charge, just like electric charge, is quantized. More precisely, we will demonstrate that every magnetic charge g obeys a relation $g = n2\pi/e$, where e is the (electric) charge of the electron and n is some integer (not explicitly known). This equation is known as the Dirac quantization condition.

To this end consider a magnetically charged particle (magnetic charge g, electrically neutral) sitting at the origin. The magnetic field produced by this particle clearly is

$$\vec{B} = \frac{1}{4\pi} \frac{g\,\vec{r}}{r^3}.$$

Contrary to the case where there are no sources for the magnetic field we cannot have $\vec{B} = \nabla \times \vec{A}$ globally anymore, since then we would have $\nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) \equiv 0$, despite the fact that there is a charge sitting at the origin.

The way out is to have $\nabla \times \vec{A} = \vec{B}$ only *locally*. For this, consider a sphere whose center is the origin and divide the sphere in a northern (N) and a southern (S) hemisphere. In standard polar coordinates local vector potentials for $\vec{B}$ on the northern and the southern hemisphere respectively are

$$\vec{A}_{\rm N} = +\frac{g}{4\pi\, r} \frac{1 - \cos\theta}{\sin\theta}\, \vec{e}_\phi, \tag{2.1}$$

and

$$\vec{A}_{\rm S} = -\frac{g}{4\pi\, r} \frac{1 + \cos\theta}{\sin\theta}\, \vec{e}_\phi. \tag{2.2}$$

[1] One might contemplate getting rid of the asymmetry introduced by the minus sign in the equation for $\nabla \times \vec{E}$. However, this is impossible, since the minus sign is dictated by energy conservation (Lenz' rule).

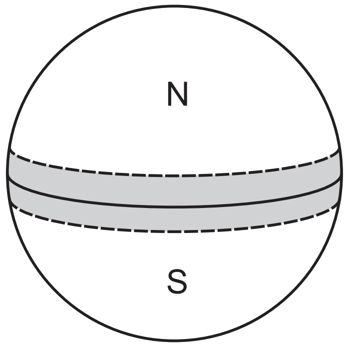

Fig. 2.1: A sphere and its northern (N) and southern (S) hemispheres. The region of overlap referred to in the text is shaded.

Now imagine that an electron (charge e, mass m) moves in the magnetic field produced by our monopole. The Schrödinger equation for the electron's wave function ψ is

$$\mathrm{i}\,\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(\frac{1}{\mathrm{i}}\nabla - e\vec{A}\right)^2 \psi. \tag{2.3}$$

Because $\vec{A}$ is defined by different expressions on the northern and southern hemispheres we really have two Schrödinger equations and they 'overlap' on and around the equator. Denote their solutions by ψ_{N} and ψ_{S} (the notation should be obvious).

If $\vec{A}_{\mathrm{N}}$ and $\vec{A}_{\mathrm{S}}$ differed only by a gauge tranformation, say $\vec{A}_{\mathrm{N}} = \vec{A}_{\mathrm{S}} + \nabla\chi$, then we would know (from quantum mechanics) that

$$\psi_{\mathrm{N}} = \exp(\mathrm{i}e\chi)\,\psi_{\mathrm{S}}. \tag{2.4}$$

Indeed,

$$\vec{A}_{\mathrm{N}} - \vec{A}_{\mathrm{S}} = \nabla\left(\frac{g}{2\pi}\phi\right), \tag{2.5}$$

so that, in fact, equation 2.4 holds with

$$\chi = \frac{g}{2\pi}\,\phi. \tag{2.6}$$

If we now consider ψ_{N} on the equator ($\theta = \pi/2$) as ϕ goes from 0 to 2π *and require that* ψ_{N} *be single-valued* we infer that

$$\frac{e\,g}{2\pi} \in \mathbb{Z}, \tag{2.7}$$

as promised.

A few remarks are in order:

- The above argument is a bit sloppy because it depends on the wave function being single-valued. This requirement is often looked upon as being unfounded, since physically relevant quantities are obtained as expectation values. However, this sort of reasoning is quite common in quantum mechanics and I am not even sure whether it is invalid.

- Concerning the last objection suffice it to say that one can also obtain the Dirac quantization condition by different lines of reasoning. For instance, one can compute the angular momentum of the electron in the field produced by the magnetic monopole and require it to be quantized in units of $\hbar/2$. This also yields the Dirac quantization condition. There is also at least one other argument leading to the desired result, see [22].

- Our discussion leading to the Dirac quantization condition was superficial in the sense that it was not based on some microscopic theory of magnetic charges.[2] This situation is remedied by the Georgi–Glashow model for which 't Hooft [20] and Polyakov [32] were able to show that it contains dyons (particles having electric charge q as well as magnetic charge g) obeying the charge quantization condition $g = 2n2\pi/q$, $n \in \mathbb{Z}$.

- In connection with the Dirac quantization condition one sometimes reads of 'minimal charges' obeying $g = 2\pi/e$ (e. g. [6]). As the example of the Georgi–Glashow model shows such minimal charges need not exist.

- The formalism we employed (vector calculus) can of course be replaced with a tensorial dicussion in which everything seems slicker. The duality map (see below) then amounts to $F_{\mu\nu} \mapsto {}^\star F_{\mu\nu}$ and ${}^\star F_{\mu\nu} \mapsto -F_{\mu\nu}$. As one can see from this, the asymmetry mentioned in footnote 1 remains. As it has a physical origin this is very much expected.

Back to the Dirac quantization condition. We call the mapping exchanging the electric and magnetic fields the *duality map*. Then, under the duality map electric and magnetic charges are also exchanged. More precisely, one charge maps to one over the other (up to a factor of 2π and possibly a minus sign). So, if one charge happens to be large it will get mapped to a small charge. Now, charge is also the coupling constant which means that by applying the duality map to a strongly coupled situation we get a weakly

[2] We took the existence of magnetic charges for granted and proceeded from there using only quantum mechanics. By the same token the result is very general and must be obeyed by any theory purporting to be a theory of magnetic monopoles.

coupled situation. This is precisely what one means by 'duality': *A map between two theories which exchanges weak and strong coupling regimes.*

Clearly, it is desirable to know dualities, because then one can learn about the strong coupling behavior of both theories involved by using perturbation theory. Of course, the two theories need not be distinct.

Besides the introductory duality we considered other dualities are known, for instance Sine-Gordon/Thirring [9, 29], Ising/Ising [26] and, essential for this thesis, duality in $N = 2$ SUSY $SU(2)$ Yang–Mills theory (plus a host of generalizations of this), known as Seiberg–Witten. This provides a nice transition to the next section.

2.2 A very quick introduction to the work of Seiberg and Witten

In this section I will review the work of Seiberg and Witten on $N = 2$ supersymmetric $SU(2)$ Yang–Mills theory. Since the original paper [33] by Seiberg and Witten is well over 40 pages long, I cannot possibly aim for completeness. On the contrary, I will try to give just enough detail, only quoting results all the way through, so that later on I can put my original work [1] (together with M. Flohr) on the scalar modes in 'Seiberg–Witten theory' in the proper context. In writing this section I found the *pedagogical introduction* by Bilal [6] very helpful. Since the contents of that work are very well suited for the aims of this section I will reproduce here essentially an abridged version of it, sometimes even quoting verbatim this source. A reader interested in a more detailed introduction to supersymmetry might have a look at, for instance, [7, 31, 28, 37] or one of the several books on the subject (e. g. the one by Bailin and Love [4]).[3] Also, there exist several other pedagogical introductions to the work of Seiberg and Witten besides the one by Bilal. To name just two: [23, 17].

2.2.1 Supersymmetric preliminaries

$N = 2$ supersymmetry combines a complex scalar field ϕ, its superpartner ψ (a two-component spinor), and a (massless gauge) vector field A_μ along with its superpartner λ (gaugino) into one supermultiplet.

The $N = 2$ $SU(2)$ super Yang–Mills action S governing the dynamics of

[3] For the reader wishing an introduction to supersymmetry in general terms, the first section ("Introduction") of the review by Sohnius [37] is especially recommendable.

this supermultiplet is

$$\operatorname{Im}\left[\frac{\tau}{16\pi}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\tilde{\theta}\,\frac{1}{2}\operatorname{Tr}(\Psi^2)\right]. \tag{2.8}$$

This formula is written in '$N = 2$ superspace language' and we will now identify a few terms in it:

- Ψ is a so-called '$N = 2$ chiral superfield.' It is a combination of ϕ, ψ, A_μ, and λ mentioned earlier.
- θ and $\tilde{\theta}$ denote a set of anticommuting variables, ubiquitous in the field of supersymmetry. They are needed for the combination of the fields ϕ, ψ, A_μ, and λ into Ψ.
- τ is a complex coupling constant.

I should mention that the $N = 2$ supersymmetry which we are considering combines two $N = 1$ supersymmetric fields, namely a vector multiplet W_α (containing (A_μ, λ)) and a chiral multiplet Φ (containing (ϕ, ψ)) into one single superfield Ψ. The 'Grassmann numbers' θ, $\tilde{\theta}$ are the notational sugar which makes this possible. One could stay entirely within the $N = 1$ formalism but this would be an unpleasant exercise.

As a statement of fact, it can be shown that any $N = 2$ supersymmetry invariant action is of the form

$$\frac{1}{16\pi}\operatorname{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\tilde{\theta}\,\mathcal{F}(\Psi), \tag{2.9}$$

where $\mathcal{F}$, the '$N = 2$ prepotential,' depends only on Ψ and not on $\Psi^\dagger$.[4] This last fact is known as 'holomorphy of the prepotential.'

Clearly, in the case of the action 2.8 one simply has

$$\mathcal{F}(\Psi) \equiv \mathcal{F}_{\text{class}}(\Psi) = \frac{1}{2}\operatorname{Tr}(\tau\Psi^2). \tag{2.10}$$

Now, what Seiberg and Witten did was to study the low energy behavior of the theory given by the action 2.8. More precisely, they determined the Wilsonian effective action of the theory—and this they did in a very remarkable way as will become clear later.

The theory under consideration (determined by the action 2.8) classically has a scalar potential $V(\phi) = \frac{1}{2}\operatorname{Tr}\{([\phi^\dagger, \phi])^2\}$. Unbroken supersymmetry requires that $V(\phi_0) = 0$ for the vacuum state ϕ_0. This happens precisely if

[4] Such short statements as this could not be obtained in the $N = 1$ formalism.

$[\phi_0^\dagger, \phi_0] = 0$. Clearly, a non-vanishing ϕ_0 is possible. (The field ϕ is an $SU(2)$ gauge field. To avoid confusion, I would like to draw the reader's attention to the fact that this entails ϕ having its values in $SU(2)$.)

Then, what are the gauge inequivalent vacua?

Every ϕ is of the form

$$\phi(x) = \frac{1}{2} \sum_{j=1}^{3} \big(a_j(x) + \mathrm{i}\, b_j(x)\big)\, \sigma_j, \tag{2.11}$$

where a_j and b_j are real, and the σ's are the Pauli matrices generating $SU(2)$.

Let us assume that not all a's vanish (otherwise exchange the roles of the a's and b's in the following). Then, by a $SU(2)$ gauge transformation we can arrange that $a_1 = a_2 = 0$. Assume this done. Then $[\phi^\dagger, \phi] = 0$ implies $b_1 = b_2 = 0$ and hence, if we put $a = a_3 + \mathrm{i}\, b_3$, we have $\phi = \frac{1}{2}\, a\sigma_3$. In the vacuum a must be a constant. Gauge transformations from the Weyl group can still change $a \mapsto -a$ so that a and $-a$ are gauge equivalent. Thus, the quantity describing gauge inequivalent vacua is $\frac{1}{2}\, a^2$, or $\mathrm{Tr}(\phi^2)$ (where the factor of $\frac{1}{2}$ has been inserted for convenience). Semi-classically it does not make a difference whether one uses $\frac{1}{2}\, a^2$ or $\mathrm{Tr}(\phi^2)$ to describe gauge inequivalent vacua. However, when quantum fluctuations are important it *does* make a difference.

Therefore, we define

$$u = \langle \mathrm{Tr}(\phi^2) \rangle, \quad \langle \phi \rangle = \frac{1}{2}\, a\sigma_3, \tag{2.12}$$

so that the complex parameter u labels the gauge inequivalent vacua. By interpreting this parameter as a coordinate on the set $\mathcal{M}$ of gauge inequivalent vacua, $\mathcal{M}$ becomes a manifold, called the *moduli space* of the theory.

2.2.2 The low-energy effective action

For non-vanishing $\langle \phi \rangle$ the $SU(2)$ gauge symmetry is broken by the Higgs mechanism.[5] If we take the participating fields ϕ, ψ, A_μ, and λ to be in the adjoint representation[6] (and this we should), then A_μ^3, ψ^3, and λ^3 as well as the mode of ϕ describing fluctuations of ϕ in the direction of σ_3 remain massless under the symmetry breaking.

