Towards logarithmic mode algebras

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It is an open question whether or not it is possible to generalize the definition of a vertex operator algebra to treat logarithmic conformal field theory already at the vacuum sector. With this task in mind, universal bracket relations for logarithmic mode algebras are given.

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Introduction. Algebraically, the most distinguishing property of logarithmic conformal field theory are indecomposable structures with respect to the action of (Virasoro) modes, while the attribute "logarithmic" derives from the appearance of logarithmic divergencies in certain correlation functions in such theories. Physically, it seems to be particularly interesting for disordered systems, but there is also a number of other physical models with indecomposable, logarithmic features, see e.g. [1, 2] and references therein.

While the algebraic aspects of "ordinary" conformal field theories, and in particular rational ones, are potently and elegantly described by the theory of vertex operator algebras in a rigorous manner (see e.g. [3, 4, 5]), several questions remain unanswered as to the logarithmic generalization so far. Expanding on Milas' earlier work [6], Huang, Lepowsky and Zhang were able to treat logarithmic conformal field theory in the language of vertex operator algebras in [7]. Their approach introduces characteristic logarithmic features through the notions of generalized modules and logarithmic intertwining operators, while the definition of the fundamental structure, the underlying vertex operator algebra itself, is left unchanged. This way, they could furthermore broaden their $P(z)$-tensor product theory to incorporate such generalized structures, which makes it possible to discuss nonmeromorphic operator product expansion rigorously, among many other things. Two recent works which investigate properties of infinite families of logarithmic conformal field theories from such a vertex algebraic perspective are [8] and [9]. Such less general but more concrete studies give further credibility to the claim that logarithmic conformal field theory may be described successfully by vertex operator algebra theory on the level of modules.

On the other hand, all known logarithmic conformal models share the property that the vacuum vector Ω has at least one logarithmic partner, i.e. Ω is an element of a nontrivial Jordan cell with respect to the action of the Virasoro mode L_0. By the operator-state-correspondence, there is also a logarithmic partner to the identity operator which typically has a logarithmic dependence on its variables. Since the vacuum vector is part of the structure of a vertex operator algebra but not necessarily of its modules, the question naturally arises whether it is possible to modify or generalize the definition of a vertex operator algebra to treat logarithmic conformal field theory already at this fundamental level.

In this note, I attempt to go one step towards an answer to this question by considering the algebras of logarithmic modes. For concreteness, the logarithmic $\theta^+\theta^-$-system is discussed, and in particular the operator product expansions of the logarithmic partner $\widetilde{\Omega}(z)$ to the identity field and the Virasoro field as well as of $\widetilde{\Omega}(z)$ and itself are derived. As these operator product expansions are believed to be valid in any logarithmic conformal field theory, the subsequent discussion of logarithmic mode algebras should be equally universal.

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The $\boldsymbol{\theta^+\theta^-}$-system. The probably most intensively studied logarithmic conformal field theory is the one with central charge $c = -2$. While the triplet algebra $\mathcal{W}(2, 3^{\times 3})$ at $c = -2$ involves three additional primary generating fields $W^a(x)$, there is also a pure Virasoro model with indecomposable but reducible structure. It has a concrete realization in terms of the $\theta^+\theta^-$-ghost system originally introduced by Zamolodchikov.

The $\theta^+\theta^-$-system is defined by two fermionic fields $\theta^+(z)$ and $\theta^-(z)$ whose mode expansions are $\theta^\pm(z) = \theta_0^\pm \log z + \xi^\pm + \sum_{m\neq 0} \theta_m^\pm z^{-m}$, and all modes anti-commute except for the following cases: $\{\xi^\pm, \theta_0^\mp\} = \pm 1$, $\{\theta_m^+, \theta_n^-\} = \frac{1}{m}\delta_{m+n,0}$ for all $m \neq 0$. The modes $\xi^\pm$ and $\theta_m^\pm$ generate a Fock space by the free action on the vacuum Ω, subject to the relations above and the condition that $\theta_m^\pm \Omega = 0$ for all $m \in \mathbb{N}$, and the normal-ordering is given by $:\theta_m^+\theta_n^-: \, := +\theta_m^+\theta_n^-$ for $m \leq n$, $:\theta_m^+\theta_n^-: \, := -\theta_n^-\theta_m^+$ for $m > n$, and $:\theta_m^\pm\xi^\pm: \, := -\xi^\pm\theta_m^\pm = -:\xi^\pm\theta_m^\pm:$. The conformal symmetry is encoded in the $\theta^+\theta^-$-system such that the energy momentum operator is realized as $T(z) = \, :(\partial\theta^+(z))(\partial\theta^-(z)):$. Indeed, the above relations can be used to express the modes of $T(z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}$ in terms of the modes of $\theta^\pm(z)$,

