Aspects of Indecomposable Vertex Operator Algebras

From C_2-cofiniteness to Nonmeromorphic Operator Product Expansion for all Triplet Algebras, and Logarithmic Mode Algebras

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Abstract

I started my studies in the framework of this thesis with the aim to gain a better understanding of the algebraic structure of (logarithmic) conformal field theory. The theory of vertex operator algebras soon was identified as the most promising and interesting setting, and most of the outcome presented here is formulated in this language. One main result is a proof of the existence and associativity of the nonmeromorphic operator product expansion for an infinite family of vertex operator algebras that are also known to be the vacuum sectors of logarithmic conformal field theories. Furthermore, all these vertex operator algebras are shown to be C_2-cofinite, and this important finiteness property is related to another one called rationality. Finally, I try to explore the algebraic roots of logarithmic conformal field theory in general and obtain some new but limited insight in logarithmic commutation relations.

Before several aspects of the theory of vertex operator algebras and its physical relevance are introduced and discussed systematically in chapter 2, the first chapter tries to sketch a bigger picture, relating much of the material in the rest of this thesis to its historical and geometrical origins and counterparts. In this way it serves as an introduction, but not one without some technical details. After a very crude positioning of two-dimensional conformal field theory within theoretical physics, the mathematical phenomenon of monstrous moonshine is touched upon, which gave birth to the study of vertex operator algebras in the second half of the last century. But vertex operator algebras are also intimately connected to the geometry of interacting strings, which motivated the abstract definition of conformal field theory by Segal and Vafa. The basic categorial and geometric ideas in these approaches are explained, and the precise way Huang's theory of geometric vertex operator algebras draws a bow from geometry to algebra in conformal field theory is described.
The second chapter provides the necessary background in vertex operator algebra theory which is needed to present and understand the main results in chapter 3, and it tries to do so in a self-contained way. In section 2.1 basic notions such as vertex operator algebras, modules, intertwining operators and fusion rules are introduced, and some of their relations and properties are studied. An exhaustive presentation of the theory of vertex operator algebras cannot be fit into one small chapter, and the focus is set on the particularly important associativity and commutativity properties which are discussed in much detail. Then

in section 2.2, the notions of rationality, regularity, C_n-cofiniteness and the Zhu algebra are introduced, and the relations among these finiteness properties for vertex operator algebras are pointed out. A brief introduction to some aspects of $\mathcal{W}$-algebras is also given in this context. Finally in section 2.3, a skeleton of $P(z)$-tensor product theory is presented. This theory allows to give conditions on the existence and associativity of the nonmeromorphic operator product expansion such that if these conditions are satisfied, the nonmeromorphic operator product expansion can be derived from first principles.

The third chapter combines results from sections 2.2 and 2.3 in a study of all triplet algebras $\{\mathcal{W}(2,(2p-1)^{\times 3})\}_{p\in\mathbb{Z}_{\geq 2}}$. Most of the material in this chapter is an extended description of the work published in [CF]. The triplet algebras are introduced and it is shown how the list of properties sufficient for the existence and associativity of the nonmeromorphic operator product expansion can almost be reduced to the C_2-cofiniteness of the vertex operator algebras $\mathcal{W}(2,(2p-1)^{\times 3})$. A careful study of singular vectors and characters for the triplet algebras is then applied to prove that they are indeed all C_2-cofinite. This together with the consequences for the nonmeromorphic operator product expansion is the main result in my thesis. While the explicitly known singular vectors and commutation relations for the triplet algebra $\mathcal{W}(2,3^{\times 3})$ make the proof rather easy in this case, obtaining the result for all other triplet algebras requires some subtle arguments. Chapter 3 ends with an application of the study of singular vectors to establish an upper bound on the dimension of the Zhu algebras for the triplet algebras.

The triplet algebras are all examples of logarithmic conformal field theories as follows from their explicitly known characters. This fact is not stressed in chapter 3 since the arguments used to establish its main results do not depend crucially on indecomposable structures – even though the triplet algebras are the first family of logarithmic conformal field theories for which C_2-cofiniteness and nonmeromorphic operator product expansion are proven. Chapter 4 on the other hand is devoted to the specific properties of logarithmic conformal field theories. More precisely, the possibility of a generalized vertex operator algebraic approach to logarithmic conformal field theory that places the defining features of such theories at the most fundamental level is investigated. This is in contrast to known attempts to describe logarithmic conformal field theory in terms of vertex operator algebra theory, which introduces the indecomposable structure and logarithms only at the level of modules and intertwining operators. Part of the reason a more fundamental approach has not yet been presented in the literature are certain expansion issues and difficulties to properly introduce logarithms into the core structure of vertex operator algebras. I am not able to master all these problems, but some results on logarithmic mode algebras are obtained which may help to describe logarithmic conformal field theory in the indicated way in the future.

Contents

Chapter 1

A Bigger Conformal Picture

Two-dimensional conformal (quantum) field theory is very special. While in more than two spacetime dimensions d, the group of transformations which leave angles invariant in Minkowski space is isomorphic to the group of Poincaré transformations in $d+2$ dimensions and is thus finite-dimensional, the symmetry algebra corresponding to conformal transformations in two dimensions is infinite-dimensional. This high degree of symmetry imposes many natural restrictions such that any field theory in two dimensions with conformal symmetry has a structure that makes it particularly clearly arranged. In many cases, such theories are completely solvable in the sense that all correlation functions, from which observable quantities are obtained in field theories, can be computed accurately in principle. It is a very satisfying result to be (at least sometimes) in a position to make exact statements in nontrivial situations instead of relying on the mysteries of perturbation theory.

All this is true, but one might object that two dimensions are not quite enough to describe what seems to be the "real world" in four spacetime dimensions. This raises the question of the significance of two-dimensional conformal field theory in physics as a language and structure to describe and substantiate measurable processes, respectively. To my knowledge, there are three main answers to this question.

Firstly, there are many established theories and models that describe physical processes that take place effectively in two dimensions. In particular, this is often the case in statistical physics and especially condensed matter physics. For example, one might be interested in phenomena that are confined to the two-dimensional surface of a three-dimensional object, or a system with one spatial dimension that evolves in time. If such situations come along with the manifestation of certain symmetries, the most important of which here is scale invariance, the power of two-dimensional conformal field theory may be utilized. A famous example is the critical Ising model whose continuum limit is described by a two-dimensional conformal field theory of central charge $c = \frac{1}{2}$.

Secondly, perturbative string theory is intimately connected to, in a sense even identical with a two-dimensional conformal field theory. String theory is the candidate for a theory that describes all known physical interactions in a unified manner that receives by far the most attention. It can be formulated in terms of an action principle, where the action is an integral over the two-dimensional surface swept out by the superstring as it propagates in

space and time. This action is invariant under conformal transformations of the worldsheet coordinates and Weyl transformations of the worldsheet metric, and this implements the conformal symmetry.
Finally, two-dimensional conformal field theory is one of the few direct contacts between physics and mathematics. As most of physics is formulated in some kind of mathematical language, this statement might seem nearly tautological, but it is meant in the stronger sense that representatives of both sciences actually come together and pursue common research with a rather open mind for the views and ideas of the other side. This is very precious. For mathematics, the intuition and insight of physicists can be an inspiring motivation to discover and develop new interesting structures or gain a deeper appreciation of known ones. On the other hand, it is desirable not only to have a merely intuitive and vague comprehension of some more or less specific physical system, but also to have a consistent language to communicate this understanding which also gives it an exact meaning and uncovers its fundamental structure, otherwise it should not be spoken of true understanding. This is why a clear formulation of physical theories with mathematical rigour should always be a goal. While in most other quantum theories mathematicians find it very hard to make sense of the concepts employed by physicists, this goal is much nearer in two-dimensional conformal field theory. And once the physics of conformal fields is understood mathematically, there is hope that at least some of the structure found can be transported or adapted to other relevant theories.

I will not attempt to give an introduction to conformal field theory as a whole here. Among the introductory expositions of conformal field theory that I have (partly) studied are [Gi], [Scho], [Sche] [Fu], [DFMS], [GG], [Gab1], [Gab3], [Nah1], [Gab4], [C] and [Gab5]. Such background and the application in string theory in mind, some of the mathematical structure will be explored and interpreted physically in this thesis.
In particular, the presence of conformal symmetry in field theory alluded to above manifests itself in the way its fundamental objects, e.g. primary fields, behave under conformal transformations, and the behaviour of less fundamental objects can be inferred. This can be translated into algebraic relations in the precise mathematical description, e.g. among the modes of a Laurent expansion of the primary fields. Making full use of the algebraic approach to conformal field theory in a consistent and appealing framework amounts to working with some variant of the notion of a vertex operator algebra. The theory of vertex operator algebras is the topic of the next chapter. Here, a brief summary will be given of how it arose historically, drawing from the material in [FLM], [Bo2], [Gan1], [Gan2] and [LL].

The Monster. The original reason to introduce the notion of a vertex operator algebra does not have an obvious connection to conformal field theory, but rather to a special part of group theory. About 25 years ago the task of classifying all finite simple groups neared its completion. This classification asserts that the following list comprises all such groups: a family of cyclic groups $\mathbb{Z}_p$ with p a prime number, a family of alternating groups or even permutation groups $\mathfrak{A}_m$ with $m \in \mathbb{Z}_{\geq 5}$, 16 families of Lie type, and 26 additional finite simple groups that are not part of an infinite family and are called sporadics. The smallest of these is the Mathieu group M_{11} whose order is 7920, and the largest sporadic simple group is the so-called Monster $\mathbb{M}$ with

$$\begin{aligned}|\mathbb{M}| &= 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \\ &= 808\,017\,424\,794\,512\,875\,886\,459\,904\,961\,710\,757\,005\,754\,368\,000\,000\,000 \\ &\approx 8 \cdot 10^{53} \ .\end{aligned}$$

The monster has a surprising relation to modular functions, i.e. functions defined on the upper-half plane $\mathbb{H} = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}$ that are invariant under the modular group $SL(2,\mathbb{Z})$ which acts on such functions via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \tau = \frac{a\tau+b}{c\tau+d}$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$. The elliptic modular function

$$j(\tau) = \frac{\left(\Theta_{E_8}(\tau)\right)^3}{\left(\eta(\tau)\right)^{24}} = q^{-1} + 744 + 196\,884q + 21\,493\,760q^2 + \ldots \quad \text{with } q = \mathrm{e}^{2\pi\mathrm{i}\tau} \ ,$$

where Θ_{E_8} is the theta function of the root lattice of the exceptional Lie algebra E_8 and η is the Dedekind eta function, is the simplest modular function in the sense that any modular function can be written as a rational function of j. McKay and Thompson made the astonishing observation that

$$\begin{aligned} 1 &= 1 \ , \\ 196\,884 &= 196\,883 + 1 \ , \\ 21\,493\,760 &= 21\,296\,876 + 196\,883 + 1 \ , \end{aligned}$$

where the first nonvanishing coefficients of the expansion of j on the left-hand side are equated to (sums of) the dimensions of the smallest irreducible representations of the Monster $\mathbb{M}$ on the right-hand side. This unexpected relation between group theory and number theory was termed "monstrous moonshine", using the English idiom to express doubts that the relation was really profound. McKay and Thompson suggested that this "numerology" is no coincidence but stems from the fact that there is an infinite-dimensional graded vector space on which the Monster acts as its symmetry group, and its homogeneous subspaces are sums of irreducible representations of $\mathbb{M}$. It was shown by Frenkel, Lepowsky and Meurman that there really is such a vector space which they called the "moonshine module" $V^\natural$. Motivated by his study of finite groups and Kac-Moody algebras, Borcherds introduced the notion of vertex algebras and stated that $V^\natural$ actually is an example of such a structure. This was proven by Frenkel, Lepowsky and Meurman. In fact, they proved that $V^\natural$ has an even more elaborate structure that also encompasses a representation of the Virasoro algebra, which they called a vertex operator algebra.

In this language, the relation between the Monster $\mathbb{M}$ and the elliptic modular function j is explained by the fact that $V^\natural$ is a vertex operator algebra $V^\natural = \coprod_{m\in\mathbb{N}} V^\natural_{(m)}$ of central charge $c = 24$ whose graded dimension satisfies the relation

$$\sum_{m\in\mathbb{N}} q^m \mathrm{dim} V^\natural_{(m)} = q\left(j(\tau) - 744\right) \ .$$

Substracting the constant 744 from $j(\tau)$ does not change its invariance under modular transformations and thus does hardly change its nature. Furthermore, the Monster $\mathbb{M}$ is the automorphism group for $V^\natural$, i.e.

$$gY(v,x)g^{-1} = Y(gv,x) \quad \text{for all } g \in \mathbb{M} \text{ and all } v \in V^\natural \ .$$

Soon after this description by mathematicians, Dixon, Ginsparg and Harvey interpreted the moonshine module in terms of string theory: $V^\natural$ can be viewed as a $\mathbb{Z}_2$-orbifold theory of free bosons compactified on $\mathbb{R}^{24}/\Lambda_{24}$, where Λ_{24} is the Leech lattice.

Segal's definition of conformal field theory. String theory was also the motivation for Segal to give his abstract definition of conformal field theory in [S]. His work has been highly influential for many mathematicians working on conformal field theory, including in particular vertex operator algebras.

In perturbative string theory, interactions of closed strings are pictured as Riemann surfaces with boundaries. The connected boundary components are one-real-dimensional manifolds and represent the incoming and outgoing strings. This setting is supposed to be a two-dimensional generalization of the Feynman diagram formalism in perturbative quantum field theory. An important difference is that while in quantum field theory the Feynman rules can be derived from first principles, this does not seem to be the case in string theory so far and the approach in perturbation theory should be seen as a postulate here.

In addition to the geometrical picture of strings propagating in spacetime, there is the description of physical states in terms of elements of vector spaces, and to each incoming or outgoing string state such a vector should correspond. Furthermore, probabilities for physical processes are typically computed in terms of quantum amplitudes, and these in turn are often given by path integrals by physicists. But it seems very difficult to talk about path integrals rigorously, and Segal tried to extract their basic properties and put them into a consistent framework to define conformal field theory. This will now be described, mostly following [S] and [Hua6].

The first step into this direction is to consider the above-mentioned string interactions from a slightly different point of view: (after a homotopic transformation) the closed strings are given by copies of circles S^1 and their combined worldsheets are imagined to represent the interaction between the strings. In more precise words, this defines a symmetric monoidal category $\mathscr{C}$ whose objects are finite ordered sets of copies of S^1, and whose morphisms are conformal equivalence classes $[\Sigma]$ of Riemann surfaces with oriented and ordered boundary components with analytic parametrization. For any two objects in $\mathscr{C}$ a morphism between them must be such that the copies of S^1 in the domain and codomain parametrize the negatively and positively oriented boundary components, respectively. Loosely speaking this says that the incoming and outgoing strings can fit on the boundary of any representative of $[\Sigma]$. The composition of two morphisms is given by the sewing of the associated Riemann surfaces; this will be explained in more detail below.

The next idea is to assign a suitable vector space V to each string described by a copy of S^1 and an operator $U_{[\Sigma]} : V_{C_1} \to V_{C_2}$, corresponding to an interaction, to each equivalence class of Riemann surfaces $[\Sigma]$ with $\partial\Sigma = C_1 \sqcup C_2$ for some representative Σ, where C_1 and C_2 denote the sets of incoming and outgoing strings, respectively. To make this assignment, a symmetric tensor category $\mathscr{V}$ is introduced whose objects are complete locally convex topological $\mathbb{C}$-vector spaces with a nondegenerate bilinear form $(\cdot\,,\,\cdot)$, and a projective functor from $\mathscr{C}$ to $\mathscr{V}$ is considered. This is a functor from $\mathscr{C}$ to the projective category whose objects are the objects of $\mathscr{V}$ and whose morphisms are one-dimensional spaces of morphisms of $\mathscr{V}$. The reason that the functor should be projective is that the above operator $U_{[\,\cdot\,]}$ should have the property that $U_{[\Sigma_1 \sqcup \Sigma_2]}$ is proportional but not necessarily equal to $U_{[\Sigma_2]} \circ U_{[\Sigma_1]}$. Also, it follows from the tensor property of $\mathscr{V}$ that if the image of one copy of S^1 under the functor is the vector space V, then the image of m copies of S^1 is equal to $V^{\otimes m}$. With this notation, the following definition can be given.

Definition. A *conformal field theory in the sense of Segal* is a projective functor $\mathscr{T}$ from $\mathscr{C}$ to $\mathscr{V}$ subject to the following axioms:

(S1) For any morphism $[\Sigma]$ in $\mathscr{C}$ from m ordered copies of S^1 to n ordered copies of S^1, let $[\Sigma_{\widehat{i,j}}]$ be the morphism from $m-1$ ordered copies of S^1 to $n-1$ ordered copies of S^1 obtained from $[\Sigma]$ by identifying the boundary component of Σ parametrized by the i-th copy of S^1 in the domain of $[\Sigma]$ with the boundary component of Σ parametrized by the j-th copy of S^1 in the codomain of $[\Sigma]$. Then the trace between the i-th tensor factor in the domain and the j-th tensor factor in the codomain of $\mathscr{T}([\Sigma])$ exists and is equal to $\mathscr{T}([\Sigma_{\widehat{i,j}}])$.

(S2) For any morphism $[\Sigma]$ in $\mathscr{C}$ from m ordered copies of S^1 to n ordered copies of S^1, let $[\Sigma_{i\to n+1}]$ be the morphism from $m-1$ ordered copies of S^1 to $n+1$ ordered copies of S^1 obtained by changing the i-th copy of S^1 of the domain of $[\Sigma]$ to the $(n+1)$-th copy of S^1 of the codomain of $[\Sigma_{i\to n+1}]$. Then $\mathscr{T}([\Sigma])$ and $\mathscr{T}([\Sigma_{i\to n+1}])$ are related by the map from $\mathrm{Hom}(V^{\otimes m}, V^{\otimes n})$ to $\mathrm{Hom}(V^{\otimes(m-1)}, V^{\otimes(n+1)})$ obtained using the map $V \to V^*$ corresponding to the bilinear form $(\cdot\,,\,\cdot)$.

While the second axiom simply formally captures the idea familiar from basic quantum theory that the relation between incoming and outgoing states is reflected by the relation between a vector space and its dual, the first axiom deserves a more detailed motivation. Perturbative string theory is described by a quantum field theory with fields Φ on Riemann surfaces, which represent the worldsheets of strings. In the path integral approach, the partition function for a quantum field theory on a Riemann surface Σ with boundary components C_k, which correspond to incoming and outgoing strings and on which the fields have fixed boundary values ϕ_k, is given by

$$\int_{\Phi|_{C_k}=\phi_k} \mathcal{D}\Phi \; \mathrm{e}^{-S_\Sigma[\Phi]} \;,$$

where S_Σ is the action on Σ. The postulate is that this integral can be computed iteratively in the sense that if Σ is sewn with itself to $\Sigma_{\widehat{i,j}}$ as indicated in the above definition, then

$$\int_{\Phi|_{C_k}=\phi_k,\, k\neq i,j} \mathcal{D}\Phi \; \mathrm{e}^{-S_{\Sigma_{\widehat{i,j}}}[\Phi]} = \int \mathcal{D}\phi_0 \left(\int_{\substack{\Phi|_{C_k}=\phi_k,\, k\neq i,j \\ \Phi|_{C_i}=\phi_0=\Phi|_{C_j}}} \mathcal{D}\Phi \; \mathrm{e}^{-S_{\Sigma}[\Phi]} \right)$$

must hold true. The first axiom above tries to state this property in terms of well-defined quantities.
Segal's definition concerns the full conformal field theory as opposed to the "chiral halves" in which the fields depend only on one variable and not also on its conjugate. But many interesting results follow already from a study of the chiral theory, and Segal also gives a corresponding definition of *weakly conformal field theories* in terms of *modular functors* and states under which conditions full theories can be constructed from weakly conformal field theories. The precise definition is as abstract as the above one and bears similarly little resemblance to vertex operator algebras, which will be introduced and studied in later chapters. Since the view on conformal field theory of many physicists is deeply influenced by concepts that also play a key role in vertex operator algebra theory, the connections between this theory and Segal's approach of sewing Riemann surfaces shall now be sketched instead of pursuing Segal's work much further. (Note that in [HK] Huang and Kong constructed full genus-zero conformal field theories from vertex operator algebras in terms of *full field algebras*.) This is supposed to illuminate the relation between geometric and algebraic aspects of conformal field theory.

The sewing operation in the moduli space of punctured Riemann surfaces. The way from the geometric theory of Riemann surfaces to the algebraic theory of vertex operator algebras starts with a precise description of how certain Riemann surfaces can be "sewn". From the discussion of the definition of vertex operator algebras in section 2.1, which relates vertex operators with "vertices" in string diagrams, one can expect that only genus-zero Riemann surfaces are relevant for a study of vertex operator algebras. This is indeed the case, and from now on only punctured *spheres*, i.e. one-complex-dimensional, compact, connected genus-zero manifolds, will be considered. The importance of punctured surfaces to conformal field theory was discussed by Vafa in [V]. This led Huang to introduce the notion of geometric vertex operator algebra and study its properties in [Hua1], [Hua2] and [Hua5]. These structures are isomorphically related to (algebraic) vertex operator algebras. In order to give the definition of a geometric vertex operator algebra, a precise handling of the sewing operation is needed.

Instead of studying Riemann surfaces with boundaries as above one can equivalently consider *spheres with tubes*. A sphere with tubes of type (m, n) has m negatively and n positively oriented, distinct and ordered points (or punctures) with $m + n$ local analytic coordinate charts vanishing at their respective points. The relation to Segal's approach in terms of Riemann surfaces with boundaries is that (after a possible rescaling of the coordinate maps) spheres with tubes of type (m, n) correspond to morphisms in $\mathscr{C}$ with m copies of S^1 in the

domain and n copies of S^1 in the codomain. Using the sewing operation described below this equivalence can be made more precise.
It turns out that only spheres with tubes of type $(1,n)$ are needed to arrive at vertex operator algebras. On the other hand, the theory of (geometric) vertex operator coalgebras developed by Hubbard in [Hub] is based on spheres with tubes of type $(m,1)$. Using the sewing operation, spheres with tubes of arbitrary type can be obtained from the special classes of types $(m,1)$ and $(1,n)$, so no sphere is left behind.
Any sphere of type $(1,n)$ can be written as $(S; p_0, p_1, \ldots, p_n; (U_0, \varphi_0), (U_1, \varphi_1), \ldots, (U_n, \varphi_n))$, where $p_i \in U_i$ are its punctures, $\varphi_i : U_i \to \mathbb{C}$ are the local coordinates, $i \in \{0, \ldots, n\}$, and the index '0' refers to the single negatively oriented puncture. Two spheres

$$(S_1; p_0, \ldots, p_{n_1}; (U_0, \varphi_0), \ldots, (U_{n_1}, \varphi_{n_1})) \quad \text{and} \quad (S_2; q_0, \ldots, q_{n_2}; (V_0, \psi_0), \ldots, (V_{n_2}, \psi_{n_2}))$$

of types $(1,n_1)$ and $(1,n_2)$ are called *conformally equivalent* if $n_1 = n_2$ and there exists a complex analytic isomorphism $F : S_1 \to S_2$ such that $F(p_i) = q_i$ and φ_i is equal to $\psi_i \circ F$ on some neighborhood of p_i for all $i \in \{0, \ldots, n_1\}$. Furthermore, one says that for two such spheres S_1 and S_2, *the i-th tube of S_1 can be sewn with the 0-th tube of S_2* if there is a positive real number r such that $\bar{B}^r \subset \varphi_i(U_i)$, $\bar{B}^{1/r} \subset \psi_0(V_0)$ and p_i, q_0 are the only punctures in $\varphi_i^{-1}(\bar{B}^r)$, $\psi_0^{-1}(\bar{B}^{1/r})$, respectively, for all $i \in \{1, \ldots, n_1\}$. Then a sphere with tubes of type $(1, n_1 + n_2 - 1)$ is obtained by cutting $\varphi_i^{-1}(\bar{B}^r)$ and $\psi_0^{-1}(\bar{B}^{1/r})$, respectively, from S_1 and S_2 and identifying the boundaries of the resulting Riemann surfaces using the map $\psi_0^{-1} \circ \frac{1}{\cdot} \circ \varphi_i^{-1}$. This sphere with tubes of type $(1, n_1 + n_2 - 1)$ has ordered punctures $(p_0, \ldots, p_{i-1}, q_1, \ldots, q_{n_2}, p_{i+1}, \ldots, p_{n_1})$ and is denoted by $S_1\,{}_i\infty_0\,S_2$. The procedure to obtain it from S_1 and S_2 is called the *sewing operation*.
In Vafa's approach the *moduli space of spheres with tubes*, which is the set of all conformal equivalence classes of spheres with tubes, plays a fundamental role. To find an appropriate representative of an element of this moduli space, one makes use of the uniformization theorem of complex analysis which says that any sphere is conformally equivalent to the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. So any conformal equivalence class of spheres with tubes of type $(1,n)$ has a representative of the form $(p_0 = z_0, \ldots, p_n = z_n; f_0, \ldots, f_n)$, where the f_i, $i \in \{0, \ldots, n\}$, are suitable analytic coordinates. Furthermore, the fact that $SL(2,\mathbb{C})$ is the automorphism group of $\widehat{\mathbb{C}}$ allows to fix two of the $n+1$ punctures as $z_0 = \infty$ and $z_n = 0$, and the remaining conformal symmetry can be used to restrict f_0 to satisfy $\lim_{z\to\infty} z f_0(z) = 1$.
These restrictions already provide a representative without redundancy, but in concrete computations related to (geometric) vertex operator algebras another realization of the maps f_i is more convenient. Any such map can be expanded into a series as

$$f_i(z) = a_0^{(i)} \left(z + \sum_{k \in \mathbb{Z}_+} a_k^{(i)} z^{k+1} \right) = a_0^{(i)} \exp \left(\sum_{k \in \mathbb{Z}_+} A_k^{(i)} z^{k+1} \frac{\mathrm{d}}{\mathrm{d}z} \right) z \, .$$

Note that the last expression has the typical appearance of a vertex operator in string theory. The space of all sequences $A = \{A_k\}_{k \in \mathbb{Z}_+}$ for which the expression $\exp(\sum_{k\in\mathbb{Z}_+} A_k z^{k+1} \frac{\mathrm{d}}{\mathrm{d}z}) z$ converges in a neighborhood of $z = 0$ is denoted by H. So by the above discussion the moduli space of spheres with tubes of types $(1,0)$ and $(1,n)$ can be identified with $K(0) = \{A \in H \mid A_1 = 0\}$ and

$$K(n) = M^{n-1} \times H \times (\mathbb{C}^\times \times H)^n$$

for $n \in \mathbb{Z}_+$ and $M^{n-1} = \{(z_1,\ldots,z_{n-1}) \in \mathbb{C}^{n-1} \mid z_i \neq z_j \text{ for all } i \neq j\}$, respectively, and an element of $K(n)$ is represented by a canonical sphere with tubes of type $(1,n)$ denoted by

$$\left(z_1,\ldots,z_{n-1}; A^{(0)}, (a_0^{(1)}, A^{(1)}), \ldots, (a_0^{(n)}, A^{(n)})\right) \ . \tag{1.1}$$

In this notation the sewing operation is a partial map ${}_i\infty_0 : K(n_1)\times K(n_2) \to K(n_1+n_2-1)$. Writing $\mathbf{0}$ for the element of H whose components are all zero, the element $I = (\mathbf{0},(1,\mathbf{0})) \in K(1)$ has the properties of a unit: for all $n \in \mathbb{N}$, $Q \in K(n)$ and $i \in \{1,\ldots,n\}$, Q can be sewn with I and I leaves Q invariant, $Q\,{}_i\infty_0 I = Q = I\,{}_1\infty_0 Q$. Finally, an element σ of the permutation group of $n-1$ objects acts on elements of the form (1.1) to give an element of the form

$$\left(z_{\sigma^{-1}(1)},\ldots,z_{\sigma^{-1}(n-1)}; A^{(0)}, (a_0^{(\sigma^{-1}(1))}, A^{(\sigma^{-1}(1))}), \ldots, (a_0^{(\sigma^{-1}(n))}, A^{(\sigma^{-1}(n))})\right) \ .$$

Geometric vertex operator algebras. After the above precise discussion of the sewing operation now the definition of geometric vertex operator algebras due to Huang can be given. The reason why this notion is presented here is that geometric vertex operator algebras are isomorphic to (algebraic) vertex operator algebras. This relation is an important connection between Segal's and Vafa's geometric approach to conformal field theory and its fundamental algebraic aspects described by (algebraic) vertex operator algebras. The latter are the objects of main study in my diploma thesis, and the discussion in this chapter is supposed to show how they fit into a bigger picture.

To give the definition of geometric vertex operator algebras, one more notion needs to be introduced. Let $V = \coprod_{m\in\mathbb{Z}} V_{(m)}$ be a $\mathbb{Z}$-graded $\mathbb{C}$-vector space with finite-dimensional homogeneous subspaces, let $V' = \coprod_{m\in\mathbb{Z}} V_{(m)}^*$ be its graded dual, let $\overline{V} = \prod_{m\in\mathbb{Z}} V_{(m)}$ denote its algebraic completion and $\langle\,\cdot\,,\,\cdot\,\rangle$ the natural pairing between V' and $\overline{V}$. Also let $\mathcal{H}_V(m)$ for any $m \in \mathbb{N}$ denote the space of homomorphisms from $V^{\otimes m}$ to $\overline{V}$. Then for all $m \in \mathbb{Z}_+$, $n \in \mathbb{N}$ and any $i \in \{1,\ldots,m\}$, the *t-contraction* is defined as the map

$$\begin{aligned}(\,\cdot\,{}_i{*}_0\,\cdot\,)_t : \mathcal{H}_V(m) \times \mathcal{H}_V(n) &\longrightarrow \mathrm{Hom}(V^{\otimes m+n-1}, V[\![t,t^{-1}]\!]) \ , \\ (f,g) &\longmapsto (f\,{}_i{*}_0\,g)_t\end{aligned}$$

which acts as

$$(f\,{}_i{*}_0\,g)_t(v_1\otimes\ldots\otimes v_{m+n-1}) = \sum_{k\in\mathbb{Z}} f(v_1\otimes\ldots\otimes v_{i-1}\otimes\pi_k g(v_i\otimes\ldots\otimes v_{i+n-1})\otimes v_{i+n}\otimes\ldots\otimes v_{m+n-1})t^k$$

for all $v_1,\ldots,v_{m+n-1} \in V$, where $\pi_k : V \to V_{(k)}$ is the natural projection for all $k \in \mathbb{Z}$. If the formal Laurent series

$$\langle v', (f\,{}_i{*}_0\,g)_t(v_1\otimes\ldots\otimes v_{m+n-1})\rangle$$

is absolutely convergent at $t = 1$ for all $v' \in V'$, the resulting element $f\,{}_i{*}_0\, g = (f\,{}_i{*}_0\, g)_1 \in \mathcal{H}_V(m+n-1)$ is defined to be the *contraction of f and g*. Note how the operation of contraction is very similar in form to the operation of sewing two Riemann surfaces. Actually the contraction imitates the sewing operation on an algebraic level, and both operations appear together in the definition of a geometric vertex operator algebra. Again, as in the case of the moduli space of punctured spheres, there is an action of the permutation group S_m of m objects on $\mathcal{H}_V(m)$: for all $\sigma \in S_m$,

$$\sigma(f)(v_1 \otimes \ldots \otimes v_m) = f(\sigma^{-1}(v_1 \otimes \ldots \otimes v_m)) .$$

Definition. A *geometric vertex operator algebra* is a $\mathbb{Z}$-graded $\mathbb{C}$-vector space

$$V = \coprod_{m \in \mathbb{Z}} V_{(m)} \quad \text{with} \quad \dim V_{(m)} < \infty \quad \text{for all } m \in \mathbb{Z}$$

together with maps

$$\nu_n : K(n) \longrightarrow \mathcal{H}_V(n)$$

for all $n \in \mathbb{N}$. These data are subject to the following axioms:

(GV1) the *positive energy axiom*

$$V_{(m)} = 0 \quad \text{for all } m \ll 0 ;$$

(GV2) the *grading axiom*

$$\langle v', \nu_1(\mathbf{0}, (a, \mathbf{0}))(v)\rangle = a^{-m}\langle v', v\rangle$$

for all $m \in \mathbb{Z}$, $v' \in V'$, $v \in V_{(m)}$ and $a \in \mathbb{C}^\times$;

(GV3) the *meromorphicity axiom*: For all $m \in \mathbb{Z}_+$, $v' \in V'$, and $v_1, \ldots, v_m \in V$, the function

$$Q \longmapsto \langle v', \nu_m(Q)(v_1 \otimes \ldots \otimes v_m)\rangle$$

is meromorphic on $K(m)$, and if z_i and z_j are the i-th and j-th puncture of Q, respectively, then for any $v_i, v_j \in V$ there exists $N_{v_i,v_j} \in \mathbb{Z}_+$ such that for any $v' \in V'$, $v_k \in V$, $k \in \{1, \ldots, m\}$ and $k \neq i, j$, the order of the pole $z_i = z_j$ of $\langle v', \nu_m(Q)(v_1 \otimes \ldots \otimes v_m)\rangle$ is less than N_{v_i,v_j};

(GV4) the *permutation axiom*

$$\sigma(\nu_n(Q)) = \nu_n(\sigma(Q))$$

for all $n \in \mathbb{N}$, $\sigma \in S_n$ and $Q \in K(n)$;

(GV5) the *sewing axiom*: There exists a unique $c \in \mathbb{C}$ (called the central charge) such that if

$$Q_1 = \Big(z_1, \ldots, z_{m-1}; A^{(0)}, (a_0^{(1)}, A^{(1)}), \ldots, (a_0^{(m)}, A^{(m)})\Big) \in K(m) ,$$

$$Q_2 = \Big(z_1, \ldots, z_{n-1}; A^{(0)}, (b_0^{(1)}, B^{(1)}), \ldots, (b_0^{(n)}, B^{(n)})\Big) \in K(n)$$

and if the i-th tube of Q_1 can be sewn with the 0-th tube of Q_2, then $\nu_m(Q_1)\, {}_i{*}_0\, \nu_n(Q_2)$ exists and satisfies the relation

$$\nu_{m+n-1}(Q_1\, {}_i\infty_0 Q_2) = \nu_m(Q_1)\, {}_i{*}_0\, \nu_n(Q_2)\, \mathrm{e}^{-\Gamma(A^{(i)}, B^{(0)}, a_0^{(i)})c}\ ,$$

where $\Gamma(A^{(i)}, B^{(0)}, a_0^{(i)})$ is a certain series in the components of $A^{(i)}$, $B^{(0)}$ and $a_0^{(i)}$ which is given in (4.2.1), (4.2.2) in [Hua5].