[5] This is one of the places where we take rather big steps in order not to get bogged down in a thicket of technicalities. The reader interested in more detail is referred to the literature mentioned earlier in this chapter.

[6] The adjoint representation of (the algebra of) $SU(2)$ has dimension 3.

These massless modes are described by the Wilsonian (low-energy) effective action.

While the $SU(2)$ gauge symmetry is broken, the theory's supersymmetry remains intact. It is possible to argue that the Wilsonian effective action S_{W} for the massless modes is of the form

$$S_{\mathrm{W}} = \frac{1}{16\pi}\,\mathrm{Im}\int \mathrm{d}^4x \left[\int \mathrm{d}^2\theta\,\mathcal{F}''(\Phi)W^\alpha W_\alpha + \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,\Phi^\dagger\mathcal{F}'(\Phi)\right], \quad (2.13)$$

where Φ is some combination of (the σ_3-mode of) ϕ and ψ^3, and W_α some combination of A^3_μ and λ^3. (Clearly, these 'combinations' are the vector and chiral multiplet mentioned earlier.) So, if one knows the prepotential $\mathcal{F}$ one also knows the effective action S_{W}.

The component form of the effective action 2.13

$$S_{\mathrm{W}} = \frac{1}{4\pi}\,\mathrm{Im}\int \mathrm{d}^4x\Big[\mathcal{F}''(\phi)\Big(-\frac{1}{4}\Big)F_{\mu\nu}(F^{\mu\nu} - \mathrm{i}\tilde{F}^{\mu\nu}) - \mathrm{i}\mathcal{F}''(\phi)\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \cdots\Big] + \frac{1}{4\pi}\,\mathrm{Im}\int \mathrm{d}^4x\Big[\mathcal{F}''(\phi)|\partial_\mu\phi|^2 - \mathrm{i}\mathcal{F}''(\phi)\psi\sigma^\mu\partial_\mu\bar{\psi} + \cdots\Big], \quad (2.14)$$

where the dots indicate non-derivative terms, motivates, by analogy with the four-dimensional sigma-model, the metric on $\mathcal{M}$

$$\mathrm{d}s^2 = \mathrm{Im}(\mathcal{F}''(a))\,\mathrm{d}a\,\mathrm{d}\bar{a} = \mathrm{Im}(\tau(a))\,\mathrm{d}a\,\mathrm{d}\bar{a}, \quad (2.15)$$

where the bar $\bar{\ }$ denotes complex conjugation.

It turns out that the metric 2.15 is not appropriate on all of $\mathcal{M}$, or, equivalently, that the fields and function $\mathcal{F}$ in the effective action 2.14 are not the appropriate description for the effective action for all u. To see this, we note that if it were appropriate on all of $\mathcal{M}$ it should be positive definite, which entails $\mathrm{Im}(\tau(a)) > 0$. This cannot be the case: $\mathcal{F}(a)$ is holomorphic, therefore $\mathrm{Im}(\tau(a)) = \mathcal{F}''(a)$ is harmonic and as such cannot have a minimum (unless it is a constant, as in the classical case). If we view $\mathrm{Im}(\tau(a))$ as defined on the (compact) set $\widehat{\mathbb{C}}$ it follows that it cannot be bounded below. Consequently, we cannot have $\mathrm{Im}(\tau(a)) > 0$ everywhere.

It is therefore necessary to consider local descriptions of $\mathcal{M}$. When a point a of $\mathcal{M}$ is reached where $\mathrm{Im}(\tau(a)) \to 0$ (a *singular point*) one has to employ a different set of coordinates $\hat{a}$.

As I had already written, classically $\mathcal{F}(\Psi) = \frac{1}{2}\tau_{\mathrm{class}}\Psi^2$. The combined perturbative result for tree-level and one-loop is

$$\mathcal{F}_{\mathrm{pert}}(\Psi) = \frac{\mathrm{i}}{2\pi}\,\Psi^2\,\log\frac{\Psi^2}{\Lambda^2}, \quad (2.16)$$

where Λ^2 fixes the normalization of $\mathcal{F}_{\text{pert}}$.

Due to non-renormalization theorems for $N = 2$ supersymmetry this is the full perturbative result. However, there are non-perturbative contributions, as will be seen later.

It is known that for $a \to \infty$ the perturbative expression 2.16 for $\mathcal{F}$ becomes an excellent approximation. It is also known that $u \sim \frac{1}{2} a^2$ in this limit. Putting this together,

$$\mathcal{F} \sim \frac{\mathrm{i}}{2\pi} a^2 \log \frac{a^2}{\Lambda}, \quad \text{as } u \to \infty, \tag{2.17}$$

and

$$\tau(a) \sim \frac{\mathrm{i}}{\pi}\Big(\log \frac{a^2}{\Lambda^2} + 3\Big), \quad \text{as } u \to \infty. \tag{2.18}$$

We see that $\mathrm{Im}(\tau(a)) \sim \frac{1}{\pi} \log \frac{|a|^2}{\Lambda^2} > 0$ for large a^2, which implies that the metric 2.15 is good for that region of the moduli space $\mathcal{M}$. As we already noted, in at least one region of $\mathcal{M}$ this cannot be the case. For a different region than a^2 large, we will obtain a description in terms of new fields by *duality*.

2.2.3 Duality

To this end, we define a field dual to Φ by

$$\Phi_D = \mathcal{F}'(\Phi), \tag{2.19}$$

and a function $\mathcal{F}_D(\Phi_D)$ dual to $\mathcal{F}(\Phi)$ by

$$\mathcal{F}_D'(\Phi_D) = -\Phi. \tag{2.20}$$

Then, one can show that the effective action 2.13 can equivalently be written as

$$S_{\mathrm{W}} = \frac{1}{16\pi} \,\mathrm{Im} \int \mathrm{d}^4x \Big[\int \mathrm{d}^2\theta\, \mathcal{F}_D''(\Phi_D) W_D^\alpha W_{D\alpha} + \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, \Phi_D^\dagger \mathcal{F}_D'(\Phi_D)\Big], \tag{2.21}$$

where there has also been introduced W_D^α, the field dual to W^α,[7] and one has the dual coupling constant

$$\tau_D(a_D) = -\frac{1}{\tau(a)}. \tag{2.22}$$

[7] The reader interested in more detail is once again referred to Bilal [6].

Note that the new, dual coupling constant $\tau_D(a_D)$ essentially is the inverse of the old coupling constant! As I had mentioned in section 2.1 this is an essential feature of duality.

The duality map effected by 2.19 and 2.20 is not the only one possible. In order to discuss the full group of duality transformations of the effective action 2.13 we rewrite it as

$$S_{\mathrm{W}} = \frac{1}{16\pi}\,\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\frac{\mathrm{d}\Phi_D}{\mathrm{d}\Phi}W^\alpha W_\alpha + \frac{1}{32\,\mathrm{i}\pi}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}(\Phi^\dagger\Phi_D - \Phi_D^\dagger\Phi). \tag{2.23}$$

Then, from this alternative form of the effective action we see that, in addition to the duality map just discussed, namely

$$\begin{bmatrix}\Phi_D\\ \Phi\end{bmatrix} \mapsto \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\Phi_D\\ \Phi\end{bmatrix}, \tag{2.24}$$

there is also a family of symmetries

$$\begin{bmatrix}\Phi_D\\ \Phi\end{bmatrix} \mapsto \begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}\begin{bmatrix}\Phi_D\\ \Phi\end{bmatrix}, \quad b \in \mathbb{Z}. \tag{2.25}$$

The duality maps 2.24 and 2.25 together generate $SL(2,\mathbb{Z})$. This is the group of duality symmetries of the effective action 2.13.

Note that the metric 2.15 can be written as

$$\mathrm{d}s^2 = \mathrm{Im}(\mathrm{d}a_D\,\mathrm{d}\bar{a}) = \frac{\mathrm{i}}{2}\,(\mathrm{d}a\,\mathrm{d}\bar{a}_D - \mathrm{d}a_D\,\mathrm{d}\bar{a}), \tag{2.26}$$

where $\langle\phi_D\rangle = \frac{1}{2}\,a_D\sigma_3$ and $a_D = \mathcal{F}'(a)$. This metric also is invariant under the duality group $SL(2,\mathbb{Z})$.

In a spontaneously broken gauge theory as the one we are considering, typically there are solitons (static, finite-energy solutions of the equations of motion) that carry magnetic charge and behave like non-singluar magnetic monopoles. The duality transformation 2.24 constructed above exchanges electric and magnetic charge degrees of freedom, hence electrically charged states (as would be described by hypermultiplets of our $N = 2$ supersymmetric version) with magnetic monopoles.

In $N = 2$ supersymmetric theories there are two types of multiplets: small (or short) ones (4 helicity states) and large (or long) ones (16 helicity states). Massless states must be in short ones if they satisfy $m^2 = 2|Z|^2$, Z being the central charge of the $N = 2$ supersymmetry algebra, or in long ones if $m^2 > 2|Z|^2$. The states that become massive by the Higgs mechanism must be in short multiplets since they were in such before the symmetry breaking (if one imagines turning on the scalar field expectation value), and the Higgs mechanism cannot generate the missing $16 - 4 = 12$ helicity states. For purely electrically charged states one has $Z = a\, n_e$ where n_e is the (integer) electric charge. Duality then implies that a purely magnetically charged state has $Z = a_D\, n_m$ where n_m is the (integer) magnetic charge. A state with both types of charge, called a dyon, has $Z = a\, n_e + a_D\, n_m$ since the central charge is additive. All this applies to states in short multiplets, so-called BPS (Bogomolnyi–Prasad–Sommerfield) states. The mass formula for these states is

$$m^2 = 2|Z|^2, \quad Z = \begin{bmatrix} n_m, & n_e \end{bmatrix} \begin{bmatrix} a_D \\ a \end{bmatrix}. \tag{2.27}$$

Under a $SL(2,\mathbb{Z})$ transformation $M = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \in SL(2,\mathbb{Z})$ acting on $\left[\begin{smallmatrix} a_D \\ a \end{smallmatrix}\right]$, the charge vector gets transformed to $\begin{bmatrix} n_m, & n_e \end{bmatrix} M = \begin{bmatrix} n'_m, & n'_e \end{bmatrix}$ which again are integer charges. In particular, one sees again at the level of the charges that the transformation 2.25 exchanges purely electrically charged states with purely magnetically charged ones.

2.2.4 Singularities and Monodromy

It is perhaps worthwile emphasizing that *if one knows a and a_D, one knows the masses of the BPS states.* Therefore it is clearly desirable to study these two quantities as functions of u, i. e. as functions on the moduli space $\mathcal{M}$. It turns out that $\mathcal{M}$ has certain 'singularities.' If one observes the behavior of $a(u)$ and $a_D(u)$ as u loops around such a singularity one finds that they do not return to their initial values but to certain linear combinations thereof: there is a non-trivial monodromy for the multivalued functions $a(u)$ and $a_D(u)$.

The first singularity is $u = \infty$. Thus, consider the region around $u = \infty$. From the expression 2.17 for $\mathcal{F}$, asymptotically valid there, we have, with $a_D = \mathcal{F}'(a)$,

$$a_D(u) = \frac{\mathrm{i}}{\pi}\, a \left(\log \frac{a^2}{\Lambda^2} + 1 \right), \quad \text{as } u \to \infty. \tag{2.28}$$

For later reference I also state the behavior

$$a = \sqrt{2u}, \quad \text{as } u \to \infty. \tag{2.29}$$

Now, if we take a and a_D clockwise around $u = \infty$, we find that $a \mapsto -a$ and $a_D \mapsto -a_D + 2a$, in matrix form:

$$\begin{bmatrix} a_D \\ a \end{bmatrix} \mapsto M_\infty \begin{bmatrix} a_D \\ a \end{bmatrix}, \quad \text{where } M_\infty = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}. \tag{2.30}$$

We remark that $u = \infty$ is a branch point of $a_D(u) \sim \frac{\mathrm{i}}{\pi}\sqrt{2u}\,(\log\frac{u}{\Lambda^2}+1)$. This explains why it is referred to as a singularity.

The point $u = \infty$ is not the only singular point of $\mathcal{M}$. As a matter of fact, $\mathcal{M}$ has *exactly three* singular points. Moreover, the two singular points besides ∞ are equal and opposite, i. e. if one is u_0 the other is $-u_0$. To see all this requires a somewhat involved argument, which I omit.