$$L_m = \begin{cases} m(\theta_0^-\theta_m^+ + \theta_m^-\theta_0^+) + \sum_{a\in\mathbb{Z}} a(m-a)\theta_{m-a}^+\theta_a^- & \text{for } m \neq 0 \, , \\ \theta_0^+\theta_0^- - \sum_{a\in\mathbb{Z}} a^2 :\theta_{-a}^+\theta_a^-: & \text{for } m = 0 \, , \end{cases} \tag{1}$$

and these modes satisfy the Virasoro algebra with central charge $c = -2$.

In addition to the energy momentum operator, a field $\widetilde{\Omega}(z)$ of generalized weight 0 can be obtained by the fields $\theta^\pm(z)$ as $\widetilde{\Omega}(z) \equiv \sum_{m\in\mathbb{Z}} \sum_{a\in\mathbb{N}} \widetilde{\Omega}_{m,a} z^{-m} (\log z)^a = -:\theta^+(z)\theta^-(z):$, and its modes can be expressed in terms of the modes of $\theta^\pm(z)$ by the relations

$$\widetilde{\Omega}_{m,0} = \begin{cases} -\xi^+\theta_m^- - \theta_m^+\xi^- + \sum_{a\neq 0,m} \theta_{m-a}^+\theta_a^- & \text{for } m \neq 0 \, , \\ -\xi^+\xi^- - \sum_{a\neq 0} :\theta_{-a}^+\theta_a^-: & \text{for } m = 0 \, , \end{cases} \tag{2a}$$

$$\widetilde{\Omega}_{m,1} = \begin{cases} -\theta^+_m\theta^-_0 - \theta^+_0\theta^-_m & \text{for } m \neq 0\ , \\ \\ -\xi^+\theta^-_0 + \xi^-\theta^+_0 & \text{for } m = 0\ , \end{cases} \tag{2b}$$

$$\widetilde{\Omega}_{m,2} = -\delta_{m,0}\theta^+_0\theta^-_0\ . \tag{2c}$$

The vector $\widetilde{\Omega} = \widetilde{\Omega}(z)\Omega|_{z=0}$ associated to the field $\widetilde{\Omega}(z)$ spans a Jordan cell of rank 2 with respect to the operator L_0 together with the vacuum Ω: $L_0\widetilde{\Omega} = \Omega$.

Commutation relations involving one logarithmic mode. With the concrete realization of a logarithmic conformal field theory given by the $\theta^+\theta^-$-system at hand, the explicit expressions in terms of the fields $\theta^\pm(z)$ can be used to study properties of the logarithmic fields $\widetilde{\Omega}(z)$.

As a first example, one can perform several calculations building on the anticommutation relations for the $\theta^\pm$- and ξ-modes and the expressions (1), (2) to obtain all commutation relations involving one Virasoro mode L_m and one logarithmic mode $\widetilde{\Omega}_{n,a}$. This yields

$$\left[L_m, \widetilde{\Omega}_{n,a}\right] = -(m+n)\widetilde{\Omega}_{m+n,a} + (a+1)\widetilde{\Omega}_{m+n,a+1} + (m+1)\delta_{m+n,-1}\delta_{a,0}\ . \tag{3}$$

The same result can also be more elegantly obtained from a naively generalized vertex operator algebra Jacobi identity, i.e. from $x_0^{-1}\delta(\frac{x_1-x_2}{x_0})Y(\omega,x_1)Y(\widetilde{\Omega},x_2,\log x_2) - x_0^{-1}\delta(\frac{x_2-x_1}{-x_0})Y(\widetilde{\Omega},x_2,\log x_2)Y(\omega,x_1) = x_2^{-1}\delta(\frac{x_1-x_0}{x_2})Y(Y(\omega,x_0)\widetilde{\Omega},x_2,\log x_2)$ where ω denotes the conformal vector. On the other hand, a consistent generalized Jacobi identity involving two logarithmic vertex operators is not available.