The above definition should be compared with the definition of a vertex operator algebra and its discussion in section 2.1. The first and second axioms above can immediately be related to the corresponding axioms of a vertex operator algebra. The permutation axiom states that the ordering of punctures of spheres with tubes is not relevant for the structure of a geometric vertex operator algebra. The central axioms are the meromorphicity axiom and the sewing axiom as they both transport most of the geometric data into an algebraic setting. It turns out that the meromorphicity axiom is needed to obtain the important associativity and commutativity properties discussed at length in the next chapter. As mentioned above, the sewing axiom consistently relates the geometric operation of sewing to an algebraic treatment, and conversely it subtly introduces the central charge into the geometric description.
One of Huang's main results concerning (geometric) vertex operator algebras is that both notions are equivalent. In fact, the following even stronger statement is true and is proven in [Hua5].

Theorem. For all $c \in \mathbb{C}$, the category of geometric vertex operator algebras with central charge c is equivalent to the category of vertex operator algebras with central charge c.

This means that there is a one-to-one correspondence between the geometric and the algebraic description of genus-one conformal field theory. For example, the vacuum vector $\Omega \in V_{(0)}$ of a vertex operator algebra V is related to the element $\mathbf{0} \in K(0)$ of the moduli space of spheres with tubes via the identity $\langle v', \nu_0(\mathbf{0})\rangle = \langle v', \Omega\rangle$ for all $v' \in V'$. Also, a vertex operator $Y(\,\cdot\,, x)$ corresponds to the canonical spheres with three tubes $P(z) = (z; \mathbf{0}, (1, \mathbf{0})) \in K(2)$ via the relation

$$\langle v', (\nu_2(P(z)))(u \otimes v)\rangle = \langle v', Y(u, x)v\rangle\,\big|_{x=z}$$

for all $v' \in V'$ and $u, v \in V$. The moduli space element $P(z)$ also plays another important role in vertex operator algebra theory as will be discussed in section 2.3.

Chapter 2

Aspects of Vertex Operator Algebra

In this chapter the notion of vertex operator algebra is introduced and its definition is discussed in some detail, in particular with respect to its physical motivation and interpretation. Then several properties of vertex operator algebras and their modules are derived, with special emphasis on commutativity and associativity properties which correspond to the physical concepts of locality and meromorphic operator product expansion, respectively. After a short review of certain finiteness properties for vertex operator algebras and their relations, as well as a brief discussion of $\mathcal{W}$-algebras, the problem of the existence and associativity of the nonmeromorphic operator product expansion is addressed. To arrive at the corresponding result the relevant aspects of the theory of $P(z)$-tensor products are presented.

2.1 Basic notions

Formal calculus. In order to introduce and work with vertex operator algebras it is convenient and natural to use the language of *formal calculus.* This completely algebraic approach avoids subtle issues of convergence that must be taken into account if operators depend on complex numbers as variables. In physical applications one is of course eventually interested in computing correlation functions and the like, and these are evaluated in terms of complex numbers. But the theory can be developed quite far with *formal variables*, and manipulating expressions and deriving results is often easier with them, emphasizing and exploiting the algebraic structure. Furthermore, all relevant results can also be expressed in terms of complex numbers, and (paying careful attention) one is free to switch between both descriptions; examples of this will be presented in this and later sections.

For a given vector space V, a basic class of objects in the theory of vertex operator algebras is the set of *formal Laurent series*

$$V[\![x, x^{-1}]\!] = \left\{ \sum_{m\in\mathbb{Z}} v_m x^{-m-1} \;\middle|\; v_m \in V \right\} ,$$

where x denotes a formal variable and the indexing is a useful convention. Among the subspaces of $V[\![x,x^{-1}]\!]$ are the sets of V-valued polynomials $V[x]$, formal Laurent polynomials $V[x,x^{-1}]$, formal power series $V[\![x]\!] = \{\sum_{m\in\mathbb{N}} v_m x^{-m-1} \,|\, v_m \in V\}$, and truncated formal Laurent series

$$V(\!(x,x^{-1})\!) = \left\{ \sum_{m\in\mathbb{Z}} v_m x^{-m-1} \;\middle|\; v_m \in V \;,\; v_m = 0 \text{ for } m \gg 0 \right\} .$$

The expression formal power series is also used for the more general formal Laurent series if either the distinction is irrelevant or if the context clarifies the meaning. Formal Laurent series with more than one variable are defined in the obvious way, e.g. $V[\![x_1,x_1^{-1},x_2,x_2^{-1}]\!] = \{\sum_{m,n\in\mathbb{Z}} v_{m,n} x_1^{-m-1} x_2^{-n-1} \,|\, v_{m,n} \in V\}$. Also, a derivative operation $\frac{\mathrm{d}}{\mathrm{d}x}$ acts on a formal series as $\frac{\mathrm{d}}{\mathrm{d}x}\sum_{m\in\mathbb{Z}} v_m x^{-m-1} = \sum_{m\in\mathbb{Z}}(-m-1)v_m x^{-m-2}$, and a residue operation Res_x extracts the coefficient of x^{-1}: $\mathrm{Res}_x \sum_{m\in\mathbb{Z}} v_m x^{-m-1} = v_0$. Because of the physical interpretation in mind, writing an element $v(x) \in V[\![x,x^{-1}]\!]$ in the form $v(x) = \sum_{m\in\mathbb{Z}} v_m x^{-m-1}$ is called a *mode expansion*.

All the above spaces are subspaces of the huge space $V\{x\}$ of formal power series with arbitrary complex powers,

$$V\{x\} = \left\{ \sum_{h\in\mathbb{C}} v_h x^{-h-1} \;\middle|\; v_h \in V \right\} .$$

The arguably most important formal power series is the *δ-function*

$$\delta(x) = \sum_{m\in\mathbb{Z}} x^m \tag{2.1}$$

with coefficients (which are all equal to one) in $\mathbb{C}$. Note that the δ-function is not called a distribution (although one can use it to define a distribution for the space of power series); this displays the power of formal calculus: the expression (2.1) could not be written down meaningfully if x were a complex number. Instead, in this context of formal calculus, the question of convergence simply does not arise.

The name of the δ-function is motivated by the fact that for any $f(x) \in V[x,x^{-1}]$, the identity $f(x)\delta(x) = f(1)\delta(x)$ holds, which follows from the relation $x^m\delta(x) = \delta(x)$ for all $m \in \mathbb{Z}$, and this imitates formally the action of the (analytic) δ-distribution. Two somewhat more intricate and even more important results are the relations

$$x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) \tag{2.2}$$

and

$$x_1^{-1}\delta\left(\frac{x_2 - x_3}{x_1}\right) f(x_1,x_2,x_3) = x_1^{-1}\delta\left(\frac{x_2 - x_3}{x_1}\right) f(x_2 - x_3, x_2, x_3) \;, \tag{2.3}$$

the latter of which is true for all those elements $f(x_1, x_2, x_3) \in (\mathrm{End}V)[\![x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}]\!]$ for which

$$\lim_{x_1 \to x_2} f(x_1, x_2, x_3) = f(x_1, x_2, x_3)\Big|_{x_1 = x_2} \tag{2.4}$$

exists and

$$f(x_1, x_2, x_3)v \in V[\![x_1, x_1^{-1}, x_2, x_2^{-1}]\!]((x_3))$$

holds for all $v \in V$. The condition of the existence of the *algebraic limit* defined in (2.4) is that for any $v \in V$, the coefficient of each power of x_2 in the mode expansion $f(x_1, x_2, x_3)v|_{x_1=x_2}$ is a finite sum of elements in $V[\![x_3]\!]$. Certainly, whenever use will be made of relation (2.3), this will be valid.

In (2.2) and (2.3), some of the arguments of the formal power series are shifted from one formal variable to the difference of two formal variables. To evaluate such expressions consistently one needs a general prescription how to expand formal power series with sums (or differences) of formal variables as their arguments. An ambiguity arises whenever a negative power of such a sum is taken because then there are usually several possible expansions. Conventually, a binomial expansion of the form

$$(x+y)^m = \sum_{k \in \mathbb{N}} \binom{m}{k} x^{m-k} y^k \quad \text{where} \quad \binom{m}{k} = \frac{1}{k!} \prod_{i=0}^{k-1} (m-i) \quad \text{for all } m \in \mathbb{C} \tag{2.5}$$

is employed, i.e. binomial expressions are expanded such that the second variable in the sum appears only with nonnegative integral powers, and care is necessary because $(x+y)^m$ is different from $(y+x)^m$ for negative m. Analytically, this amounts to an expansion of $(z+w)^m$ in the domain $|z| > |w|$ for complex numbers z and w, and certainly in other domains the expansion can be different.

As an example of the application of the binomial expansion convention, part of the left-hand side of (2.3) is expanded into modes,

$$x_1^{-1}\delta\left(\frac{x_2 - x_3}{x_1}\right) = x_2^{-1}\delta\left(\frac{x_1 + x_3}{x_2}\right) = x_2^{-1} \sum_{m \in \mathbb{Z}} (x_1 + x_3)^m x_2^{-m} = \sum_{m \in \mathbb{Z}} \sum_{k \in \mathbb{N}} \binom{m}{k} x_1^{m-k} x_2^{-m-1} x_3^k\,,$$

where the first equality is due to relation (2.2).

It turns out to be useful to have a precise algebraic prescription of expanding expressions in several formal variables. Let ι_{ij} be the linear map

$$\iota_{ij} : \left\langle x_i, x_i^{-1}, x_j, x_j^{-1}, (x_i + x_j)^{-1}, (x_i - x_j)^{-1} \right\rangle \subset \mathbb{C}(x_i, x_j) \longrightarrow \mathbb{C}[\![x_i, x_i^{-1}, x_j, x_j^{-1}]\!]$$

such that $\iota_{ij}(f(x_i, x_j))$ is the power series expansion of $f(x_i, x_j)$ involving only finitely many negative powers of x_j, where $\mathbb{C}(x_i, x_j)$ denotes the space of rational functions in x_i and x_j. It is possible to generalize this definition to arbitrarily many formal variables, but this will not be needed in the following.

Definition of a vertex operator algebra. With the basic tools of formal calculus at hand, it is now possible to define the notion of a *vertex operator algebra*. This is the rigorous mathematical pendant to the physical notion of an "operator algebra" in the sense of a "chiral algebra", which became a widely used term after the seminal paper [BPZ] by Belavin, Polyakov and Zamolodchikov in 1984. The first axiomatic approach to the notion of a vertex operator algebra was given by Frenkel, Huang and Lepowsky in [FHL]. Further excellent introductions to the topic are the books [FLM], [Kac], [FBZ] and [LL], which all stress different aspects of the theory. Much of the material in this section is based on these references.

Definition. A vertex operator algebra is a $\mathbb{Z}$-graded $\mathbb{C}$-vector space

$$V = \coprod_{m\in\mathbb{Z}} V_{(m)} \quad \text{with} \quad \dim V_{(m)} < \infty \quad \text{for all } m \in \mathbb{Z} \tag{2.6}$$

together with a linear map $V \otimes V \to V[\![x, x^{-1}]\!]$, or equivalently

$$\begin{aligned} V &\longrightarrow (\mathrm{End}V)[\![x, x^{-1}]\!] \ , \\ v &\longmapsto Y(v, x) = \sum_{m\in\mathbb{Z}} v_m x^{-m-1} \ , \end{aligned}$$

where the formal power series $Y(v, x)$ is called the *vertex operator* associated to the element $v \in V$. Furthermore, there is a special element $\Omega \in V_{(0)}$ called the *vacuum* and another special element $\omega \in V_{(2)}$ called the *conformal vector*.
These data are subject to the following axioms for all $u, v \in V$:

(V1) the *truncation condition*

$$u_m v = 0 \quad \text{for all } m \gg 0 \ ; \tag{2.7}$$

(V2) the *vacuum property*

$$Y(\Omega, x) = \mathbb{1}_V \ ; \tag{2.8}$$

(V3) the *creation property*

$$Y(v, x)\Omega \in V[\![x]\!] \quad \text{and} \quad Y(v, x)\Omega\Big|_{x=0} = v \ , \tag{2.9}$$

which in terms of modes reads

$$v_m\Omega = 0 \quad \text{for all } m \in \mathbb{N} \quad \text{and} \quad v_{-1}\Omega = v \ ; \tag{2.10}$$

(V4) the *Jacobi identity*

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)Y(u, x_1) \\ &= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2) \ ; \end{aligned} \tag{2.11}$$

(V5) the modes L_m, $m \in \mathbb{Z}$, defined by

$$Y(\omega, x) = \sum_{m\in\mathbb{Z}} L_m x^{-m-2}$$

satisfy the commutation relations of the *Virasoro algebra* Vir,

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n,0} \quad \text{with } c \in \mathbb{C}\ , \tag{2.12}$$

and the homogeneous subspaces $V_{(m)}$ are exactly the eigenspaces of the operator L_0 with eigenvalues m;

(V6) the L_{-1}*-derivative property*

$$\frac{\mathrm{d}}{\mathrm{d}x} Y(v, x) = Y(L_{-1}v, x)\ . \tag{2.13}$$

The data and axioms of a vertex operator algebra can be thought of as the vacuum sector of *meromorphic conformal quantum field theory*, clearly stating its assumptions and general structure. Certainly, physicists dealt with such and similar structures long before Borcherds introduced the notion of a vertex algebra in [Bo1] in 1983. One prime example of a (speculative) physical theory which heavily relies on such conformal structures is perturbative string theory. Here, one pictures the worldsheets of interacting strings as Riemann surfaces embedded into higher dimensional spacetime, and correlation functions corresponding to physical observables in spacetime are computed in terms of fields that "live on" the two-dimensional worldsheet. So because of the conformal symmetry of perturbative string theory, processes in spacetime (with a dimension larger than two) are supposed to be described by a two-dimensional conformal field theory.
On the other hand, excited string states are represented by elements of a vector space, and there should be a correspondence between these vectors and the fields in the geometrical picture of interacting strings. This *operator-state-correspondence* is given by the above vertex operators $Y(\,\cdot\,, x)$ which associate operator-valued fields to all states in the space V. Furthermore, these operators capture the essence of a basic three-string-interaction, where two closed strings join into one. To first order in perturbation theory, this process is described by a sphere with three tubes corresponding to the incoming and outgoing strings attached to it, or equivalently (via a conformal map) three punctures, two of which are conventionally placed at 0 and ∞ using the conformal invariance, while the third puncture is at some point $z \in \mathbb{C}^\times$. This so-called "pair-of-paints" is mapped by Segal's functor $\mathscr{T}$ to a z-dependent homomorphism $V \times V \to V$. Using a formal variable x instead of a complex variable z, this homomorphism is the vertex operator $Y(\,\cdot\,, x)\,\cdot$. If the symbol 'Y' is rotated into '$\rightarrowtail$' it looks similar to a one-dimensional projection of such a pair-of-paints string diagram. This is why these linear maps are called vertex operators.
So in string theory, and also in statistical physics, the concept of vertex operators was

already used before it found its way into mathematics. This situation is quite similar to the one of vector spaces. The properties of vectors (represented by arrows) were already intuitively used to describe physical processes such as objects moving in space long before the modern axiomatic definition of vector spaces was formulated by Peano in 1888. And just as most people feel a little uneasy the very first time they are confronted with this axiomatic definition of vector spaces, the above definition of a vertex operator algebra might seem unnatural and too abstract at first. But just like vector spaces soon become quite familiar and useful objects to physicists, vertex operator algebras are a very natural and useful language to discuss chiral conformal field theories. The following brief discussion attempts to clarify some of the aspects of the definition of vertex operator algebras.

The relevance of vertex operators $Y(\,\cdot\,,x)$ has already been mentioned, and the existence of a vacuum vector Ω is also a familiar attribute. The vacuum property (2.8) can be thought of as the correspondence that the identity operation $\mathbb{1}_V$ which is the operator associated to the vacuum state reflects its simple and fundamental structure. The creation property (2.9) in a way says that any state v can be created out of the vacuum by acting on it with the appropriate operator. This operator is evaluated at $x = 0$, which corresponds to the infinite past in *radial coordinates* $z = \mathrm{e}^{\mathrm{i}(x^0+x^1)}$, and this is what is to be expected for an asymptotic incoming state in perturbation theory. In the theory of vertex operator algebras, formal variables x are used most of the time, but when complex numbers z appear as arguments in correlation functions or intertwining operators (to be defined below), radial coordinates and quantization should be assumed throughout. The reason that vertex operator algebras are said to describe meromorphic conformal field theories is that $Y(\,\cdot\,,x)$ depends only on x and not on a second conjugate variable.

The fact that in (2.6) the dimensions of all homogeneous subspaces $V_{(m)}$ are bounded is a simple finiteness condition, which is actually sometimes dropped. Another finiteness condition which is physically more important is the truncation condition (2.7). Formally, this is a technical assumption that ensures that certain operations like the multiplication of two vertex operators can be defined. But in its physical interpretation, it also encodes a restriction of a lower-bounded energy spectrum in the theory, as the operator L_0 is directly related to the Hamiltonian.

The operator L_0 is part of a representation of the Virasoro algebra on the vertex operator algebra V. The modes L_m, $m \in \mathbb{Z}$, of the Virasoro field or *energy momentum operator* $T(x) = Y(\omega, x)$ which corresponds to the conformal vector ω generate this representation together with the central element $c\mathbb{1}_V$. This representation implements the conformal symmetry in a two-dimensional quantum theory for the following reason: The conformal group $\mathrm{Conf}(\mathbb{R}^{1,1})$ in two-dimensional Minkowski space (whose compactification is $S^1 \times S^1$) is isomorphic to the product $\mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1)$ of the group of orientation preserving diffeomorphisms of S^1, as discussed e.g. in [Scho]. The Lie algebra of this group is the space of smooth vector fields on S^1, whose complexification $\mathrm{Vect}(S^1) \otimes \mathbb{C}$ has a dense subalgebra called the Witt algebra W. The Witt algebra is the conformal symmetry algebra of classical theories in two dimensions. Instead of studying its projective representations in the context of quantum theories, one is interested in its central extensions. Since the second cohomology group of the Witt algebra is one-dimensional, $H^2(\mathsf{W}, \mathbb{C}) \cong \mathbb{C}$, its central extensions on irre-

ducible representations are unique up to a complex parameter c. For a given c, this central extension is the Virasoro algebra (2.12). The appearance of the circle S^1 in this construction is directly related to its role in the objects of the category $\mathscr{C}$ in Segal's definition of conformal field theory.
The global subgroup of $\mathrm{Conf}(\mathbb{R}^{1,1})$ generated by L_0 and $L_{\pm 1}$ is the Möbius group. The element L_{-1} is the generator of spacetime translations, and this explains the L_{-1}-derivative property (2.13).
While the Virasoro algebra associated to the conformal vector ω is an important part of the definition of a vertex operator algebra, the axiom of the Jacobi identity (2.11) carries by far most of the structure as will be argued below. First, an explanation of its name is due. Let A, B, C be any elements of a Lie algebra. Then the (ordinary) Jacobi identity states that $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$, but this can also be expressed by

$$(\mathrm{ad}A)(\mathrm{ad}B)C - (\mathrm{ad}B)(\mathrm{ad}A)C = \mathrm{ad}((\mathrm{ad}A)B)C \ , \tag{2.14}$$

where $(\mathrm{ad}A)B = [A, B]$. This is strikingly similar to (2.11): in both cases there is some kind of a commutator on the left-hand side, while the right-hand side displays some kind of associativity, but in the case of vertex operator algebras, the situation is complicated by the appearance of δ-functions in several formal variables, which might be attributed to the "quantum" character of the theory. So the Jacobi identity (2.11) can be thought of as a generalization of the ordinary Jacobi identity (2.14).
Actually, it follows from the Jacobi identity (2.11) that vertex operator algebras appear as "quantum analogs" of both Lie algebras and associative commutative algebras. Also, in the presence of the other axioms, the Jacobi identity is equivalent to certain commutativity and associativity properties, which in turn represent fundamental physical concepts. This will be explained later in this section.

Commutation relations. In order to get a first idea of how the axioms of a vertex operator algebra and in particular the Jacobi identity can be used to arrive at interesting results, several commutation relations will be derived here. The method used in such calculations is to apply appropriate residue operations to the Jacobi identity to extract the relevant information.

To obtain commutation relations involving the Virasoro modes L_m, one sets $u = \omega$ while leaving v arbitrary in (2.11) and applies $\mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_1^{m+1}$ with $m \in \mathbb{Z}$ to both sides of the equation. For the left-hand side, this yields

$$\begin{aligned}
&\mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_1^{m+1}\left(x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(\omega, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(\omega, x_1)\right)\\
&= \mathrm{Res}_{x_1} x_1^{m+1}\sum_{n\in\mathbb{Z}}\left[x_1^{-n-2}L_n, Y(v, x_2)\right]\\
&= [L_m, Y(v, x_2)] \ .
\end{aligned} \tag{2.15}$$

Applying the same operation to the right-hand side gives

$$\begin{aligned}
&\mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_1^{m+1} x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) Y(Y(\omega,x_0)v,x_2)\\
=&\mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_1^{m}\delta\left(\frac{x_2+x_0}{x_1}\right) Y(Y(\omega,x_0)v,x_2)\\
=&\mathrm{Res}_{x_0}\mathrm{Res}_{x_1} x_1^{m}\sum_{k\in\mathbb{Z}}(x_2+x_0)^k x_1^{-k} Y(Y(\omega,x_0)v,x_2)\\
=&\mathrm{Res}_{x_0}\mathrm{Res}_{x_1}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\binom{k}{l} x_2^{k-l}x_0^{l}x_1^{m-k} Y\left(\sum_{n\in\mathbb{Z}} L_n v x_0^{-n-2},x_2\right)\\
=&\mathrm{Res}_{x_0}\sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\binom{m+1}{l} x_0^{l-n-2}x_2^{m+1-l} Y(L_n v,x_2)\\
=&\sum_{l\in\mathbb{N}}\binom{m+1}{l} x_2^{m+1-l} Y(L_{l-1}v,x_2)\ .
\end{aligned}\tag{2.16}$$

Thus, by comparing (2.15) and (2.16) one arrives at

$$[L_m,Y(v,x)]=\sum_{l\in\mathbb{N}}\binom{m+1}{l} x^{m+1-l} Y(L_{l-1}v,x)\ ,\tag{2.17}$$

which is true for any $v\in V$. Note that the sum on the right-hand side it finite because of the truncation condition (2.7). For the special case $m=0$, the equation (2.17) reads

$$[L_0,Y(v,x)]=Y(L_0v,x)+x\frac{\mathrm{d}}{\mathrm{d}x}Y(v,x)\ .$$

From this relation one can infer the *weight* of the modes of a homogeneous element v, which is denoted by $\mathrm{wt}v$. To do this, assume that $L_0v=hv$, i.e. $v\in V_{(h)}$. Then by expanding $Y(v,x)$ in the above equation into modes on both sides and comparing coefficients, one arrives at

$$[L_0,v_n]=(h-n-1)v_n \quad \text{which implies that} \quad v_nV_{(m)}\subset V_{(h+m-n-1)}\ ,\tag{2.18}$$

and this means that the modes v_n of a homogeneous vector $v\in V_{(h)}$ are of weight $h-n-1$. Another important consequence of (2.17) is the following: Suppose that v is a *primary vector of weight* h, i.e. $L_mv=0$ for all $m>0$ and $L_0v=hv$. Then by a reparametrization of the summation index it follows that

$$[L_m,Y(v,x)]=x^m\left(h(m+1)+x\frac{\mathrm{d}}{\mathrm{d}x}\right)Y(v,x)\ ,\tag{2.19}$$

which is a well-known equivalent characterization of a primary vector by the corresponding vertex operator or primary field.

More generally, by performing residue operations similarly to those that led to (2.17), one obtains the commutator relations for the modes of two arbitrary elements $u, v \in V$:

$$[u_m, v_n] = \sum_{k\in\mathbb{N}} \binom{m}{k} (u_k v)_{m+n-k} \,. \tag{2.20}$$

This equation is not only useful in explicit calculations, but it also states that the modes associated to the elements of a vertex operator algebra V form a Lie algebra with the commutator as its bracket. This is so because $u_k v$ is again an element in V, and by the truncation condition (2.7) the sum in (2.20) is finite. This is one of the reasons why vertex operator algebras can be seen as analogs of Lie algebras.
The commutation relation (2.20) is complemented by the following iterate relation for modes, which is obtained by equating the coefficients of $x_0^{-m-1}x_1^{-1}x_2^{-n-1}$ on both sides of the Jacobi identity:

$$(u_m v)_n = \sum_{i\in\mathbb{N}} (-1)^i \binom{m}{i} u_{m-i} v_{n+i} - \sum_{i\in\mathbb{N}} (-1)^{i+m} \binom{m}{i} v_{m+n-i} u_i \,. \tag{2.21}$$

This is also called the *Borcherds identity* as it appeared as an axiom of his original definition of vertex algebras.
Before proceeding to state further consequences of the axioms of vertex operator algebras, now a more general setting will be presented.

Modules for vertex operator algebras. In many cases the physically interesting properties of a theory are not directly described in terms of its fundamental mathematical structure, but rather does this structure manifest itself in terms of representations. This is certainly also the case in conformal field theory where one explicitly distinguishes between the vacuum sector and the remaining part of the theory, and the corresponding notion in the language used here is that of a *module for a vertex operator algebra V*. Given a vector space W, it is called a V-module if it is possible to transport a maximal amount of the vertex operator algebra structure of V to W. The precise definition is as follows.

Definition. Let V be a vertex operator algebra. A V-module is a $\mathbb{C}$-graded $\mathbb{C}$-vector space

$$W = \coprod_{h\in\mathbb{C}} W_{(h)} \quad \text{with} \quad \dim W_{(h)} < \infty \quad \text{for all } h \in \mathbb{C} \tag{2.22}$$

together with a linear map $V \otimes W \to W[\![x, x^{-1}]\!]$, or equivalently

$$\begin{aligned} V &\longrightarrow (\mathrm{End}W)[\![x, x^{-1}]\!] \,, \\ v &\longmapsto Y_W(v, x) = \sum_{m\in\mathbb{Z}} v_m^W x^{-m-1} \,, \end{aligned} \tag{2.23}$$

where the formal power series $Y_W(v, x)$ is called the *vertex operator acting on* W associated to the element $v \in V$.
These data are subject to the following axioms for all $u, v \in V$ and $w \in W$:

(M1) the *truncation condition*

$$u_m^W w = 0 \quad \text{for all } m \gg 0\ ; \tag{2.24}$$

(M2) the *vacuum property*

$$Y_W(\Omega, x) = \mathbb{1}_W\ ; \tag{2.25}$$

(M3) the *Jacobi identity*

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(u, x_1)Y_W(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W(v, x_2)Y_W(u, x_1)$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(Y(u, x_0)v, x_2)\ ; \tag{2.26}$$

(M4) the modes L_m^W, $m \in \mathbb{Z}$, defined by

$$Y_W(\omega, x) = \sum_{m\in\mathbb{Z}} L_m^W x^{-m-2}$$

satisfy the commutation relations of the Virasoro algebra

$$[L_m^W, L_n^W] = (m - n)L_{n+m}^W + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \quad \text{with } c \in \mathbb{C}\ ,$$

and the homogeneous subspaces $W_{(h)}$ are exactly the eigenspaces of the operator L_0^W with eigenvalues h;

(M5) the L_{-1}*-derivative property*

$$\frac{\mathrm{d}}{\mathrm{d}x}Y_W(v, x) = Y_W(L_{-1}v, x)\ . \tag{2.27}$$

Note that while the grading in (2.22) is now by $\mathbb{C}$,[1] the weights of the modes v_m^W are graded by $\mathbb{Z}$, so V-modules are still part of meromorphic conformal field theory. If there is no confusion the notation is simplified such that the index 'W' for the vertex operators on W and their modes is not displayed.

Any vertex operator algebra is a module for itself. Other prime examples are the minimal Virasoro models and the WZW theories, which are modules for the vertex operator algebras generated by the conformal vector and the generating vectors of Kac-Moody algebras, respectively; this will be made more precise in section 2.2. More generally, given any V-module $W = \coprod_{h\in\mathbb{C}} W_{(h)}$, one can construct another module W' from it in the following way. As a

[1] I am not aware of any physically meaningful conformal field theory with weights that have a nonzero imaginary part, and one might take the grading to be given by $\mathbb{R}$ or even $\mathbb{Q}$. But the general theory can also be developed with a complex grading, and there is no reason not to be general here.

vector space, W' is given by the *restricted dual*

$$W' = \coprod_{h\in\mathbb{C}} W^*_{(h)} \ .$$

Since all the homogeneous subspaces of W' are finite-dimensional, there is a natural pairing between W and W' which is denoted by $\langle\,\cdot\,,\,\cdot\,\rangle$. With this, one can define a linear map

$$\begin{aligned} V &\longrightarrow (\mathrm{End}W')[\![x,x^{-1}]\!] \ , \\ v &\longmapsto Y'(v,x) = \sum_{m\in\mathbb{Z}} v'_m x^{-m-1} \ , \end{aligned}$$

via the relation

$$\Big\langle Y'(v,x)w', w\Big\rangle = \Big\langle w', Y\left(\mathrm{e}^{xL_1}\left(-x^{-2}\right)^{L_0} v, x^{-1}\right) w\Big\rangle$$

for all $v \in V$, $w \in W$ and $w' \in W'$, where $Y^{\mathrm{o}}(v,x) = Y(\mathrm{e}^{xL_1}\left(-x^{-2}\right)^{L_0} v, x^{-1})$ is called the *opposite vertex operator* to $Y(v,x)$. An important result is the following theorem which is obtained by an explicit check of the axioms of a module for a vertex operator algebra.