For loops around u_0 and $-u_0$ one has the monodromies

$$\begin{bmatrix} a_D \\ a \end{bmatrix} \mapsto M_{u_0} \begin{bmatrix} a_D \\ a \end{bmatrix}, \quad \text{where } M_{u_0} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \tag{2.31}$$

and

$$\begin{bmatrix} a_D \\ a \end{bmatrix} \mapsto M_{-u_0} \begin{bmatrix} a_D \\ a \end{bmatrix}, \quad \text{where } M_{-u_0} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}, \tag{2.32}$$

respectively.

What are the physical interpretations of the three singular points ∞, u_0, $-u_0$? As $a \to \infty$ our effective theory becomes asymptotically free (that is the reason why 2.17 is valid) and the singular points $u = \pm u_0$ are strong coupling singularities. The theory at u_0 is $N = 2$ supersymmetric QED with very light electrons and the point $-u_0$ corresponds to a $\begin{bmatrix} 1, & -1 \end{bmatrix}$ dyon becoming massless. (Of course this requires choosing one of $\pm u_0$ as being u_0.)

So far we know not much about $a_D(u)$ and $a(u)$, namely their asymptotics and monodromies around the singular points ∞ and $\pm u_0$. Surprisingly, the 'power of complex analysis' allows us to determine $a_D(u)$ and $a(u)$ from the known monodromies up to normalization. The normalization can then be fixed from the known asymptotics, thereby giving us full knowledge of $a_D(u)$ and $a(u)$.

2.2.5 Determination of the low-energy effective action

There are at least two ways to determine $a_D(u)$ and $a(u)$. One uses differential equations, the other a certain elliptic curve together with a differential defined on it. The latter approach was the one Seiberg and Witten originally took and it is this approach which is basic to my work discussed in a later chapter.

Here I discuss both approaches sequentially.

For technical simplicity we assume that $u_0 = 1$. This is possible because its precise value depends on the renormalization conditions [33]. If one wants to be indecisive about u_0, one simply need replace $u \pm 1$ by $\frac{u}{u_0} \pm 1$ in ensuing equations.

The approach using differential equations.

Monodromies typically arise from differential equations with periodic coefficients. One instance of this occurs in solid state physics where one considers the Schrödinger equation with periodic potential

$$-\psi''(x) + V(x)\,\psi(x) = 0, \quad V(x + 2\pi) = V(x). \tag{2.33}$$

There are two independent solutions $\psi_1(x)$ and $\psi_2(x)$. One wants to compare solutions at x and at $x + 2\pi$. Since, due to the periodicity of the potential V, the differential equation at $x + 2\pi$ is exactly the same as at x, the set of solutions must be the same. In other words, $\psi_1(x+2\pi)$ and $\psi_2(x+2\pi)$ must be linear combinations of $\psi_1(x)$ and $\psi_2(x)$:

$$\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}(x + 2\pi) = M\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}(x), \tag{2.34}$$

where M is a (constant) monodromy matrix.

The same situation arises for differential equations in the complex plane with meromorphic coefficients. Consider again the Schrödinger-type equation

$$-\psi''(z) = V(z)\,\psi(z) = 0 \tag{2.35}$$

with meromorphic $V(z)$, having poles at $z_1, \dots, z_p$ and (in general) also at ∞. The periodicity of the previous example is now replaced by the single-valuedness of $V(z)$ as z goes around any of its poles. So, as z goes once around any one of the z_i the differential equation 2.35 does not change. Then, by the same argument as above, the two solutions $\psi_1(z)$ and $\psi_2(z)$, when continued along the path around z_i, must again be linear combinations of $\psi_1(z)$ and $\psi_2(z)$:

$$\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}\big(z + \mathrm{e}^{2\pi\mathrm{i}}(z - z_i)\big) = M_i\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}(z) \tag{2.36}$$

with a constant (monodromy) matrix M_i for each pole z_i of the potential $V(z)$.

In our problem the (multivalued) functions $a_D(z)$ and $a(z)$ have three singularities with non trivial monodromies at -1, 1, and ∞ (recall our assumption that $u_0 = 1$). From results of complex analysis it follows that they can be interpreted as solutions of a second order differential equation 2.35 with potential V given by

$$V(z) = -\frac{1}{4}\left[\frac{1-\lambda_1^2}{(z+1)^2} + \frac{1-\lambda_2^2}{(z-1)^2} + \frac{1-\lambda_1^2-\lambda_2^2+\lambda_3^2}{(z+1)(z-1)}\right]. \tag{2.37}$$

It can be shown that the solutions of the resulting differential equation 2.35 which have the correct asymptotic behavior are

$$a_D(u) = \mathrm{i}\frac{u-1}{2}\,{}_2F_1\Big(\frac{1}{2},\frac{1}{2};2;\frac{1-u}{2}\Big), \tag{2.38}$$

and

$$a(u) = \sqrt{2}\,(u+1)^{\frac{1}{2}}\,{}_2F_1\Big(-\frac{1}{2},\frac{1}{2};1;\frac{2}{u+1}\Big), \tag{2.39}$$

where ${}_2F_1$ is the Gaussian hypergeometric function.

These solutions can also be nicely written as

$$a_D(u) = \frac{\sqrt{2}}{\pi}\int_1^u \frac{\sqrt{x-u}}{\sqrt{x^2-1}}\,\mathrm{d}x, \tag{2.40}$$

and

$$a(u) = \frac{\sqrt{2}}{\pi}\int_{-1}^1 \frac{\sqrt{x-u}}{\sqrt{x^2-1}}\,\mathrm{d}x. \tag{2.41}$$

One can invert equation 2.41 to obtain $u(a)$ and insert the result into $a_D(u)$ to obtain $a_D(u)$. Integrating with respect to a yields $\mathcal{F}(a)$ and hence the low-energy effective action. The thus obtained prepotential $\mathcal{F}(a)$ is not globally valid but only on a certain portion of the moduli space $\mathcal{M}$. Different analytic continuations must be used on other portions.

To sum up, this completely solves the problem of determining the low-energy effective action.

The approach using elliptic curves

In their paper, Seiberg and Witten do not use the differential equation approach just described, but rather introduce an auxiliary construction: a certain elliptic curve by means of which two functions with correct monodromy properties are constructed.

Namely, they consider the elliptic curve given by the equation (y and x are *complex*)

$$y^2 = (x^2-1)(x-u). \tag{2.42}$$

This is a one-dimensional *complex* manifold (hence the term *curve*). Interpreted as a two-dimensional *real* manifold it is a torus, see figure 2.2. The modulus u is a parameter for the exact geometry of the torus.

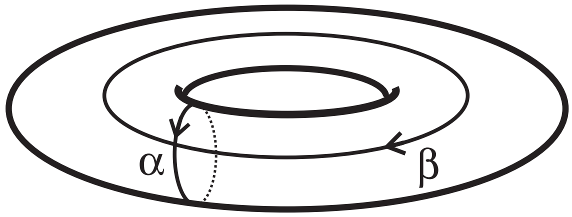

Fig. 2.2: A torus. Also shown are the two cycles α and β referred to in the text.

On the curve 2.42 there is also defined a differential

$$\lambda_{\mathrm{SW}} = \frac{\sqrt{2}}{2\pi} \frac{\sqrt{x-u}}{\sqrt{x^2-1}}\, \mathrm{d}x. \tag{2.43}$$

The solutions $a_D(u)$ and $a(u)$ are obtained by integrating this differential along a basis of the homology of the torus. Any homology basis of the torus has exactly two cycles, say α and β. One particular choice for these cycles is shown in figure 2.2. Thus, we have

$$a_D = \int_\beta \lambda_{\mathrm{SW}}, \tag{2.44}$$

and

$$a = \int_\alpha \lambda_{\mathrm{SW}}. \tag{2.45}$$

These integrals are called period integrals. They are known to satisfy a second-order differential equation, the so-called Picard–Fuchs equation, which is nothing else than our Schrödinger-type equation 2.35 with V given by equation 2.37.

To motivate the curve 2.42 Seiberg and Witten remark that the monodromy matrices M_∞, M_{u_0} and M_{-u_0} are very special. They do not generate all of $SL(2,\mathbb{Z})$ but only a certain subgroup $\Gamma(2) \subset SL(2,\mathbb{Z})$ of matrices congruent to 1 modulo 2. Furthermore they remark that the u-plane with punctures at 1, -1, ∞ can be thought of as the quotient of the upper half plane H by $\Gamma(2)$, and that $H/\Gamma(2)$ naturally parametrizes (i. e. is the moduli space of) elliptic curves described by equation 2.42.

3. Generalizations to other gauge groups

This is a short chapter. I describe the generalization of the work of Seiberg and Witten to other gauge groups. Actually I only discuss the generalization to gauge group $SU(N_c)$ with N_f massive hypermultiplets. $N = 2$ supersymmetric Yang–Mills theory starts out as a bosonic theory. Massive hypermultiplets are fermions which are put in 'by hand' (for further details see, for instance, [34]).

As in the last chapter I only quote results most of the time. The idea is to not spend much time on the foundations but to move quickly so that applications can be discussed. This is important because the generalizations are not solutions in themselves. Rather, they set up problems to be solved. Namely, determining the periods of certain differentials.

The first section covers the case where the gauge group is $SU(N)$ and there are no massive hypermultiplets present. In the second section the modifications brought about by massive hypermultiplets are communicated. I have organized the chapter like this because I found it easier to understand that way.

3.1 The Spectral Curve and SW Differential for gauge group $SU(N)$

In the last chapter we considered the determination of the low-energy effective action of $N = 2$ supersymmetric $SU(2)$ Yang–Mills theory. The group $SU(2)$ is not the only possible gauge group. One could consider an analogous theory with a different gauge group. The question arises: Can the effective action for such a theory be obtained in an analogous fashion as that of the $SU(2)$ case? For a wide variety of gauge groups the answer is positive. For a small selection see [25, 24, 3, 10, 8]. Lerche [27] gives further references to relevant publications.

Since there seems to be no general agreement about the form and normalization of the spectral curves and Seiberg–Witten differential it might be helpful for the reader to know that the conventions I use are identical with those of [15].

For the present I only explain the generalization to gauge group $SU(N)$ (the N has nothing to do with the N in $N = 2$). In the next section I will also consider adding extra matter—so-called massive hypermultiplets.

We are dealing with a theory analogous to the $SU(2)$ case discussed in chapter 2, only that now the gauge group is $SU(N)$ with N unrestricted. One is interested in its effective action. In particular there are again *scalar modes* which give the masses of the BPS states of the theory. To begin with, just as in the $SU(2)$ case the scalar superfield component ϕ labels a continuous family of equivalent ground states, and one has the moduli (vev's) u_k, $k = 2, \dots, N$, defined by

$$u_k = \langle \mathrm{Tr}(\phi^k) \rangle. \tag{3.1}$$

Instead of two scalar modes a and a_D dual to each other, one now has $2(N-1)$ scalar modes a^i, a_D^i, $i = 1, \dots, N-1$. They are functions of the moduli u_k.

As before, the scalar modes give the masses of BPS states by the formula

$$m_{(\mathbf{q},\mathbf{g})} \propto |a_i q^i + a_D^i g_i|, \tag{3.2}$$

where now such a state has an electric-magnetic charge *vector* $(\mathbf{q}, \mathbf{g})$ and $\mathbf{q}$ and $\mathbf{g}$ are $(N-1)$-tuples of integers.

As in the case of gauge group $SU(2)$ all the information about the scalar modes of the theory is encoded in a certain hyperelliptic curve and a differential λ_{SW} defined on it.

This hyperelliptic curve, called the *spectral curve* of the theory, is given by

$$y^2 = A(x)^2 - B := \left(x^N - \sum_{k=2}^{N} u_k x^{N-k} \right)^2 - \Lambda^{2N} = \prod_{i=1}^{2N} (x - e_i), \tag{3.3}$$

where Λ is the cut-off parameter which comes into play because we are dealing with an effective theory. The fundamental theorem of algebra makes possible the last equality. One sees that the moduli u_k determine the exact geometry of the curve.

The variety given by equation 3.3 is a one-dimensional complex manifold. Interpreted as a real manifold it is two-dimensional. In fact, it is a genus $g = N - 1$ Riemann surface. For instance, $N = 3$, i.e. $g = 2$, corresponds to a pretzel like object,[1] or if you will, a 'torus with two holes,' see figure 3.1.

[1] This terminology is a little strange because a Bavarian, as well as a Swabian pretzel has *three* holes. One can imagine a Mathematical pretzel as having two and only two holes.

The general case looks like figure 3.2: the resulting manifold is homeomorphic to a sphere with g handles.