The operator product expansion $\widetilde{\Omega}(z)\widetilde{\Omega}(w)$. In ordinary conformal field theory, the commutation relations between modes of two fields are equivalent to the operator product expansion of the two fields, so one may try to obtain information about the bracket relations between the $\widetilde{\Omega}$-modes by studying the operator product expansion of $\widetilde{\Omega}(z)\widetilde{\Omega}(w)$. This product can be computed with the help of a variant of Wick's theorem. If the normal-ordered product of two arbitrary logarithmic fields $f(z) = \sum_{m\in\mathbb{Z},a\in\mathbb{N}} z^{-m}(\log z)^a f_{m,a}$ and $g(z) = \sum_{n\in\mathbb{Z},b\in\mathbb{N}} z^{-n}(\log z)^b g_{n,b}$ is defined as $:f(z)g(w): = f(z)_+g(w) + g(w)f(z)_-$ with

$$f(z)_+ = \sum_{m\leq 0} z^{-m}f_{m,0} + \sum_{m\leq -1}\sum_{a>0} z^{-m}(\log z)^a f_{m,a}\ ,$$
$$f(z)_- = \sum_{m>0} z^{-m}f_{m,0} + \sum_{m>-1}\sum_{a>0} z^{-m}(\log z)^a f_{m,a}\ ,$$

Wick's theorem as stated in [10] can be generalized to the logarithmic case and subsequently applied to the present case of interest to compute the operator product expansion of $\widetilde{\Omega}(z)\widetilde{\Omega}(w) = :\theta^+(z)\theta^-(z)::\theta^+(w)\theta^-(w):$. This yields

$$\widetilde{\Omega}(z)\widetilde{\Omega}(w) = -(\log(z-w))^2 - 2\log(z-w)\widetilde{\Omega}(w) + (\text{terms regular in } (z-w))\ . \tag{4}$$

Bracket relations from the operator product expansion. The way commutation relations for modes are often obtained in conformal field theory is to compute contour integrals of the corresponding operator product expansion, but this method can certainly not be directly applied to the logarithmic case because the contour integral of a logarithm is simply not defined. Given two logarithmic fields, in order to infer the commutators of their modes from their operator product expansion without having to compute contour integrals or (formal) residues, one can compare coefficients of monomials of all the variables on both sides of the operator product expansion. To do this one has to consider the non-normal-ordered part of the product $Y(u,x_1)Y(v,x_2) = \sum_{i=0}^{N_{uv}-1}(x_1-x_2)^{-i-1}Y(u_i v,x_2) + \mathbf{:}\,Y(u,x_1)Y(v,x_2)\,\mathbf{:}$ of the two fields on one side of the equation, while on the other side the singular part of the expansion in the difference of the variables has to be considered, see e.g. [4, 11] for the non-logarithmic case. This method in particular circumvents the problem of having to deal with ill-defined residues of logarithms.

Using this method one can easily extract all commutation relations from operator product expansions of fields in an arbitrary meromorphic conformal field theory. But more interesting from the present point of view is the case of the operator product expansion $T(z)\widetilde{\Omega}(w)$ which involves one ordinary quantum field and one logarithmic field. This operator product expansion can be computed using the $\theta^+\theta^-$-system, but it can also be inferred using the general relation $L_0\widetilde{\Omega} = \Omega$, and it is given by $T(z)\widetilde{\Omega}(w) \sim \frac{1}{(z-w)^2} + \frac{1}{(z-w)}\partial\widetilde{\Omega}(w)$. Proceeding as described above, one arrives at exactly the same result as the one obtained in (3), which gives further credibility to the method used here and provides a third independent possibility to compute the commutators $[L_m, \widetilde{\Omega}_{n,b}]$. Nevertheless, it should be stressed that this method is only assumed to work also in the logarithmic case where it cannot be obtained from first principles so far due to the lack of a consistent definition of a Jordan vertex operator algebra.