Theorem. (W',Y') is a V-module.

The module (W',Y') is called the *contragredient module* since the map $(\,\cdot\,)'$ is a contravariant functor on the category of modules for vertex operator algebras. Also, there is a natural isomorphism between W and its double-contragredient module, and it follows from the definition that

$$\langle \psi'_m w', w\rangle = \langle w', \psi_{-m} w\rangle$$

for all $m \in \mathbb{Z}$ and all primary vectors ψ whose associated vertex operators are expanded into modes as $\sum_{m\in\mathbb{Z}} \psi_m x^{-m-\mathrm{wt}\psi}$.
The natural pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ is fundamental as it allows to introduce correlation functions, and the important associativity and commutativity properties discussed below will be formulated in terms of this pairing and the contragredient module.

Intertwining operators. Modules for vertex operator algebras alone are not enough to describe conformal field theories. What is needed is a way for different modules to "communicate" or "interact" with each other, but this relationship should respect the relevant vertex operator algebra structure. The corresponding generalization of vertex operators is the notion of *intertwining operators* which are also called "chiral operators".

Definition. Let V be vertex operator algebra and let (W_1,Y_1), (W_2,Y_2) and (W_3,Y_3) be V-modules. An intertwining operator of type $\binom{W_3}{W_1\,W_2}$ is a linear map $W_1 \otimes W_2 \to W_3\{x\}$, or equivalently

$$W_1 \longrightarrow (\mathrm{Hom}(W_2,W_3))\{x\} \ ,$$

$$w_{(1)} \longmapsto \mathcal{Y}(w_{(1)}, x) = \sum_{m \in \mathbb{C}} (w_{(1)})^{\mathcal{Y}}_m x^{-m-1} .$$

These data are subject to the following axioms for all $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$:

(IO1) the *truncation condition*

$$(w_{(1)})_m w_{(2)} = 0 \quad \text{for all } m \text{ with } \mathrm{Re}(m) \gg 0 ;$$

(IO2) the *Jacobi identity*

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_3(v, x_1)\mathcal{Y}(w_{(1)}, x_2)w_{(2)} \\ & \quad - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) \mathcal{Y}(w_{(1)}, x_2)Y_2(v, x_1)w_{(2)} \\ = {} & x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}(Y_1(u, x_0)w_{(1)}, x_2)w_{(2)} ; \end{aligned}$$

(IO3) the L_{-1}*-derivative property*

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{Y}(w_{(1)}, x) = \mathcal{Y}(L_{-1}w_{(1)}, x) .$$

The dimension of the vector space formed of all intertwining operators of type $\binom{W_3}{W_1\,W_2}$ is denoted by $N^{W_3}_{W_1 W_2}$ and is called the *fusion rule* for W_1, W_2 and W_3.

The fusion rules have two important symmetry properties. Writing $N^{W_k}_{W_i W_j} \equiv N^k_{ij}$, they read $N^k_{ij} = N^k_{ji}$ and $N^k_{ij} = N^{j'}_{ik'}$, as can be proven with the help of the so-called *skew-symmetry* of vertex operators and the properties of the contragredient module. By the definition $N_{ijk} = N^{i'}_{jk}$ this can also be expressed as

$$N_{ijk} = N_{\sigma(i)\sigma(j)\sigma(k)}$$

for all permutations σ of $\{i, j, k\}$.

Associativity and commutativity properties. For vertex operators acting on modules, certain duality and locality properties follow rather straightforwardly from the definitions. In several approaches to vertex operator algebras, these properties are actually taken to replace the Jacobi identity. This is understandable as they are easier to state and their physical relevance is much more obvious. Nevertheless here the Jacobi identity is taken to be the main axiom because of the way it encodes nearly all of the important properties of vertex operator algebra theory, and because in some more advanced applications it seems to be indispensable.

Theorem. Let V be a vertex operator algebra, W a V-module, and $u, v \in V$, $w \in W$, $w' \in W'$. Then the following *associativity property* holds between the product and the iterate of two vertex operators:

$$\iota_{12}^{-1}\langle w', Y(u,x_1)Y(v,x_2)w\rangle = \left(\iota_{20}^{-1}\langle w', Y(Y(u,x_0)v,x_2)w\rangle\right)\Big|_{x_0=x_1-x_2} . \tag{2.28}$$

Because the proof of this statement further illustrates how to use the Jacobi identity to extract nontrivial results, it is given here in detail.[2]
The identity (2.28) can be proven in three steps. Firstly, one uses the Jacobi identity together with relations (2.2) and (2.3) to find

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1)w \\
&= x_1^{-1}\delta\left(\frac{x_0+x_2}{x_1}\right)Y(u,x_1)Y(v,x_2)w - x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y(Y(u,x_0)v,x_2)w \\
&= x_1^{-1}\delta\left(\frac{x_0+x_2}{x_1}\right)Y(u,x_0+x_2)Y(v,x_2)w - x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y(Y(u,x_0)v,x_2)w .
\end{aligned}$$

This is now multiplied on both sides with x_1^N for some as yet unspecified $N \in \mathbb{N}$; taking then Res_{x_1}, one arrives at

$$\begin{aligned}
&\mathrm{Res}_{x_1}x_1^N x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1)w \\
&= (x_0+x_2)^N\left(Y(u,x_0+x_2)Y(v,x_2)w - Y(Y(u,x_0)v,x_2)w\right) .
\end{aligned}$$

Because of the truncation property (2.24) one can now choose N such that $u_m w = 0$ for all $m \geq N$. Then the left-hand side of the above equation vanishes, with the result

$$(x_0+x_2)^N Y(u,x_0+x_2)Y(v,x_2)w = (x_0+x_2)^N Y(Y(u,x_0)v,x_2)w . \tag{2.29}$$

This is called the *weak associativity* of two vertex operators.
In a second step it will be shown that

$$\langle w', Y(u,x_0+x_2)Y(v,x_2)w\rangle = \iota_{02}p(x_0,x_2) \tag{2.30}$$

and

$$\langle w', Y(Y(u,x_0)v,x_2)w\rangle = \iota_{20}p(x_0,x_2) \tag{2.31}$$

for a rational function $p(x_0,x_2)$ of the form

$$p(x_0,x_2) = \frac{q(x_0,x_2)}{x_0^k(x_0+x_2)^l x_2^m}$$

[2]Though it would be satisfactory to prove all statements made here, limitations of space and time advise against it.

where $q(x_0, x_2)$ is a polynomial and $k, l, m \in \mathbb{Z}$. To do this, use is made of the weak associativity (2.29) in the form

$$(x_0 + x_2)^l \langle w', Y(u, x_0 + x_2) Y(v, x_2) w \rangle = \langle w', (x_0 + x_2)^l Y(Y(u, x_0)v, x_2) w \rangle \ .$$

Because of the truncation property (2.24) and the binomial expansion convention, the left-hand side of this equation involves only finitely many negative powers of x_2. From this, together with the relation (2.18), it follows that there appear only finitely many positive powers of x_0 on the left-hand side. Using an analogous argument, the right-hand side involves only finitely many negative powers of x_0 and only finitely many positive powers of x_2. Because they are equal, both are simply given by a formal Laurent polynomial $r(x_0, x_2) \in \mathbb{C}[x_0, x_0^{-1}, x_2, x_2^{-1}]$. The rational function $p(x_0, x_2) = r(x_0, x_2)(x_0 + x_2)^l$ then satisfies the stated conditions.
The final step aims at translating the formal variable x_0 to $x_1 - x_2$ in the above expressions. This substitution can readily be made on the left-hand side of (2.30). Noting that

$$\iota_{12} p(x_1 - x_2, x_2) = \left(\iota_{02} p(x_0, x_2)\right)\Big|_{x_0 = x_1 - x_2} \ ,$$

this yields

$$\langle w', Y(u, x_1) Y(v, x_2) w \rangle = \iota_{12} p(x_1 - x_2, x_2) \ .$$

Together with (2.31) this gives the desired result (2.28).

The above theorem on associativity can be rephrased in a way that is more familiar from physical applications of conformal field theory. Indeed, in terms of complex variables z_1, z_2 it states that

$$\langle w', Y(u, z_1) Y(v, z_2') w \rangle \quad \text{and} \quad \langle w', Y(Y(u, z_1 - z_2)v, z_2) w \rangle$$

are absolutely convergent to a common rational function in the domains

$$|z_1| > |z_2| > 0 \quad \text{and} \quad |z_2| > |z_1 - z_2| > 0 \ ,$$

respectively. It is important to observe that such relations really only hold within the natural pairing $\langle \cdot \, , \cdot \rangle$ in this context, i.e. (2.28) is not true in general if the product and iterate of two vertex operators are equated as formal series in x_1 and x_2 outside the "matrix elements". But when calculating physical observables in concrete theories, one is really only interested in these matrix elements or *correlation functions*, and these are also the objects of interest in other axiomatic approaches such as those by Osterwalder and Schrader in [OS1], [OS2] or Gaberdiel and Goddard in [GG]. In this context, the associativity above is also called *duality* and plays a central role in the theory.

Similarly to the proof of weak associativity (2.29) one can show that *weak commutativity* holds in the setting of a vertex operator algebra V: for all $u, v \in V$ there exists $N \in \mathbb{N}$ such that

$$(x_1 - x_2)^N [Y(u, x_1), Y(v, x_2)] = 0 \ . \tag{2.32}$$

Replacing the Jacobi identity by this relation yields an equivalent definition of a vertex operator algebra. It is important to note that it is not valid to obtain $[Y(u,x_1),Y(v,x_2)] = 0$ from (2.32) by multiplying it with $(x_1-x_2)^{-N}$ because the tripel product $(x_1-x_2)^{-N}(x_1-x_2)^N[Y(u,x_1),Y(v,x_2)]$ fails to exist in the sense alluded to in conjunction with (2.4).
More generally, with slightly less effort than in the case of associativity, one can prove the following result.

Theorem. Let V be a vertex operator algebra, W a V-module, and $u, v \in V$, $w \in W$, $w' \in W'$. Then the following *commutativity property* holds for two vertex operators:

$$\iota_{12}^{-1}\langle w', Y(u,x_1)Y(v,x_2)w\rangle = \iota_{21}^{-1}\langle w', Y(v,x_2)Y(u,x_1)w\rangle \ . \tag{2.33}$$

This property is also referred to as *locality* because of its relevance in the Wightman axioms discussed in [SW] and the fact that in a one-dimensional quantum theory, any nonidentical two points are spacelike separated.

The associativity and commutativity properties in the above two theorems are among the reasons why vertex operator algebras cannot only be seen as quantum analogs of Lie algebras, but also of associative commutative algebras, despite the fact that these two notions cannot coincide in a nontrivial way classically.

Meromorphic operator product expansion. Physically, the associativity property of vertex operators describes the short distance behaviour of two quantum fields as it only holds for sufficiently close complex variables z_1 and z_2. It turns out that the form (2.28) of the associativity is also the one that appears in more general and deeper parts of the theory to be described below.
On the other hand, this is not exactly the form in which it is usually used in two-dimensional physics, where it appears as the *meromorphic operator product expansion.* The notion of operator product expansion was originally introduced by Wilson in [Wi] and Kadanoff in [Kad] and is very powerful since products of fields occur in particular in correlation functions, which eventually allow to compute observables which can then be compared with experimental data. One important advantage of operator product expansion, viewed as a tool that expands the product of two fields into a series in which each summand involves only one single field, is that in this way n-point functions can be expressed in terms of $(n-1)$-point functions. This does not only tremendously facilitate concrete computations, but it also structures the theory conceptually and makes it completely solvable in many cases. This is why a reformulation of associativity will be sketched here.

To start with, the associativity property can also be expressed by stating that for all $u, v \in V$ and $w \in W$ the three expressions

$$\begin{aligned} Y(u,x_1)Y(v,x_2)w &\in W((x_1))((x_2)) \ , \\ Y(v,x_2)Y(u,x_1)w &\in W((x_2))((x_1)) \ , \\ Y(Y(u,x_1-x_2)v,x_2)w &\in W((x_2))((x_1-x_2)) \end{aligned}$$

are the expansions of one and the same element of

$$W[\![x_1,x_2]\!][x_1^{-1},x_2^{-1},(x_1-x_2)^{-1}]$$

in their respective domains, see [FBZ]. Note that here $Y(Y(u,x_1-x_2)v,x_2)w$ is not considered as an element of $W[\![x_1,x_1^{-1},x_2,x_2^{-1}]\!]$ (as which it might not even exist) but as an element of $W(\!(x_2)\!)(\!(x_1-x_2)\!)$. Having these subtleties in mind, one may write the meromorphic operator product expansion as

$$Y(u,x_1)Y(v,x_2)w = \sum_{m\in\mathbb{Z}}(x_1-x_2)^{-m-1}Y(u_m v,x_2)w\ . \tag{2.34}$$

In order to proceed the following results obtained by Kac in [Kac] are very helpful. A similar treatment by Matsuo and Nagatomo can be found in [MN1], [MN2].

Proposition. The following two assertions are equivalent to the locality

$$(x_1-x_2)^N[f(x_1),g(x_2)]=0 \quad \text{for } N\gg 0$$

of two arbitrary formal power series $f(x_1)$ and $g(x_2)$.

(i) There exist N formal power series $h_i(x_2)$, $i\in\{0,\dots,N-1\}$, such that

$$[f(x_1),g(x_2)] = \sum_{i=0}^{N-1}\frac{1}{i!}\,h_i(x_2)\left(\frac{\mathrm{d}}{\mathrm{d}x_2}\right)^i\tilde{\delta}(x_1-x_2)\ ,$$

where $\tilde{\delta}(x_1-x_2)=x_2^{-1}\delta(x_1/x_2)$ has the propery $t(x_1)\tilde{\delta}(x_1-x_2)=t(x_2)\tilde{\delta}(x_1-x_2)$ for all formal power series $t(x_1)$.

(ii) There exist N formal power series $h_i(x_2)$, $i\in\{0,\dots,N-1\}$, such that

$$f(x_1)g(x_2) = \sum_{i=0}^{N-1}\left(\iota_{12}(x_1-x_2)^{-i-1}\right)h_i(x_2) + :f(x_1)g(x_2):\ ,$$

$$g(x_2)f(x_1) = \sum_{i=0}^{N-1}\left(\iota_{21}(x_1-x_2)^{-i-1}\right)h_i(x_2) + :f(x_1)g(x_2):\ ,$$

where $:\cdot:$ denotes the *normal-ordered product* defined by

$$:f(x_1)g(x_2): = f(x_1)_+g(x_2)+g(x_2)f(x_1)_- \tag{2.35}$$

with $f(x_1)_\pm$ denoting the regular and singular part of $f(x_1)$ in x_1, respectively.

Using this proposition, it follows from the weak commutativity (2.32) that the product of two vertex operators can also be written as

$$Y(u,x_1)Y(v,x_2) = \sum_{i=0}^{N-1} (x_1-x_2)^{-n-1} y_i(x_2) + :Y(u,x_1)Y(v,x_2):$$

where $y_i(x_2)$, $i \in \{0,\dots,N-1\}$, are some formal power series. But this means that $Y(u,x_1)Y(v,x_2)w \in W((x_1))((x_2))$ is an expansion of

$$\left(\sum_{i=0}^{N-1} (x_1-x_2)^{-n-1} y_i(x_2) + :Y(u,x_1)Y(v,x_2):\right) w \in W[\![x_1,x_2]\!][x_1^{-1},x_2^{-1},(x_1-x_2)^{-1}] \,.$$

Comparing this with the corresponding expansion in (2.34) allows to identify the formal power series $y_i(x_2)$ with $Y(u_iv,x_2)$. Using the above proposition again then yields

$$Y(u,x_1)Y(v,x_2) = \sum_{i=0}^{N-1} \left(\iota_{12}(x_1-x_2)^{-i-1}\right) Y(u_iv,x_2) + :Y(u,x_1)Y(v,x_2):\,, \tag{2.36a}$$

$$Y(v,x_2)Y(u,x_1) = \sum_{i=0}^{N-1} \left(\iota_{21}(x_1-x_2)^{-i-1}\right) Y(u_iv,x_2) + :Y(u,x_1)Y(v,x_2):\ . \tag{2.36b}$$

This is another form of the meromorphic operator product expansion, stating clearly that it actually involves two crucial pieces of information. In the physics literature, this is often shortened by suppressing the second part, not specifying the domain of expansion and discarding the normal-ordered terms which are regular in x_1-x_2:

$$Y(u,x_1)Y(v,x_2) \sim \sum_{i=0}^{N-1} (x_1-x_2)^{-i-1} Y(u_iv,x_2)\,.$$

It has already been stressed that the interaction between different modules described by intertwining operators is particularly interesting, so a natural question to ask is whether there is also an associativity property for intertwining operators. In contrast to the case of vertex operators acting on modules discussed above, one should expect the corresponding result to be a *nonmeromorphic operator product expansion*, where also noninteger powers of the variables may appear. Indeed, such a result can be obtained, but it involves a lot more effort than the meromorphic case. Several aspects of this together with the precise statement of the nonmeromorphic operator product expansion will be reviewed in section 2.3.

2.2 Finiteness properties and $\mathcal{W}$-algebras

Any nontrivial vertex operator algebra is a very large structure, and even though the Jacobi identity (or equivalently duality and locality) imposes certain restrictions, it is not always easy to organize a vertex operator algebra into concise substructures. As in conformal field theory all the possible modules also have to be taken into account, one might feel overstrained. But fortunately, in many interesting cases vertex operator algebras display various kinds of finiteness properties which make it more manageable to deal with the whole structure. Prominent among such properties are the notions of rationality, regularity, C_2-cofiniteness and the Zhu algebra, which also have an interesting interdependence.

Rationality and regularity. Let V be a vertex operator algebra. Instead of asking under which circumstances the set of V-modules is particularly "small", one can also be interested in more general classes of such modules, and many relevant results also hold in these more general cases. A *weak V-module* is a structure that necessarily satisfies all the axioms of a V-module, except those related to the grading of the vector space. This means that (2.22) might not hold, and there is no additional condition on the operator L_0. Certainly any module is also a weak module.
A weak V-module W is called *admissible* if it admits an $\mathbb{N}$-grading $W = \coprod_{n\in\mathbb{N}} W_n$ such that $v_m W_n \subset W_{\mathrm{wt}v+n-m-1}$ for all $v \in V$. This is parallel to the situation of an ordinary module besides the fact that the grading must not necessarily coincide with the L_0-eigenvalues. Admissible modules are also called $\mathbb{N}$-*gradable*.

Now it is possible to define an important finiteness property. A vertex operator algebra V is called *weakly rational* if every admissible V-module is a direct sum of irreducible V-modules. Actually, there are a number of different notions of rationality in the literature, and the one just introduced is the one defined by Dong, Li and Mason in [DLM]. They also showed that a rational vertex operator algebra has only finitely many isomorphism classes of irreducible admissible modules. This is obviously a strong finiteness condition, and this is why it appears in the definition of Zhu given in [Z].
Additional requirements for rationality are often imposed in the physics literature, though it is not always precisely stated what notion of rationality is actually used. For example because of the importance of conformal field theory on a torus for the general construction, the convergence of (generalized) characters and their closure with respect to modular transformations is sometimes taken to be part of the definition of rational vertex operator algebras. Such properties are not relevant for the part of the theory that is considered here, and therefore they are not discussed any further here. Instead, another kind of finiteness is assumed for rationality, namely that all fusion rules N_{ij}^k (i.e. the dimensions of the spaces of intertwining operators) are finite. This means that here the definition of Huang and Lepowsky from [HL3] is adopted: A vertex operator algebra V is defined to be *rational* if

(i) there exist only finitely many irreducible V-modules up to equivalence;

(ii) every V-module is completely reducible;

(iii) the fusion rules for all triplets of V-modules are finite.

It has been stated above that the first condition is redundant, but it is still taken to be part of the definition in order to stress its importance. Also, for certain natural classes of vertex operator algebras, the third condition is automatically satisfied as will be discussed below.

An apparently somewhat stronger requirement than rationality is regularity. A vertex operator algebra V is called *regular* if any weak V-module is a direct sum of irreducible V-modules.

The vertex operator algebras associated to the minimal Virasoro models, WZW models and the moonshine module are among the best-studied vertex operator algebras and have received much attention regarding possible physical applications. This is partly so because of the fact that they are all rational and regular. In order to get an idea of how the finiteness property of rationality can manifest itself the first two cases are briefly reviewed here.
The Virasoro algebra (2.12) has two subalgebras $\mathcal{L}_\pm = \coprod_{m\in\pm\mathbb{Z}_+} \mathbb{C}L_m$. Consider the trivial one-dimensional $\mathcal{L}_+$-module $\mathbb{C}\Omega_{c,h}$ on which the central charge element and the operator L_0 act by multiplication with the complex numbers c and h, respectively. Then the *Verma module* $M(c,h)$ is defined to be the free $\mathcal{L}_-$-module generated by $\Omega_{c,h}$. It has a unique maximal proper submodule denoted by $J(c,h)$, so the quotient $L(c,h) = M(c,h)/J(c,h)$ is irreducible. One can show that $L(c,0)$ has the structure of a vertex operator algebra whose vacuum and conformal vector are $\Omega_{c,0}$ and $L_{-2}\Omega_{c,0}$, respectively. Using the Kac determinant formula and results of Feigin and Fuchs, Wang proved in [Wa] in which cases this vertex operator algebra is rational:

Theorem. The vertex operator algebra $L(c,0)$ is rational if and only if c is either zero or equal to

$$c_{p,q} = 1 - 6\frac{(p-q)^2}{pq} \quad \text{with } p,q \in \mathbb{Z}_{\geq 2} \text{ relatively prime.} \tag{2.37}$$

Furthermore, a (finite) set of representatives of equivalence classes of irreducible modules for $L(c_{p,q},0)$ is

$$\{L(c_{p,q}, h_{r,s}(p,q))\}_{0<r<p,0<s<q;r,s\in\mathbb{N}} \quad \text{with} \quad h_{r,s}(p,q) = \frac{(sp-rq)^2-(p-q)^2}{4pq} \ .$$

For the case of WZW models, let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra whose Killing form is denoted by $(\,\cdot\,,\,\cdot\,)$ and normalized such that $(\theta,\theta)=2$ for the highest root θ, and let $\mathfrak{h}\subset\mathfrak{g}$ be a Cartan subalgebra. The associated *untwisted affine Lie algebra* or Kac-Moody algebra $\widehat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$ has the bracket

$$[a\otimes t^m, b\otimes t^n] = [a,b]\otimes t^{m+n} + (a,b)m\delta_{m+n,0}K \ .$$

$\widehat{\mathfrak{g}}$ has two subalgebras $\widehat{\mathfrak{g}}_\pm = \mathfrak{g}\otimes t^{\pm1}\mathbb{C}[t^{\pm1}]$. If for $\lambda\in\mathfrak{h}^*$, $L(\lambda)$ denotes the irreducible highest-weight $\mathfrak{g}$-module with highest weight λ, then $L(\lambda)$ can be taken to be a trivial $\widehat{\mathfrak{g}}_+$-module on which K acts as a complex number k. From this one obtains a $\widehat{\mathfrak{g}}$-module $\widehat{M}(k,\lambda)$ by letting $\widehat{\mathfrak{g}}_-$ act freely on $L(\lambda)$. Let $\widehat{J}(k,\lambda)$ be the maximal proper graded submodule of $\widehat{M}(k,\lambda)$ and $\widehat{L}(k,\lambda) = \widehat{M}(k,\lambda)/\widehat{J}(k,\lambda)$. Frenkel and Zhu showed in [FZ] that for $k\neq h^\vee$, $\widehat{L}(k,0)$ has the structure of a vertex operator algebra whose vacuum and conformal vector are given by 1 and

$$\omega = \frac{1}{2(k+h^\vee)}\sum_{i=1}^{\dim\mathfrak{g}}(a^i_{-1})^2 1 \ ,$$

respectively, where $h^\vee$ is the dual Coxeter number and $\{a^i\}_{i\in\{1,\dots,\dim\mathfrak{g}\}}$ is an orthonormal basis for $\mathfrak{g}$. This realization of the Virasoro algebra in terms of affine Lie algebras is also called the *Sugawara construction.*

Theorem. If $k \neq h^\vee$ and $k \in \mathbb{N}$, the vertex operator algebra $\widehat{L}(k,0)$ is rational, and a (finite) set of representatives of equivalence classes of irreducible modules for $\widehat{L}(k,0)$ is

$$\left\{\widehat{L}(k,\lambda) \;\middle|\; \lambda \in \mathfrak{h}^*,\, \dim L(\lambda) < \infty,\, (\lambda,\theta) \le k\right\} .$$

***C_n*-cofiniteness.** Rationality and regularity are related in several ways to another form of "finiteness", C_2-cofiniteness, which will also play an important role in the next chapter.

Let V be a vertex operator algebra and W a V-module. If the subspaces

$$\begin{aligned} C_1(W) &= \operatorname{span}\left\{u_{-1}w \;\middle|\; u \in \coprod_{m>0} V_{(m)},\, w \in W\right\} , \\ C_n(W) &= \operatorname{span}\left\{u_{-n}w \;\middle|\; u \in V,\, w \in W\right\} \quad \text{for } n \ge 2, \end{aligned}$$

are of finite codimension in W, i.e. $\dim(W/C_n(W)) < \infty$ for $n \in \{1,\dots,n\}$, then W is called C_n*-cofinite.*
Because of the L_{-1}-derivative property written in the form

$$Y\left(L_{-1}^m v, x\right) = \frac{\mathrm{d}^m}{\mathrm{d}x^m} Y(v,x) ,$$

it directly follows by comparing coefficients that

$$v_{-m-1} = \frac{1}{m}\left(L_{-1}v\right)_{-m} = \frac{1}{m!}\left(L_{-1}^m v\right)_{-1} \qquad \text{for all } m \in \mathbb{Z}_+ , \tag{2.38}$$

and hence every C_m-cofinite module W is also C_n-cofinite for all $m \ge n \ge 1$. For $n = 1$, this can also be expressed by writing

$$C_1(W) = \operatorname{span}\left\{u_{-m}w \;\middle|\; u \in \coprod_{n>0} V_{(n)},\, w \in W,\, m \in \mathbb{Z}_+\right\} . \tag{2.39}$$

Another relevant property of the subspaces $C_n(W)$ is that they are invariant under the action of v_m for all $v \in V$ and $m \le 0$. To prove this, the commutation relation (2.20) for modes is written in the alternate form

$$v_m u_{-n} w = u_{-n} v_m w + \sum_{i\in\mathbb{N}} \binom{m}{i} \left(v_i u\right)_{m-n-i} w . \tag{2.40}$$

For $m \leq 0$ the right-hand side of this equation obviously is an element of $C_n(W)$ because of the relation (2.38), and so this must also be true for the left-hand side. The result is

$$v_m C_n(W) \subset C_n(W) \quad \text{for all } v \in V,\ m \in \mathbb{Z}_{\leq 0},\ n \in \mathbb{Z}_+ \,. \tag{2.41}$$

The condition of C_2-cofiniteness was introduced by Zhu in [Z] and subsequently used to prove the convergence and modular invariance of characters for certain vertex operator algebras, and it is also related to his famous associative algebra $A(V)$ which is introduced below.
But C_2-cofiniteness is also important because of its close relation to rationality and regularity. Indeed, it was proven by Li in [L] that any regular vertex operator algebra is also C_2-cofinite, and Abe, Buhl and Dong were able to show in [ABD] that regularity is equivalent to rationality and C_2-cofiniteness together for vertex operator algebras $V = \coprod_{m\in\mathbb{N}} V_{(m)}$ with $V_{(0)} = \mathbb{C}\Omega$ (such vertex operator algebras are also said to be of *CFT type*). Furthermore, it was proven by theses authors and Gaberdiel and Neitzke in [GN] that for a C_2-cofinite vertex operator algebra, all fusion rules for irreducible weak modules are finite, and as a consequence the third condition for rationality on page 28 is also redundant.
What makes this relationship particularly interesting is the fact that while rationality explicitly concerns the modules for a vertex operator algebra, the C_2-cofiniteness condition can be studied solely in terms of the vertex operator algebra itself, without reference to any modules.

Zhu's algebra $A(V)$. In his analysis of correlation functions in conformal field theories defined on genus-one Riemann surfaces, Zhu also introduced a certain associative algebra which he used to construct an explicit basis for the so-called conformal block on the torus under suitable conditions. In many cases of interest, this algebra is particularly useful to classify all irreducible modules for a vertex operator algebra.

Let V be a vertex operator algebra. Then the vector space $O(V) \subset V$ is defined to be the linear span of all elements of the form

$$\mathrm{Res}_x \left(x^{-2}(x+1)^{\mathrm{wt}\,u} Y(u,x)v \right)$$

where $u, v \in V$ with u a homogeneous element. The space $O(V)$ can be shown to be a two-sided ideal for the product operation $*$ defined by

$$u * v = \mathrm{Res}_x \left(x^{-1}(x+1)^{\mathrm{wt}\,u} Y(u,x)v \right)$$

and linearity, where $u, v \in V$ as above. Thus, $*$ is defined on the *Zhu algebra* $A(V) = V/O(V)$, and it turns out that $(A(V), *)$ is an associative algebra whose unit element is the equivalence class $[\Omega]$ of the vacuum vector.
It was pointed out by Brungs and Nahm in [BN] that the Zhu algebra is naturally isomorphic to the *zero mode algebra* familiar to physicists working on conformal field theory.

In the context of the Zhu algebra, the grading of any V-module W is written such that $W = \coprod_{m\in\mathbb{N}} W_{(m)}$ with $W_{(0)} \neq 0$. This is always possible because of the truncation condition

(2.24). The following result due to Zhu already illustrates the relevance of the algebra $A(V)$ for the representation theory of vertex operator algebras.

> *Theorem.* Let V be a vertex operator algebra and W a V-module. Then $W_{(0)}$ forms a representation space for the algebra $A(V)$ given by the linear extension of the map
>
> $$\begin{aligned} A(V) &\longrightarrow \mathrm{End}W_{(0)} \ , \\ [v] &\longmapsto v_{\mathrm{wt}v-1} \ , \end{aligned}$$
>
> where $v \in V$ is a homogeneous element. Conversely, for every representation $A(V) \rightarrow \mathrm{End}R$ of the algebra $A(V)$, there exists a V-module W such that $W_{(0)} = R$. Furthermore, the set of equivalence classes of irreducible V-modules is in one-to-one correspondence to the set of equivalence classes of irreducible representations of $A(V)$.

Certainly this theorem is particularly interesting if $A(V)$ is finite-dimensional because in this case it follows that there are only finitely many inequivalent irreducible V-modules. Indeed, it is true in general that irreducible representations of finite-dimensional algebras are finite-dimensional themselves, and there are only finitely many equivalence classes of such irreducible representations. This is why the following result by Dong, Li and Mason obtained in [DLM], which places an upper bound on the dimension of $A(V)$ in terms of C_2-cofiniteness, is of considerable practical use.

> *Proposition.* Let V be a C_2-cofinite vertex operator algebra. Then the algebra $A(V)$ is finite-dimensional and $\dim A(V) \leq \dim\tilde{V}$ for all subspaces $\tilde{V} \subset V$ such that $V = C_2(V) + \tilde{V}$.