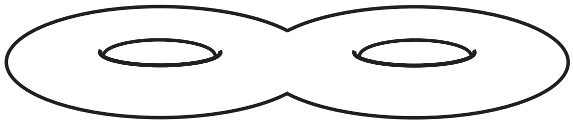

Fig. 3.1: The pretzel

Fig. 3.2: The pretzel with g holes

We will always consider the spectral curve to be equipped with a canonical homology basis, say $B = \{\alpha_i, \beta^i\}_{1\leq i\leq N-1}$, with $\alpha_i \cap \beta^j = \delta_i^j$. We had already encountered one instance of such an homology basis in the previous chapter, namely the cycles α and β on the torus. The cycles of the homology basis will encircle pairs of the e's in equation 3.3. In the applications I will always indicate explicitly the chosen homology basis. Choosing a homology basis amounts to saying which particles one is going to call electrically charged and magnetically charged, respectively. Due to duality this is conventional. For more on *homology* see Nakahara [30], especially p. 115. What we have called a canonical homology basis, Nakahara calls a *canonical system of curves* (on the Riemann surface under consideration).

On the spectral curve 3.3 there is defined the Seiberg–Witten differential

$$\lambda_{\mathrm{SW}} = \frac{1}{2\pi \mathrm{i}} \frac{\prod_{\ell=0}^{N-1}(x - z_\ell)}{\prod_{i=1}^{2N}\sqrt{x - e_i}}\, \mathrm{d}x, \tag{3.4}$$

where the z_ℓ ($\ell > 0$) are the zeros of $2A'(x)B$, $z_0 = 0$ and the e's are the zeros of y^2. Usually the e's are called *branch points*. There are other, equivalent forms of the Seiberg–Witten differential because one can always add terms which would not affect the evaluation of period integrals. The particular form we have given is singled out in string-theoretic derivations of Seiberg–Witten low-energy effective field-theories (see, e. g. [27]).

As in the $SU(2)$ case, the scalar modes a_D^i and a_i, $i = 1, \dots, N-1$ are given by period integrals:

$$a_D^i = \int_{\beta^i} \lambda_{\mathrm{SW}}, \quad a_i = \int_{\alpha_i} \lambda_{\mathrm{SW}}, \tag{3.5}$$

where α_i and β^i are (dual) cycles of the chosen homology basis for our curve.

A few words about the 'operational definition'[2] of the period integrals 3.5 are in order: At first sight the formulas look rather formidable. How on earth is one to turn *that* into something which, at least in principle, looks calculable. The problem seems to lie in the hyperelliptic curve which serves as the manifold carrying the curves along which to take the integrals. Indeed, if one thinks about it as a 2-dimensional real manifold this is difficult. Luckily, this approach is not necessary. Rather, one interprets the manifold as a 'ramified branched covering of the complex plane.' This amounts to stacking two copies of the complex plane[3] atop of each other and marking off on both the branch points (the e's in eq. 3.3). Then one introduces cuts between certain branch points and for every cut identifies one side of the cut with the opposite side of the corresponding cut on the other plane (for this identifying business see figure 3.3). Cycles on the hyperelliptic curve then correspond to cycles on this branched covering. A cycle can completely be situated on one plane, or it can run on both planes. In the latter case it crosses a cut on one side and emerges on the opposite side of the corresponding cut on the other plane. In figures, if part of a cycle runs on the underlying plane (with the plane on top of it displayed in the figure), this is indicated by representing that part of the cycle as a dashed line (while the part which runs on the top plane is shown as a solid line). For an example see figure 5.1.

The last paragraph has intentionally been rather sketchy. For more details I refer the reader to [14], which is modern and rather difficult or to the classic [21], which explains with many illustrations and in a nice 'cut and paste' fashion how to construct 'branched ramified coverings.' Unfortunately, this approach appears to be so much outmoded in mathematics that it seems impossible to find in the modern mathematical literature—not even as an informal visualization.

The scalar modes are known to be solutions to certain differential equations in the complex plane (so-called Picard–Fuchs equations). This fact makes it possible to forego the explicit evaluation of the period integrals 3.5 and instead retreat to studying differential equations. This has been the

[2] I apologize to any reader who finds this term objectionable.

[3] To be precise I should remark that one really has to use two complex spheres—alas, the description of the construction process becomes less vivid when one is more precise.

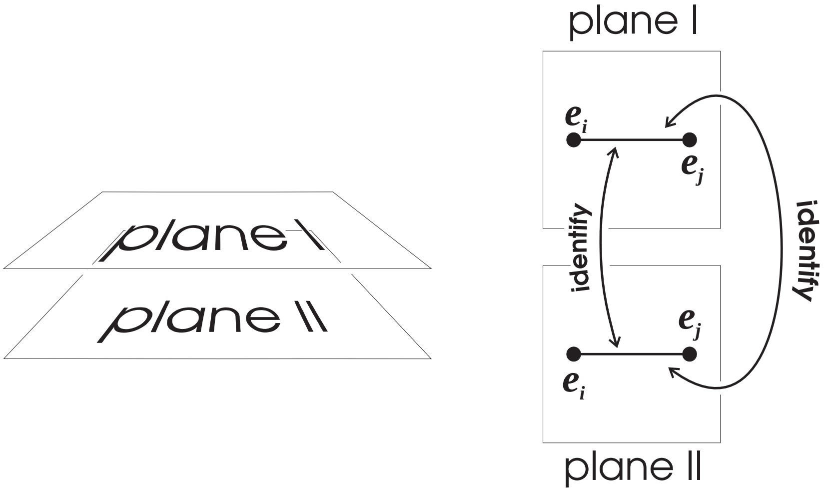

Fig. 3.3: An illustration of the identifying process explained in the text.

method of most publications on the subject (one notable exception is [15]). If I am allowed an opinionated remark here, I must say that I find this method of solution rather cumbersome for two reasons: It is hard work inasmuch the method quickly gets out of hand and it fails to yield a general solution since one has a new problem to solve for each N.

In a way the last remark has been preparatory for the following sales talk about the method discovered by myself and M. Flohr, to be explained in a later chapter: Our approach is to tackle the scalar modes head-on using a class of special functions known as *Lauricella functions of type D*. This is rather convenient because it gives general formulas for the scalar modes once and for all—the fact that not very much is known about these functions is a different story.

Nevertheless, using our method, we have been able to reproduce some standard, known results, and we could ask a rather fundamental question about the interpretation of the scalar modes everywhere in moduli space. At any rate, as will be seen later, the study of the scalar modes reduces to studying analytic continuations of these special functions.

One instance of the differential equations approach to the scalar modes can be found in [24].

3.2 Adding extra matter

In this section I will consider adding extra matter, so-called massive hypermultiplets, to the situation of the previous section.

Now, the changes introduced by this are comparatively minor. The spectral curve, as well as the differential is changed. Furthermore one also has to enlarge the homology basis, cf. below. Otherwise everything stays the same.

If we denote the gauge group by $SU(N_c)$ and there are present N_f hypermultiplets with masses m_r, then the spectral curve reads

$$\begin{aligned} y^2 &= A(x)^2 - B(x) \\ &:= \left(x^{N_c} - \sum_{k=2}^{N_c} u_k x^{N_c-k}\right)^2 - \Lambda^{2N_c-N_f} \prod_{r=1}^{N_f}(x-m_r) = \prod_{i=1}^{2N_c}(x-e_i), \end{aligned} \tag{3.6}$$

and the Seiberg–Witten differential on this curve is

$$\lambda_{\mathrm{SW}} = \frac{1}{2\pi \mathrm{i}} \frac{\prod_{\ell=0}^{N_c+N_f-1}(x-z_\ell)}{\prod_{i=1}^{2N}\sqrt{x-e_i}\,\prod_{j=1}^{N_f}(x-m_j)}\,\mathrm{d}x, \tag{3.7}$$

where the z_ℓ ($\ell > 0$) now denote the zeros of $2A'(x)B(x) - A(x)B'(x)$, and again $z_0 = 0$.

The masses of BPS states of the theory involve its scalar modes and these are given by

$$a_i = \int_{\alpha_i} \lambda_{\mathrm{SW}}, \quad a_D^i = \int_{\beta^i} \lambda_{\mathrm{SW}}, \tag{3.8}$$

for (dual) cycles α_i, β^i of an homology basis for the curve 3.6. We also have to enlarge the homology by cycles round the poles of the Seiberg–Witten differential. The corresponding scalar modes are suppressed here, since their evaluation reduces to the calculation of residues.

4. Explicit formulas for the scalar modes

In this chapter I derive explicit formulas for the scalar modes which were introduced in the last chapter.[1] I had previously remarked that the traditional approach to this is solving Picard–Fuchs equations for the scalar modes. This method is rather cumbersome and fails, to the best of my knowledge, to yield truly general formulas, i. e. ones that are valid for any $SU(N)$, let alone an arbitrary number of massive hypermultiplets.

The formulas to be derived involve the function *Lauricella* $F_D^{(n)}$, a function of n complex variables. Parts of the theory of this function are given in the Appendix.

Perhaps the first time Lauricella D made an appearance in connection with Seiberg–Witten theory was in the paper [15] by M. Flohr. Flohr interpreted some results in Seiberg–Witten theory in terms of CFT and derived formulas for the scalar modes in the cases $SU(2)$ and $SU(3)$ in terms of Lauricella D. Due to the nature of the approach via CFT the resulting formulas had less 'degrees of freedom' one might have wished for. By this I mean the fact that some of the branch points were *fixed*, with reference to conformal invariance, to 0, 1, and ∞. For some applications (e. g. the Argyres–Douglas point, to be considered later) this was inappropriately inconvenient.

Following up on this work I took a rather different route and tried to calculate the period integrals for the scalar modes directly without any appeal to the methods of CFT. The resulting formulas had enough 'degrees of freedom,' namely no branch points were fixed. This fact made convenient another point of view on what kind of functions the scalar modes are: instead of thinking of them as functions of the moduli one could now interpret them as functions of the branch points. Since the spectral curves are completely determined by the branch points this is quite appealing. The moduli enter the spectral curve in a rather complicated fashion. This is not true of the branch points which are just the zeroes of y^2. Also, in string-theoretic

[1] By this I mean, of course, formulas which facilitate the study of these functions on moduli space.

derivations of Seiberg–Witten models the spectral curve appears through intersecting branes and so there the branch points are the ‘primordial objects.’ For an introduction see [27].

The first section gives the results in the case without massive hypermultiplets, whereas the second section adds massive hypermultiplets.

4.1 The formulas: case without massive hypermultiplets

Let us evaluate the period integrals 3.5 in the case without massive hypermultiplets. We choose a homology basis $B = \{\alpha_i, \beta^i\}$ for our curve 3.3, such that for any $\gamma \in B$ we have

$$\int_\gamma \lambda_{\mathrm{SW}} = 2 \int_{e_i}^{e_j} \lambda_{\mathrm{SW}}, \tag{4.1}$$

for some branch points e_i, e_j.

Geometrically speaking, the existence of such a basis means that if $\gamma \in B$ encircles e_i and e_j, then no other e lies on the straight line connecting those two points, so that the original contour integral can be converted into twice the integral along that line (this last statement is a logical implication of the nature of the contour, i. e. if the basis is as described, the contour integral can always be converted to a line integral).

However, for some configurations of the e's no such basis might exist and therefore, unless otherwise stated, we shall explicitly *assume* its existence, treating it as an *hypothesis* for what follows.

As to the promised formula we have

$$
\begin{aligned}
&a_i \quad \text{or} \quad a_D^i \\
&= \int_\gamma \lambda_{\mathrm{SW}} = 2\int_{e_i}^{e_j} \lambda_{\mathrm{SW}} \\
&= \frac{1}{\pi \mathrm{i}} \int_{e_i}^{e_j} \prod_{k=0}^{N-1}(x-z_k) \prod_{\ell=1}^{2N}(x-e_\ell)^{-\frac{1}{2}}\,\mathrm{d}x \\
&= \frac{1}{\pi \mathrm{i}}(e_j-e_i) \int_0^1 \prod_{k=0}^{N-1}\big(e_i-(e_i-e_j)t-z_k\big) \prod_{\ell=1}^{2N}\big(e_i-(e_i-e_j)t-e_\ell\big)^{-\frac{1}{2}}\,\mathrm{d}t \\
&= \frac{1}{\pi \mathrm{i}}(e_j-e_i) \prod_{k=0}^{N-1}(e_i-z_k)(e_i-e_j)^{-\frac{1}{2}}(e_j-e_i)^{-\frac{1}{2}} \prod_{\substack{\ell=1\\ \ell\neq i,j}}^{2N}(e_i-e_j)^{-\frac{1}{2}} \times \\
&\quad \times \int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}} \prod_{k=0}^{N-1}\left(1-t\frac{e_i-e_j}{e_i-z_\ell}\right) \prod_{\substack{\ell=1\\ \ell\neq i,j}}^{2N}\left(1-t\frac{e_i-e_j}{e_i-e_\ell}\right)^{-\frac{1}{2}}\,\mathrm{d}t \\
&= (e_i-e_j)^{\frac{1}{2}} \prod_{k=0}^{N-1}(e_i-z_k) \prod_{\substack{\ell=1\\ \ell\neq i,j}}^{2N}(e_i-e_\ell)^{-\frac{1}{2}} \times \\
&\quad \times F_D^{(3N-2)}\Big(\frac{1}{2}, \underbrace{-1,\dots,-1}_{N\,\text{parameters}}, \underbrace{\frac{1}{2},\dots,\frac{1}{2}}_{\substack{2N-2\\ \text{parameters}}}; 1; \underbrace{\left\{\frac{e_i-e_j}{e_i-z_k}\right\}_k}_{N\,\text{variables}}, \underbrace{\left\{\frac{e_i-e_j}{e_i-e_\ell}\right\}_{\ell\neq i,j}}_{2N-2\,\text{variables}}\Big),
\end{aligned}
\tag{4.2}
$$

where in the last line we have made use of the integral identity A.2. The function $F_D^{(n)}$ is the Lauricella function of type D, alluded to earlier. See the Appendix for details.