After this reassuring example now the case of $\widetilde{\Omega}(z)\widetilde{\Omega}(w)$ is addressed which is given by $\widetilde{\Omega}(z)\widetilde{\Omega}(w) \sim -(\log(z-w))^2 - 2\log(z-w)\widetilde{\Omega}(w)$. Proceeding as before, the left-hand side can be written as

$$\begin{aligned}\widetilde{\Omega}(z)\widetilde{\Omega}(w) = \mathbf{:}\,\widetilde{\Omega}(z)\widetilde{\Omega}(w)\,\mathbf{:} &+ \sum_{m\geq 1}\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}} z^{-m}w^{-n}(\log w)^b \left[\widetilde{\Omega}_{m,0}, \widetilde{\Omega}_{n,b}\right] \\ &+ \sum_{m\geq 0}\sum_{n\in\mathbb{Z}}\sum_{a>0}\sum_{b\in\mathbb{N}} z^{-m}w^{-n}(\log z)^a(\log w)^b \left[\widetilde{\Omega}_{m,a}, \widetilde{\Omega}_{n,b}\right] , \end{aligned} \tag{5}$$

while the relevant part $-(\log(z-w))^2 - 2\log(z-w)\widetilde{\Omega}(w)$ of the right-hand side can be expanded in z, w, $\log z$ and $\log w$ as

$$\begin{aligned} &-(\log z)^2 + \sum_{m\geq 1}\frac{2}{m+1}z^{-m}w^m\log z - \sum_{m\geq 2}\sum_{i=1}^{m-1}\frac{2}{mi}z^{-m}w^m \\ &\quad - \sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}} 2\widetilde{\Omega}_{n,b}w^{-n}\log z(\log w)^b + \sum_{m\geq 1}\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}}\frac{2}{m}\widetilde{\Omega}_{m+n,b}z^{-m}w^{-n}(\log w)^b . \end{aligned} \tag{6}$$

Comparing (5) and (6) may at first suggest the following bracket:

$$\left[\widetilde{\Omega}_{m,a},\widetilde{\Omega}_{n,b}\right] \stackrel{?}{=} -\delta_{a,2}\delta_{b,0}\delta_{m,0}\delta_{n,0} + \delta_{a,1}\delta_{b,0}\delta_{m+n,0}(1-\delta_{m,0})\frac{2}{m} - \delta_{a,0}\delta_{b,0}\delta_{m+n,0}(1-\delta_{m,1})\sum_{i=1}^{m-1}\frac{2}{mi} - \delta_{a,1}\delta_{m,0}2\widetilde{\Omega}_{n,b} + \delta_{a,0}\frac{2}{m}\widetilde{\Omega}_{m+n,b}\,. \quad (7)$$

Here in the third term on the right-hand side the factor $(1-\delta_{m,1})$ can be discarded if the convention is imposed that $\sum_{i=k}^{l} s_i \equiv 0$ for all $l < k$, i.e. one only counts in the positive direction. This convention is employed in the following.

According to the above reasoning, the relation (7) can only be possibly true for $(m, a) \in (\mathbb{Z}_+ \times \{0\}) \cup (\mathbb{N} \times \mathbb{Z}_+)$. But the right-hand side of (7) does not have the same symmetry as the left-hand side: a permutation of the kind $(m, a) \leftrightarrow (n, b)$ should have the same effect as a mere multiplication by -1. Obviously this is not the case, and so one may expect that an expansion in the domain $|w| > |z|$ will lead to additional terms such that the full bracket has the correct symmetry. This would be in contrast to the cases considered before, where "half of" the commutator was actually already the "full" commutator.