$\mathcal{W}$-algebras. A *$\mathcal{W}$-algebra* of type $\mathcal{W}(2, h_1, \ldots, h_m)$ is a vertex operator algebra which has a minimal generating set consisting of the vacuum Ω, the conformal vector ω of weight 2 and m additional primary vectors W^i of weight h_i, $i \in \{1, \ldots, m\}$. The vertex operators or fields associated to these vectors are simple in the sense that they are not normal-ordered products of other fields. Sometimes the term $\mathcal{W}$-algebra is also used to refer to the algebra of modes and their normal-ordered products instead of the vertex operator algebra. In this situation the term *maximally extended symmetry algebra* is synonymous for $\mathcal{W}$-algebra because it is the maximal structure of a given type that contains the Virasoro algebra and satisfies the (ordinary) Jacobi identity.

$\mathcal{W}$-algebras by themselves do not necessarily satisfy finiteness conditions as those discussed above, but still much of their structure can be inferred from the properties of the finitely many vectors W^i. This becomes particularly apparent using Nahm's results on the *quasi-primary normal-ordered product.* For a more extensive introduction to $\mathcal{W}$-algebras, see e.g. [Nah2] or [Fl1].

A formal power series is called *quasi-primary* if the identity (2.19) holds for $m \in \{\pm 1, 0\}$. The usual normal-ordered product $:\phi_i(x)\phi_j(y): = \phi_i(x)_+\phi_j(y) + \phi_j(y)\phi_i(x)_-$ of two quasi-primary fields $\phi_i(x)$ and $\phi_j(y)$ is not necessarily quasi-primary for $x = y$. One of Nahm's results is that it is always possible to add certain correction terms, yielding a quasi-primary normal-ordered product denoted by $\mathcal{N}(\,\cdot\,,\,\cdot\,)$:

$$\begin{aligned}\mathcal{N}\left(\phi_j, \partial^n \phi_i\right) = & \sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{2(h_i+h_j+n-1)}{r}^{-1} \binom{2h_i+n-1}{r} \\ & \cdot \partial^r N^{(h_i+n+r)}\left(\phi_j, \partial^{n-r}\phi_i\right) \\ & - (-1)^n \sum_{\{k \,|\, h(ijk)\geq 1\}} C_{ij}^k \binom{h(ijk)+n-1}{n} \\ & \cdot \binom{2(h_i+h_j+n-1)}{n}^{-1} \binom{2h_i+n-1}{h(ijk)+n} \binom{\sigma(ijk)-1}{h(ijk)-1}^{-1} \\ & \cdot \frac{\partial^{h(ijk)+n}\phi_k}{(\sigma(ijk)+n)(h(ijk)-1)} \,. \end{aligned} \tag{2.42}$$

Here, $\{\phi_k\}_k$ is the family of quasi-primary fields of the corresponding $\mathcal{W}$-algebra, h_k are their respective weights, $h(ijk) := h_i + h_j - h_k$ and $\sigma(ijk) := h_i + h_j + h_k - 1$. The *structure constants* C_{ij}^k are defined such that $\sum_l C_{ij}^l d_{lk} = C_{ijk}$ with

$$C_{ijk} = \left\langle \Omega', (\phi_k)_{+h_k} (\phi_i)_{-h_k+h_j} (\phi_j)_{-h_j}\Omega \right\rangle \quad \text{and} \quad d_{ij} = \left\langle \Omega', (\phi_i)_{+h_i}(\phi_j)_{-h_j}\Omega\right\rangle \ ,$$

and the $N^{(\,\cdot\,)}$-product is defined by the relations

$$N^{(m)}(\phi,\psi)(x) = \sum_{n\in\mathbb{Z}} x^{-n-h_\phi-h_\psi} N^{(m)}(\phi,\psi)_n \ , \tag{2.43a}$$

$$N^{(m)}(\phi,\psi)_n = \sum_{k<m} \phi_{n+k}\psi_{-k} + \sum_{k\geq m} \psi_{-k}\phi_{n+k} \tag{2.43b}$$

for any $m \in \mathbb{Z}$. The quasi-primary normal-ordered product of more than two fields is defined recursively, for example $\mathcal{N}(\phi_i,\phi_j,\phi_k) = \mathcal{N}(\phi_i, \mathcal{N}(\phi_j,\phi_k))$. If the product of a field with itself is considered the notation is simplified, for example $\mathcal{N}(\psi,\psi) = \mathcal{N}(\psi^2)$.
Furthermore, in this notation the commutators of modes are given by

$$\left[(\phi_i)_m, (\phi_j)_n\right] = d_{ij}\delta_{m+n,0}\binom{h_i+m-1}{2h_i-1} + \sum_{\{k \,|\, h(ijk)\geq 1\}} C_{ij}^k \, p_{h_i,h_j,h_k}(m,n)(\phi_k)_{m+n} \tag{2.44}$$

in terms of the *universal polynomials*

$$p_{h_i,h_j,h_k}(m,n) = \sum_{r,s\in\mathbb{N}} \delta_{r+s,h(ijk)-1} a_{ijk}^r \binom{m+n-h_k}{r}\binom{h_i-n-1}{s}$$

with

$$a_{ijk}^r = \binom{2h_k+r-1}{r}^{-1}\binom{h_i+h_k-h_j+r-1}{r} \, .$$

Note that in the definition of the structure constants C_{ij}^k and in the relation (2.44), the convention for the indices of modes associated to the fields ϕ_i is not the one introduced before, i.e. any vertex operator is expanded into a series $\sum_{n\in\mathbb{Z}} v_n x^{-n-1}$ regardless of the weight of the associated vector v. This is the convention that is used in most of the mathematics literature. On the other hand, in the physics literature it is common to expand a field that is associated to a vector u of weight h into a series $\sum_{n\in\mathbb{Z}} u_n^{\text{phys}} x^{-n-h}$. The latter convention is used here only in the context of $\mathcal{W}$-algebras. When comparing results expressed in differing notations, the relation $u_n = u_{n-h+1}^{\text{phys}}$ is used.

In conformal field theory in general and in the study of $\mathcal{W}$-algebras in particular, the formal power series known as the *character*

$$\chi_V(q) = \mathrm{tr}_V q^{L_0 - c/24} = q^{-c/24} \sum_{n\in\mathbb{N}} \dim V_{(n)}\, q^n$$

of the vertex operator algebra $V = \mathcal{W}(2, h_1, \ldots, h_m)$ is of fundamental importance. In the next chapter, this character will be compared with the character of the vacuum Verma module of the $\mathcal{W}$-algebra. This is the induced module

$$U(\mathcal{W}(2, h_1, \ldots, h_m)) \otimes_{U(\mathcal{W}(2,h_1,\ldots,h_m)_{(+)})} \mathbb{C}_c \ ,$$

where $U(\,\cdot\,)$ denotes the universal enveloping algebra of the $\mathcal{W}$-algebra, the space $\mathcal{W}(2, h_1, \ldots, h_m)_{(+)}$ is defined by

$$\mathcal{W}(2, h_1, \ldots, h_m)_{(+)} = \coprod_{n\le 1} \mathbb{C}L_{-n} \oplus \coprod_{i=1}^{m} \coprod_{n_i \le h_i - 1} \mathbb{C}W^i_{-n_i} \ ,$$

and $\mathbb{C}_c$ is the trivial $\mathcal{W}(2, h_1, \ldots, h_m)_{(+)}$-module of central charge c. In other words, the vacuum Verma module is generated freely by the action of the modes L_n and W_n^i on a nonzero element Ω in $\mathbb{C}_c$, subject to the restrictions

$$L_n\Omega = 0 \text{ for all } n \ge -1 \quad \text{and} \quad W_n^i\Omega = 0 \text{ for all } n \ge -h_i + 1 \ . \tag{2.45}$$

Because of these restrictions, the dimensions of the homogeneous subspaces $V_{(n)}$ are smaller than $p(n)$, where $p(n)$ is the number of partitions of n into sums of positive integers, generated by the function

$$(\varphi(q))^{-1} = \prod_{n\ge 1} (1 - q^n)^{-1} = \sum_{n\in\mathbb{N}} p(n) q^n \ .$$

Taking the restrictions (2.45) into account, the vacuum Verma module character is given by

$$\chi_V^{\text{Verma}}(q) = q^{-c/24} (\varphi_2(q))^{-1} \prod_{i=1}^{m} (\varphi_{h_i}(q))^{-1} \ , \tag{2.46}$$

where the generating functions φ_k, $k \ge 2$, have been introduced as truncated φ-functions:

$$\varphi_k(q) = \prod_{n\ge k} (1 - q^n) = \varphi(q) \prod_{l=1}^{k-1} \left(1 - q^l\right)^{-1} \ . \tag{2.47}$$

2.3 $P(z)$-tensor product theory

This section takes up the question of an associativity property for intertwining operators, i.e. of the nonmeromorphic operator product expansion. The first problem is that there does not seem to be a natural way to make sense of an identity like (2.28) in the case of intertwining operators. The reason is that in order for the product of two such operators to exist, they must be of type $\binom{W_4}{W_1\, M}$ and $\binom{M}{W_2\, W_3}$, respectively, for some modules M and W_i, $i \in \{1, 2, 3, 4\}$. But in general, two operators of such types cannot be iterated.
To find appropriate associated intertwining operators that can be iterated, Huang and Lepowsky developed a theory of certain tensor products of modules for vertex operator algebras in [HL3], [HL4], [HL5] and [Hua3] which was generalized by Huang, Lepowsky and Zhang in [HLZ]. This theory is already very interesting for its own sake and can easily be motivated: For a vertex operator algebra V and n V-modules $W_1, \ldots, W_n$, one can naturally endow $V^{\otimes n}$ with a vertex operator algebra structure which has a module $W_1 \otimes \ldots \otimes W_n$. But this vector space is not a module for V itself. Finding the correct tensor product operation that leaves the space of V-modules invariant also eventually solves the problem of an associative nonmeromorphic operator product expansion.

Generalized modules and logarithmic intertwining operators. The theory just advertised can actually be developed in a more general setting than that presented in section 2.1. This is of special interest because this generalization aims at formulating *logarithmic conformal field theory*[3] in terms of vertex operator algebras at the level of modules.

Let V be a vertex operator algebra. A *generalized V-module W* is a structure that satisfies all the axioms of a V-module except that its grading is not given by L_0-eigenspaces but by generalized L_0-eigenspaces, so that W can be written as

$$W = \coprod_{h \in \mathbb{C}} W_{[h]} \quad \text{with} \quad W_{[h]} = \{w \in W \mid (L_0 - h)^m\, w = 0 \text{ for } m \gg 0\} \ .$$

From now on, all generalized modules will be assumed to be objects of a full subcategory $\mathcal{C}$ (of the category whose objects are $\mathbb{R}$-graded generalized modules for any fixed vertex operator algebra V) that is closed under the contragredient functor $(\,\cdot\,)'$.
Next, following [Mil] and [HLZ], a new formal variable denoted by $\log x$ is introduced which is not exactly to be thought of as the logarithm of the formal variable x, because this would lead to inconsistencies regarding the existence of certain formal power series. Despite the suggestive notation, $\log x$ is a formal variable independent of the formal variable x in most respects. But still, this variable is defined to behave in the expected way under differentiation: On a formal power series $f(x, \log x) = \sum_{m,n\in\mathbb{C}} f_{m,n}(\log x)^m x^n \in W\{x, \log x\}$,

[3]Logarithmic conformal field theory owes its name to the existence of logarithmic divergencies in correlation functions of such theories, but certain indecomposability properties seem to be even more fundamental features – certainly from an algebraic point of view, which also suggests (chiral) logarithmic conformal field theory to be some sort of a generalization of ordinary (chiral) conformal field theory. Physically, it seems to be particularly interesting for disordered systems, but there is also a number of other physical models with "logarithmic" features, see e.g. [Fl3], [Gab2] and references therein.

the operator $\frac{\mathrm{d}}{\mathrm{d}x}$ acts as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{m,n\in\mathbb{C}} f_{m,n}(\log x)^m x^n\right) = \sum_{m,n\in\mathbb{C}} \left((n+1)f_{m,n+1} + (m+1)f_{m+1,n+1}\right)(\log x)^m x^n \ . \quad (2.48)$$

To have a convenient notation, the dependence of a formal power series $f(x, \log x)$ on $\log x$ is suppressed. In order to do this consistently, the following notational conventions are imposed:

$$\begin{aligned} f(x+y) &= \sum_{m,n\in\mathbb{C}} f_{m,n}(x+y)^n \left(\log x + \sum_{i\in\mathbb{Z}_+} \frac{(-1)^{i-1}}{i}\left(\frac{y}{x}\right)^i\right)^m , \\ f(x\mathrm{e}^y) &= \sum_{m,n\in\mathbb{C}} f_{m,n}x^n \mathrm{e}^{ny}(\log x + y)^m , \\ f(xy) &= \sum_{m,n\in\mathbb{C}} f_{m,n}x^n y^n(\log x + \log y)^m . \end{aligned}$$

In the first line a logarithm as a series in the formal variables x and y was expanded, and the binomial expansion convention applies as usual. The notation is chosen such that the expected properties with respect to differentiation hold; for example, by an involved computation it can be shown that $\mathrm{e}^{y\frac{\mathrm{d}}{\mathrm{d}x}}f(x) = f(x+y)$ and $\mathrm{e}^{xy\frac{\mathrm{d}}{\mathrm{d}x}}f(x) = f(x\mathrm{e}^y)$.

Definition. Let (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) be generalized V-modules. A *logarithmic intertwining operator* of type $\binom{W_3}{W_1\ W_2}$ is a linear map $W_1 \otimes W_2 \to W_3[\log x]\{x\}$, or equivalently

$$\begin{aligned} W_1 &\longrightarrow (\mathrm{Hom}(W_2, W_3))[\log x]\{x\} , \\ w_{(1)} &\longmapsto \mathcal{Y}(w_{(1)}, x) = \sum_{m\in\mathbb{C}}\sum_{a\in\mathbb{N}} (w_{(1)})^{\mathcal{Y}}_{m,a} x^{-m-1}(\log x)^a , \end{aligned}$$

with only finitely many nontrivial powers of $\log x$. These data are subject to the following axioms for all $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$:

(ĩO1) the *truncation condition*

$$(w_{(1)})_{m,a} w_{(2)} = 0 \quad \text{for all } m \text{ with } \mathrm{Im}(m) \gg 0, \text{ independently of } a;$$

(ĩO2) the *Jacobi identity*

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_3(v, x_1)\mathcal{Y}(w_{(1)}, x_2)w_{(2)} \\ & \quad - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) \mathcal{Y}(w_{(1)}, x_2)Y_2(v, x_1)w_{(2)} \\ &= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}(Y_1(u, x_0)w_{(1)}, x_2)w_{(2)} ; \end{aligned}$$

(ĨO3) the L_{-1}-*derivative property*

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{Y}(w_{(1)},x) = \mathcal{Y}(L_{-1}w_{(1)},x) \ .$$

The vector space formed of all logarithmic intertwining operators of type $\binom{W_3}{W_1\ W_2}$ is denoted by $\mathcal{L}^{W_3}_{W_1W_2}$.

Definition of the $P(z)$-tensor product. The abstract definition of a kind of tensor product that leaves the space of modules for a vertex operator algebra invariant is motivated by the exact relation of vertex operator algebra theory to the geometrical formulation of conformal field theory in terms of (the sewing of) Riemann surfaces with punctures which describe string interactions. More precisely, the canonical element $P(z)$ of the moduli space of spheres with punctures which has three ordered punctures at ∞, $z \in \mathbb{C}^\times$ and 0 is the geometric object that corresponds to intertwining operators.

Definition. Let (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) be generalized V-modules. A $P(z)$-*intertwining map* of type $\binom{W_3}{W_1\ W_2}$ is a linear map

$$F : W_1 \otimes W_2 \longrightarrow \overline{W_3} \equiv \prod_{h\in\mathbb{C}} (W_3)_{[h]} = (W_3')^*$$

subject to the following axioms for all $n \in \mathbb{C}$, $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$:

(IM1) the *truncation condition*

$$\pi_{n-m}F(w_{(1)} \otimes w_{(2)}) = 0 \quad \text{for } m \gg 0 \ ,$$

where $\pi_h : \overline{W_3} \to (W_3)_{[h]}$ denotes the natural projection;

(IM2) the *Jacobi identity*

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right) Y_3(v, x_1)F(w_{(1)} \otimes w_{(2)}) - x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right) F(w_{(1)} \otimes Y_2(v, x_1)w_{(2)}) \\ &= z^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) F(Y_1(u, x_0)w_{(1)} \otimes w_{(2)}) \ . \end{aligned}$$

The vector space formed of all intertwining maps of type $\binom{W_3}{W_1\ W_2}$ is denoted by $\mathcal{M}[P(z)]^{W_3}_{W_1W_2}$ or simply $\mathcal{M}^{W_3}_{W_1W_2}$ if the dependence on z is clear from the context. Furthermore, a $P(z)$-*product of* W_1 *and* W_2 is a module W_3 together with an intertwining map F of type $\binom{W_3}{W_1\ W_2}$ which is denoted by $(W_3, Y_3; F)$, and a *morphism* between two $P(z)$-products $(W_3, Y_3; F)$ and $(W_4, Y_4; G)$ is a module map $\eta : W_3 \to W_4$, i.e. $\eta(v_n^{W_3}w_{(3)}) = v_n^{W_4}\eta(w_{(3)})$, with $G = \overline{\eta} \circ F$.

There is an isomorphic correspondence between logarithmic intertwining operators and $P(z)$-intertwining maps: To any $\mathcal{Y} \in \mathcal{L}^{W_3}_{W_1 W_2}$ one can associate an element $F_{\mathcal{Y}} \in \mathcal{M}^{W_3}_{W_1 W_2}$ by setting $F_{\mathcal{Y}}(w_{(1)} \otimes w_{(2)}) = \mathcal{Y}(w_{(1)}, z)w_{(2)}$ for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, and which in general is an element of the algebraic completion $\overline{W_3}$ of W_3. Conversely, for an intertwining map $F \in \mathcal{M}^{W_3}_{W_1 W_2}$ one obtains a logarithmic intertwining operator $\mathcal{Y}_F$ of the same type by

$$\mathcal{Y}_F(w_{(1)}, x)w_{(2)} = y^{L_0} x^{L_0} F\left(y^{-L_0} x^{-L_0} w_{(1)} \otimes y^{-L_0} x^{-L_0} w_{(2)}\right)\Big|_{y=z^{-1}} .$$

Definition. Let W_1 and W_2 be generalized V-modules. A *$P(z)$-tensor product of W_1 and W_2* is a $P(z)$-product $(W_0, Y_0; F_0)$ of W_1 and W_2 such that for all $P(z)$-products $(W, Y; F)$ of W_1 and W_2 there is a unique morphism from $(W_0, Y_0; F_0)$ to $(W, Y; F)$. If this $P(z)$-tensor product exists then it is denoted by

$$\left(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)}\right) ,$$

and $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ is called the *$P(z)$-tensor product module of W_1 and W_2*.

Note that while $\boxtimes_{P(z)}$ is an intertwining map, $W_1 \boxtimes_{P(z)} W_2$ does not denote the image of this map because by definition the former is a generalized module. The image of $W_1 \otimes W_2$ under $\boxtimes_{P(z)}$ is in the algebraic completion $\overline{W_1 \boxtimes_{P(z)} W_2}$.

Constructing $\boxtimes_{P(z)}$. It does not follow obviously from the above definition of the $P(z)$-tensor product that it actually exists. The strategy to construct it uses the notion of the contragredient module.

The first step is to define an operator $Y'_{P(z)}(\,\cdot\,, x)$ whose restriction to a suitable subspace of $(W_1 \otimes W_2)^*$ will be the contragredient vertex operator. Define the linear map $\tau_{P(z)}$: $V \otimes \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}] \to \mathrm{End}(W_1 \otimes W_2)^*$ implicitly via the relation

$$\begin{aligned}
&\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1} - z}{x_0}\right) Y_t(v, x_1)\kappa\right) \\
&= z^{-1}\delta\left(\frac{x_1^{-1} - x_0}{z}\right) \kappa\left(Y_1\left(e^{x_1 L_1}\left(-x_1^{-2}\right)^{L_0} v, x_0\right) w_{(1)} \otimes w_{(2)}\right) \\
&\quad + x_0^{-1}\delta\left(\frac{z - x_1^{-1}}{x_0}\right) \kappa\left(w_{(1)} \otimes Y_2^{\mathrm{o}}(v, x_1) w_{(2)}\right) ,
\end{aligned}$$

for all $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $\kappa \in (W_1 \otimes W_2)^*$, where $Y_t(v, x) \in (V \otimes \mathbb{C}[t, t^{-1}])[\![x, x^{-1}]\!]$ is the formal power series $\sum_{m\in\mathbb{Z}} (v \otimes t^m) x^{-m-1}$. Then $Y'_{P(z)}(v, x)$ is defined to be $\tau_{P(z)}(Y_t(v, x))$, and it follows by taking a residue that $Y'_{P(z)}(\,\cdot\,, x)$ acts as

$$\begin{aligned}
&\left(Y'_{P(z)}(v, x_1)\kappa\right)\left(w_{(1)} \otimes w_{(2)}\right) \\
&= \kappa\left(w_{(1)} \otimes Y_2^{\mathrm{o}}(v, x_1) w_{(2)}\right) \\
&\quad + \mathrm{Res}_{x_0} z^{-1}\delta\left(\frac{x_1^{-1} - x_0}{z}\right) \kappa\left(Y_1\left(e^{x_1 L_1}\left(-x_1^{-2}\right)^{L_0} v, x_0\right) w_{(1)} \otimes w_{(2)}\right)
\end{aligned}$$

and is a linear map $Y'_{P(z)}(\,\cdot\,, x) : V \to \mathrm{End}(W_1 \otimes W_2)^*[\![x, x^{-1}]\!]$. Since the full and not the restricted dual of the tensor product appears here, $Y'_{P(z)}(\,\cdot\,, x)$ in general is not a vertex operator and it will be necessary to restrict it to a suitable subspace of $(W_1 \otimes W_2)^*$.
To proceed, let $(W, Y; F)$ be any $P(z)$-product of W_1 and W_2 and define the linear map $F' : W' \to (W_1 \otimes W_2)^*$ by $\langle F'(w'), w_{(1)} \otimes w_{(2)}\rangle = \langle w', F(w_{(1)} \otimes w_{(2)})\rangle$ with $w' \in W'$. Then the following intermediate result holds.

Proposition. If

$$W_1 \boxbslash_{P(z)} W_2 = \bigcup_{P(z)\text{-products }(W,Y;F)} F'(W') \subset (W_1 \otimes W_2)^* \tag{2.49}$$

together with $Y'_{P(z)}$ is an object in $\mathcal{C}$, then by denoting $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ its contragredient module, the $P(z)$-tensor product exists and is given by $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; i')$ where $i : W_1 \boxbslash_{P(z)} W_2 \hookrightarrow (W_1 \otimes W_2)^*$ is the natural inclusion.

From the definition (2.49) it follows that every $\kappa \in W_1 \boxbslash_{P(z)} W_2 \subset (W_1 \otimes W_2)^*$ satisfies the following nontrivial conditions.

The $P(z)$-*compatibility condition*:

(i) The *lower-truncation condition*: Only finitely many negative powers of x appear in the formal power series $Y'_{P(z)}(v, x)\kappa$ for all $v \in V$.

(ii) $\tau_{P(z)}\left(x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right) Y_t(v, x_1)\right)\kappa = x_0^{-1}\delta\left(\frac{x_1^{-1}-z}{x_0}\right) Y'_{P(z)}(v, x_1)\kappa$ holds for all $v \in V$.

The $P(z)$-*local grading restriction condition*:

(i) The *grading condition*: κ is a finite sum of generalized eigenvectors in $(W_1 \otimes W_2)^*$ for the operator $(L'_{P(z)})_0$.

(ii) The minimal subspace W_κ of $(W_1 \otimes W_2)^*$ which satisfies $\kappa \in W_\kappa$ and $\tau_{P(z)}(v \otimes t^m)W_\kappa \subset W_\kappa$ for all $v \in V$ and $m \in \mathbb{Z}$ (which states stability with respect to the modes of $Y'_{P(z)}(v, x)$) has finite-dimensional homogeneous subspaces with respect to the $(L'_{P(z)})_0$-grading, and these subspaces vanish for generalized $(L'_{P(z)})_0$-eigenvalues with sufficiently small real parts.

These conditions give rise to an equivalent characterization of the $P(z)$-tensor product:

Proposition. If for any element $\kappa \in (W_1 \otimes W_2)^*$ which satisfies the $P(z)$-compatibility condition and the $P(z)$-local grading restriction condition the generalized module generated by the action of the modes of $Y'_{P(z)}(v, x)$ on κ is an object in $\mathcal{C}$ for all $v \in V$, then the subspace of $(W_1 \otimes W_2)^*$ consisting of all these elements is equal to $W_1 \boxbslash_{P(z)} W_2$.

Associativity isomorphism. The theory of $P(z)$-tensor products allows to make a precise statement on the associativity of logarithmic intertwining operators, or equivalently intertwining maps. The idea is to express the iterate of two intertwining maps as a single intertwining map which corresponds to an element of a suitable $P(z)$-tensor product.

When working with products and iterates of intertwining maps it is necessary to have a valid definition of these notions as intertwining maps are maps from the tensor product of two modules to the algebraic completion of another module, and intertwining maps are not defined on such huge spaces. Let W_i, $i \in \{1,2,3,4\}$, and M be objects in $\mathcal{C}$ and let F and G be $P(z_1)$- and $P(z_2)$-intertwining maps of type $\binom{W_4}{W_1\, M}$ and $\binom{M}{W_2\, W_3}$, respectively. If the series

$$\sum_{m\in\mathbb{C}} \left\langle w'_{(4)}, F(w_{(1)} \otimes \pi_m(G(w_{(2)} \otimes w_{(3)})))\right\rangle$$

absolutely converges for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W_4$, then the product of F and G is defined to *exist*, and the resulting map from $W_1 \otimes W_2 \otimes W_3$ to $\overline{W_4}$ is called the *product of F and G*. This product is denoted by $\gamma(F; \mathbb{1}_{W_1}, G)$. This notation is due to the formulation of the $P(z)$-tensor product theory in terms of *operads* which is a particularly useful concept in the study of the geometric aspects of conformal field theory, see [HL1], [HL2], [Hua5] and [Schl].
For two intertwining maps $\tilde{F}$ and $\tilde{G}$ of type $\binom{\tilde{M}}{\tilde{W}_1\, \tilde{W}_2}$ and $\binom{\tilde{W}_4}{\tilde{M}\, \tilde{W}_3}$, respectively, the *iterate of $\tilde{F}$ and $\tilde{G}$* is defined analogously and is denoted by $\gamma(\tilde{F}; \tilde{G}, \mathbb{1}_{\tilde{W}_3})$.
The conditions of existence of products and iterates of intertwining maps are actually mutually dependent, and one has the following result. For all complex numbers z_1 and z_2 satisfying $|z_1| > |z_2| > 0$, and for any $P(z_1)$-intertwining map F of type $\binom{W_4}{W_1\, M}$ and any $P(z_2)$-intertwining map G of type $\binom{M}{W_2\, W_3}$, the product $\gamma(F; \mathbb{1}_{W_1}, G)$ exists for all modules M and W_i, $i \in \{1,2,3,4\}$, in $\mathrm{ob}\,\mathcal{C}$ if and only if: for all complex numbers z_0 and z_2 satifying $|z_2| > |z_0| > 0$, and for any $P(z_2)$-intertwining map $\tilde{F}$ of type $\binom{\tilde{W}_4}{\tilde{M}\, \tilde{W}_3}$ and any $P(z_0)$-intertwining map $\tilde{G}$ of type $\binom{\tilde{M}}{\tilde{W}_1\, \tilde{W}_2}$, the product $\gamma(\tilde{F}; \tilde{G}, \mathbb{1}_{W_1})$ exists for all modules $\tilde{M}$ and $\tilde{W}_i$, $i \in \{1,2,3,4\}$, in $\mathrm{ob}\,\mathcal{C}$. If one of these two equivalent statements on the existence of all products or iterates holds true, then the *convergence condition* is said to be satisfied in the category $\mathcal{C}$.

Given that the convergence condition is satisfied a first statement on the existence and associativity of the nonmeromorphic operator product expansion can be made. To do this, two further conditions are needed which are parallel to the $P(z)$-local grading restriction condition that led to the alternate characterization of the $P(z)$-tensor product. To formulate these conditions, the following notation is convenient: For any $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$, define $\mu^{(1)}_{\lambda, w_{(1)}}$ to be the linear functional $\lambda(w_{(1)} \otimes \cdot\,) \in (W_2 \otimes W_3)^*$ for $w_{(1)} \in W_1$, and define $\mu^{(2)}_{\lambda, w_{(3)}}$ to be the linear functional $\lambda(\,\cdot \otimes w_{(3)}) \in (W_1 \otimes W_2)^*$ for $w_{(3)} \in W_3$. Then the following conditions on an element $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ will be relevant.

The $P^{(1)}(z)$*-local grading restriction condition*:

(i) The $P^{(1)}(z)$-*grading condition*: For all $w_{(1)} \in W_1$, the element $\mu^{(1)}_{\lambda,w_{(1)}} \in (W_2 \otimes W_3)^*$ is the limit, in the locally convex topology defined by the pairing between $(W_2 \otimes W_3)^*$ and $W_2 \otimes W_3$, of an absolutely convergent series of generalized eigenvectors in $(W_2 \otimes W_3)^*$ with respect to the operator $(L'_{P(z)})_0$.

(ii) For all $w_{(1)} \in W_1$, the minimal subspace $W^{(1)}_{\lambda,w_{(1)}}$ of $(W_2 \otimes W_3)^*$ which contains the terms in the series in (i) and satisfies $\tau_{P(z)}(v \otimes t^m)W^{(1)}_{\lambda,w_{(1)}} \subset W^{(1)}_{\lambda,w_{(1)}}$ for all $v \in V$ and $m \in \mathbb{Z}$ (which states stability with respect to the modes of $Y'_{P(z)}(v,x)$) has finite-dimensional homogeneous subspaces with respect to the $(L'_{P(z)})_0$-grading, and these subspaces vanish for generalized $(L'_{P(z)})_0$-eigenvalues with sufficiently small real parts.

The $P^{(2)}(z)$-*local grading restriction condition*:

(i) The $P^{(2)}(z)$-*grading condition*: For all $w_{(3)} \in W_3$, the element $\mu^{(2)}_{\lambda,w_{(3)}} \in (W_1 \otimes W_2)^*$ is the limit, in the locally convex topology defined by the pairing between $(W_1 \otimes W_2)^*$ and $W_1 \otimes W_2$, of an absolutely convergent series of generalized eigenvectors in $(W_1 \otimes W_2)^*$ with respect to the operator $(L'_{P(z)})_0$.

(ii) For all $w_{(3)} \in W_3$, the minimal subspace $W^{(2)}_{\lambda,w_{(3)}}$ of $(W_1 \otimes W_2)^*$ which contains the terms in the series in (i) and satisfies $\tau_{P(z)}(v \otimes t^m)W^{(2)}_{\lambda,w_{(3)}} \subset W^{(2)}_{\lambda,w_{(3)}}$ for all $v \in V$ and $m \in \mathbb{Z}$ (which states stability with respect to the modes of $Y'_{P(z)}(v,x)$) has finite-dimensional homogeneous subspaces with respect to the $(L'_{P(z)})_0$-grading, and these subspaces vanish for generalized $(L'_{P(z)})_0$-eigenvalues with sufficiently small real parts.

Now the important result is the following.

Theorem. Let W_1, W_2, W_3, W'_4 and M be objects in $\mathcal{C}$ and let F_1 and F_2 be $P(z_1)$- and $P(z_2)$-intertwining maps of type $\binom{W_4}{W_1\,M}$ and $\binom{M}{W_2\,W_3}$, respectively, such that their product $\gamma(F_1; \mathbb{1}_{W_1}, F_2)$ exists. If $(\gamma(F_1; \mathbb{1}_{W_1}, F_2))'(w'_{(4)})$ satisfies the $P^{(2)}(z_1 - z_2)$-local grading restriction condition for all $w'_{(4)} \in W'_4$, then there exists a $P(z_2)$-intertwining map F of type $\binom{W_4}{W_1 \boxtimes_{P(z_1-z_2)} W_2\; W_3}$ such that

$$\left\langle w'_{(4)}, F_1(w_{(1)}, z_1)F_2(w_{(2)}, z_2)w_{(3)} \right\rangle = \left\langle w'_{(4)}, F\left(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}, z_2\right) w_{(3)} \right\rangle \qquad (2.50)$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$.