I must admit that the preceding calculation looks haphazard, but it is easily explained. The cycle γ encircles two branch points, e_i, e_j, say. The second equality results from the hypothesis concerning our homology basis. The fourth equality results from the standard parametrization of the line segment $[e_i, e_j]$ and from there on follow only manipulations and identities.

I believe it is worthwile emphasizing that we have obtained an expression for the scalar modes almost without effort,[2] in particular *without studying Picard–Fuchs equations.* Moreover, the 'dimension' of the gauge group

[2] Of course, this is an over-exaggeration because the workload has been shifted to the identification of a certain integral as some Lauricella function. Still, it looks as if trouble is not conserved since we only have to go through this calculation *exactly once*, in contrast to the usual 'Picard–Fuchs game.'

$SU(N)$ enters the equations as a parameter. The problem of determining the scalar modes *is solved* once and for all in an explicit fashion.

It is clear that one could now, by conformal invariance, fix some of the e's, sending the others to certain crossing ratios. For the reasons explained in the introduction I do not wish to do so. But I could.[3]

4.2 The formulas: case with massive hypermultiplets

The calculation of the scalar modes in the presence of massive hypermultiplets works analogously to the case without massive hypermultiplets. Therefore I will omit it and only state the result. In any case, the calculation is best carried out by inspecting the individual steps in the case without massive hypermultiplets.

Let B be a homology basis for the curve 3.6 of the same type as the one for the curve 3.3 described in the last section 4.1. In addition to the properties stated there we now also have to make sure that if $\gamma \in B$ encircles e_i, e_j no mass m_r lies on the straight line connecting e_i and e_j.

Let $\gamma \in B$. The cycle γ encircles two branch points, e_i and e_j, say. We have

$$\begin{aligned} a_i \quad \text{or} \quad a_D^i & \\ = (e_i - e_j)^{\frac{1}{2}} & \prod_{k=0}^{N_c+N_f-1} (e_i - z_k) \prod_{\substack{\ell=1 \\ \ell \neq i,j}}^{2N} (e_i - e_\ell)^{-\frac{1}{2}} \prod_{r=1}^{N_f} (e_i - m_r)^{-1} \times \\ \times F_D^{(3N_c+2N_f-2)} & \Big(\frac{1}{2}, \underbrace{-1, \ldots, -1}_{N_c+N_f \text{ parameters}}, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{\substack{2N_c-2 \\ \text{parameters}}}, \underbrace{1, \ldots, 1}_{N_f \text{ parameters}}; 1; \\ & \underbrace{\left\{ \frac{e_i - e_j}{e_i - z_k} \right\}_k}_{N_c+N_f \text{ variables}}, \underbrace{\left\{ \frac{e_i - e_j}{e_i - e_\ell} \right\}_{\ell \neq i,j}}_{2N_c-2 \text{ variables}}, \underbrace{\left\{ \frac{e_i - e_j}{e_i - m_r} \right\}}_{N_f \text{ variables}} \Big). \end{aligned} \tag{4.3}$$

This is the result for the case with massive hypermultiplets.

As a reminder to the symbols in the last equation: the gauge group is $SU(\underline{N_c})$ and there are present $\overline{N_f}$ massive hypermultiplets.

In keeping with our remark in section 3.2 we have suppressed the scalar modes corresponding to cycles round poles of the Seiberg–Witten differential.

[3] ;-)

4.3 A more geometric viewpoint

In the introduction to this chapter I had written something to the effect that equation 4.2 afforded the interpretation of the scalar modes as functions of the branch points alone. Inspection of this equation shows that still another viewpoint is possible whose emphasis is on the 'geometry' of the moduli space.

To see this clearly we make the following definitions

$$\zeta := e_i - e_j, \tag{4.4}$$

$$\xi_\nu := \frac{e_i - e_j}{e_i - z_{\nu-1}}, \quad \nu \in \{1, \ldots, N\}; \tag{4.5}$$

$$x_\nu := \frac{e_i - e_j}{e_i - e_\nu}, \quad \nu \in \{1, \ldots, i-1, i+1, \ldots, 2N\}. \tag{4.6}$$

Note that x_i is *not defined.*

Furthermore, let

$$b_\xi := \underbrace{(-1, \ldots, -1)}_{N\,\text{components}}, \tag{4.7}$$

$$b_x := \underbrace{(\frac{1}{2}, \ldots, \frac{1}{2})}_{2N-2\,\text{components}}. \tag{4.8}$$

Then the result 4.2 reads

$$a_i \quad \text{or} \quad a_D^i$$

$$= \zeta \prod_{\nu=1}^{N} \xi_\nu^{-1} \prod_{\substack{\nu=1 \\ \nu \neq i}}^{2N} x_\nu^{\frac{1}{2}} \; F_D^{(3N-2)}\big(\frac{1}{2}, b_\xi, b_x; 1; \{\xi_\nu\}_{\nu=1}^{N}, x_1, \ldots, \widehat{x_j}, \ldots, x_{2N}\big), \tag{4.9}$$

where $\widehat{\;}$ denotes omission. We remark that in total there are $3N-2$ 'x'-arguments to the F_D, x_i being undefined and x_j being omitted.

The promised shift in emphasis takes place if we now consider a_i and a_D^i as functions solely of ζ and the ξ's and x's, the domain of these 'new' functions being the subset of $\mathbb{C} \times \mathbb{C}^N \times \mathbb{C}^{2N-2}$ which results if we regard ζ and the ξ's and x's as functions of the u's—the vacuum expectation values—and let the latter vary freely over the whole of $\mathbb{C}^{N-1}$.

Unfortunately it turns out that the described domain is rather complicated. Nevertheless, I believe that this viewpoint sheds some additional light on the structure of the moduli space.

5. Applications

In this chapter I discuss applications of the formulas for the scalar modes which were derived in the last chapter. The first section is on the $u \to \infty$ case in the $SU(2)$ setting and the second section discusses the Argyres–Douglas point in the $SU(3)$ setting.

5.1 $SU(2)$

As a test on the validity of equation 4.2 for the scalar modes, I would like to show that it gives the correct asymptotic behavior 2.28 and 2.29 for the scalar modes as $u \to \infty$ when the gauge group is $SU(2)$ (no massive hypermultiplets present).

Let us first gather the data we have for this case. Referring to the curve 3.3, here $N = 2$ and there is exactly one vev u.

The z's (cf. the line below eq. 3.4) are

$$z_0 = 0, \quad z_1 = 0 \tag{5.1}$$

and the branch points are

$$e_1 = \sqrt{u - \Lambda^2}, \tag{5.2}$$
$$e_2 = -\sqrt{u + \Lambda^2}, \tag{5.3}$$
$$e_3 = -\sqrt{u - \Lambda^2}, \tag{5.4}$$
$$e_4 = \sqrt{u + \Lambda^2}. \tag{5.5}$$

Taking this into account, we have the situation depicted in figure 5.1 where I have also indicated the homology basis I will choose for the following.

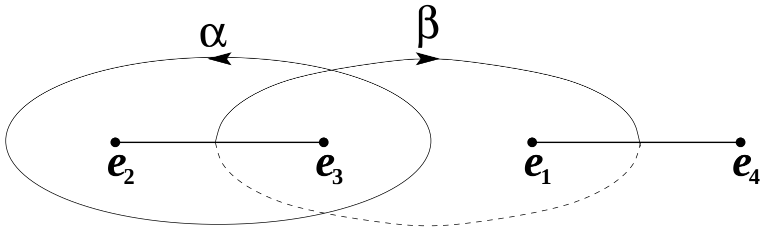

Fig. 5.1: Homology basis for the $SU(2)$ case

Therefore, using eq. 4.2 for the scalar modes,

$$\begin{aligned} a(u) &= \int_\alpha \lambda_{\mathrm{SW}} = -2\int_{e_2}^{e_3} \lambda_{\mathrm{SW}} \\ &= -\frac{e_2^2}{(e_2-e_1)^{\frac{1}{2}}(e_2-e_4)^{\frac{1}{2}}} F_D^{(4)}\Big(\dots; \frac{e_2-e_3}{e_2-0}, \frac{e_2-e_3}{e_2-0}, \frac{e_2-e_3}{e_2-e_1}, \frac{e_2-e_3}{e_2-e_4}\Big) \\ &\sim \frac{1}{\sqrt{8}}\sqrt{2u}\,, \quad u \to \infty, \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} a_D(u) &= \int_\beta \lambda_{\mathrm{SW}} = -2\int_{e_3}^{e_1} \lambda_{\mathrm{SW}} \\ &= -\frac{e_3^2}{(e_3-e_2)^{\frac{1}{2}}(e_3-e_4)^{\frac{1}{2}}} F_D^{(4)}\Big(\dots; \frac{e_3-e_1}{e_3-0}, \frac{e_3-e_1}{e_3-0}, \frac{e_3-e_1}{e_3-e_2}, \frac{e_3-e_1}{e_3-e_4}\Big) \\ &\sim -\frac{1}{\pi}\frac{e_3^2}{(e_1-e_3)^{\frac{1}{2}}(e_3-e_4)^{\frac{1}{2}}} \log\left(\frac{e_1-e_3}{e_3-e_2}\right), \quad u \to \infty \\ &\sim \frac{1}{\sqrt{8}}\frac{\mathrm{i}}{\pi}\sqrt{2u}\,\log u, \quad u \to \infty, \end{aligned} \tag{5.7}$$

where for $a_D(u)$ we have made use of equation A.11 in the appendix.

Comparison with 2.28 and 2.29 reveals that in our results there is an extraneous factor of $\frac{1}{\sqrt{8}}$. This does *not* indicate the falsity of formula 4.2 for the scalar modes since any *overall* prefactor can be absorbed in a redefinition of the Seiberg–Witten differential or by sticking in an appropriate factor in the BPS mass formula. The literatur is pervaded by inconsistencies and different choices with regards to the normalization of the Seiberg–Witten differential and the scalar modes.

The rigorous minded reader might want to consider the following paragraph, where I explain in a little more detail how I applied the continuation formula A.11 to obtain the asymptotic behavior of $a_D(u)$.

Details of previous derivation. For the following it is useful to write out in full the F_D in formula 5.7:

$$F_D^{(4)}\Big(\frac{1}{2}, -1, -1, \frac{1}{2}, \frac{1}{2}; 1; \frac{e_3 - e_1}{e_3 - 0}, \frac{e_3 - e_1}{e_3 - 0}, \frac{e_3 - e_1}{e_3 - e_2}, \frac{e_3 - e_1}{e_3 - e_4}\Big). \tag{5.8}$$

Since the first two arguments are identical and equal to 2 this reduces to

$$F_D^{(3)}\Big(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; 2, \frac{e_3 - e_1}{e_3 - e_2}, \frac{e_3 - e_1}{e_3 - e_4}\Big), \tag{5.9}$$

as is easily seen from the power series representation A.1 for Lauricella $F_D^{(n)}$. Using the power series even though $\frac{e_3-e_1}{e_3-0} \equiv 2$ is legal, since the corresponding parameters are negative integers so that this part of the series terminates after a finite number of terms (see the point just below eq. A.4), giving something polynomial in the argument.