But instead of finding the correct expansion for $|w| > |z|$, one may also propose to argue in the following way: only one expansion (for $|z| > |w|$) has to be carried out as this already gives all the crucial information on the bracket (as in the case of meromorphic fields and $T(z)\widetilde{\Omega}(w)$). The missing terms are simply added such that antisymmetry is warranted. This suggests that the bracket should be

$$\left[\widetilde{\Omega}_{m,a},\widetilde{\Omega}_{n,b}\right] = (\delta_{a,0}\delta_{b,2} - \delta_{a,2}\delta_{b,0})\,\delta_{m,0}\delta_{n,0} + (\delta_{a,1}\delta_{b,0} + \delta_{a,0}\delta_{b,1})\,(1-\delta_{m,0})\delta_{m+n,0}\frac{2}{m} - \left(\sum_{i=1}^{m-1}\frac{1}{i} + \sum_{i=1}^{-m-1}\frac{1}{i}\right)\delta_{a,0}\delta_{b,0}\delta_{m+n,0}\frac{2}{m} - \delta_{a,1}\delta_{m,0}2\widetilde{\Omega}_{n,b} + \delta_{b,1}\delta_{n,0}2\widetilde{\Omega}_{m,a} + \delta_{a,0}(1-\delta_{m,0})\frac{2}{m}\widetilde{\Omega}_{m+n,b} - \delta_{b,0}(1-\delta_{n,0})\frac{2}{n}\widetilde{\Omega}_{m+n,a}\,, \quad (8)$$

where only the minimal number of new terms was added to the relation (7) to secure antisymmetry; additional terms are not to be expected "because of symmetry".

This makes (8) the best proposal for the bracket of two logarithmic $\widetilde{\Omega}$-modes so far. As the operator product expansion (4) is believed to be correct in all logarithmic conformal theories (and not only in the $\theta^+\theta^-$-system), this relation would also apply universally. On the other hand, the reasoning leading to the anti-symmetrized form (8) seems to be imperfect, and a more thorough argument would be welcome.

Final remarks. It is interesting to note that the relation (8) cannot be the bracket for a Lie algebra spanned by the modes $\widetilde{\Omega}_{m,a}$. Indeed, explicitly calculating $[[\widetilde{\Omega}_{l,a},\widetilde{\Omega}_{m,b}],\widetilde{\Omega}_{n,c}] + [[\widetilde{\Omega}_{m,b},\widetilde{\Omega}_{n,c}],\widetilde{\Omega}_{l,a}] + [[\widetilde{\Omega}_{n,c},\widetilde{\Omega}_{l,a}],\widetilde{\Omega}_{m,b}]$, one finds that this satisfies the ordinary Lie algebra Jacobi identity for arbitrary $l, m, n \in \mathbb{Z}$ only if the logarithmic indices a, b, c are positive integers. An analogous statement is true for the double-bracket $[[L_l,\widetilde{\Omega}_{m,b}],\widetilde{\Omega}_{n,c}]$ and its cyclic permutations, using in addition the commutation relation (3) for $[L_l,\widetilde{\Omega}_{m,b}]$. On the other hand, the identity

$[[L_m, L_n], \widetilde{\Omega}_{l,a}] + [[L_n, \widetilde{\Omega}_{l,a}], L_m] + [[\widetilde{\Omega}_{l,a}, L_m], L_n] = 0$ is satisfied for all $l, m, n \in \mathbb{Z}$ and $a \in \mathbb{N}$. The failure of the bracket (8) to satisfy the Lie algebra Jacobi identity may not be reason enough for it to be disqualified, as not much is known on the algebras of modes in logarithmic conformal field theory. In particular, $[L_m, \widetilde{\Omega}_{n,a}]$ is really a commutator while the object $[\widetilde{\Omega}_{m,a}, \widetilde{\Omega}_{n,b}]$ computed here is a more abstract bracket. At the present stage there seems to be no necessity for the modes to span a Lie algebra, and the relation between the operator product expansion and commutation relations of modes in logarithmic theories would differ from the ordinary case. In the general setting of logarithmic conformal field theory it might simply not be true that all modes are elements of a Lie algebra as in the case of ordinary vertex operator algebras – if at all it is possible to treat logarithmic conformal field theory at the level of vertex operator algebras and not only their modules in the first place; this remains an open question.

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