The relation (2.50) is the exact statement of the nonmeromorphic operator product expansion and its associativity, and it should be compared to the meromorphic case (2.28) to which it is very similar in form. The subtlety is that Huang, Lepowsky and Zhang really proved under which conditions in the operator product expansion of two intertwining maps there only appear powers of the variables and their logarithms (with no further dependence on the variables), while in the physics literature this is usually assumed without proof. It takes

all the sophistication of the $P(z)$-tensor product theory (which has only been touched upon very lightly here) to really prove this point.
Similarly to the above theorem on the product of two intertwining maps, there is also an analogous result on the iterate of two intertwining maps which involves the $P^{(1)}(z)$-local grading restriction condition.
Developing the theory still further, it can be shown that the associativity property of the nonmeromorphic operator product expansion has an analog in the concise isomorphism

$$W_1 \boxtimes_{P(z_1)} \big(W_2 \boxtimes_{P(z_2)} W_3\big) \longrightarrow \big(W_1 \boxtimes_{P(z_1-z_2)} W_2\big) \boxtimes_{P(z_2)} W_3$$

for all generalized modules W_1, W_2 and W_3 in $\mathrm{ob}\,\mathcal{C}$.

Convergence and extension properties. The above central result on the nonmeromorphic operator product expansion relies on the technical assumptions of the $P^{(2)}(z)$-local grading restriction condition. Verifying this condition for concrete vertex operator algebras "in real life" is not very convenient, and it is desirable to have to check another set of conditions that is more manageable.

Let W_i, $i \in \{1, 2, 3, 4\}$, and M be generalized modules in $\mathrm{ob}\,\mathcal{C}$ and let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be two logarithmic intertwining operators of type $\binom{W_4}{W_1\,M}$ and $\binom{M}{W_2\,W_3}$, respectively. Then the following is certainly a comprehensive condition from the point of view of logarithmic conformal field theory.

The *convergence and extension property for products*:

> There exists $N \in \mathbb{Z}$ depending only on $\mathcal{Y}_1$ and $\mathcal{Y}_2$, and for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$, there exists $M \in \mathbb{N}$, $r_k, s_k \in \mathbb{R}$, $i_k, j_k \in \mathbb{N}$ and analytic functions $f_{i_k j_k}(z)$ on $|z| < 1$, $k \in \{1, \ldots, M\}$, satisfying
>
> $$\mathrm{wt} w_{(1)} + \mathrm{wt} w_{(2)} + s_k > N \quad \text{for all } k \in \{1, \ldots, M\}$$
>
> such that
>
> $$\left\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\right\rangle\Big|_{x_1=z_z, x_2=z_2}$$
>
> is convergent for $|z_1| > |z_2| > 0$ and can be analytically extended to the multi-valued analytic function
>
> $$\sum_{k=1}^{M} z_2^{r_k}(z_1 - z_2)^{s_k}(\log z_2)^{i_k}(\log(z_1 - z_2))^{j_k} f_{i_k j_k}\left(\frac{z_1 - z_2}{z_2}\right)$$
>
> in the domain $|z_2| > |z_1 - z_2| > 0$.

This condition can be applied to yield the following result which is a first step to reformulate the sufficient conditions for the existence and associativity of the nonmeromorphic operator product expansion.

Theorem. If every finitely-generated lower-truncated generalized V-module is an object in $\mathcal{C}$ and the convergence and extension property holds in $\mathcal{C}$, then the convergence condition holds in $\mathcal{C}$, i.e. all relevant products and iterates of $P(z)$-intertwining maps exist, see page 40. Furthermore, let W_1, W_2, W_3, W_4' and M be any objects in $\mathcal{C}$ and let F_1 and F_2 be $P(z_1)$- and $P(z_2)$-intertwining maps of type $\binom{W_4}{W_1\, M}$ and $\binom{M}{W_2\, W_3}$, respectively, with $|z_1| > |z_2| > |z_1 - z_2| > 0$. Then $(\gamma(F_1; \mathbb{1}_{W_1}, F_2))'(w'_{(4)}) \in (W_1 \otimes W_2 \otimes W_3)^*$ satisfies the $P^{(2)}(z_1 - z_2)$-local grading restriction condition for all $w'_{(4)} \in W_4'$.

The lower-truncation condition is of course satisfied by definition for all (generalized) modules for a vertex operator algebra. The reason that it is mentioned here explicitly is that all results summarized in this section actually hold for the wider class of *conformal vertex algebras* and their modules. These satisfy all the axioms of the vertex operator algebra case except that their homogeneous subspaces need not vanish for sufficiently small (generalized) L_0-eigenvalues, and they also need not be finite-dimensional. This is also why the condition of quasi-finite-dimensionality (to be defined below) appears in the next theorem.
Using Huang's results on differential equations for matrix elements of products of intertwining maps obtained in [Hua7] and the theory of differential equations with regular singular points, the convergence and extension property can be replaced in the above theorem by a simple finiteness property.

Theorem. If all generalized V-modules $W = \coprod_{h\in\mathbb{R}} W_{[h]}$ in $\mathrm{ob}\,\mathcal{C}$ are C_1-cofinite and quasi-finite-dimensional, i.e. $\coprod_{h<R} W_{[h]}$ is finite-dimensional for all $R \in \mathbb{R}$, then the convergence and extension property holds in $\mathcal{C}$.

This ends the discussion of $P(z)$-tensor product theory and its application to the nonmeromorphic operator product expansion. It was shown by Huang and Lepowsky in [Hua4] and [HL6] that the conditions for its existence and associativity hold for the vertex operator algebras associated to the minimal Virasoro models and the WZW models. In the next chapter, it will be proven that these conditions also hold for another class of vertex operator algebras.

Chapter 3

Properties of the Triplet Algebras

In this chapter the results on associativity in the nonmeromorphic case are very briefly summarized and applied to an infinite family of vertex operator algebras, the triplet $\mathcal{W}$-algebras. These are introduced and it is shown that they satisfy all the sufficient conditions for the existence and associativity of the nonmeromorphic operator product expansion. This is done by firstly examining in detail one such triplet algebra, which has been studied extensively before, and then generalizing the arguments made in the special case to all triplet algebras. The main effort to succeed in this is to analyze certain singular vectors and prove that all triplet algebras are C_2-cofinite, which is an interesting result independent of its application in the context of associativity. Finally, an upper bound on the dimension of the Zhu algebras associated to the triplet algebras is given. Most of this chapter is based on the work described in [CF], though the exposition presented here is more detailed.

The results of the $P(z)$-tensor product theory concerning the nonmeromorphic operator product expansion as presented in section 2.3 are not yet in a very concise form. Combining all three theorems of that section, one finds that for a given conformal vertex algebra V, the nonmeromorphic operator product expansion exists and is associative as stated in (2.50) if V satisfies the following conditions, where $\mathcal{C}$ is an as yet unspecified full subcategory of the category whose objects are $\mathbb{R}$-graded generalized V-modules and that is closed under the contragredient functor:

(1) All generalized V-modules in $\mathrm{ob}\,\mathcal{C}$ are C_1-cofinite, i.e. for all $W \in \mathrm{ob}\,\mathcal{C}$,

$$\dim\big(W/C_1(W)\big) < \infty \quad \text{with} \quad C_1(W) = \mathrm{span}\left\{ u_{-1}w \;\middle|\; u \in \coprod_{m>0} V_{(m)},\, w \in W \right\} .$$

(2) All generalized V-modules in $\mathrm{ob}\,\mathcal{C}$ are quasi-finite-dimensional, i.e. for all $W \in \mathrm{ob}\,\mathcal{C}$,

$$\dim \coprod_{m<R} W_{[m]} < \infty \qquad \text{for all } R \in \mathbb{R} .$$

(3) Every object which is a finitely generated lower-truncated generalized V-module, except that it may have infinite-dimensional homogeneous subspaces, is an object in $\mathcal{C}$.

In the case that the conformal vertex algebra V has the full structure of a vertex operator algebra, then all its generalized modules are quasi-finite-dimensional and lower-truncated by definition, as was already noted in the last chapter.

3.1 The triplet algebra at $c = -2$

It will now be shown that the above conditions hold for the $\mathcal{W}$-algebra of type $\mathcal{W}(2, 3^{\times 3})$ and a suitable choice of the category $\mathcal{C}$. This $\mathcal{W}$-algebra is generated by the modes L_m of the Virasoro field $T(x) = \sum_{m\in\mathbb{Z}} L_m x^{-m-2}$ associated to the vector ω of weight 2 which implements the conformal symmetry, and the modes W^a_m of a triplet (under the action of the group $SO(3)$) of primary fields of weight 3, $W^a(x) = \sum_{m\in\mathbb{Z}} W^a_m x^{-m-3}$ with $a \in \{\pm 1, 0\}$, which "maximally extend" the conformal symmetry. Because of the $SO(3)$-symmetry, which is realized such that the structure constants C^k_{ij} of the $\mathcal{W}$-algebra involve the structure constants ε_{ijk} of the Lie algebra $\mathfrak{so}(3)$, $\mathcal{W}(2, 3^{\times 3})$ is also called a *triplet algebra*. It has a Virasoro central charge of $c = -2$ and its relevant commutation relations can be computed from (2.44) to be

$$[L_m, L_n] = (m-n)L_{m+n} - \frac{1}{6}\left(m^3 - m\right)\delta_{m+n,0}\ , \tag{3.1a}$$

$$[L_m, W^a_n] = (2m-n)W^a_{m+n}\ , \tag{3.1b}$$

$$\begin{aligned}\left[W^a_m, W^b_n\right] = \delta_{ab}\Bigg(&2(m-n)\Lambda_{m+n} + \frac{1}{20}(m-n)\left(2m^2+2n^2-mn-8\right)L_{m+n}\\ &-\frac{1}{120}m(m^2-1)(m^2-4)\delta_{m+n,0}\Bigg)\\ &+\mathrm{i}\varepsilon_{abc}\left(\frac{5}{14}\left(2m^2+2n^2-3mn-4\right)W^c_{m+n} + \frac{12}{15}V^c_{m+n}\right)\ ,\end{aligned} \tag{3.1c}$$

where

$$\begin{aligned}\Lambda &= \mathcal{N}(T,T) = N^{(2)}(T,T) - \frac{3}{10}\partial^2 T\ ,\\ V^a &= \mathcal{N}(W^a,T) = N^{(2)}(W^a,T) - \frac{3}{14}\partial^2 W^a\ .\end{aligned}$$

In section 2.2 the conditions for the smallest nontrival vertex operator algebras (generated only by the vacuum and the conformal vector) to be rational were given, among which is the relation (2.37) for the central charge, $c_{p,q} = 1 - 6(p-q)^2/(pq)$ with $p, q \in \mathbb{Z}_{\geq 2}$ relatively prime. In this notation the central charge of the triplet algebra considered here can be written as $c = c_{2,1} = -2$. Since $1 \notin \mathbb{Z}_{\geq 2}$, the Virasoro algebra with $c = -2$ alone cannot lead to a rational vertex operator algebra, but the question might arise whether or not the extended $\mathcal{W}$-algebra satisfies the conditions of rationality. The answer is that this is not the case, at least not in the meaning of "rational" adopted here. The reason is that the triplet algebra at $c = -2$ has modules that are reducible but indecomposable and thus does not satisfy the definition of rationality given in section 2.2. Indeed, it was noted by Gaberdiel and Kausch in [GK1] and by Rohsiepe in [Roh] that the triplet algebra at $c = -2$ has only four (generalized) highest weight modules, with (generalized) highest weights $-\frac{1}{8}$, 0, $\frac{3}{8}$ and 1, respectively. While the modules corresponding to the values $-\frac{1}{8}$ and $\frac{3}{8}$ are irreducible, this is not the case for the other two. For example, the module with generalized highest weight 1 has an irreducible submodule generated by a vector φ, but it also contains another vector ϕ which is not an element of this submodule but has the property that

$$L_0\phi = \phi + \varphi \ .$$

This Jordan cell structure in the action of L_0 is typical of a logarithmic conformal field theory, and it rules out rationality.
Nevertheless, four (generalized) highest weight modules are not so much; in particular, these are only finitely many modules, and in this sense the triplet algebra at $c = -2$ certainly has a finiteness property. An even stronger statement is true: The set of finitely many (generalized) highest weight modules *closes under fusion*, i.e. for two elements W_i and W_j of this set, the fusion rules N_{ij}^k vanish for all W_k that are not elements of this set. In this sense the triplet algebra at $c = -2$ is called "rational" in [GK1].

Now that the vertex operator algebra $V = \mathcal{W}(2, 3^{\times 3})$ has been presented, the choice for the category $\mathcal{C}$ is made. It is taken to be the category whose objects are precisely all finitely generated lower-truncated $\mathbb{R}$-graded generalized V-modules. In particular, this choice includes all (generalized) highest weight modules for V, but also those on which L_1 acts only nilpotently (and not necessarily trivially) on the generating vector. The restriction to finitely generated modules does not seem to be very limiting from the point of view of a physicist.
By this choice of $\mathcal{C}$, condition (**3**) above is satisfied. The fact that the homogeneous subspaces of the (generalized) modules in $\mathrm{ob}\,\mathcal{C}$ are really finite-dimensional follows from results of Buhl in [Bu] on a module spanning set, using the fact that they are finitely generated and all triplet algebras are C_2-cofinite, which will be proven below.
As the triplet algebra is a vertex operator algebra, condition (**2**) is automatically satisfied, which is easily verified in this concrete example: by the action of any mode v_n with $v \in V$, the weight of a homogeneous element to which v_n is applied to changes by an integer value, and there are only finitely many vectors that generate V, namely the vacuum Ω, the conformal vector ω and W^a, $a \in \{\pm 1, 0\}$.
In order to see that condition (**1**) is satisfied as well, i.e. $C_1(W)$ is finite-codimensional for all $W \in \mathrm{ob}\,\mathcal{C}$, one can assume without loss of generality that W is generated by some element $w = w^{(0)}$ together with its finitely many "logarithmic partners" $w^{(i)}$, i.e. $L_0 w^{(i)} = (\mathrm{wt} w^{(i)}) w^{(i)} + w^{(i+1)}$ and $w^{(i)} = 0$ for sufficiently large i. Then by (2.21) and (2.40), every vector in W is a linear combination of elements of the form

$$\mathcal{M}_{-m_1} \ldots \mathcal{M}_{-m_k} L_{-1}^M \prod_{a \in \{\pm 1, 0\}} \left(\left(W_{-2}^a\right)^{N_2^a} \left(W_{-1}^a\right)^{N_1^a} \left(W_0^a\right)^{N_0^a} \right) \mathcal{M}_{n_1} \ldots \mathcal{M}_{n_l} w^{(i)} \tag{3.2}$$

where $\mathcal{M}$ is a placeholder for either L or W^a; M, N_0^a, N_1^a, $N_2^a \in \mathbb{N}$, $n_1, \ldots, n_l \in \mathbb{Z}_+$ and $m_1, \ldots, m_k \in \mathbb{Z}_{\geq 2}$ for $\mathcal{M} = L$ while $m_1, \ldots, m_k \in \mathbb{Z}_{\geq 3}$ for $\mathcal{M} = W^a$.

It follows from (2.39) that the vectors

$$L_{-m+1}w \text{ and } W^a_{-m}w \text{ are in } C_1(W) \text{ for all } w \in W \text{ and all } m \geq 3 ,$$

while for all other values of m, this is not necessarily so. Thus in the case that k is strictly larger than zero, any element of the form (3.2) is in $C_1(W)$. On the other hand, for $k = 0$ there are only finitely many possibilities for terms of the form $\mathcal{M}_{n_1} \ldots \mathcal{M}_{n_l} w^{(i)}$ not to vanish because of the lower-truncatedness of W. The factor $(W^a_0)^{N^a_0}$ can also do no harm as it does not change the generalized weight of the element it is applied to, and each Jordan cell is finite-dimensional by the definition of $\mathcal{C}$.
So what deserves special attention are the powers of L_{-1}, W^a_{-1} and W^a_{-2} in the case $k = 0$, because when applied to an element of W, the result need not be in $C_1(W)$, but each of these modes strictly increases the generalized weight. As there is certainly no "upper-truncation condition" for the module W, the appearance of these modes in (3.2) makes it seem possible that the complement of $C_1(W)$ in W is infinite-dimensional.
But fortunately, in this situation a theorem obtained by Buhl in [Bu] applies, which is a generalization of an earlier result of Gaberdiel and Neitzke in [GN].

Theorem. Let V be a C_2-cofinite vertex operator algebra, i.e. $\dim(V/C_2(V)) < \infty$ with $C_2(V) = \text{span}\{u_{-2}v \,|\, u, v \in V\}$, and let W be a weak V-module which is generated by $w \in W$. Then W is spanned by elements of the form

$$x^1_{-n_1} \ldots x^k_{-n_k} w \tag{3.3}$$

with $n_1 \geq \ldots \geq n_k > -L$, where L is some fixed real number, and the vectors $x^1, \ldots, x^k \in V$ are representatives of the elements of a basis of $V/C_2(V)$. In addition, if $n_j \leq 0$, then $n_i = n_j$ for at most Q indices i, where Q is another fixed real number.

Note that this statement is true for general vertex operator algebras and not only in the case of $\mathcal{W}$-algebras, so the convention used for the indices of modes is not the one that is preferred in the physics literature, as has been explained in section 2.2; to switch between both conventions, the relation $u_n = u^{\text{phys}}_{n-\text{wt}u+1}$ is employed.
The last part of the above theorem is the most important one for the present situation of the triplet algebra V as it implies that only a limited number of powers of L_{-1}, W^a_{-1} and W^a_{-2} has to be considered in (3.2) if V is C_2-cofinite. This is indeed the case:

Proposition. The triplet algebra at $c = -2$ is C_2-cofinite.

Several authors (see [GN] and [Miy]) have been aware of this fact for some time, and it was recently proven by Abe in [A]. The following proof uses a completely different method.
The key to this proof is the existence of certain *singular vectors.* These are homogeneous vectors in the Verma module generated by the action of the modes L_m and W^a_m with $m \in \mathbb{Z}$ and $a \in \{\pm 1, 0\}$, such that they are annihilated by all modes that lower the (generalized)

weight of the vector they are applied to. Indeed, an explicit expression for six singular vectors at level 6 was obtained in [GK1] and is given by

$$\begin{aligned} N^{ab} =& W^a_{-3} W^b_{-3}\Omega - \delta_{ab}\left(\frac{8}{9}\, L^3_{-2} + \frac{19}{36}\, L^2_{-3} + \frac{14}{9}\, L_{-4}L_{-2} - \frac{16}{9}\, L_{-6}\right)\Omega \\ &+ \mathrm{i}\varepsilon_{abc}\left(-2W^c_{-4}L_{-2} + \frac{5}{4}\, W^c_{-6}\right)\Omega\ , \end{aligned} \tag{3.4}$$

as can be checked with the help of the commutation relations (3.1).
In order to prove that V is really C_2-cofinite, one first observes that in the expression (3.4) for the singular vector N^{ab}, because of (2.38) each term that it is made of is manifestly in $C_2(V)$ except for $W^a_{-3}W^b_{-3}\Omega$ and $L^3_{-2}\Omega$. Similarly to the maximal ideals in the case of the minimal Virasoro models and the WZW models discussed in section 2.2, any singular vector is divided out of the Verma module to give the triplet algebra the structure of a vertex operator algebra. So it follows that for $a \neq b$,

$$W^a_{-3}W^b_{-3}\Omega \in C_2(V)$$

and

$$\left(\left(W^a_{-3}\right)^2 - \left(W^b_{-3}\right)^2\right)\Omega \in C_2(V)\ .$$

Since by (2.41), W^a_{-3} leaves the space $C_2(V)$ invariant, $W^a_{-3}((W^a_{-3})^2 - (W^b_{-3})^2)\Omega$ is an element of $C_2(V)$ as well. But this element can also be written as

$$\left(W^a_{-3}\right)^3\Omega - W^a_{-3}\left(W^b_{-3}\right)^2\Omega = \left(W^a_{-3}\right)^3\Omega - W^b_{-3}W^a_{-3}W^b_{-3}\Omega + Y_{-6}W^b_{-3}\Omega\ , \tag{3.5}$$

where $Y_{-6} = [W^a_{-3}, W^b_{-3}]$ applied to any vector $v \in V$ yields an element of $C_2(V)$ because in the commutator of modes of primary fields of weight 3 there can only appear modes corresponding to fields of weight less than or equal to 5, as follows from the general commutation relation (2.44) for $\mathcal{W}$-algebras. So in particular, the last term in (3.5) is in $C_2(V)$. In addition, the second last term in this equation also is in $C_2(V)$ as W^b_{-3} leaves this space invariant. Hence, it follows that $(W^a_{-3})^3\Omega \in C_2(V)$, and thus

$$\left(W^a_{-3}\right)^m\Omega \in C_2(V) \quad \text{for all } m \geq 3\ . \tag{3.6}$$

From this and the fact that $((W^a_{-3})^2 - \frac{8}{9}\, L^3_{-2})\Omega$ is in $C_2(V)$ it follows that $(W^a_{-3})^2L^3_{-2}\Omega \in C_2(V)$. Now using the invariance of $C_2(V)$ under L_{-2} and W^a_{-3} one more time it is easy to see that

$$\left(\left(W^a_{-3}\right)^2 - \frac{8}{9}\, L^3_{-2}\right)^2\Omega = \left(\left(W^a_{-3}\right)^4 + \frac{64}{81}\, L^6_{-2} - \frac{8}{9}\left(W^a_{-3}\right)^2 L^3_{-2} - \frac{8}{9}L^3_{-2}\left(W^a_{-3}\right)^2\right)\Omega$$

is an element of $C_2(V)$. But from the above discussion it is also clear that each term on the right-hand side apart from $\frac{64}{81}\, L^6_{-2}\Omega$ is in $C_2(V)$, and so it follows that $L^6_{-2}\Omega$ must be an element of $C_2(V)$ as well.

It has just been shown that sufficiently large powers of L_{-2} and W^a_{-3} (6 or maybe less in the first case, 3 or maybe less in the latter) applied to any element in V yield elements in $C_2(V)$. Thus it is proven that $C_2(V)$ is finite-codimensional.

Now that it has been shown that the main prerequisite of Buhl's theorem is satisfied for the triplet algebra at $c=-2$, it can be used since by definition any object in $\mathcal{C}$ is a weak module and the elements in (3.2) are of the same form as those in (3.3). This means that if ω and W^a are not in $C_2(V)$ and can thus be taken to be representatives of elements in a basis for $V/C_2(V)$, there actually is some sort of an "upper-truncation condition", but for the exponents of the modes L_{-1}, W^a_{-1} and W^a_{-2} in (3.2). So for $k=0$, it follows that only finitely many elements of the form (3.2) span the "$(k=0)$-part" of W. This is exactly the statement that W is C_1-cofinite.
It remains to be verified that ω and W^a are not in $C_2(V)$. For the moment, consider the possibility that ω is in $C_2(V)$. Then there must be $u, v \in V$ such that $u_{-2}v = \omega$. By comparing weights on both sides, one arrives at the condition $\mathrm{wt}u + \mathrm{wt}v + 1 = 2$. But since the vertex operator algebra V under consideration is of CFT type, i.e. it is of the form $V = \coprod_{n\in\mathbb{N}} V_{(n)}$ with $V_{(0)} = \mathbb{C}\Omega$, this condition says that either u or v must be (a scalar multiple of) the vacuum (and the other one of weight 1). This is not possible for the conformal vector, leading to a contradiction. By a similar reasoning, one also sees that $W^a \notin C_2(V)$.
As W was taken to be an arbitrary element of $\mathrm{ob}\,\mathcal{C}$, it is now established that all generalized V-modules of interest here are C_1-cofinite and thus condition (**1**) is satisfied.

Finally, it needs to be shown that the chosen category is closed with respect to the contragredient functor. By the definition of the graded dual $W' = \coprod_{n\in\mathbb{Z}} (W_{[n]})^*$ it is clear that it is lower-truncated. In order to establish that it is also finitely generated, choose a minimal generating set $\{w_1, \dots, w_N\} \subset W \in \mathrm{ob}\,\mathcal{C}$ from a basis $\bigcup_{n\in\mathbb{Z}} B_n$ of W, where B_n is a basis of $W_{[n]}$ for all $n \in \mathbb{Z}$. Then all $w \in W$ are linear combinations of elements of the form

$$\mathcal{M}_{n_1} \dots \mathcal{M}_{n_k} w_i \ ,$$

where $\mathcal{M}$ denotes the same as in (3.2). Let $w'_1, \dots, w'_N$ be the elements of the dual basis in W' such that $\langle w'_i, w_j \rangle = \delta_{ij}$. Because of this, all $w' \in W'$ that may give a nonvanishing matrix element with some $w \in W$ must be linear combinations of elements of the form $\mathcal{M}'_{n_k} \dots \mathcal{M}'_{n_1} w'_i$. To see this, assume that there is an element $\tilde{w}' \notin \{w'_1, \dots, w'_N\}$ in W' such that $\{\tilde{w}', w'_1, \dots, w'_N\}$ is a subset of a minimal set of generating vectors of W'. It follows that $\langle \tilde{w}', w_i \rangle = 0$ for all $i \in \{1, \dots, N\}$ and thus

$$\begin{aligned}
\left\langle \mathcal{M}'_{-m_1} \dots \mathcal{M}'_{-m_k} \tilde{w}', \mathcal{M}_{-n_1} \dots \mathcal{M}_{-n_l} w_i \right\rangle &= \left\langle \tilde{w}', \mathcal{M}'_{m_k} \dots \mathcal{M}'_{m_1} \mathcal{M}_{-n_1} \dots \mathcal{M}_{-n_l} w_i \right\rangle \\
&= \delta_{\sum_i m_i, \sum_j n_j} \left\langle \tilde{w}', \sum_{\{I \,|\, \mathrm{wt}\, w_I = \mathrm{wt}\, w_i\}} a_I w_I \right\rangle \\
&= 0 \ ,
\end{aligned}$$

where $a_I \in \mathbb{C}$ are the coefficients that result from applying the commutation relations of the $\mathcal{M}$-modes. This means that the subspace of generalized weight $\mathrm{wt}\, \tilde{w}'$ has a dimension that is strictly larger than the dimension of the corresponding subspace in W. But by the definition of the graded dual of W, these finite-dimensional subspaces must have the same dimension, so there cannot be an element $\tilde{w}'$ as above, and $\mathcal{C}$ is closed under the contragredient functor.

To summarize the results of this section, the triplet algebra at $c = -2$ satisfies all the conditions for the existence and associativity of the nonmeromorphic operator product expansion. While proving this fact another interesting property of this vertex operator algebra was established: it is also C_2-cofinite. As has been mentioned above, the triplet algebra at $c = -2$ is not rational in the strict sense adopted here because of the existence of reducible but indecomposable generalized modules, and thus its C_2-cofiniteness contradicts a conjecture that rationality, regularity and C_2-cofiniteness are equivalent properties of vertex operator algebras. Nevertheless, a set of finitely many generalized modules for the triplet algebra at $c = -2$ closes under fusion, and it may be conjectured that the equivalence of C_2-cofiniteness and rationality holds in this more general sense.

3.2 The triplet algebras at $c_{p,1}$

The triplet algebra at $c = -2$ is only the first member of an infinite family of triplet $\mathcal{W}$-algebras $\{\mathcal{W}(2,(2p-1)^{\times 3})\}_{p\in\mathbb{Z}_{\geq 2}}$ with central charge $c_{p,1} = 1 - 6(p-1)^2/p$, where for each $p \in \mathbb{Z}_{\geq 2}$ the three primary fields of weight $2p-1$ are a triplet under the action of the group $SO(3)$, which means that the structure constants $C^{W^c}_{W^a,W^b}$ are proportional to ε_{abc}, see [Kau1] and [Fl2]. It will be shown in this section that the above conditions (**1**), (**2**) and (**3**) are also satisfied in this general case.

If one defines the category $\mathcal{C}$ analogously to the special case of $p = 2$ in the previous section, one immediately sees that the conditions of quasi-finite-dimensionality and of finitely generated lower-truncated modules in $\mathrm{ob}\,\mathcal{C}$ hold in the same way as before with the obvious generalization of the arguments. What requires additional work is to establish the C_1-cofiniteness of all objects in $\mathcal{C}$.
Let V_Δ denote the vertex operator algebra associated to the $\mathcal{W}$-algebra $\mathcal{W}(2,\Delta^{\times 3})$ for a fixed $\Delta := 2p-1$ with $p \in \mathbb{Z}_{\geq 3}$. If V_Δ is C_2-cofinite, one can apply Buhl's theorem as in the case $p = 2$, and any V_Δ-module under consideration would be C_1-cofinite, which together with the other properties of V_Δ gives the existence and associativity of the nonmeromorphic operator product expansion. Compared with the case $p = 2$, the difficulty of proving the C_2-cofiniteness of V_Δ stems from the lack of explicit expressions for singular vectors that are crucial for a proof of C_2-cofiniteness. A priori, it is not even clear whether such singular vectors at all exist for arbitrary $p \in \mathbb{Z}_{\geq 3}$.
In principle, it is possible to determine whether or not such singular vectors exist by trying to construct them explicitly. First, one has to obtain the commutation relations of the fundamental modes by the general identity (2.44). In order to do that, all the structure constants C^k_{ij} have to be computed, which soon becomes a tremendous task with increasing p. But for an examination of singular vectors the commutation relations of the modes are only the beginning, and much more effort is needed to come to a conclusion. Finally, all

these calculations have to be carried out separately for each $p \in \mathbb{Z}_{\geq 3}$. That is why another approach is desirable to examine the infinitely many triplet algebras all at once.
As it turns out, one can argue for the existence of certain singular vectors of weight $2(2p-1)$ with the help of the explicitly known character of V_Δ that was obtained by Flohr in [Fl2]. By analyzing this character in detail, one obtains the following result.

Proposition. For all $\Delta \in (2\mathbb{Z}_+ + 1)$, the triplet algebra $\mathcal{W}(2, \Delta^{\times 3})$ has six singular vectors at level 2Δ of the form

$$N^{ab} = W^a_{-\Delta} W^b_{-\Delta}\Omega + \delta_{ab}\big(\text{Virasoro-polynomial}\big)\Omega + \varepsilon_{abc}\big(\text{Virasoro-}W^c_m\text{-polynomial}\big)\Omega\,, \tag{3.7}$$

where in the last term only monomials with exactly one W^c-mode appear.