Following up on the remark 'something polynomial' we obtain

$$\begin{aligned} F_D^{(3)}\Big(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; \mathbf{2}, \frac{e_3 - e_1}{e_3 - e_2}, \frac{e_3 - e_1}{e_3 - e_4}\Big) & \\ = F_D^{(2)}\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1; \frac{e_3 - e_1}{e_3 - e_2}, \frac{e_3 - e_1}{e_3 - e_4}\Big) & \\ -\mathbf{2}F_D^{(2)}\Big(\frac{1}{2}+1, \frac{1}{2}, \frac{1}{2}; 1+1; \frac{e_3 - e_1}{e_3 - e_2}, \frac{e_3 - e_1}{e_3 - e_4}\Big) & \\ +\frac{3}{8}\mathbf{2}^2 F_D^{(2)}\Big(\frac{1}{2}+2, \frac{1}{2}, \frac{1}{2}; 1+2; \frac{e_3 - e_1}{e_3 - e_2}, \frac{e_3 - e_1}{e_3 - e_4}\Big). & \end{aligned} \tag{5.10}$$

Now, in the limit $u \to \infty$, $|\frac{e_3-e_1}{e_3-e_2}| \to \infty$ and $|\frac{e_3-e_1}{e_3-e_4}| \nearrow 1$, the divergence being faster than the convergence. Therefore we can employ the analytic continuation formula A.11. Doing so, one easily verifies that the first asymptotic in 5.7 is correct. The expression in the fourth line is asymptotically equal to the one before and since the relation $\sim$ is transitive everything is in perfect order.

5.2 $SU(3)$, Argyres–Douglas' $\mathbb{Z}_3$-point

In this section I apply our main result eq. 4.2 to the $\mathbb{Z}_3$-point discovered by Argyres and Douglas (cf. [2]). For more references see [1].

Roughly speaking, the Argyres–Douglas point is interesting because it provides us with an example of a theory in which the BPS spectrum contains a pair of dual particles, i. e. particles where one is electrically and the other magnetically charged, in which both particles simultaneously become massless.

5.2.1 Vanishing of Scalar Modes

The $\mathbb{Z}_3$-point is the e-configuration in the $SU(3)$ case determined by the choice of vev's $u = 0$, $v = \Lambda^3$ (for completeness, we mention that there exist other choices which also lead to '$\mathbb{Z}_3$-points'). Our aim is to apply our formula 4.2 to this case.

Therefore, following Argyres and Douglas, we write

$$u = \delta u, \quad v = \Lambda^3 + \delta v. \tag{5.11}$$

We shall restrict ourselves to (real) $\delta u < 0$ and (real) $\delta v > 0$. We remark on these hypotheses below.

Referring to the curve 3.3, the e's (branch points) are the zeros of

$$\big(x^3 - \delta u\, x - (\Lambda^3 + \delta v)\big)^2 - \Lambda^6. \tag{5.12}$$

They are easily seen to be separated into two classes: Those which are the zeros of $p_1(x) = x^3 - \delta u\, x - \delta v$ and those which are the zeros of $p_2(x) = x^3 - \delta u\, x - \delta v - 2\Lambda^3$.

Put $P = \mathrm{sgn}(-\delta v/2)\,\sqrt{|-\delta u/3|}$ and $\beta = \frac{1}{3}\mathrm{arsinh}\frac{-\delta v/2}{P^3}$.

Then, under our assumptions on the δ's, the zeros of p_1 are (see e. g. [39])

$$\begin{aligned} e_1 &= -2P\sinh\beta, \\ e_2 &= P(\sinh\beta - \mathrm{i}\sqrt{3}\cosh\beta), \\ e_3 &= P(\sinh\beta + \mathrm{i}\sqrt{3}\cosh\beta). \end{aligned} \tag{5.13}$$

We shall not need explicit formulas for the zeros of p_2. It will suffice to know that whenever the δ's are small, the zeros of p_2 (call them e_4,e_5,e_6) will be near the third roots of $2\Lambda^3$. The resulting configuration, along with our choice of homology basis, is visualized in Figure 5.2. Before writing down our expressions for the a's and a_D's we also need to know that the z's (recall the definition just below eq. 3.4) now are

$$z_0 = 0, \quad z_1 = \sqrt{\delta u/3}, \quad z_2 = -\sqrt{\delta u/3}. \tag{5.14}$$

In the study of the $\mathbb{Z}_3$-point one is mostly interested in a_1 and a_D^1 because it are these two dual quantities which simultaneously vanish at the $\mathbb{Z}_3$-point.

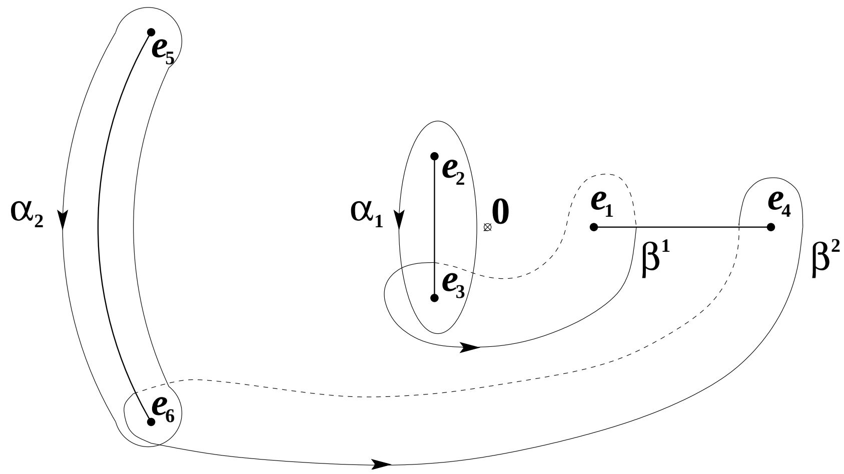

Fig. 5.2: Homology basis for $SU(3)$

This, of course, is related to the fact that as the δ's tend to 0 the cycles α_1 and β^1 contract to points because e_1,e_2,e_3 tend to 0.

As functions of δu and δv, i. e. near the Argyres–Douglas point, the scalar modes a_1, a_D^1 are given by (cf. eq. 4.2)

$$
\begin{aligned}
a_1 = 2\int_{e_2}^{e_3} \lambda_{\mathrm{SW}} = & \frac{e_2(e_2^2-\delta u/3)}{(e_2-e_1)^{\frac{1}{2}}(e_2-e_4)^{\frac{1}{2}}(e_2-e_5)^{\frac{1}{2}}(e_2-e_6)^{\frac{1}{2}}} \times \\
& \times F_D^{(7)}\Big(\ldots; \frac{e_2-e_3}{e_2-0}, \frac{e_2-e_3}{e_2-\sqrt{\delta u/3}}, \frac{e_2-e_3}{e_2+\sqrt{\delta u/3}}, \\
& \frac{e_2-e_3}{e_2-e_1}, \frac{e_2-e_3}{e_2-e_4}, \frac{e_2-e_3}{e_2-e_5}, \frac{e_2-e_3}{e_2-e_6}\Big), \qquad (5.15)
\end{aligned}
$$

and

$$
\begin{aligned}
a_D^1 = 2\int_{e_3}^{e_1} \lambda_{\mathrm{SW}} = & \frac{e_3(e_3^2-\delta u/3)}{(e_3-e_2)^{\frac{1}{2}}(e_3-e_4)^{\frac{1}{2}}(e_3-e_5)^{\frac{1}{2}}(e_3-e_6)^{\frac{1}{2}}} \times \\
& \times F_D^{(7)}\Big(\ldots; \frac{e_3-e_1}{e_3-0}, \frac{e_3-e_1}{e_3-\sqrt{\delta u/3}}, \frac{e_3-e_1}{e_3+\sqrt{\delta u/3}}, \\
& \frac{e_3-e_1}{e_3-e_2}, \frac{e_3-e_1}{e_3-e_4}, \frac{e_3-e_1}{e_3-e_5}, \frac{e_3-e_1}{e_3-e_6}\Big). \qquad (5.16)
\end{aligned}
$$

The next thing we shall do is examine the behavior of a_1 and a_D^1 as the δ's both tend to 0. There are several meanings one can attach to the phrase 'tend to 0.' One possible meaning is the usual one from the calculus of several (complex) variables, where both variables are considered as *independent* of each other. Another possible meaning, and this is often done in physics, is to consider both variables as *dependent*, say $\delta v = -\delta u$ (recall our assumptions about the δ's). We will follow the latter course (cf. our general remarks below). This ensures that both variables will be of the same order of smallness during the limiting process.

In figure 5.3 I have drawn how the inner branch points e_1, e_2, e_3 will move as $\delta u \to 0$ through negative values.

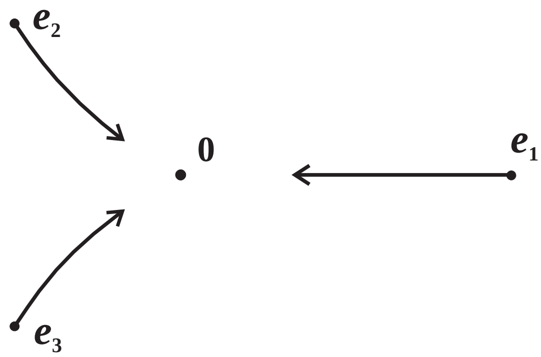

Fig. 5.3: Motion of the inner e's as $\delta u \to 0$ through negative values

Let us now examine the behavior of the scalar modes as $\delta u \to 0$ through negative values.

We first consider a_1: As $\delta u \to 0$, $\delta u < 0$, the arguments to the F_D in eq. 5.15 behave as follows.

$$\begin{aligned}
\frac{e_2 - e_3}{e_2 - 0} &\to \frac{1}{2}(3 - \mathrm{i}\sqrt{3}), \\
\frac{e_2 - e_3}{e_2 - \sqrt{\delta u/3}} &\to \frac{1}{2}(3 - \mathrm{i}\sqrt{3}), \\
\frac{e_2 - e_3}{e_2 + \sqrt{\delta u/3}} &\to \frac{1}{2}(3 - \mathrm{i}\sqrt{3}), \\
\frac{e_2 - e_3}{e_2 - e_1} &\to \frac{1}{2}(1 - \mathrm{i}\sqrt{3}), \\
\frac{e_2 - e_3}{e_2 - e_4} &\to 0, \\
\frac{e_2 - e_3}{e_2 - e_5} &\to 0, \\
\frac{e_2 - e_3}{e_2 - e_6} &\to 0.
\end{aligned} \tag{5.17}$$

Furthermore, *the prefactor of the* F_D *in eq. 5.15 tends to* 0. These limits were obtained using MATHEMATICA. We will now prove that the value of the F_D at the limits just written down is a complex number (i. e. that it is defined for that particular constellation of arguments).

First observe that

$$F_D^{(7)}(\ldots, 0, 0, 0) = F_D^{(4)}(\ldots), \tag{5.18}$$

and

$$F_D^{(4)}\Big(\frac{1}{2}, -1, -1, -1, \frac{1}{2}; 1; \frac{1}{2}(3 - \mathrm{i}\sqrt{3}), \frac{1}{2}(3 - \mathrm{i}\sqrt{3}), \frac{1}{2}(3 - \mathrm{i}\sqrt{3}), \frac{1}{2}(1 - \mathrm{i}\sqrt{3})\Big)$$
$$= F_D^{(2)}\Big(\frac{1}{2}, -3, \frac{1}{2}; 1; \frac{1}{2}(3 - \mathrm{i}\sqrt{3}), \frac{1}{2}(1 - \mathrm{i}\sqrt{3})\Big). \tag{5.19}$$

Since -3 is a negative integer this last F_D reduces to a polynomial in $\frac{1}{2}(3 - \mathrm{i}\sqrt{3})$ (see Appendix A), the coefficients being rational numbers multiplied by expressions of the form ${}_2F_1\left(\frac{1}{2} + n, \frac{1}{2}; 1 + n; \frac{1}{2}(1 - \mathrm{i}\sqrt{3})\right)$, where n is some nonnegative integer (${}_2F_1$ is the Gaussian hypergeometric function). Since for our constellation of parameters the only singularity of ${}_2F_1$ on the unit circle is at 1, these 'expressions' reduce to (finite) complex numbers. Thus, as promised, the F_D in eq. 5.15 converges at the limits of its arguments.

Therefore (recall that the prefactor tends to 0*),* a_1 *tends to* 0*, as* $\delta u \to 0$*,* $\delta u < 0$.

We now consider a_D^1: Referring to eq. 5.16 one finds that the arguments of the F_D in that equation, taken in the same order as those of the F_D in eq. 5.15, tend to the same limits as indicated in eq. 5.17. Following exactly the same arguments as for a_1 we find that the F_D in the expression for a_D^1 tends to the same (finite) complex number as the one in eq. 5.15. Also, the prefactor in eq. 5.16 tends to 0 as $\delta u \to 0$, $\delta u < 0$.

Therefore, a_D^1 *tends to* 0 *as* $\delta u \to 0$*,* $\delta u < 0$.