Note that the singular vectors (3.4) in the special case $\Delta = 3$ are of the form indicated in the proposition. In order to prove that singular vectors as in (3.7) exist in general, the character

$$\chi_{V_\Delta}(q) = \frac{q^{-1/24}}{\varphi(q)} \sum_{n\in\mathbb{Z}} (2n+1) q^{(2np+p-1)^2/(4p)} \tag{3.8}$$

from [Fl2] is needed. If one expands both this character and the vacuum Verma module character $\chi^{\text{Verma}}_{V_\Delta}(q)$ given by (2.46) into formal power series in q and compares the coefficients of $q^{(2p-1)+3}$ (times $q^{-c_{p,1}/24}$), one finds that the dimensions of the homogeneous subspaces of weight $(2p-1)+3$ of the vacuum Verma module and the $\mathcal{W}$-algebra itself differ by 3. The reason for this is the following: From the Kac determinant it follows that the Virasoro algebra of central charge $c_{p,1}$ has an infinite set of highest weight modules where the highest weights are given by $h_{2k-1,1} = (k-1)(kp-1)$, $k \in \mathbb{Z}_+$. By a standard argument it follows that these modules have singular vectors at level $2k-1$. In particular, for $k = 2$ the highest weight vectors of weight $2p-1$ can be identified with the vectors $W^a_{-\Delta}\Omega$ due to $\Delta = 2p-1$. So because of the additional structure of the $\mathcal{W}$-algebra with its fields W^a, pure Virasoro modules are embedded into the full vertex operator algebra $\mathcal{W}(2, \Delta^{\times 3})$, and the difference of the dimensions above is due to the three singular vectors of weight $(2p-1)+3$.
If these three vectors are divided out of the vacuum Verma module, a structure is obtained to which the character

$$\tilde{\chi}_\Delta(q) = q^{-c_{p,1}/24}\left(\frac{1}{\varphi_2(q)} + \frac{3q^{2p-1}(1-q^3)}{\varphi(q)(\varphi_{2p-1}(q))^2}\right) \tag{3.9}$$

pertains, where the notation introduced in (2.47) is used. The first term in this expression accounts for the action of the Virasoro algebra on the vacuum alone. The second term reflects the fact that beginning at level $2p-1$, the modes associated to the three distinct W^a-fields act nontrivially on the vacuum. With respect to the Virasoro algebra, this is a highest weight vector, which explains the factor $q^{2p-1}/\varphi(q)$. Furthermore, the factor $1-q^3$ is due to the singular vectors of weight $(2p-1)+3$ discussed above, and the term $(\varphi_{2p-1}(q))^{-2}$ comes from the action of the W^a-modes on the vacuum. The second power (and not the

third) has to be taken here in order not to doubly count the contribution from the W^a-modes because of the 3-fold multiplicity.
Partially expanding both (3.8) and (3.9) into a formal power series yields

$$\chi_{V_\Delta}(q) = \frac{q^{-c_{p,1}/24}}{\varphi(q)} \left(1 - q + 3q^{2p-1} - 3q^{2p+2} + \mathcal{O}(q^{6p-2})\right) , \tag{3.10}$$

$$\tilde{\chi}_\Delta(q) = \frac{q^{-c_{p,1}/24}}{\varphi(q)} \left(1 - q + 3q^{2p-1} - 3q^{2p+2} + 6q^{4p-2} + \mathcal{O}(q^{4p-1})\right) . \tag{3.11}$$

Now the dimensions of the homogeneous subspaces of weight $2\Delta = 4p-2$ described by these characters are examined. Comparing the coefficients of q^{4p-2} (times $q^{-c_{p,1}/24}$) by taking the relevant contributions from $(\varphi(q))^{-1} = \sum_{n\in\mathbb{N}} p(n)q^n$ into account, one immediately sees from (3.10) and (3.11) that these dimensions differ by 6. This means that six additional singular vectors of weight 2Δ are divided out in $\mathcal{W}(2,\Delta^{\times 3})$. The reason that these vectors must involve a term with two W^a-modes is that there are no pure Virasoro singular vectors of weight 2Δ, as follows from the Kac determinant. Finally, the form of (3.7) is a direct consequence of the $SO(3)$-structure of $\mathcal{W}(2,\Delta^{\times 3})$. This proves the proposition.

To gain a better understanding of how the comparison of coefficients works in the above proof, it is helpful to write down the expansions (3.10) and (3.11) in detail for some explicit values of Δ:

$$\begin{aligned}
q^{c_{2,1}/24}\,\chi_{V_{\Delta=3}}(q) &= 1 + q^2 + 4q^3 + 5q^4 + 8q^5 + 10q^6 + 16q^7 + 22q^8 + 32q^9 + \dots\ , \\
q^{c_{2,1}/24}\,\tilde{\chi}_{\Delta=3}(q) &= 1 + q^2 + 4q^3 + 5q^4 + 8q^5 + 16q^6 + 28q^7 + 46q^8 + 77q^9 + \dots\ , \\
q^{c_{3,1}/24}\,\chi_{V_{\Delta=5}}(q) &= 1 + q^2 + q^3 + 2q^4 + 5q^5 + 7q^6 + 10q^7 + 13q^8 + 20q^9 + 27q^{10} + 38q^{11} \\
&\quad + 51q^{12} + 69q^{13} + \dots\ , \\
q^{c_{3,1}/24}\,\tilde{\chi}_{\Delta=5}(q) &= 1 + q^2 + q^3 + 2q^4 + 5q^5 + 7q^6 + 10q^7 + 13q^8 + 20q^9 + 33q^{10} + 50q^{11} \\
&\quad + 75q^{12} + 105q^{13} + \dots\ , \\
q^{c_{4,1}/24}\,\chi_{V_{\Delta=7}}(q) &= 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 7q^7 + 10q^8 + 14q^9 + 18q^{10} + 26q^{11} \\
&\quad + 36q^{12} + 48q^{13} + 64q^{14} + 86q^{15} + 112q^{16} + \dots\ , \\
q^{c_{4,1}/24}\,\tilde{\chi}_{\Delta=7}(q) &= 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 7q^7 + 10q^8 + 14q^9 + 18q^{10} + 26q^{11} \\
&\quad + 36q^{12} + 48q^{13} + 70q^{14} + 98q^{15} + 136q^{16} + \dots\ .
\end{aligned}$$

This way the appearance of six singular vectors in V_3, V_5 and V_7 at level 6, 10 and 14, respectively, is obvious.

Now with the knowledge of singular vectors of the form (3.7) the proof of the C_2-cofiniteness of the $\mathcal{W}$-algebras $\mathcal{W}(2,\Delta^{\times 3})$ can be continued. As in the special case $p = 2$ it is clear that nearly all possible vectors in the expression (3.7) for the singular vector N^{ab} are elements of $C_2(V_\Delta)$ because of the fact that $W^a_m\Omega = 0$ for all $m \geq -\Delta + 1$. The only vectors for which this might not be true are $W^a_{-\Delta}W^b_{-\Delta}\Omega$ and $\alpha L^\Delta_{-2}\Omega$, the latter appearing in the δ_{aa}-term in N^{ab}. If it can be shown that the coefficient α is not zero, the exact same reasoning as in the case $p = 2$ can be applied to see that V_Δ is C_2-cofinite. So the question that remains to be answered is whether or not $\alpha \neq 0$. The answer is the following.

> *Lemma.* For all $\Delta \in (2\mathbb{Z}_+ + 1)$, the term $L_{-2}^{\Delta}\Omega$ appears in the singular vectors N^{aa} in (3.7) with a nonzero coefficient α, possibly depending on Δ.

Conceptually, this result does not seem to be particularly significant, but computationally, establishing its truth is the most laborious part of the proof of the C_2-cofiniteness of all triplet algebras. It turns out to be quite subtle to find the correct way to make use of the scarce pieces of information available. This befits the fact that C_2-cofiniteness can be ascertained for infinitely many vertex operator algebras all at once, without having to compute all structure constants and construct singular vectors explicitly. It is actually remarkable that such a fundamental property can be established from relatively little data.

To prove the above lemma, one first observes that the vertex operator to which a singular vector corresponds necessarily is a primary field. In particular, it is a quasi-primary field. As the vector $W_{-\Delta}^a W_{-\Delta}^b\Omega$ appears in the expression for the singular vector N^{ab}, the corresponding quasi-primary null-field can be written as a linear combination of quasi-primary fields, and one of these must be the normal-ordered product $\mathcal{N}(W^a, W^b)$.
The next step is to note that the quasi-primary field $\mathcal{N}(W^a, W^b)$ alone cannot be the null-field. To see this, the fact that the mode L_1 annihilates the vector N^{ab} can be used. Indeed, by expanding the null-field into modes,

$$\begin{aligned}
L_1 N^{aa} &= L_1\left(W_{-\Delta}^a W_{-\Delta}^a\Omega + \big(\text{Virasoro-polynomial}\big)\Omega\right) \\
&\overset{\Delta-1}{=} L_1\left(W_{-\Delta}^a W_{-\Delta}^a\Omega + \beta L_{-4}L_{-2}^{\Delta-2}\Omega + \gamma L_{-3}^2 L_{-2}^{\Delta-3}\Omega\right) \\
&\overset{\Delta-1}{=} 0\,.
\end{aligned} \tag{3.12}$$

Here, the symbol $\overset{\Delta-1}{=}$ has been introduced, which means "equal to, modulo vectors with less than $\Delta - 1$ modes applied to the vacuum Ω". For example,

$$\beta L_{-4}L_{-2}^{\Delta-2}\Omega + \gamma L_{-3}^2 L_{-2}^{\Delta-3}\Omega \overset{\Delta-1}{=} \beta L_{-4}L_{-2}^{\Delta-2}\Omega + \gamma L_{-3}^2 L_{-2}^{\Delta-3}\Omega + \delta L_{-4}^2 L_{-2}^{\Delta-4}\Omega\,.$$

So far, the values of the constants β and γ are unknown. If the null-field were equal to $\mathcal{N}(W^a, W^b)$, the coefficients $\beta = \beta_{WW}$ and $\gamma = \gamma_{WW}$ could be computed from the formula (2.42) for quasi-primary normal-ordered products in terms of the structure constant $C_{W^a,W^a}^{\mathcal{N}(T^{\Delta-1})}$. In principle, this constant can be computed for each $p \in \mathbb{Z}_{\geq 3}$ separately, but neither are such computations carried out easily nor is it necessary to know the exact value of the constant; only the information that it is not zero is crucial.
With this, a straight-forward calculation using (3.12) shows that

$$L_2\left(W_{-\Delta}^a W_{-\Delta}^a + \beta_{WW} L_{-4}L_{-2}^{\Delta-2} + \gamma_{WW} L_{-3}^2 L_{-2}^{\Delta-3}\right)\Omega \overset{\Delta-1}{\neq} 0\,.$$

So the field $\mathcal{N}(W^a, W^b)$ is quasi-primary but not primary and can thus not be the null-field. Instead, other quasi-primary fields must be added to $\mathcal{N}(W^a, W^b)$ to get the null-field. Of all these fields, only those are of immediate interest that yield primarity of the null-field *at length* $\Delta-1$, i.e. $L_2 N^{aa} \stackrel{\Delta-1}{=} 0$. Define $\mathcal{X}$ to be the set of all quasi-primary fields of weight 2Δ except $\mathcal{N}(T^\Delta)$ in whose mode expansion appear Virasoro-monomials up to degree $\Delta-1$; in particular, $L_{-4}L_{-2}^{\Delta-2}$ is such a monomial. For example, $\mathcal{N}(\partial^2 T, \mathcal{N}(T^{\Delta-2})) \in \mathcal{X}$. Then the singular vector associated to the null-field satisfies the identity

$$\begin{aligned} N^{aa} &\stackrel{\Delta-1}{=} \left((\mathcal{N}(W^a,W^a))_{-2\Delta} + \alpha \left(\mathcal{N}(T^\Delta)\right)_{-2\Delta} + \sum_{X\in\mathcal{X}} k_X X_{-2\Delta} \right) \Omega \\ &\stackrel{\Delta-1}{=} \Bigg(W^a_{-\Delta} W^a_{-\Delta} + \alpha L^\Delta_{-2} + (\beta_{T^\Delta} + \beta_{WW})\, L_{-4} L_{-2}^{\Delta-2} \\ &\qquad + (\gamma_{T^\Delta} + \gamma_{WW})\, L_{-3}^2 L_{-2}^{\Delta-3} + \sum_{X\in\mathcal{X}} \left(\beta_X L_{-4}L_{-2}^{\Delta-2} + \gamma_X L_{-3}^2 L_{-2}^{\Delta-3}\right) \Bigg) \Omega\,. \end{aligned}$$

Note that there are no vectors of length $\Delta-1$ in $L_2 L_{-3}^2 L_{-2}^{\Delta-3}\Omega$, so the γ-terms do not have to be considered when L_2 acts on N^{aa}.

Now the assumption is made that $\alpha=0$. Then one can use the fact that $L_2 N^{aa}=0$ to find an explicit expression for the parameter

$$B := \sum_{X\in\mathcal{X}} \beta_X$$

in terms of the structure constant $C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a}$. (Fields $\mathcal{F}$ of weight $2\Delta-1$ like $\mathcal{N}(\partial T, T^{\Delta-2})$ with one derivative term need not be taken into account since the structure constants $C^{\mathcal{F}}_{W^a,W^a}$ for such fields vanish, see [BFKNRV].) To determine B, one needs to know in which exact way β_{WW} is proportional to $C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a}$, so that β_{WW} can be written as $\beta_{WW} = \beta'_{WW} C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a}$ with β'_{WW} a nonzero constant whose exact value can be calculated to be $\beta'_{WW} = -\frac{(2\Delta-1)(\Delta-1)}{2(4\Delta-3)}$ by equation (2.42). With this notation it follows that

$$\begin{aligned} 0 = L_2 N^{aa} &\stackrel{\Delta-1}{=} L_2 \left(W^a_{-\Delta} W^a_{-\Delta} + \left(\beta'_{WW} C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a} + B \right) L_{-4} L_{-2}^{\Delta-2} \right) \Omega \\ &\stackrel{\Delta-1}{=} \left(\left[L_2, W^a_{-\Delta} W^a_{-\Delta}\right] + 6 \left(\beta'_{WW} C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a} + B \right) L_{-2}^{\Delta-1} \right) \Omega \\ &\stackrel{\Delta-1}{=} \left(\left[L_2, W^a_{-\Delta}\right] W^a_{-\Delta} + 6 \left(\beta'_{WW} C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a} + B \right) L_{-2}^{\Delta-1} \right) \Omega \\ &\stackrel{\Delta-1}{=} \left((2(\Delta-1)+\Delta)\, W^a_{2-\Delta} W^a_{-\Delta} + 6 \left(\beta'_{WW} C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a} + B \right) L_{-2}^{\Delta-1} \right) \Omega \\ &\stackrel{\Delta-1}{=} \left((3\Delta-2) \left[W^a_{2-\Delta}, W^a_{-\Delta}\right] + 6 \left(\beta'_{WW} C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a} + B \right) L_{-2}^{\Delta-1} \right) \Omega \\ &\stackrel{\Delta-1}{=} \left((3\Delta-2) C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a} + 6 \left(\beta'_{WW} C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a} + B \right) \right) L_{-2}^{\Delta-1} \Omega\,, \end{aligned}$$

where it has been used in the last line that $p_{\Delta,\Delta,2\Delta-2}(2-\Delta,-\Delta)=1$. The above equation holds if and only if

$$B = -\frac{6\Delta^2 - 8\Delta + 3}{6(4\Delta-3)}\, C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a}\,. \tag{3.13}$$

Before going on with the proof of the lemma, it shall now be explained how exactly the value of β_{WW}, which is the coefficient of the term $L_{-4}L_{-2}^{\Delta-2}\Omega$ of length $\Delta-1$ in the expression for $\mathcal{N}(W^a,W^a)_{-2\Delta}\Omega$, is computed from (2.42). This kind of calculation will also be relevant further on.
The only possibility for a term in the mode expansion of $\mathcal{N}(W^a,W^a)$ to be of length $\Delta-1$ is that it is contributed from the field $\phi_k=\mathcal{N}(T^{\Delta-1})$, which appears as $\partial^2\mathcal{N}(T^{\Delta-1})$ in the second sum in (2.42). All other terms in the expression for $\mathcal{N}(W^a,W^a)$ involve terms of length $\Delta-2$ or less.
To compute the contribution of terms of length $\Delta-1$ from $\mathcal{N}(T^{\Delta-1})$ in $\mathcal{N}(W^a,W^a)$, only the $N^{(\cdot)}$-product part of $\mathcal{N}(T^{\Delta-1})$ has to be taken into account as again all other terms in the corresponding version of (2.42) do not contribute at length $\Delta-1$. The nested $N^{(\cdot)}$-product is calculated recursively, and suppressing the dependence on the formal variable x in most places, the first few steps are

$$
\begin{aligned}
&N^{(2\Delta-4)}\left(\dots N^{(6)}\left(T,N^{(4)}\left(T,N^{(2)}(T,T)\right)\right)\dots\right)\\
&=N^{(2\Delta-4)}\left(\dots N^{(6)}\left(T,N^{(4)}\left(T,\sum_{n\in\mathbb{Z}}x^{-n-4}\left\{\sum_{k_1=-\infty}^{1}L_{k_1+n}L_{-k_1}+\sum_{k_1=2}^{\infty}L_{-k_1+n}\right\}\right)\right)\dots\right)\\
&=N^{(2\Delta-4)}\left(\dots N^{(6)}\left(T,\sum_{n\in\mathbb{Z}}x^{-n-6}\left\{\sum_{k_2=-\infty}^{3}L_{k_2+n}\left[\sum_{k_1=-\infty}^{1}L_{k_1-k_2}L_{-k_1}+\sum_{k_1=2}^{\infty}L_{-k_1}L_{k_1-k_2}\right]\right.\right.\right.\\
&\qquad\left.\left.\left.+\sum_{k_2=4}^{\infty}\left[\sum_{k_1=-\infty}^{1}L_{k_1-k_2}L_{-k_1}+\sum_{k_1=2}^{\infty}L_{-k_1}L_{k_1-k_2}\right]L_{k_2+n}\right\}\right)\dots\right)\\
&=N^{(2\Delta-4)}\left(\dots N^{(8)}\left(T,\sum_{n\in\mathbb{Z}}x^{-n-8}\left\{\sum_{k_3=-\infty}^{5}L_{k_3+n}\left[\sum_{k_2=-\infty}^{3}L_{k_2-k_3}\left(\sum_{k_1=-\infty}^{1}L_{k_1-k_2}L_{-k_1}\right.\right.\right.\right.\right.\\
&\qquad\left.+\sum_{k_1=2}^{\infty}L_{-k_1}L_{k_1-k_2}\right)+\sum_{k_2=4}^{\infty}\left(\sum_{k_1=-\infty}^{1}L_{k_1-k_2}L_{-k_1}+\sum_{k_1=2}^{\infty}L_{-k_1}L_{k_1-k_2}\right)L_{k_2-k_3}\Bigg]\\
&\qquad+\sum_{k_3=6}^{\infty}\left[\sum_{k_2=-\infty}^{3}L_{k_2-k_3}\left(\sum_{k_1=-\infty}^{1}L_{k_1-k_2}L_{-k_1}+\sum_{k_1=2}^{\infty}L_{-k_1}L_{k_1-k_2}\right)\right.\\
&\qquad\left.\left.\left.\left.+\sum_{k_2=4}^{\infty}\left(\sum_{k_1=-\infty}^{1}L_{k_1-k_2}L_{-k_1}+\sum_{k_1=2}^{\infty}L_{-k_1}L_{k_1-k_2}\right)L_{k_2-k_3}\right]L_{k_3+n}\right\}\right)\dots\right)\,. \qquad (3.14)
\end{aligned}
$$

Multiplying by the correct coefficient from (2.42), setting $n=-2\Delta$ in (3.14) and applying the result to the vacuum Ω yields the contribution to $\mathcal{N}(W^a,W^a)_{-2\Delta}\Omega$. All terms but those in the last sum in (3.14) either vanish when applied to Ω or they involve at least one mode L_m with $m\geq-1$ such that these terms are of length $\Delta-2$ or less by use of the commutation relations for L_m. Thus by induction, the contribution at length $\Delta-1$ is

$$\sum_{\substack{k_i\geq 2i\\ i\in\{1,\dots,\Delta-2\}}}L_{-k_1}L_{k_1-k_2}L_{k_2-k_3}\dots L_{k_{\Delta-3}-k_{\Delta-2}}L_{k_{\Delta-2}-2\Delta}\Omega \qquad (3.15)$$

multiplied by $(-n-2\Delta+2)(-n-2\Delta+1)|_{n=-2\Delta} = 2$ (because of the second derivative in $\partial^2\mathcal{N}(T^{\Delta-1})$) and $-\frac{1}{4}C_{W^a,W^a}^{\mathcal{N}(T^{\Delta-1})}\frac{2\Delta-1}{4\Delta-3}$ (which is the coefficient of $\partial^2\mathcal{N}(T^{\Delta-1})$ in the expression for $\mathcal{N}(W^a,W^a)$ in (2.42)). Counting the nonvanishing terms in (3.15) an rearranging it by use of the commutation relations by which only irrelevant contributions of length less than $\Delta-1$ are modified, one thus arrives at

$$\mathcal{N}(W^a,W^a)_{-2\Delta}\Omega = \beta_{WW}L_{-4}L_{-2}^{\Delta-2}\Omega + \gamma_{WW}L_{-3}^2L_{-2}^{\Delta-3}\Omega$$

with

$$\beta_{WW} = 2\cdot\left(-\frac{1}{4}C_{W^a,W^a}^{\mathcal{N}(T^{\Delta-1})}\frac{2\Delta-1}{4\Delta-3}\right)\cdot(\Delta-1)\ , \tag{3.16a}$$

$$\gamma_{WW} = 2\cdot\left(-\frac{1}{4}C_{W^a,W^a}^{\mathcal{N}(T^{\Delta-1})}\frac{2\Delta-1}{4\Delta-3}\right)\cdot\left((\Delta-2)^2-\frac{1}{2}(\Delta-2)(\Delta-3)\right)\ . \tag{3.16b}$$

Now the proof of the lemma is continued. The idea to prove that $\alpha\neq 0$ is to find another way to explicitly compute the value of B that does not agree with the one given in (3.13). For this it is convenient to express the parameters $\beta=\beta_{WW}+B$ and $\gamma=\gamma_{WW}+\sum_{X\in\mathcal{X}}\gamma_X$ in terms of the structure constant $C_{W^a,W^a}^{\mathcal{N}(T^{\Delta-1})}$ and B alone: β_{WW} and γ_{WW} are given in (3.16) and from the fact that each field X in $\mathcal{X}$ is quasi-primary (which means $L_1X_{-2\Delta}\Omega \overset{\Delta-1}{=} 0$ among other things) it follows that

$$\begin{aligned} 0 &\overset{\Delta-1}{=} L_1\sum_{X\in\mathcal{X}}k_X X_{-2\Delta}\Omega \\ &\overset{\Delta-1}{=} L_1\left(BL_{-4}L_{-2}^{\Delta-2}+\sum_{X\in\mathcal{X}}\gamma_X L_{-3}^2L_{-2}^{\Delta-3}\right)\Omega \\ &\overset{\Delta-1}{=} \left(5BL_{-3}L_{-2}^{\Delta-2}+8\sum_{X\in\mathcal{X}}\gamma_X L_{-3}L_{-2}^{\Delta-2}\right)\Omega \end{aligned}$$

and thus $\sum_{X\in\mathcal{X}}\gamma_X = -\frac{5}{8}B$. This yields

$$\beta = -\frac{1}{2}\frac{2\Delta-1}{4\Delta-3}C_{W^a,W^a}^{\mathcal{N}(T^{\Delta-1})}(\Delta-1)+B\ , \tag{3.17a}$$

$$\gamma = -\frac{1}{2}\frac{2\Delta-1}{4\Delta-3}C_{W^a,W^a}^{\mathcal{N}(T^{\Delta-1})}\left((\Delta-2)^2-\frac{1}{2}(\Delta-2)(\Delta-3)\right)-\frac{5}{8}B\ . \tag{3.17b}$$

These relations will be made use of without explicit mention in the following.
The vector $N_{-2\Delta}^{aa}\Omega$ is already completely known at length $\Delta-1$ up to the structure constant $C_{W^a,W^a}^{\mathcal{N}(T^{\Delta-1})}$, and the same situation will now be achieved for the vector $N_{-2\Delta-1}^{aa}\Omega$ as an intermediate step. For this, the relation

$$[L_m,\phi_n] = ((h-1)m-n)\phi_{m+n}$$

with $m \in \{\pm 1, 0\}$ for a quasi-primary field ϕ of weight h is employed. On the one hand, at length $\Delta - 1$ the term $[L_{-1}, N^{aa}_{-2\Delta}]\Omega = L_{-1}N^{aa}$ is equal to

$$
\begin{aligned}
& L_{-1}\left(W^a_{-\Delta}W^a_{-\Delta} + \beta L_{-4}L_{-2}^{\Delta-2} + \gamma L_{-3}^2 L_{-2}^{\Delta-3}\right)\Omega \\
\overset{\Delta-1}{=} & \left(W^a_{-\Delta}\left[L_{-1}, W^a_{-\Delta}\right] + \left[L_{-1}, W^a_{-\Delta}\right]W^a_{-\Delta} + 3\beta L_{-5}L_{-2}^{\Delta-2}\right. \\
& \quad \left. +(\Delta-2)\beta L_{-4}L_{-3}L_{-2}^{\Delta-3} + 4\gamma L_{-4}L_{-3}L_{-2}^{\Delta-3} + (\Delta-3)\gamma L_{-3}^3 L_{-2}^{\Delta-4}\right)\Omega \\
\overset{\Delta-1}{=} & \left(2W^a_{-\Delta-1}W^a_{-\Delta} + \left[W^a_{-\Delta}, W^a_{-\Delta-1}\right]\right. \\
& \quad \left. +3\beta L_{-5}L_{-2}^{\Delta-2} + \left((\Delta-2)\beta + 4\gamma\right)L_{-4}L_{-3}L_{-2}^{\Delta-3} + (\Delta-3)\gamma L_{-3}^3 L_{-2}^{\Delta-4}\right)\Omega \\
\overset{\Delta-1}{=} & C_{W^a,W^a}^{\mathcal{N}(T^{\Delta-1})} p_{\Delta,\Delta,2\Delta-2}(-\Delta, -\Delta-1)\Bigg((\Delta-1)L_{-5}L_{-2}^{\Delta-2} \\
& \quad +(\Delta-1)(\Delta-2)L_{-4}L_{-3}L_{-2}^{\Delta-3} + \binom{\Delta-1}{3}L_{-3}^3 L_{-2}^{\Delta-4}\Bigg)\Omega \\
& \quad + \left(3\beta L_{-5}L_{-2}^{\Delta-2} + \left((\Delta-2)\beta + 4\gamma\right)L_{-4}L_{-3}L_{-2}^{\Delta-3}\right)\Omega \\
& \quad + (\Delta-3)\gamma L_{-3}^3 L_{-2}^{\Delta-4}\Omega\,.
\end{aligned} \tag{3.18}
$$

But because of the quasi-primarity of the vector N^{aa}, this must also be equal to $N^{aa}_{-2\Delta-1}\Omega$. Of course the latter is not known explicitly, but at length $\Delta - 1$ the relevant parameters can be inferred. Firstly, there is a contribution to $N^{aa}_{-2\Delta-1}\Omega$ from $\mathcal{N}(W^a, W^a)_{-2\Delta-1}\Omega$, and only the terms of length $\Delta - 1$ will be of importance here. Secondly, the contribution of the fields in $\mathcal{X}$ has to be taken into account. Computing this contribution exactly would require the knowledge of the exact values of the parameters k_X in

$$
N^{aa} \overset{\Delta-1}{=} \mathcal{N}(W^a, W^a)_{-2\Delta}\Omega + \sum_{X\in\mathcal{X}} k_X X_{-2\Delta}\Omega\,.
$$

These are not available, but all one really needs to know in this case are the coefficients of the relevant monomials at length $\Delta - 1$. Denoting these coefficients by ξ_i, $i \in \{1, 2, 3\}$, (3.18) is also equal to

$$
\begin{aligned}
& \mathcal{N}(W^a, W^a)_{-2\Delta-1}\Omega + \left(\xi_1 L_{-5}L_{-2}^{\Delta-2} + \xi_2 L_{-4}L_{-3}L_{-2}^{\Delta-3} + \xi_3 L_{-3}^3 L_{-2}^{\Delta-4}\right)\Omega \\
\overset{\Delta-1}{=} & -\frac{1}{4}C_{W^a,W^a}^{\mathcal{N}(T^{\Delta-1})}\frac{2\Delta-1}{4\Delta-3}(-n-2\Delta+2)(-n-2\Delta+1)\Big|_{n=-2\Delta-1} \\
& \quad \cdot\left((\Delta-1)L_{-5}L_{-2}^{\Delta-2} + (\Delta-1)(\Delta-2)L_{-4}L_{-3}L_{-2}^{\Delta-3} + \binom{\Delta-1}{3}L_{-3}^3 L_{-2}^{\Delta-4}\right)\Omega \\
& \quad + \left(\xi_1 L_{-5}L_{-2}^{\Delta-2} + \xi_2 L_{-4}L_{-3}L_{-2}^{\Delta-3} + \xi_3 L_{-3}^3 L_{-2}^{\Delta-4}\right)\Omega\,,
\end{aligned} \tag{3.19}
$$

where the coefficients are computed similarly to the way that led to (3.16). Now comparing the coefficients of the vectors $L_{-5}L_{-2}^{\Delta-2}\Omega$, $L_{-4}L_{-3}L_{-2}^{\Delta-3}\Omega$ and $L_{-3}^3 L_{-2}^{\Delta-4}\Omega$ in (3.18) and (3.19) yields

$$
\xi_1 = \frac{1}{2}\left(6B + C_{W^a,W^a}^{\mathcal{N}(T^{\Delta-1})}(\Delta-1)\right)\,, \tag{3.20a}
$$

$$\xi_2 = \frac{1}{2}\left(-9B + 2B\Delta + C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a}\left(\Delta^2 - 3\Delta + 2\right)\right) , \tag{3.20b}$$

$$\xi_3 = \frac{1}{24}\left(45B - 15B\Delta + C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a}\left(2\Delta^3 - 12\Delta^2 + 22\Delta - 12\right)\right) . \tag{3.20c}$$

With this knowledge of both vectors $N^{aa}_{-2\Delta}\Omega$ and $N^{aa}_{-2\Delta-1}\Omega$ at length $\Delta - 1$, now one last piece of information can be utilized in order to find another way to compute B. Until now, only the quasi-primarity of the null-field has been used. But actually it is also primary, i.e. the relation

$$[L_m, N^{aa}_n] = ((2\Delta - 1)m - n)N^{aa}_{m+n}$$

holds for all integers m and n. In particular, this is true for $m = 2$ and $n = -2\Delta - 1$, and thus

$$\begin{aligned}
0 &= (6\Delta - 1)N^{aa}_{-2\Delta+1}\Omega = \left[L_2, N^{aa}_{-2\Delta-1}\right]\Omega = L_2 N^{aa}_{-2\Delta-1}\Omega \\
&\overset{\Delta-1}{=} -\frac{3}{2}C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a}\frac{2\Delta-1}{4\Delta-3}\left(7(\Delta-1) + 6(\Delta-1)(\Delta-2)\right)L_{-3}L^{\Delta-2}_{-2}\Omega \\
&\qquad + \left(7\xi_1 + 6\xi_2\right)L_{-3}L^{\Delta-2}_{-2}\Omega + \left[L_2, N^{(\Delta)}(W^a, W^a)_{-2\Delta-1}\right]\Omega \\
&\overset{\Delta-1}{=} -\frac{3}{2}C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a}\frac{2\Delta-1}{4\Delta-3}\left(7(\Delta-1) + 6(\Delta-1)(\Delta-2)\right)L_{-3}L^{\Delta-2}_{-2}\Omega \\
&\qquad + \left(7\xi_1 + 6\xi_2\right)L_{-3}L^{\Delta-2}_{-2}\Omega + \left[L_2, W^a_{-\Delta}W^a_{-\Delta-1} + W^a_{-\Delta-1}W^a_{-\Delta}\right]\Omega \\
&\overset{\Delta-1}{=} -\frac{3}{2}C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a}\frac{2\Delta-1}{4\Delta-3}\left(7(\Delta-1) + 6(\Delta-1)(\Delta-2)\right)L_{-3}L^{\Delta-2}_{-2}\Omega \\
&\qquad + \left(7\xi_1 + 6\xi_2\right)L_{-3}L^{\Delta-2}_{-2}\Omega \\
&\qquad + \left(\left[L_2, W^a_{-\Delta}\right]W^a_{-\Delta-1} + \left[L_2, W^a_{-\Delta-1}\right]W^a_{-\Delta}\right)\Omega \\
&\overset{\Delta-1}{=} -\frac{3}{2}C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a}\frac{2\Delta-1}{4\Delta-3}\left(7(\Delta-1) + 6(\Delta-1)(\Delta-2)\right)L_{-3}L^{\Delta-2}_{-2}\Omega \\
&\qquad + \left(7\xi_1 + 6\xi_2\right)L_{-3}L^{\Delta-2}_{-2}\Omega + \left(2(\Delta-1) + \Delta\right)\left[W^a_{-\Delta+2}, W^a_{-\Delta-1}\right]\Omega \\
&\qquad + \left(2(\Delta-1) + \Delta + 1\right)\left[W^a_{-\Delta+1}, W^a_{-\Delta}\right]\Omega \\
&\overset{\Delta-1}{=} -\frac{3}{2}C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a}\frac{2\Delta-1}{4\Delta-3}\left(7(\Delta-1) + 6(\Delta-1)(\Delta-2)\right)L_{-3}L^{\Delta-2}_{-2}\Omega \\
&\qquad + \left(7\xi_1 + 6\xi_2\right)L_{-3}L^{\Delta-2}_{-2}\Omega \\
&\qquad + (3\Delta-2)C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a}p_{\Delta,\Delta,2\Delta-2}(2-\Delta, -\Delta-1)(\Delta-1)L_{-3}L^{\Delta-2}_{-2}\Omega \\
&\qquad + (3\Delta-1)C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a}p_{\Delta,\Delta,2\Delta-2}(1-\Delta, -\Delta)(\Delta-1)L_{-3}L^{\Delta-2}_{-2}\Omega ,
\end{aligned} \tag{3.21}$$

where in this case the term $N^{(\Delta)}(W^a, W^a)_{-2\Delta-1}\Omega$ (using the notation introduced in (2.43)) does lead to a contribution at length $\Delta - 1$, in contrast to the situation in equation (3.19). Now using (3.17) and (3.20) in (3.21) yields the following alternate expression for the parameter B:

$$B = -\frac{12\Delta^2 - 18\Delta + 7}{4(4\Delta - 3)}\,C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a} .$$

This can only be in agreement with (3.13) for $C^{\mathcal{N}(T^{\Delta-1})}_{W^a,W^a} = 0$, which is not the case. Thus, the assumption $\alpha = 0$ leads to a contradiction and the lemma is proven.

Now all the conditions are satisfied to immediately carry over the steps in the proof of the proposition in section 3.1, and it follows that the vertex operator algebra $\mathcal{W}(2,(2p-1)^{\times 3})$ is C_2-cofinite for all $p \in \mathbb{Z}_{\geq 2}$. Consequently, Buhl's theorem can be applied just as in the special case $\Delta = 3$, and the three conditions (**1**), (**2**) and (**3**) given at the beginning of this chapter are satisfied for all triplet algebras. Thus, the following main result is proven.

> *Theorem.* For all $p \in \mathbb{Z}_{\geq 2}$, the nonmeromorphic operator product expansion exists and is associative for the vertex operator algebra $\mathcal{W}(2,(2p-1)^{\times 3})$. Furthermore, all these vertex operator algebras are C_2-cofinite.

The fact that all triplet algebras are C_2-cofinite adds credibility to the conjecture that "rationality" in the sense of Gaberdiel and Kausch, i.e. a finite set of generalized modules closes under fusion, is equivalent to C_2-cofiniteness. Indeed, from the C_2-cofiniteness of V_{2p-1} it follows that the Zhu algebra $A(V_{2p-1})$ is finite-dimensional, and because of this there are only finitely many equivalence classes of indecomposable $A(V_{2p-1})$-modules. This together with the strong restrictions coming from the structure of $\mathcal{W}$-algebras suggests that the assumedly equivalent properties both hold for all triplet algebras.
The above theorem is also interesting for a different reason. Not only the triplet algebra at $c = -2$ but also all other triplet algebras are known to have indecomposable modules with respect to the action of L_0, and thus they constitute the vacuum sectors of a family of logarithmic conformal field theories. The proof that they are all C_2-cofinite and satisfy the associated finiteness conditions shows that such congenial properties may just as well appear in the general framework of logarithmic conformal field theory, not only in a few exotic cases. Likewise, the precise statement on nonmeromorphic operator product expansion contributes to the program to formulate and treat logarithmic conformal field theory rigorously.

An upper bound on dim $\boldsymbol{A(V_\Delta)}$. As another application of the results above, one can obtain an upper bound on the dimension of the Zhu algebra $A(V_\Delta)$ for any triplet algebra V_Δ. The strategy is to use the proposition in section 2.2, i.e. to find a minimal subspace $\tilde{V}_\Delta \subset V_\Delta$ such that $V_\Delta = C_2(V_\Delta) + \tilde{V}_\Delta$. Then the dimension of $\tilde{V}_\Delta$ dominates the dimension of the Zhu algebra.

From the existence of the singular vectors N^{ab} and from the lemma above it follows that for all $a, b \in \{\pm 1, 0\}$ and $\Delta \in (\mathbb{Z}_+ + 1)$,

$$\left(W^a_{-\Delta} W^b_{-\Delta} + \alpha L^{\Delta}_{-2}\right) \Omega \in C_2(V_\Delta) \tag{3.22}$$

and

$$L^{2\Delta}_{-2}\Omega \in C_2(V_\Delta)\,, \quad \left(W^a_{-\Delta}\right)^3 \Omega \in C_2(V_\Delta)\,.$$

With this information one can argue that

$$\left\{L^k_{-2}\Omega,\, L^l_{-2} W^a_{-\Delta}\Omega \mid k \in \{0,\dots,2\Delta-1\},\, l \in \{0,\dots,\Delta-1\},\, a \in \{\pm 1, 0\}\right\} \tag{3.23}$$

is a generating system for $\tilde{V}_\Delta$. Note that this is a more precise statement than was necessary above to prove that all triplet algebras are C_2-cofinite. To verify the claim that (3.23) is a generating system, one can first observe that the vectors $L_{-2}^{\Delta+i}W_{-\Delta}^a\Omega$, $i \in \mathbb{N}$, are in $C_2(V_\Delta)$ because of $((W_{-\Delta}^a)^2 + \alpha L_{-2}^{\Delta})\Omega \in C_2(V_\Delta)$, $(W_{-\Delta}^a)^3\Omega \in C_2(V_\Delta)$ and the fact that this space is invariant under the action of both L_{-2} and $W_{-\Delta}^a$. Furthermore, the elements $L_{-2}^j(W_{-\Delta}^a)^2\Omega$, $j \in \mathbb{N}$, need not be taken as part of a generating system for $\tilde{V}_\Delta$ since by acting on (3.22) with L_{-2}^j, one finds that these elements can be written as a linear combination of some vector in $C_2(V_\Delta)$ and $\alpha L_{-2}^{\Delta+j}\Omega$, and the latter is already accounted for in (3.23).
From (3.23) and the proposition in section 2.2 it follows that

$$\dim A(V_\Delta) \leq 5\Delta \quad \text{for all } \Delta \in (2\mathbb{Z}_+ + 1)\ . \tag{3.24}$$

At least for $\Delta = 3$, the dimension of the Zhu algebra is strictly lower than this bound as Abe proved in [A] that $\dim A(V_3) = 11$ by explicitly calculating relations between elements of the Zhu algebra. Such calculations are quite laborious and would have to be done for each triplet algebra separately, so the upper bounds in (3.24) are certainly of interest.

Chapter 4

Mode Algebras in Logarithmic Conformal Field Theory

In this chapter some aspects of the possibility of realizing "logarithmic" properties in two-dimensional conformal field theory at the most fundamental algebraic level are investigated. Such a structure may be called a "Jordan vertex operator algebra" as indecomposability is among the defining characteristics of logarithmic conformal field theory. The difficulties of establishing such a hypothetical structure are discussed and a certain concrete logarithmic model (the $\theta^+\theta^-$-system) is studied in detail, seeking to identify generic features to understand logarithmic conformal field theory in general. In particular, the commutation relations between Virasoro modes L_m and modes $\widetilde{\Omega}_{n,a}$ pertaining to the logarithmic partner field of the identity operator are successfully obtained in various ways. On the other hand, the commutator $[\widetilde{\Omega}_{m,a}, \widetilde{\Omega}_{n,b}]$ of two logarithmic modes resists a complete and rigorous treatment but nevertheless, an ansatz for it is proposed and thoroughly motivated.

In the discussion of $P(z)$-tensor product theory in section 2.3 the notions of generalized module and logarithmic intertwining operator were introduced. With these notions Huang, Lepowsky and Zhang were able to treat aspects of logarithmic conformal field theory in the language of vertex operator algebras. Their approach places generalized, logarithmic features at the level of modules, while the definition of the fundamental structure, the vertex operator algebra, is left unchanged.
On the other hand, all logarithmic conformal models I am aware of share the property that the vacuum vector Ω has at least one *logarithmic partner*, i.e. Ω is an element of a nontrivial Jordan cell with respect to the action of L_0. By the operator-state-correspondence, there is also a logarithmic partner to the identity operator which typically has a logarithmic dependence on its variables. Since the vacuum vector is part of the structure of a vertex operator algebra but not necessarily of its modules, the question arises whether it is possible to modify or generalize the definition of a vertex operator algebra to treat logarithmic conformal field theory already at this fundamental level.

Jordan vertex operator algebras? In order to explore the possibility of a *Jordan vertex operator algebra* J, the formal logarithmic variable $\log x$ introduced in section 2.3 is used.

It appears that most of the axioms of a vertex operator algebra as presented in section 2.1 are naturally generalized to incorporate properties of logarithmic conformal field theory: J is $\mathbb{Z}$-graded by generalized L_0-eigenvalues; the vertex operator is a map

$$\begin{aligned} J &\longrightarrow (\mathrm{End} J)[\log x][\![x, x^{-1}]\!] , \\ v &\longmapsto Y(v, x, \log x) = \sum_{m\in\mathbb{Z}} \sum_{a\in\mathbb{N}} x^{-m-1} (\log x)^a v_{m,a} ; \end{aligned}$$

there are $d+1 \in \mathbb{Z}_+$ vectors $\Omega^{(i)}$ that span a Jordan cell with respect to L_0 with $i \in \{0, \ldots, d\}$ and the generalized vacuum property associates $\Omega = \Omega^{(0)}$ to the identity operator $\mathbb{1}_J$; the truncation condition (2.7) is extended by the additional condition $u_{m,a}v = 0$ for $a \gg 0$ and arbitrary $m \in \mathbb{Z}$; the creation property (2.9) is modified to $Y(v, x, \log x)\Omega \in J[\![x]\!]$ and $Y(v, x, \log x)\Omega|_{x=0} = v$; the operator $T(x) = Y(\omega, x)$ associated to the conformal vector ω has no logarithmic dependence (this choice is motivated by the known operator product expansion of $T(x)$ with itself in general logarithmic models and the concrete realization in the $\theta^+\theta^-$-system discussed below) and its modes give a representation of the Virasoro algebra; the L_{-1}-derivative property is kept in the same form as in (2.13) but the natural definition (2.48) is employed on the left-hand side.
The difficulty in establishing the definition of a Jordan vertex operator algebra is to find an appropriate generalization of the Jacobi identity (2.11). The naive ansatz

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1, \log x_1) Y(v, x_2, \log x_2) \\ & \quad - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2, \log x_2) Y(u, x_1, \log x_1) \\ & \stackrel{?}{=} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0, \log x_0)v, x_2, \log x_2) \end{aligned} \tag{4.1}$$

quickly leads to consequences that are not consistent with known logarithmic models. For example, all modes $u_{m,a}$ with $a > 0$ associated to nontrivial powers of the formal logarithmic variable commute with all other modes. The reason for this is that the left-hand side of the above relation involves $\log x_1$ but not $\log x_0$, while the right-hand side involves $\log x_0$ but not $\log x_1$. Explicitly, the operation Res_{x_0} applied to a mode expansion of (4.1) yields that $\sum_{m,n\in\mathbb{Z}} \sum_{a,b\in\mathbb{N}} x_1^{-m-1} x_2^{-n-1} (\log x_1)^a (\log x_2)^b [u_{m,a}, v_{n,b}]$ would be equal to

$$\sum_{m,n\in\mathbb{Z}} \sum_{a,b\in\mathbb{N}} \sum_{k\in\mathbb{N}} \binom{m}{k} x_1^{-m-1} x_2^{-n-1} (\log x_0)^a (\log x_2)^b \left(u_{k,a}v\right)_{m+n-k,b} .$$

But by comparing coefficients of both expressions one finds that $[u_{m,a}, v_{n,b}] = 0$ for all $a > 0$. This is not in agreement with known logarithmic models where logarithmic modes can have nontrivial commutation relations; an example of such a case will be given below. Because of this failure, the ansatz (4.1) must be discarded.

Another possibility to enhance the Jacobi identity one might think of is to include factors like

$$(\log x_0)^{-1}\delta\left(\frac{\log x_1 - \log x_2}{\log x_0}\right) \quad \text{or} \quad (\log x_0)^{-1}\delta\left(\frac{\log x_1 + \log(1 - x_2/x_1)}{\log x_0}\right) \tag{4.2}$$

in (4.1) in analogy to the factors of the form $x_0^{-1}\delta(\frac{x_1-x_2}{x_0})$ in the original Jacobi identity. Note the difference between the above two δ-function expressions: the first involves the formal variables $\log x_1$, $\log x_2$ and $\log x_0$, the second involves the formal variables $\log x_1$, $\log x_0$, x_1 and x_2. The difference is that $\log x_2$ cannot be expanded into a series because it is an independent formal variable, while $\log(1 - x_2/x_1)$ is a series in x_1 and x_2 defined by the general relation

$$\log(x + y) = \log x + \log(1 + y/x) = \log x + \sum_{i\in\mathbb{Z}_+} \frac{(-1)^{i-1}}{i} x^{-i}y^i \ . \tag{4.3}$$

But adding factors like in (4.2) to the ansatz (4.1) is not a successful generalization as well because the case of an ordinary vertex operator algebra (which is the special case of a logarithmic genus-zero conformal field theory with trivial Jordan cells) cannot be recovered from it. Indeed, applying the operation $\mathrm{Res}_{x_0}\mathrm{Res}_{\log x_0}$ and comparing coefficients in the case that u and v are linear combinations of ordinary eigenvectors of L_0 yields the commutator $[u_m, v_n]$ on the left-hand side, while the right-hand side vanishes since there are no negative powers of the formal variable $\log x_0$. As this result does not agree with relation (2.20) in the case of ordinary vertex operator algebras, extending the Jacobi identity by terms like in (4.2) is not the right thing to do as well.

Instead of adding further δ-functions to the ansatz (4.1) one could try to eliminate its inconsistencies by changing the logarithmic arguments of the vertex operators as suggested by the δ-functions that are already present. As illustrated by its many applications in chapter 2, a δ-function "virtually" acts by equating formal variables such that its argument is equal to 1, see e.g. (2.3). This might suggest that in (4.1) for example the term $x_0^{-1}\delta(\frac{x_1-x_2}{x_0})Y(u, x_1, \log x_1)Y(v, x_2, \log x_2)$ should be changed to $x_0^{-1}\delta(\frac{x_1-x_2}{x_0})Y(u, x_1, \log(x_0 + x_2))Y(v, x_2, \log(-x_0 + x_1))$. But modifying all three terms in (4.1) this way immediately leads to the same problem as before, i.e. logarithmic variables do not match on both sides. Because of this one may propose the ansatz

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) & Y(u, x_1, \log x_1)Y(v, x_2, \log x_2) \\ & - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2, \log x_2)Y(u, x_1, \log x_1) \\ \stackrel{?}{=} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) & Y(Y(u, x_0, \log(-x_2 + x_1))v, x_2, \log(x_1 - x_0)) \end{aligned} \tag{4.4}$$

where only the right-hand side was changed. This also seems plausible since at least when extracting commutation relations from the Jacobi identity, the crucial information is found on the right-hand side while the left-hand side simply gives the commutator as discussed starting on page 17. Applying Res_{x_0} to the left-hand side of (4.4) gives a generating function for the commutators $[u_{m,a}, v_{n,b}]$, while on the right-hand side after a computation using (4.3) and the relation[1]

$$(-\log(1-x))^k = k! \sum_{n_k=k}^{\infty} \sum_{n_{k-1}=k-1}^{n_k-1} \cdots \sum_{n_1=1}^{n_2-1} \frac{x^{n_k}}{n_k n_{k-1} \ldots n_1} \tag{4.5}$$

several times it yields

$$\sum_{m,n\in\mathbb{Z}} \sum_{a,b\in\mathbb{N}} \sum_{i,j,k\in\mathbb{N}} \sum_{r\in\mathbb{N}_{\geq i}} \sum_{s\in\mathbb{N}_{\geq k}} \sum_{m_{i-1}=i-1}^{r-1} \cdots \sum_{m_1=1}^{m_2-1} \sum_{n_{k-1}=k-1}^{s-1} \cdots \sum_{n_1=1}^{n_2-1} \frac{i!k!}{m_i m_{i-1} \ldots m_1 n_k n_{k-1} \ldots n_1}$$
$$\cdot \binom{m+r-s}{j} \binom{a}{i} \binom{b}{k} x_1^{-m-1} x_2^{-n-1} (\log(-x_2))^{a-i} (\log x_1)^{b-k} (u_{j+s}v)_{m+n-j-s} \,. \tag{4.6}$$

For fixed indices m, n, a, b the above sums over i, j, k, s are all finite when applied to an arbitrary element in J by the generalized truncation property, but the sum over r is infinite and thus the above expression is not in J but in its algebraic completion $\overline{J}$. This does certainly not speak in favor of ansatz (4.4), but one might still be inclined not to discard it right away, as a logarithmic generalization of vertex operator algebras might have some unexpected properties like this. What really disqualifies the above ansatz is the fact that it leads to vanishing commutation relations between the logarithmic partners $\Omega^{(p)}$ of the vacuum, $p \in \{1, \ldots, d\}$, in many cases. To see this, one considers possible Jordan vertex operator algebras of the form $J = \coprod_{n\in\mathbb{N}} J_{[n]}$ such that if $u = \Omega^{(p)}$ and $v = \Omega^{(q)}$ are substituted into (4.6), the term $\Omega^{(p)}_{j+s,a}\Omega^{(q)}$ identically vanishes for all $j, s \in \mathbb{N}$ as the result would have to be of negative weight. Again, there are logarithmic models in which such commutators do not vanish and thus (4.4) is not the sought-after generalization.

Other ansätze similar to those above or motivated by alternate approaches to vertex operator algebras are plagued by related difficulties, and I have not been able to find a consistent definition of a Jordan vertex operator algebra. Before generally withdrawing to the position that logarithmic conformal field theory cannot be described on the vacuum level in terms of a vertex operator algebraic structure, the possibility remains that a detailed study of concrete physical models displaying logarithmic features may reveal a structure that suggests a general treatment. Such a concrete model will now be investigated.

The $\boldsymbol{\theta^+\theta^-}$-system. The probably most intensively studied logarithmic conformal field theory is the one with central charge $c = -2$, see e.g. [Gu], [Kau2], [GK1], [FGN], [GK2], [Kau3], [Fl3], [Gab2]. While the triplet algebra $\mathcal{W}(2, 3^{\times 3})$ at $c = -2$ discussed in section 3.1 involves three additional generating fields $W^a(x)$, there is also a pure Virasoro model with indecomposable but reducible structure. It has a concrete realization in terms of the $\theta^+\theta^-$-*system*.

[1] I found this relation in [Nag].

The $\theta^+\theta^-$-system is defined by two fermionic fields $\theta^+(z)$ and $\theta^-(z)$ whose mode expansions are

$$\theta^\pm(z) = \theta_0^\pm \log z + \xi^\pm + \sum_{m\neq 0} \theta_m^\pm z^{-m} \ , \tag{4.7}$$

and all modes anti-commute except for the following cases:

$$\left\{\xi^\pm, \theta_0^\mp\right\} = \pm 1 \ , \quad \left\{\theta_m^+, \theta_n^-\right\} = \frac{1}{m}\,\delta_{m+n,0} \quad \text{for all } m \neq 0 \ . \tag{4.8}$$

Note that there appears a logarithm in (4.7) and that all indices m and n are integer if not stated otherwise. The modes $\xi^\pm$ and $\theta_m^\pm$ generate a Fock space by the free action on the vacuum Ω, subject to the condition that $\theta_m^\pm\Omega = 0$ for all $m \in \mathbb{N}$, and the normal-ordering is given analogously to the definition in section 2.1,

$$\begin{aligned} :\theta^+(z)\theta^-(z): &= \theta^+(z)_+\theta^-(z) - \theta^-(z)\theta^+(z)_- \\ &= \left(\xi^+ + \sum_{m<0}\theta_m^+ z^{-m}\right)\theta^-(z) - \theta^-(z)\left(\theta_0^+\log z + \sum_{m>0}\theta_m^+ z^{-m}\right) \ , \end{aligned}$$

where the fermionic nature has been taken into account, and the terms ξ^+ and $\theta_0^+\log z$ are naturally taken to be in the regular and singular part of $\theta^+(z)$, respectively. The normal-ordering of the fields translates to a normal-ordering of the modes given by

$$:\theta_m^+\theta_n^-: = \begin{cases} +\theta_m^+\theta_n^- & \text{for } m \leq n \ , \\ -\theta_n^-\theta_m^+ & \text{for } m > n \ , \end{cases} \qquad \text{and} \qquad :\theta_m^\pm\xi^\pm: = -\xi^\pm\theta_m^\pm = -:\xi^\pm\theta_m^\pm: \ . \tag{4.9}$$

The conformal symmetry is encoded in the $\theta^+\theta^-$-system such that the energy momentum operator is realized as

$$T(z) = :(\partial\theta^+(z))(\partial\theta^-(z)): \ .$$

Indeed, (4.8) and (4.9) can be used to express the modes L_m of $T(z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}$ in terms of the modes of $\theta^\pm(z)$,

$$L_m = \begin{cases} m(\theta_0^-\theta_m^+ + \theta_m^-\theta_0^+) + \sum_{a\in\mathbb{Z}} a(m-a)\theta_{m-a}^+\theta_a^- & \text{for } m \neq 0 \ , \\ \theta_0^+\theta_0^- - \sum_{a\in\mathbb{Z}} a^2 :\theta_{-a}^+\theta_a^-: & \text{for } m = 0 \ , \end{cases} \tag{4.10}$$

and these modes satisfy the Virasoro algebra with central charge $c = -2$. To verify this one easily checks that $[L_m, L_n] = (m-n)L_{m+n}$ for $m \neq -n$, and for $m = -n < 0$ the following calculation proves that $c = -2$: the commutator $[L_m, L_{-m}]$ is equal to

$$\begin{aligned} &\left[m(\theta_0^-\theta_m^+ + \theta_m^-\theta_0^+) + \sum_{a\in\mathbb{Z}} a(m-a)\theta_{m-a}^+\theta_a^-, -m(\theta_0^-\theta_{-m}^+ + \theta_{-m}^-\theta_0^+) + \sum_{b\in\mathbb{Z}} b(-m-b)\theta_{-m-b}^+\theta_b^-\right] \\ &= -m^2\left(\theta_0^-\theta_m^+\theta_0^-\theta_{-m}^+ - \theta_0^-\theta_{-m}^+\theta_0^-\theta_m^+ + \theta_0^-\theta_m^+\theta_{-m}^-\theta_0^+ - \theta_{-m}^-\theta_0^+\theta_0^-\theta_m^+\right. \end{aligned}$$

$$
\begin{aligned}
&\quad +\theta^-_m\theta^+_0\theta^-_0\theta^+_{-m} - \theta^-_0\theta^+_{-m}\theta^-_m\theta^+_0 + \theta^-_m\theta^+_0\theta^-_{-m}\theta^+_0 - \theta^-_{-m}\theta^+_0\theta^-_m\theta^+_0\big) \\
&\quad - \sum_{b\neq 0} bm(b+m)\left(\theta^-_0\theta^+_m\theta^+_{-m-b}\theta^-_b - \theta^-_0\theta^+_m\theta^+_{-m-b}\theta^-_b + \theta^-_m\theta^+_0\theta^+_{-m-b}\theta^-_b - \theta^+_{-m-b}\theta^-_b\theta^-_m\theta^+_0\right) \\
&\quad - \sum_{a\neq 0} am(m-a)\left(\theta^+_{m-a}\theta^-_a\theta^-_0\theta^+_{-m} - \theta^-_0\theta^+_{-m}\theta^+_{m-a}\theta^-_a + \theta^+_{m-a}\theta^-_a\theta^-_{-m}\theta^+_0 - \theta^-_{-m}\theta^+_0\theta^+_{m-a}\theta^-_a\right) \\
&\quad - \sum_{a\neq 0}\sum_{b\neq 0} ab(m-a)(b+m)(\theta^+_{m-a}\theta^-_a\theta^+_{-m-b}\theta^-_b - \theta^+_{m-a}\theta^-_a\theta^+_{-m-b}\theta^-_b) \\
&= 2m\theta^+_0\theta^-_0 + \sum_{b\neq 0} bm(b+m)\frac{1}{m}\,\delta_{b+m,0}\theta^-_0\theta^+_{-m-b} + \sum_{a\neq 0} am(m-a)\frac{1}{m}\,\delta_{a,m}\theta^-_0\theta^+_{m-a} \\
&\quad - \sum_{a\neq 0}\sum_{b\neq 0} ab(m-a)(b+m)\left(\frac{1}{-b-m}\,\delta_{a,b+m}\theta^+_{m-a}\theta^-_b - \frac{1}{m-a}\,\delta_{a,b+m}\theta^+_{-b-m}\theta^+_a\right) \\
&= 2m\theta^+_0\theta^-_0 - \sum_{b>0} b^2(b+m)\theta^+_{-b}\theta^-_b - \sum_{b<0} b^2(b+m)\left(-\theta^-_b\theta^+_{-b} - \frac{1}{b}\right) \\
&\quad + \sum_{b>-m} b(b+m)^2\theta^+_{-b-m}\theta^-_{b+m} + \sum_{b<-m} b(b+m)^2\left(-\theta^-_{b+m}\theta^+_{-b-m} - \frac{1}{b+m}\right) \\
&= 2m\theta^+_0\theta^-_0 - \sum_{b>0} b^2(b+m)\theta^+_{-b}\theta^-_b + \sum_{b<0} b^2(b+m)\theta^-_b\theta^+_{-b} + \sum_{b<0} b(b+m) \\
&\quad + \sum_{b>0}(b-m)b^2\theta^+_{-b}\theta^-_b - \sum_{b<0}(b-m)b^2\theta^-_b\theta^+_{-b} - \sum_{b<-m} b(b+m) \\
&= 2m\theta^+_0\theta^-_0 - 2m\sum_{b>0} b^2\theta^+_{-b}\theta^-_b + 2m\sum_{b<0} b^2\theta^-_b\theta^+_{-b} - \sum_{b=1}^{-m-1} b(b+m) \\
&= 2mL_0 - \frac{1}{6}(m^3-m)\,. \qquad (4.11)
\end{aligned}
$$

In addition to the energy momentum operator, a field $\widetilde{\Omega}(z)$ of generalized weight 0 can be obtained by the fields $\theta^\pm(z)$ as

$$
\widetilde{\Omega}(z) = \sum_{m\in\mathbb{Z}}\sum_{a\in\mathbb{N}} \widetilde{\Omega}_{m,a} z^{-m}(\log z)^a = -\,{:}\,\theta^+(z)\theta^-(z)\,{:}\;,
$$

and its modes can be expressed in terms of the modes of $\theta^\pm(z)$ by the relations

$$
\widetilde{\Omega}_{m,0} = -\begin{cases} \xi^+\theta^-_m + \theta^+_m\xi^- + \sum_{a\neq 0,m}\theta^+_{m-a}\theta^-_a & \text{for } m\neq 0\,, \\ \xi^+\xi^- + \sum_{a\neq 0}{:}\,\theta^+_{-a}\theta^-_a\,{:} & \text{for } m=0\,, \end{cases} \qquad (4.12a)
$$

$$
\widetilde{\Omega}_{m,1} = -\begin{cases} \theta^+_m\theta^-_0 + \theta^+_0\theta^-_m & \text{for } m\neq 0\,, \\ \xi^+\theta^-_0 - \xi^-\theta^+_0 & \text{for } m=0\,, \end{cases} \qquad (4.12b)
$$

$$
\widetilde{\Omega}_{m,2} = -\delta_{m,0}\theta^+_0\theta^-_0\,. \qquad (4.12c)
$$

The vector $\Omega^{(1)} \equiv \widetilde{\Omega} = \widetilde{\Omega}(z)\Omega|_{z=0}$ associated to the field $\widetilde{\Omega}(z)$ spans a Jordan cell of rank 2 with respect to the operator L_0 together with the vacuum Ω. This follows from (4.10), (4.12), the anti-commutation relations (4.8) and the fact that the modes $\theta^{\pm}_m$ with $m \in \mathbb{N}$ annihilate the vacuum:

$$L_0\widetilde{\Omega} = \left(\theta_0^+\theta_0^- - \sum_{a\in\mathbb{Z}} a^2 :\theta^+_{-a}\theta^-_a:\right)\widetilde{\Omega}(z)\Omega\,\big|_{z=0} = \theta_0^+\theta_0^-\left(-\xi^+\xi^-\Omega\right) = -\theta_0^+\xi^-\Omega = +\Omega\,.$$

Commutation relations involving logarithmic modes. Now that it has been shown how the $\theta^+\theta^-$-system realizes a logarithmic conformal field theory where $\widetilde{\Omega}(z)$ denotes the logarithmic partner of the identity operator, the explicit expressions in terms of the modes $\xi^{\pm}$ and $\theta^{\pm}_m$ can be used to study properties of the logarithmic fields. There is hope that such results may also help to find a general formulation using vertex operator algebra methods as discussed above.

As a first example, one can perform several calculations similar to (4.11) to obtain all commutation relations involving one Virasoro mode L_m and one logarithmic mode $\widetilde{\Omega}_{n,a}$. This yields

$$\left[L_m, \widetilde{\Omega}_{n,0}\right] = -(m+n)\widetilde{\Omega}_{m+n,0} + \widetilde{\Omega}_{m+n,1} + (m+1)\delta_{m+n,0}\,, \tag{4.13a}$$

$$\left[L_m, \widetilde{\Omega}_{n,1}\right] = -(m+n)\widetilde{\Omega}_{m+n,1} + 2\widetilde{\Omega}_{m+n,2}\,, \tag{4.13b}$$

$$\left[L_m, \widetilde{\Omega}_{n,2}\right] = 0\,. \tag{4.13c}$$

The above result can be compared with what a computation starting at the naively generalized Jacobi identity (4.1) would end up with. It was argued above that (4.1) leads to inconsistencies for generic vectors u and v; but in the special case that the vertex operator associated to the first vector u does not involve logarithmic variables such problems do not arise. In particular, for the choice $u = \omega$ and $v = \widetilde{\Omega}$ the operation Res_{x_0} can be applied to (4.1) without difficulties and a comparison of coefficients yields

$$\begin{aligned}\left[L_m, \widetilde{\Omega}_{n,a}\right] &= \sum_{l\in\mathbb{N}} \binom{m+1}{l}\left(L_{l-1}\widetilde{\Omega}\right)_{m+n-l,a} \\ &= \left(L_{-1}\widetilde{\Omega}\right)_{m+n,a} + (m+1)\Omega_{m+n,a} \\ &= -(m+n)\widetilde{\Omega}_{m+n,a} + (a+1)\widetilde{\Omega}_{m+n,a+1} + (m+1)\delta_{m+n,-1}\delta_{a,0}\,,\end{aligned} \tag{4.14}$$

where use was made of the L_{-1}-derivative property $\frac{\mathrm{d}}{\mathrm{d}x}Y(\widetilde{\Omega}, x, \log x) = Y(L_{-1}\widetilde{\Omega}, x, \log x)$ and the relation $L_0\widetilde{\Omega} = \Omega$. The short vertex operator algebraic calculation (4.14) exactly reproduces all three relations (4.13) simultaneously, where the latter could only be obtained by page-filling case differentiations. This may be interpreted as a hint that a Jordan vertex operator algebra can actually be defined consistently such that its generalized Jacobi identity reduces to (4.1) in the case that only the second vertex operator (associated to v) may involve logarithms. The above example illustrates that such a Jordan vertex operator algebra would be a superior language to describe logarithmic conformal field theory, allowing to swiftly and elegantly deriving the desired results.