This vanishing of a_1 and a_D^1 reproduces the result stated by Argyres and Douglas [2] in the special limit (recall our restrictions on the δ's) we have considered.

General Discussion

The reader might ask why in the previous section we imposed so peculiar restrictions on δu and δv. In the present section, we wish to address precisely this question.

Partly the answer is simple: We imposed these conditions so that we were able to consider the limit of the scalar modes at the Argyres–Douglas point in a straight-forward manner. For instance, we know for sure that it is possible to consider the case $\delta u > 0$, $\delta u = \delta v$, $\delta u \to 0$,[1] and the analysis of this is exactly the same as in the case we have presented in detail (even the arguments of the F_D's tend to the same limits). However, there is a complication involved. Now one has to look at the discriminant of p_1; if it is 0, a degeneration of the 'inner' e's (cf. Fig. 5.2) takes place ($e_2 = e_3$, moreover, $e_2 = -\sqrt{-\delta u/3}$), and if it is negative, the homology basis of Fig. 5.2 is inadequate for our formulas because then all 'inner' e's are real. One does away with the trouble of the discriminant in this case by setting $\delta u = \delta v$ (this implies positivity of the discriminant). The moral is that it is possible to impose different restrictions on the δ's than we have done in this paper.

In the general case of arbitrary complex $\delta u, \delta v$ with an arbitrary approach to 0 (as in standard calculus) the complications pile up: Firstly, there is the technical problem of keeping track of third roots of complex numbers. This is very difficult, since the expressions involved in the usual Cardano's formula are intrinsically discontinuous due to branch cuts; also, multivaluedness becomes a burden. Secondly, there is the problem that one cannot choose a homology basis once and for all. For instance, if $\delta v > 0$, $\delta u < 0$ the configuration of the 'inner' e's looks like that of those in Fig. 5.2, only reflected through the origin. Of course this necessitates the use of a different homology basis than the one shown in Figure 5.2. If the δ's can arbitrarily approach 0, then it is clear that one could start with a homology basis like the one in Fig. 5.2 and end up with the need to chose a different one. Our formulas are not suited for such a case. Besides, as remarked in section 3.1 choosing a homology basis means choosing what to call electric and magnetic charge respectively. One can imagine that it is very difficult to switch homology bases in a consistent way.

The question arises: Do a_1,a_D^1 vanish at the Argyres–Douglas point regardless of exactly how the δ's tend to 0? If they should happen to be continuous functions the answer is affirmative: Yes! However, we do not deem their continuity as self-evident, since the e's depend on the δ's in a rather complicated fashion. On some Riemann-surface a_1 and a_D^1 most likely *are* continuous. But that seems to be something different than what one usually thinks of.

It is tautological to say that if $a_1(\delta u, \delta v)$, $a_D^1(\delta u, \delta v)$ should happen to be discontinuous at $(0, 0)$, then there would exist some approach of the δ's to 0 which would not yield a vanishing limit.

Even if the scalar modes should turn out to be continuous it is quite puzzling to observe that in the special case we have considered the Lauri-

[1] Again, a_1 and a_D^1 vanish.

cella functions involved in our formulas tend to the same values, so that the vanishing of the scalar modes is determined by the vanishing of the prefactors. What is puzzling about this is that it looks as if a_1, a_D^1 become linearly dependent at the Argyres–Douglas point.

In light of the theory of Lauricella functions one might make the general conjecture that the set of BPS periods alone is not exhaustive to define the scalar modes everywhere in moduli space. This conjecture is mainly supported by the fact, explained in the Appendix, that the set of Lauricella functions does not close under analytic continuation (which is in marked contrast to the ordinary Gaussian hypergeometric function, cf. [5]). Roughly, what is meant by this is that one starts out with the definition of the period integrals, given as eq. 3.5. One evaluates this in the case of some 'generic' homology basis (by this is meant some 'nice' homology basis; e. g. the one shown in 5.2). Then the scalar modes are defined everywhere in moduli space by analytic continuation. It turns out that some analytic continuations cannot be interpreted anymore as simple period integrals. Rather, one needs integrals along more complicated contours (so-called 'three-foil loops'). For more details along these lines see [1].

It is worthwhile to repeat the last two questions which have arisen from the explicit formulas for the scalar modes: Are the scalar modes continuous functions of the moduli? Can the scalar modes always be interpreted as BPS periods?

These are, perhaps, very difficult questions to answer (the second it seems is even difficult to understand). Therefore, at the time of writing, I do not have an answer to them (nor do I believe anyone has). Still I would like to stress that these questions are novel and, I think, of not entirely negligible interest.

6. Discussion

In this thesis, explicit formulas for the scalar modes in the Seiberg–Witten $SU(N)$ setting in the cases with and without extra matter (massive hypermultiplets) were derived. These formulas were successfully applied in a derivation of the asymptotics for the scalar modes in the asymptotically free region ($u \to \infty$) of moduli space in the $SU(2)$ case and they were applied in a study of the Argyres–Douglas point in the $SU(3)$ case.

The last of these applications raised questions about the continuity of the scalar modes as functions of the moduli and also about their interpretation in certain limiting cases (namely, at the Argyres–Douglas point).

The appearance of these questions was intimately related to the class of special functions termed Lauricella functions of type D (Lauricella $F_D^{(n)}$) and their properties under analytic continuation. Although this discussion precedes the Appendix where this function[1] is discussed, by custom I must include it here. In order to make progress in the direction of answering these questions (indeed, to even understand that there are questions to be answered), analytic continuation formulas for Lauricella $F_D^{(n)}$ were derived which seemed to be missing from the literature. Generally speaking, the theory of Lauricella functions is rather incomplete. For instance, there is lacking a general theory concerning analytic continuations of Lauricella functions. This means that the continuation formulas given in this thesis could not be obtained from general results but had to be derived 'from scratch.'

The explicit formulas for the scalar modes appear to be a quite powerful tool in Seiberg–Witten theory. Firstly, they render unnecessary finding and solving Picard–Fuchs equations (or going through other procedures) every time one considers a new gauge group—they hold for any $SU(N)$ (moreover, see the next paragraph). Secondly, they have proven to be of considerable help in addressing sharply questions in Seiberg–Witten theory which otherwise are difficult to put a finger on, as in the case of the Argyres–Douglas point. Also, they provide a general means of dealing with the case where massive hypermultiplets are added. By inspection of the formulas it is clear

[1] It is convenient to speak of Lauricella $F_D^{(n)}$ as a single function although it is clearly a *family* of functions (parametrized by $n \in \mathbb{N}_{>0}$).

that this case can be handled in the same framework as the case without massive hypermultiplets, as the formulas have *very* closely related forms. In short, it all comes down to investigating properties (mainly under analytic continuation) of Lauricella functions.

There are several interesting questions one might ask which point to possible future investigations. There is the question what the interpretation of the scalar modes in all of moduli space (cf. the discussion at the end of chapter 5) will turn out to be. Then there is the question whether generalizations to other simply laced gauge groups of the formulas given in this thesis might yield interesting results—I am absolutely sure (in fact, I know) that such generalizations are possible. And, on a technical side, it would be interesting to know whether the given continuation formulas for Lauricella $F_D^{(n)}$ also have 'nicer' looking forms—generally, it would be most welcome to see a more complete theory of this class of special functions. As mentioned above, studying the scalar modes for simply laced gauge groups can—by virtue of the explicit formulas and generalizations thereof—always be reduced to studying Lauricella functions.

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Appendix

A. Lauricella $F_D^{(n)}$

The purpose of this appendix is to collect various information pertaining to Lauricella $F_D^{(n)}$. The major reference for Lauricella $F_D^{(n)}$ and other 'multiple hypergeometric functions' is [13], from which we will cite freely.

A.1 The Definition

Lauricella $F_D^{(n)}$ is a function of n complex variables and $n+2$ parameters, defined by the power series

$$\begin{aligned} &F_D^{(n)}(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n) \\ &\qquad = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m1+\cdots+m_n} m_1! \cdots m_n!} \, x_1^{m_1} \cdots x_n^{m_n}, \end{aligned} \quad \text{(A.1)}$$

whenever $|x_1|, \ldots, |x_n| < 1$ and by analytic continuation elsewhere. The symbol $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the so-called Pochhammer symbol.

The function $F_D^{(n)}$ has the integral representation

$$\begin{aligned} &\int_0^1 t^{a-1}(1-t)^{c-a-1} \prod_{i=1}^{n} (1-tx_i)^{-b_i} \, \mathrm{d}t = \\ &\qquad \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} \, F_D^{(n)}(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n), \end{aligned} \quad \text{(A.2)}$$

if $\mathrm{Re}(a)$ and $\mathrm{Re}(c-a)$ are positive. This is proved using the binomial theorem. See [16], Appendix B for a detailed derivation.

A number of facts can be read off the power series representation A.1:

1. If one of the variables of $F_D^{(n)}$, say x_i, is equal to 0, then the $F_D^{(n)}$ reduces to a $F_D^{(n-1)}$:

$$\begin{aligned} &F_D^{(n)}(a, b_1, \ldots, b_i, \ldots, b_n; c; x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \\ &\qquad = F_D^{(n-1)}(a, b_1, \ldots, \widehat{b_i}, \ldots, b_n; c; x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \end{aligned} \quad \text{(A.3)}$$

where $\widehat{\ }$ denotes omission.

2. If two variables have equal values, say $x_i = x_j$, a similar reduction takes place:

$$\begin{aligned} F_D^{(n)}&(a, b_1, \dots, b_i, \dots, b_j, \dots, b_n; c; x_1, \dots, x_i, \dots, x_j, \dots, x_n) \\ &= F_D^{(n-1)}(a, b_1, \dots, b_{i-1}, b_i + b_j, b_{i+1}, \dots, \widehat{b_j}, \dots, b_n; c; \\ &\qquad x_1, \dots, x_i, \dots, \widehat{x_j}, \dots, x_n). \end{aligned} \quad (A.4)$$

3. If b_i is a negative integer, then the part of the series corresponding to x_i terminates after a finite number of terms (because $(b_i)_n = 0$ for $n > |b_i|$) and thus reduces to a polynomial in x_i. In this case, the modulus of x_i is immaterial for the validity of A.1.

If $|x_i| \geq 1$, for some $i \in \{1, \dots, n\}$, then the power series representation A.1 is not valid anymore. Rather, one must have recourse to an analytic continuation of $F_D^{(n)}$. This is effected by writing

$$\begin{aligned} F_D^{(n)}&(a, b_1, \dots, b_n; c; x_1, \dots, x_n) \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_{i-1}=0}^{\infty} \sum_{m_{i+1}=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_{i-1}+m_{i+1}+\dots+m_n} \prod_{\ell\neq i}(b_\ell)_{m_\ell}}{(c)_{m_1+\dots+m_{i-1}+m_{i+1}+\dots+m_n} \prod_{\ell\neq i} m_\ell!} \times \\ &\qquad \times \prod_{\ell\neq i} x_\ell^{m_\ell} {}_2F_1(a + \sum_{\ell\neq i} m_\ell, b_i; c + \sum_{\ell\neq i} m_\ell; x_i), \end{aligned} \quad (A.5)$$

and employing a suitable continuation formula for the Gaussian hypergeometric function ${}_2F_1$. Erdélyi et al. [12] as well as Becken and Schmelcher [5] give several such continuation formulas. Especially the latter reference appears to be quite exhaustive. Also, see the next section A.2.

A.2 Analytic continuation of Lauricella functions

The Seiberg–Witten periods are analytic functions ‘everywhere’ in the moduli space, i. e. for generic values of either the vacuum expectation values u_k or the branch points e_ℓ. However, it is clear that this is not necessarily the case in the singular regions where one or more branch points e_ℓ become identical. In fact, a typical feature of a dual pair (a, a_D) of Seiberg–Witten periods corresonding to a dual pair of homology cycles (α, β) is that with a vanishing cycle $\alpha \to 0$, only $a(u)$ becomes small, while the dual period $a_D(u)$ diverges logarithmically.[1]

[1] Such behavior can be seen, for instance, in the $u \to \infty$ case for gauge group $SU(2)$.

Exactly this does not happen at the Argyres–Douglas point and this is what makes it so interesting: a pair of dual periods *simultaneously* become small, i. e. two particles dual to each other under some sort of electromagnetic duality, simultaneously become massless.

There is another interesting fact about such points in moduli space, where intersecting homology cycles vanish at the same time. If one uses the branch points as the natural coordinates to parametrize the theory, it is known from the theory of generalized hypergeometric functions that a complete set of analytic continuations cannot be given entirely in terms of the same class of functions one starts out with (cf. [13]). This is to be contrasted with the well-known result that the ordinary Gaussian hypergeometric function admits analytic continuations everywhere in the complex plane, which again can be expressed as linear combinations of Gaussian hypergeometric functions (see, e. g. [5]). This can be used, for example, to find analytic continuation formulas for the Lauricella F_D functions, as long as we only need to continue one of its arguments outside the unit circle of convergence. As soon as we wish to have more than one argument outside the unit circle, things become complicated. In the case of the Lauricella F_D functions, one is confronted with the following problem:

Within its region of convergence, the Lauricella F_D series can be represented in the form of an Euler–type integral, i. e. an integral along a simple loop, which we choose to be one of the homology cycles. More generally, Pochhammer double loops might be admitted as well. The point is, that only two of the singular points of the differential are enclosed by the loop. The thus defined functions possess analytic continuations which, for generic values of the parameters, can again be given in terms of multi-variable power series (perhaps multiplied by a common fractional power). However, such analytic continuations are only valid outside the unit ball within cones delimited by the singular hyperplanes, given by coinciding singular points. Thus, one needs a considerably larger set of analytic continuations than in the single-variable case. Morevoer, not all of these analytic continuations can be represented by Euler–type integrals. As mentioned earlier, namely near the end of section 5.2.1, this raises the physically relevant question what the meaning of the Seiberg–Witten periods then is. As long as they can be understood as contour integrals around homology basis elements, they represent the mass of particles having charge fixed by the corresponding homology element. But what would the meaning be, if no such simple contour existed? Exton [13] mentions that at least so-called three-foil loops are necessary to be able to represent a full set of analytic continuations in terms of integrals. Three-foil loops are three times self-intersecting loops which enclose three different sets of singular points.

In order to study the analytic continuations of the Lauricella functions of type F_D, one needs a further class of related functions, defined by the expansions

$$D_{p,q}^{(n)}(a, b_1, \ldots, b_n; c, c'; x_1, \ldots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_{p+1}+\cdots+m_n-m1-\cdots-m_p}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_{q+1}+\cdots+m_n-m_1-\cdots-m_p}c'_{m_{p+1}+\cdots+m_q}m_1!\cdots m_n!} x_1^{m_1}\cdots x_n^{m_n}, \tag{A.6}$$

where $0 \leq p \leq q \leq n$. It is important to note that these functions, which appear in the analytic continuations of the Lauricella $F_D^{(n)}$ functions, do not possess Euler–type integral representations. The simplest known integral representation is in fact a Pochhammer double loop integral involving a Lauricella function in its kernel, namely

$$\begin{aligned}\frac{(2\pi\mathrm{i})^2}{\Gamma(a)\Gamma(a')\Gamma(2-a-a')} & D_{p,q}^{(n)}(a+a'-1, b1, \ldots, b_n; c, c'; x_1, \ldots, x_n) \\ = \int \mathrm{d}u\,(-u)^{-a'}(u-1)^{-a} & F_D^{q-p}(a', b_{p+1}, \ldots, b_q; c'; \frac{x_{p+1}}{u}, \ldots, \frac{x_q}{u}) \times \\ & \times D_{n-q+p,n-q+p}^{(p)}(a, b_{q+1}, \ldots, b_n, b_1, \ldots, b_p; c, c; \\ & \frac{x_{q+1}}{1-u}, \ldots, \frac{x_n}{1-u}, \frac{x_1}{1-u}, \ldots, \frac{x_p}{1-u}),\end{aligned} \tag{A.7}$$

where the Pochhammer double loop encircles 0 and 1. We will now give three cases of analytic continuations, other cases can be obtained in a similar way. One starts with the still simple case that only one argument is either close to 1 or ∞. This case can be developed along the lines set out in [13] by rewriting the multiple series in such a way that the innermost summations is replaced by ordinary Gaussian hypergeometric series, for which the analytic continuation is known, see eq. A.5 above. This yields the following results: For the region $1/|x_n| < 1$ near infinity, the analytic continuation reads

$$\begin{aligned} F_D^{(n)}(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n) & \\ = \frac{\Gamma(c)\Gamma(b_n-a)}{\Gamma(b_n)\Gamma(c-a)}(-x_n)^{-a} & \times \\ \times F_D^{(n)}(a, b_1, \ldots, b_{n-1}, 1-c+a; 1-b_n+a; \frac{x_1}{x_n}, \ldots \frac{x_{n-1}}{x_n}, \frac{1}{x_n}) & \\ + \frac{\Gamma(c)\Gamma(a-b_n}{\Gamma(a)\Gamma(c-b_n)}(-x_n)^{-b_n} & \times \\ \times D_{1,1}^{(n)}(a-b_n, b_n, b_1, \ldots, b_{n-1}; c-b_n, c-b_n; \frac{1}{x_n}, x_1, \ldots, x_{n-1}). & \end{aligned} \tag{A.8}$$

If $|1-x_n| < 1$, we are in the region close to one, and the analytic continuation now reads

$$
\begin{aligned}
F_D^{(n)}&(a, b_1, \dots, b_n; c; x_1, \dots, x_n) \\
&= \frac{\Gamma(c)\Gamma(c-a-b_n)}{\Gamma(c-a)\Gamma(c-b_n)} \times \\
&\times D_{0,n-1}^{(n)}(a, b_1, \dots, b_n; a+b_n-c+1, c-b_n; x_1, \dots, x_{n-1}, 1-x_n) \\
&+ \frac{\Gamma(c)\Gamma(a+b_n-c)}{\Gamma(a)\Gamma(b_n)}(1-x_n)^{c-a-b_n} \times \\
&\times D_{0,n-1}^{(n)}(c-b_n, b_1, \dots, b_{n-1}, c-a; c-a-b_n+1, c-b_n; x_1, \dots, x_{n-1}, 1-x_n).
\end{aligned}
\tag{A.9}
$$

The first of the two continuations has the advantage that, in the region of large $|x_n|$, the first term on the right hand side is convergent even if all the other x_i, $i \neq n$, are close to one. Thus, to find the analytic continuation in the case that one argument is large and another is close to one, one only need to seek the analytic continuation of the second term on the right hand side. We will give here the result for the case that $|x_1|$ is large and $|x_n|$ is close to one, since other cases can easily be obtained by permutations:

$$
\begin{aligned}
F_D^{(n)}&(a, b_1, \dots, b_n; c; x_1, \dots, x_n) \\
&= \frac{\Gamma(c)\Gamma(b_1-a)}{\Gamma(b_1)\Gamma(c-a)}(-x_1)^{-a} F_D^{(n)}(a, 1-c+a, b_2, \dots, b_n; 1-b_1+a; \frac{1}{x_1}, \frac{x_2}{x_1}, \dots \frac{x_n}{x_1}) \\
&+ \frac{\Gamma(c)}{\Gamma(a)}(-x_1)^{-b_1}\Big(\frac{\Gamma(c-a-b_n)}{\Gamma(c-b_1-b_n)} \times \\
&\times D_{1,2}^{(n)}(a-b_1, b_1, b_n, b_2, \dots, b_{n-1}; c-b_1-b_n, a+b_n-c+1; \frac{1}{x_1}, 1-x_n, x_2, \dots, x_{n-1}) \\
&+ \frac{\Gamma(a+b_n-c)}{\Gamma(b_n)}(1-x_n)^{c-a-b_n} \times \\
&\times D_{1,2}^{(n)}(c-b_1-b_n, b_1, c-a, b_2, \dots, b_{n-1}; c-b_1-b_n, c-a-b_n+1; \\
&\qquad \frac{1}{x_1}, 1-x_n, x_2, \dots, x_{n-1})\Big).
\end{aligned}
\tag{A.10}
$$

Unfortunately, these formulas are valid only for generic values of the parameters. In the cases relevant for the Seiberg–Witten periods, we have certain relations such as $a = b_i$ for some i which will cause singularities, if one attempts to analytically continue in the coordinate x_i. To obtain the correct answer, one has to take one further step, namely a limiting procedure. This is the well known Frobenius process, which essentially is nothing else than to consider the limit of $b_i = a + \varepsilon$ for $\varepsilon \to 0$. As an example, we present here one particular instance, namely

$$
\begin{aligned}
F_D^{(n)}&(a, b_1, \dots, b_{n-1}, a; c; x_1, \dots, x_n) \\
&= \Gamma\begin{bmatrix} c \\ a, c-a \end{bmatrix} (-x_n)^{-a} \times \\
&\times \sum_M \sum_{m_n=0}^{\infty} \Gamma\begin{bmatrix} c-a-|M| \\ c-a+|M| \end{bmatrix} \frac{(a)_{|M|+m_n}(1-c+a)_{2|M|+m_n}}{(|M|+m_n)!m_n!} \prod_{i=1}^{n-1} \frac{(b_i)_{m_i}}{m_i!} \times \\
&\times (\log(-x_n) + h_{m_n}) \left(\frac{x_1}{x_n}\right)^{m_1} \cdots \left(\frac{x_{n-1}}{x_n}\right)^{m_n-1} \left(\frac{1}{x_n}\right)^{m_n} \\
&+ \Gamma\begin{bmatrix} c, c-a \\ a \end{bmatrix} (-x_n)^{-a} \times \\
&\times \sum_M \sum_{m_n=0}^{|M|-1} \frac{(a)_{m_n}\Gamma(|M|-m_n)}{m_n!(c-a)_{|M|-m_n}} \prod_{i=1}^{n-1} \frac{(b_i)_{m_i}}{m_i!} x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} \left(\frac{1}{x_n}\right)^{m_n}, \quad \text{(A.11)}
\end{aligned}
$$

where

$$
h_{m_n} = \psi(1+|M|+m_n) + \psi(1+m_n) - \psi(a+|M|+m_n) - \psi(c-a-m_n), \quad \text{(A.12)}
$$

and we have made use of multindex notation, so that $M = (m_1, \dots, m_{n-1})$, $|M| = \sum_{i=1}^{m_n} m_i$ and summation over M means summation over each m_i ($i = 1, \dots, n-1$) from 0 to ∞. The symbol ψ denotes the digamma function.

In fact, this result was first obtained 'directly' by using a suitable continuation formula for the Gaussian hypergeometric function.

Since such limiting procedures make the formulas extremely cumbersome, it is easier to work with the generic formulas, do the explicitly needed expansions with a computer algebra package and then take the limit. The Lauricella functions which we encountered in our study of the $SU(2)$ case (cf. sect. 5.1) can be written in the following form: For $a(u)$, one has $F_D^{(3)}(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; x, \frac{x}{2-x}, \frac{x}{2})$, where $x = 1 - \frac{e_1}{e_4}$ with the notations used there, i.e. $e_1 = \sqrt{u - \Lambda^2}$ and $e_4 = \sqrt{u + \Lambda^2}$. For small x, this has a good power series expansion, namely

$$
\begin{aligned}
F_D^{(3)}&(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; x, \frac{x}{2-x}, \frac{x}{2}) \\
&= 1 - \frac{3}{4}x + \frac{5}{32}x^2 + \frac{3}{123}x^3 - \frac{169}{262144}x^4 - \frac{1131}{1048576}x^5 + \dots \quad \text{(A.13)}
\end{aligned}
$$

Now, again for small x, the dual period is proportional to $F_D^{(3)}(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; 2, 2\frac{x-1}{x}, 2\frac{x-1}{x-2})$ which calls for an analytic continuation valid near the point $(0, \infty, 1)$. Actually, the first argument has modulus greater than 1, but since the Lauricella function is only polynomial in its first argument, we do not have to perform an analytic continuation for it. We find thus for the dual period

$$
\begin{aligned}
&F_D^{(3)}(\frac{1}{2}, \frac{1}{2}, -2, \frac{1}{2}; 1; 2\frac{x-1}{x}, 2, 2\frac{x-1}{x-2}) \\
&\quad = -\frac{\sqrt{2x}}{\pi}\Big(2 + 3x + \frac{27}{8}x^2 + \frac{19}{6}x^3 + \frac{9559}{4096}x^4 + \frac{12019}{12288}x^5 + \dots \\
&\quad + \log(2)\big(3 + \frac{15}{4}x + \frac{135}{32}x^2 + \frac{561}{128}x^3 + \frac{989}{256}x^4 + \frac{11169}{4096}x^5 + \dots\big) \\
&\quad - \log(x)\big(1 + \frac{5}{4}x + \frac{41}{32}x^2 + \frac{147}{128}x^3 + \frac{193}{256}x^4 + \frac{575}{4096}x^5 + \dots\big)\Big). \qquad \text{(A.14)}
\end{aligned}
$$

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Selbstständigkeitserklärung

Hiermit versichere ich, die vorliegende Diplomarbeit selbstständig und unter ausschließlicher Verwendung der angegebenen Hilfsmittel angefertigt zu haben.

Hannover, den 26. Oktober 2004,