On the other hand, a Jacobi identity for two logarithmic vertex operators is still out of reach, so as a next step the commutation relations for the logarithmic modes $\widetilde{\Omega}_{m,a}$ will be studied using the $\theta^+\theta^-$-system. But while the commutators $[L_m, L_n]$ and $[L_m, \widetilde{\Omega}_{n,a}]$ follow straight-forwardly from the anti-commutation relations (4.8) and the explicit expressions (4.10) and (4.12) for the Virasoro and $\widetilde{\Omega}$-modes, respectively, the analogous computation of the commutator $[\widetilde{\Omega}_{m,a}, \widetilde{\Omega}_{n,b}]$ leads to unexpected properties: For some ranges of the indices $(m, a), (n, b) \in \mathbb{Z} \times \mathbb{N}$, the commutator of two logarithmic $\widetilde{\Omega}$-modes does not close. Indeed, only in the following four cases the expressions for the commutators given in terms of the modes $\theta^{\pm}_m$ and $\xi^{\pm}$ can be arranged such that all these modes can be collected into $\widetilde{\Omega}$-modes:

$$\begin{aligned}
\left[\widetilde{\Omega}_{0,2}, \widetilde{\Omega}_{m,1}\right] &= -2\widetilde{\Omega}_{m,2} && \text{for all } m \in \mathbb{Z}\ , \\
\left[\widetilde{\Omega}_{m,1}, \widetilde{\Omega}_{n,1}\right] &= -\frac{2}{n}\widetilde{\Omega}_{m+n,2} && \text{for all } m, n \in \mathbb{Z}^{\times}\ , \\
\left[\widetilde{\Omega}_{0,1}, \widetilde{\Omega}_{m,1}\right] &= +2\widetilde{\Omega}_{m,1} && \text{for all } m \in \mathbb{Z}^{\times}\ , \\
\left[\widetilde{\Omega}_{0,2}, \widetilde{\Omega}_{m,0}\right] &= -2\widetilde{\Omega}_{m,1} - \delta_{m,0} && \text{for all } m \in \mathbb{Z}\ .
\end{aligned}$$

In all other cases, the right-hand side of such commutators cannot be written solely in terms of $\widetilde{\Omega}$-modes given by (4.12) and constant summands. For example one finds that

$$\begin{aligned}
\left[\widetilde{\Omega}_{0,1}, \widetilde{\Omega}_{0,0}\right] &= \left[\xi^+\theta^-_0 - \xi^-\theta^+_0, \xi^+\xi^- + \sum_{a>0}(\theta^+_{-a}\theta^-_a - \theta^-_{-a}\theta^+_a)\right] \\
&= \xi^+\theta^-_0\xi^+\xi^- - \xi^+\xi^-\xi^+\theta^-_0 - \xi^-\theta^+_0\xi^+\xi^- + \xi^+\xi^-\xi^-\theta^+_0 \\
&\qquad + \sum_{a>0}(\xi^+\theta^-_0\theta^+_{-a}\theta^-_a - \theta^+_{-a}\theta^-_a\xi^+\theta^-_0 - \xi^-\theta^+_0\theta^+_{-a}\theta^-_a + \theta^+_{-a}\theta^-_a\xi^-\theta^+_0 \\
&\qquad\qquad - \xi^+\theta^-_0\theta^-_{-a}\theta^+_a + \theta^-_{-a}\theta^+_a\xi^+\theta^-_0 + \xi^-\theta^+_0\theta^-_{-a}\theta^+_a - \theta^-_{-a}\theta^+_a\xi^-\theta^+_0) \\
&= \xi^+(1 - \xi^+\theta^-_0)\xi^- + \xi^-(-1 - \xi^-\theta^+_0)\xi^+ \\
&= 2\xi^+\xi^-\ ,
\end{aligned} \tag{4.15}$$

and the product $\xi^+\xi^-$ appears in (4.12) only in the mode $\widetilde{\Omega}_{0,0}$, but the sum $2\sum_{a\neq 0} : \theta^+_{-a}\theta^-_a :$, which together with $2\xi^+\xi^-$ would give $-2\widetilde{\Omega}_{0,0}$, is missing on the right-hand side above. Similar mismatchings occur also in all other remaining cases; either a sum seems to be missing in order to obtain full $\widetilde{\Omega}$-modes, or there are sums that cannot be collected into expressions of such $\widetilde{\Omega}$-modes. An example for such a case is the commutator $[\widetilde{\Omega}_{m,0}, \widetilde{\Omega}_{0,0}]$ for $m \neq 0$, which can be calculated to be equal to

$$-\frac{1}{m}(\xi^+\theta^-_m - \xi^-\theta^+_m) + \sum_{a\neq 0,m}\left(\frac{1}{a-m} - \frac{1}{a}\right)\theta^+_{m-a}\theta^-_a\ .$$

If the coefficients in the sum were equal to $-1/m$, the above expression would be exactly $\widetilde{\Omega}_{m,0}$, but as they are not, the commutator $[\widetilde{\Omega}_{m,0}, \widetilde{\Omega}_{0,0}]$ cannot be written as a linear combination of $\widetilde{\Omega}$-modes.
These results are alarmingly different from the general fact that for ordinary vertex operator algebras the modes span a Lie algebra, see (2.20). Actually, this can be interpreted as another manifestation of the difficulty to understand the connection between two logarithmic vertex operators as opposed to one logarithmic vertex operator and one ordinary vertex operator. (4.13) shows that commutation relations can be given for the modes in the latter case.

The operator product expansion $\widetilde{\Omega}(z)\widetilde{\Omega}(w)$. In ordinary conformal field theory, the commutation relations between modes of two fields are equivalent to the operator product expansion of the two fields, so one may try to obtain information about the commutation relations between the $\widetilde{\Omega}$-modes by studying the operator product expansion of $\widetilde{\Omega}(z)\widetilde{\Omega}(w)$. This product can be computed with the help of a variant of Wick's theorem. To formulate this result the normal-ordered product of two logarithmic fields $f(z) = \sum_{m\in\mathbb{Z},a\in\mathbb{N}} z^{-m}(\log z)^a f_{m,a}$ and $g(z) = \sum_{n\in\mathbb{Z},b\in\mathbb{N}} z^{-n}(\log z)^b g_{n,b}$ is defined as

$$\boldsymbol{:} f(z)g(w) \boldsymbol{:} = f(z)_+ g(w) + g(w) f(z)_- \ ,$$

where $f(z)_+$ denotes the regular part and $f(z)_-$ denotes the singular part, using the following generalization of the meromorphic case given by (2.35):

$$\begin{aligned} f(z)_+ &= \sum_{m\le 0} z^{-m} f_{m,0} + \sum_{m\le -1} \sum_{a>0} z^{-m}(\log z)^a f_{m,a} \ , \\ f(z)_- &= \sum_{m>0} z^{-m} f_{m,0} + \sum_{m>-1} \sum_{a>0} z^{-m}(\log z)^a f_{m,a} \ . \end{aligned}$$

Proposition. For two collections $\phi^1(z), \ldots, \phi^M(z)$ and $\psi^1(w), \ldots, \psi^N(w)$ of logarithmic fields, define the *contraction of $\phi^i(z)$ and $\psi^j(w)$* as $[\phi^i\psi^j] = [\phi^i(z)_-, \psi^j(w)]$ for all $i \in \{1, \ldots, M\}$ and $j \in \{1, \ldots, N\}$, where $[\,\cdot\,,\,\cdot\,]$ denotes the superbracket. Suppose that the following properties hold:

(i) $[[\phi^i(z)_-, \psi^j(w)], f^k(z)_\pm] = 0$ for f denoting either ϕ or ψ and for all i, j, k in their respective domains;

(ii) $[\phi^i(z)_\pm, \psi^j(w)_\pm] = 0$ for all $i \in \{1, \ldots, M\}$ and $j \in \{1, \ldots, N\}$.

Then the normal-ordered product $\boldsymbol{:} \phi^1(z) \ldots \phi^M(z) \boldsymbol{::} \psi^1(w) \ldots \psi^N(w) \boldsymbol{:}$ is given by

$$\begin{aligned} \sum_{k=0}^{\min(M,N)} \sum_{\substack{i_1<\ldots<i_k \\ j_1\neq\ldots\neq j_k}} & (-1)^{p_{i_1,\ldots,i_k;j_1\ldots,j_k}} [\phi^{i_1}\psi^{j_1}] \ldots [\phi^{i_k}\psi^{j_k}] \\ & \cdot \boldsymbol{:} \phi^1(z) \ldots \phi^M(z)\psi^1(w) \ldots \psi^N(w) \boldsymbol{:}_{(\hat{\imath}_1,\ldots,\hat{\imath}_k;\hat{\jmath}_1\ldots,\hat{\jmath}_k)} \ , \end{aligned} \tag{4.16}$$

where $(-1)^{p_{i_1,\ldots,i_k;j_1\ldots,j_k}}$ is the sign obtained by commuting the contracted fields out of the normal-ordered product, and the subscript $(\hat{\imath}_1, \ldots, \hat{\imath}_k; \hat{\jmath}_1 \ldots, \hat{\jmath}_k)$ means that the fields $\phi^{i_1}(z), \ldots, \phi^{i_k}(z)$ and $\psi^{j_1}(w), \ldots, \psi^{j_k}(w)$ are not included in the product.

This is the logarithmic generalization of Wick's theorem as presented in the book [Kac] of Kac, and given the notation for the normal-ordered product for logarithmic fields as introduced above, the proof is exactly the same as the one given by Kac.
The above proposition can be used to compute the operator product expansion $\widetilde{\Omega}(z)\widetilde{\Omega}(w) = (-:\theta^+(z)\theta^-(z):)(-:\theta^+(w)\theta^-(w):)$. To verify the conditions (i) and (ii) for the fields $\phi^1(z) = \psi^1(z) = \theta^+(z)$ and $\phi^2(w) = \psi^2(w) = \theta^-(w)$, one may first determine the contractions $[\theta^\pm\theta^\pm]$, $[\theta^+\theta^-]$ and $[\theta^-\theta^+]$:

$$\begin{aligned}
[\theta^\pm\theta^\pm] &= \left\{\theta^\pm \log z + \sum_{m>0} \theta^\pm_m z^{-m}, \theta^\pm \log z + \xi^\pm \sum_{n\neq 0} \theta^\pm_n w^{-n}\right\} = 0\ ,\\
[\theta^+\theta^-] &= \left\{\theta^+_0 \log z + \sum_{m>0} \theta^+_m z^{-m}, \theta^-_0 \log w + \xi^- \sum_{n\neq 0} \theta^-_n w^{-n}\right\}\\
&= \sum_{m>0}\sum_{n\neq 0} \frac{1}{m}\delta_{m+n,0} z^{-m} w^{-n} - \log z = -\log\left(1-\frac{w}{z}\right) - \log z = -\log(z-w)\ ,\\
[\theta^-\theta^+] &= +\log(z-w)\ .
\end{aligned}$$

Then condition (i) is satisfied because the contraction $[\theta^+(z)_-, \theta^-(w)] = -\log(z-w)$ commutes with everything, and condition (ii) follows from

$$\begin{aligned}
\{\theta^+(z)_+, \theta^-(w)_+\} &= \left\{\xi^+ + \sum_{m\leq 0} \theta^+_m z^{-m}, \xi^- + \sum_{n\leq 0} \theta^+_n w^{-n}\right\} = \{\theta^+_0, \xi^-\} + \{\xi^+, \theta^-_0\} = 0\ ,\\
\{\theta^+(z)_-, \theta^-(w)_-\} &= \left\{\theta^+_0 \log z + \sum_{m>0} \theta^+_m z^{-m}, \theta^-_0 \log w + \sum_{n>0} \theta^-_n w^{-n}\right\} = 0\ .
\end{aligned}$$

As a consequence, (4.16) can be applied to yield

$$\begin{aligned}
\widetilde{\Omega}(z)\widetilde{\Omega}(w) &= :\theta^+(z)\theta^-(z)::\theta^+(w)\theta^-(w):\\
&= :\theta^+(z)\theta^-(z)\theta^+(w)\theta^-(w): - [\theta^+\theta^+] :\theta^-(z)\theta^-(w): + [\theta^+\theta^-] :\theta^-(z)\theta^+(w):\\
&\quad + [\theta^-\theta^+] :\theta^+(z)\theta^-(w): - [\theta^-\theta^-] :\theta^+(z)\theta^+(w): + [\theta^+\theta^-][\theta^-\theta^+]\\
&\quad - [\theta^+\theta^+][\theta^-\theta^-]\\
&= :\theta^+(z)\theta^-(z)\theta^+(w)\theta^-(w): + [\theta^+\theta^-] :\theta^-(z)\theta^+(w): + [\theta^-\theta^+] :\theta^+(z)\theta^-(w):\\
&\quad + [\theta^+\theta^-][\theta^-\theta^+]\ .
\end{aligned}$$

The remaining normal-ordered products can be expressed as

$$\begin{aligned}
:\theta^+(z)\theta^-(w): &= \sum_{k\in\mathbb{N}} \frac{1}{k!} :(\partial^k\theta^+(w))\theta^-(w): (z-w)^k\\
&= + :\theta^+(w)\theta^-(w): + \text{(terms regular in } (z-w))\ ,\\
:\theta^-(z)\theta^+(w): &= - :\theta^+(w)\theta^-(w): + \text{(terms regular in } (z-w))\ ,
\end{aligned}$$

such that one arrives at the following operator product expansion for the logarithmic partner of the identity:

$$\widetilde{\Omega}(z)\widetilde{\Omega}(w) = -(\log(z-w))^2 - 2\log(z-w)\widetilde{\Omega}(w) + (\text{terms regular in } (z-w)) \,. \tag{4.17}$$

Commutation relations from the operator product expansion. The way commutation relations for modes are often obtained in conformal field theory is to compute contour integrals of the corresponding operator product expansion, but this method can certainly not be directly applied to the logarithmic case because the contour integral of a logarithm is simply not defined. Given two logarithmic fields, in order to infer the commutators of their modes from their operator product expansion without having to compute contour integrals or (formal) residues, one can compare coefficients of monomials of all the variables on both sides of the operator product expansion. To do this one has to consider the non-normal-ordered part of the product of the two fields on one side of the equation, while on the other side the singular part of the expansion in the difference of the variables has to be considered. This method in particular circumvents the problem of having to deal with ill-defined residues of logarithms, and it is an application of the alternate point of view on operator product expansions given by (2.36). Following the treatment presented in section 2.3, the logarithm $\log z$ is interpreted as a variable formally independent of z.

To get to know this method in detail it will first be applied to two examples that are already well-understood, such that its application to the case of the commutators $[\widetilde{\Omega}_{m,a}, \widetilde{\Omega}_{n,b}]$ will be carried out more easily later on.
The first example is the operator product expansion of the Virasoro field $T(z)$ with itself. Let $|z| > |w|$ as a relation between nonzero complex numbers. Alternatively, all variables can also be considered as formal together with the convention that any function of $(z-w)$ is always expanded into a series with only finitely many negative powers of the second variable w in the difference. Then the operator product expansion is given by

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial T(w) \,, \tag{4.18}$$

using the sloppy notation explained in section 2.1. The left-hand side can be written as

$$\begin{aligned} T(z)T(w) &= T(z)_+T(w) + T(z)_-T(w) \\ &= T(z)_+T(w) + T(w)T(z)_- + [T(z)_-, T(w)] \\ &= \,:T(z)T(w): + \sum_{m>-2}\sum_{n\in\mathbb{Z}} z^{-m-2}w^{-n-2}\,[L_m, L_n] \,. \end{aligned} \tag{4.19}$$

On the other hand, the singular part of the right-hand side is

$$\begin{aligned} &\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial T(w) \\ &= \sum_{m\in\mathbb{N}} \frac{c}{2}\binom{-4}{m} z^{-4-m}(-w)^m + \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}} 2\binom{-2}{m} z^{-2-m}(-w)^m w^{-n-2} L_n \end{aligned}$$

$$
\begin{aligned}
&+\sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}}\binom{-1}{m}(-n-2)z^{-1-m}(-w)^m w^{-n-3}L_n \\
=&\sum_{m\geq 2}\frac{c}{2}\binom{-4}{m-2}(-1)^m z^{-m-2}w^{m-2}+\sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}}2(m+1)z^{-m-2}w^{-n-2}L_{m+n} \\
&+\sum_{m\geq -1}\sum_{n\in\mathbb{Z}}(-m-n-2)z^{-m-2}w^{-n-2}L_{m+n} \\
=&\sum_{m\geq 2}\frac{c}{2}\binom{m+1}{3}z^{-m-2}w^{m-2}+\sum_{m\geq -1}\sum_{n\in\mathbb{Z}}(m-n)z^{-m-2}w^{-n-2}L_{m+n}\ ,
\end{aligned}
\tag{4.20}
$$

where the identity $\binom{k}{l}=(-1)^l\binom{-k+l-1}{l}$ for $k\in\mathbb{R}_{<0}$ was used in the last step. The first sum in the last line can be extended to the domain $\{m\geq -1\}$ because ± 1 and 0 are exactly the roots of m^3-m, so that comparison with (4.19) yields the Virasoro algebra

$$
[L_m,L_n]=(m-n)L_{n+m}+\frac{c}{12}(m^3-m)\delta_{m+n,0}\quad \text{for all } m\in\mathbb{Z}_{\geq -1} \text{ and } n\in\mathbb{Z}\ . \tag{4.21}
$$

To arrive at this result also for the case $m<-1$ and thus obtain the full Virasoro algebra, let now $|w|>|z|$. According to (2.36) the left-hand side of the operator product expansion is now

$$
T(w)T(z)=\boldsymbol{:}T(z)T(w)\boldsymbol{:}+\sum_{m\leq -2}\sum_{n\in\mathbb{Z}}z^{-m-2}w^{-n-2}[L_n,L_m]\ , \tag{4.22}
$$

where the domain of one sum is different from the one in equation (4.19). Instead of repeating the calculation in (4.20), the right-hand side of the operator product expansion has to be expanded in the domain $|w|>|z|$ – or in a more algebraic language: applied to some vector in the underlying vertex operator algebra V one now considers the expansion of an element of $U(\mathsf{Vir})V[\![w]\!][w^{-1},(z-w)^{-1}]$ in $U(\mathsf{Vir})V(\!(w)\!)(\!(z)\!)$ instead of $U(\mathsf{Vir})V(\!(z)\!)(\!(w)\!)$. Effectively this means that z and $-w$ have to be permuted before doing the binomial expansion:

$$
\begin{aligned}
&\frac{c/2}{(-w+z)^4}+\frac{2}{(-w+z)^2}T(w)+\frac{1}{(-w+z)}\partial T(w) \\
=&\sum_{m\in\mathbb{N}}\frac{c}{2}\binom{-4}{m}(-w)^{-4-m}z^m+\sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}}2\binom{-2}{m}(-w)^{-2-m}z^m w^{-n-2}L_n \\
&+\sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}}\binom{-1}{m}(-n-2)(-w)^{-1-m}z^m w^{-n-3}L_n \\
=&\sum_{m\geq 2}\frac{c}{2}\binom{-4}{m-2}(-1)^m z^{m-2}w^{-m-2}+\sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}}2(m+1)z^m w^{-m-n-4}L_n \\
&-\sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}}(-n-2)z^m w^{-m-n-4}L_n \\
=&\sum_{m\geq 2}\frac{c}{12}(m^3-m)z^{m-2}w^{-m-2}+\sum_{m\leq -2}\sum_{n\in\mathbb{Z}}2(-m-1)z^{-m-2}w^{-n-2}L_{m+n}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{m\leq -2}\sum_{n\in\mathbb{Z}}(-m-n-2)z^{-m-2}w^{-n-2}L_{m+n}\\
&= \sum_{m\geq -1}\frac{c}{12}(m^3-m)z^{m-2}w^{-m-2} + \sum_{m\leq -2}\sum_{n\in\mathbb{Z}}(n-m)z^{-m-2}w^{-n-2}L_{m+n}\\
&= -\sum_{m\leq -2}\frac{c}{12}(m^3-m)z^{-m-2}w^{m-2} + \sum_{m\leq -2}\sum_{n\in\mathbb{Z}}(n-m)z^{-m-2}w^{-n-2}L_{m+n}\ .
\end{aligned}
$$

Comparing this result with equation (4.22) yields exactly those commutators that are still missing in (4.21) for the full Virasoro algebra, so this standard result has now been obtained by the method of comparing coefficients in the correct expansions. By computations very similar to the one above one can extract all other commutation relations from operator product expansions of fields in a meromorphic conformal field theory.

The second example is the case of the operator product expansion $T(z)\widetilde{\Omega}(w)$ which involves one ordinary quantum field and one logarithmic field, the logarithmic partner of the identity. This operator product expansion can be computed using the $\theta^+\theta^-$-system, but it can also be inferred using the general relation $L_0\widetilde{\Omega} = \Omega$, and it is given by

$$
T(z)\widetilde{\Omega}(w) \sim \frac{1}{(z-w)^2} + \frac{1}{(z-w)}\partial\widetilde{\Omega}(w)\ . \tag{4.23}
$$

Similarly to the first example, the left-hand side can be written as

$$
T(z)\widetilde{\Omega}(w) = \boldsymbol{:}\,T(z)\widetilde{\Omega}(w)\,\boldsymbol{:} + \sum_{m>-2}\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}} z^{-m-2}w^{-n}(\log w)^b\left[L_m,\widetilde{\Omega}_{n,b}\right]\ , \tag{4.24}
$$

and in the appropriate expansion the singular part of the right-hand side is equal to

$$
\begin{aligned}
&\frac{1}{(z-w)^2} + \frac{1}{(z-w)}\partial\widetilde{\Omega}(w)\\
&= \sum_{m\in\mathbb{N}}\binom{-2}{m}z^{-2-m}(-w)^m + \sum_{m\in\mathbb{N}}\binom{-1}{m}z^{-1-m}(-w)^m\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}}\partial\left(w^{-n}(\log w)^b\right)\widetilde{\Omega}_{n,b}\\
&= \sum_{m\in\mathbb{N}}(m+1)z^{-m-2}w^m + \sum_{m\in\mathbb{N}}z^{-m-1}w^m\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}}\left((-n)(\log w)^b + b(\log w)^{b-1}\right)w^{-n-1}\widetilde{\Omega}_{n,b}\\
&= \sum_{m\geq -1}(m+1)z^{-m-2}w^m + \sum_{m\geq -1}\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}}z^{-m-2}w^{-n}(\log w)^b(-m-n)\widetilde{\Omega}_{m+n,b}\\
&\quad + \sum_{m\geq -1}\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}}z^{-m-2}w^{-n}(\log w)^b(b+1)\widetilde{\Omega}_{m+n,b+1}\ .
\end{aligned} \tag{4.25}
$$

Comparing (4.24) and (4.25) leads to

$$
\left[L_m,\widetilde{\Omega}_{n,b}\right] = (m+1)\delta_{b,0}\delta_{m+n,-1} - (m+n)\widetilde{\Omega}_{m+n,b} + (b+1)\widetilde{\Omega}_{m+n,b+1} \tag{4.26}
$$

for $m \in \mathbb{Z}_{\geq -1}$. This is exactly the same result as the one obtained in (4.13) or (4.14), which gives further credibility to the method used here and provides a third independent possibility to compute the commutators $[L_m, \widetilde{\Omega}_{n,b}]$. Indeed, analogously to the case of two ordinary quantum fields one can show that the relation (4.26) also holds true for $m \in \mathbb{Z}_{\leq -2}$. In view of the difficulties that arise in the case of two logarithmic fields discussed below, it should be noted that the reason for the absence of such difficulties in the current case is that the modes L_m in the commutator are not labeled by an additional logarithmic index a.

After the familiar operator product expansions $T(z)T(w)$ and $T(z)\widetilde{\Omega}(w)$ now the case of $\widetilde{\Omega}(z)\widetilde{\Omega}(w)$ is addressed which involves two logarithmic fields and serves as the prime example in the task to study the possibility of Jordan vertex operator algebras. The operator product expansion $\widetilde{\Omega}(z)\widetilde{\Omega}(w)$ was derived in (4.17) and is given by

$$\widetilde{\Omega}(z)\widetilde{\Omega}(w) \sim -\left(\log(z-w)\right)^2 - 2\log(z-w)\widetilde{\Omega}(w) \ . \tag{4.27}$$

Proceeding as before, the left-hand side can be written as

$$\begin{aligned}
\widetilde{\Omega}(z)\widetilde{\Omega}(w) &= \widetilde{\Omega}(z)_+\widetilde{\Omega}(w) + \widetilde{\Omega}(z)_-\widetilde{\Omega}(w) \\
&= \widetilde{\Omega}(z)_+\widetilde{\Omega}(w) + \widetilde{\Omega}(w)\widetilde{\Omega}(z)_- + \left[\widetilde{\Omega}(z)_-, \widetilde{\Omega}(w)\right] \\
&= :\widetilde{\Omega}(z)\widetilde{\Omega}(w): + \sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}} w^{-n}(\log w)^b \left[\sum_{m\geq 1} z^{-m}\widetilde{\Omega}_{m,0}\right. \\
&\qquad \left. + \sum_{m\geq 0}\sum_{a>0} z^{-m}(\log z)^a\widetilde{\Omega}_{m,a}, \widetilde{\Omega}_{n,b}\right] \\
&= :\widetilde{\Omega}(z)\widetilde{\Omega}(w): + \sum_{m\geq 1}\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}} z^{-m}w^{-n}(\log w)^b \left[\widetilde{\Omega}_{m,0}, \widetilde{\Omega}_{n,b}\right] \\
&\qquad + \sum_{m\geq 0}\sum_{n\in\mathbb{Z}}\sum_{a>0}\sum_{b\in\mathbb{N}} z^{-m}w^{-n}(\log z)^a(\log w)^b \left[\widetilde{\Omega}_{m,a}, \widetilde{\Omega}_{n,b}\right] \ ,
\end{aligned} \tag{4.28}$$

while the relevant part of the right-hand side can be expanded in z, w, $\log z$ and $\log w$ as

$$\begin{aligned}
&-\left(\log(z-w)\right)^2 - 2\log(z-w)\widetilde{\Omega}(w) \\
&= -\left(\log z - \sum_{i\geq 1}\frac{1}{i}z^{-i}w^i\right)^2 - 2\left(\log z - \sum_{i\geq 1}\frac{1}{i}z^{-i}w^i\right)\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}}\widetilde{\Omega}_{n,b}w^{-n}(\log w)^b \\
&= -(\log z)^2 + 2\log z\sum_{i\geq 1}\frac{1}{i}z^{-i}w^i - 2!\sum_{n_2=2}^{\infty}\sum_{n_1=1}^{n_2-1}\frac{1}{n_1 n_2}z^{-n_2}w^{n_2} \\
&\qquad - \sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}} 2\widetilde{\Omega}_{n,b}w^{-n}\log z(\log w)^b + \sum_{i\geq 1}\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}}\frac{2}{i}\widetilde{\Omega}_{n,b}z^{-i}w^{i-n}(\log w)^b \\
&= -(\log z)^2 + \sum_{m\geq 1}\frac{2}{m+1}z^{-m}w^m\log z - \sum_{m\geq 2}\sum_{i=1}^{m-1}\frac{2}{mi}z^{-m}w^m \\
&\qquad - \sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}} 2\widetilde{\Omega}_{n,b}w^{-n}\log z(\log w)^b + \sum_{m\geq 1}\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}}\frac{2}{m}\widetilde{\Omega}_{m+n,b}z^{-m}w^{-n}(\log w)^b
\end{aligned} \tag{4.29}$$

where use was made of the relation (4.5) in the second step. Comparing (4.28) and (4.29) may at first suggests the following commutator:

$$\left[\widetilde{\Omega}_{m,a},\widetilde{\Omega}_{n,b}\right] \stackrel{?}{=} -\delta_{a,2}\delta_{b,0}\delta_{m,0}\delta_{n,0} + \delta_{a,1}\delta_{b,0}\delta_{m+n,0}(1-\delta_{m,0})\frac{2}{m} \\ - \delta_{a,0}\delta_{b,0}\delta_{m+n,0}(1-\delta_{m,1})\sum_{i=1}^{m-1}\frac{2}{mi} - \delta_{a,1}\delta_{m,0}2\widetilde{\Omega}_{n,b} + \delta_{a,0}\frac{2}{m}\widetilde{\Omega}_{m+n,b}\ . \tag{4.30}$$

Here in the third term on the right-hand side the factor $(1-\delta_{m,1})$ can be discarded if the convention is imposed that $\sum_{i=k}^{l} s_i \equiv 0$ for all $l < k$, i.e. one only counts in the positive direction. This convention is employed in the following.

According to the above reasoning, the relation (4.30) can only be true for $(m,a) \in (\mathbb{Z}_+ \times \{0\}) \sqcup (\mathbb{N} \times \mathbb{Z}_+)$. But the right-hand side of (4.30) does not have the same symmetry as the left-hand side: a permutation of the kind $(m,a) \leftrightarrow (n,b)$ should have the same effect as a mere multiplication by -1. Obviously this is not the case, and so one may expect that an expansion in the domain $|w| > |z|$ will lead to additional terms such that the full commutator has the correct symmetry. This would be in contrast to the two cases considered before, where "half of" the commutator was actually already the "full" commutator.

With this in mind, let now $|w| > |z|$. Then the left-hand side of the operator product expansion is

$$\widetilde{\Omega}(w)\widetilde{\Omega}(z) = \boldsymbol{:}\,\widetilde{\Omega}(z)\widetilde{\Omega}(w)\,\boldsymbol{:} + \sum_{m\leq 0}\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}} z^{-m}w^{-n}(\log w)^b\left[\widetilde{\Omega}_{n,b},\widetilde{\Omega}_{m,0}\right] \\ + \sum_{m\leq -1}\sum_{n\in\mathbb{Z}}\sum_{a>0}\sum_{b\in\mathbb{N}} z^{-m}w^{-n}(\log z)^a(\log w)^b\left[\widetilde{\Omega}_{n,b},\widetilde{\Omega}_{m,a}\right]\ . \tag{4.31}$$

The right-hand side of the operator product expansion (4.27) now is somewhat problematic: if one considers the variables z, w, $\log z$ and $\log w$ to be formally independent, there can be no factors of $\log w$ other than those that appear in $\widetilde{\Omega}(w)$ that can be compared with the $(\log w)$-powers in (4.31), since by definition $\log(z-w) = \log z - \sum_{i\geq 1} w^i z^{-i}/i$. This would lead to an inconsistency.

If one is prepared to partially abandon this point of view there are several possibilities: at first one might be tempted to expand $\log(z-w)$ at $z = 2w$ such that the quantity $\log w$ appears,

$$\log(z-w) = \log w + \sum_{i\geq 1}\frac{(-1)^{i-1}}{i}\left(\frac{z-2w}{w}\right)^i\ .$$

But an expansion around $z = 2w$ is certainly questionable under the assumption $|w| > |z|$. What is more, because of the binomial expansion of $(z-2w)^i$, after some calculation this approach for $m < 0$ and $a = 0$ leads to the term

$$\sum_{l\in\mathbb{N}}\frac{(-1)^{1-m}2^l}{(l-m)}\binom{l-m}{l}\widetilde{\Omega}_{m+n,b}\ , \tag{4.32}$$

which should correspond the term $\delta_{a,0}\frac{2}{m}\widetilde{\Omega}_{m+n,b}$ in the case of $(m,a) \in (\mathbb{Z}_+ \times \{0\}) \sqcup (\mathbb{N} \times \mathbb{Z}_+)$. Apart from the fact that both terms have little resemblance, the sum in (4.32) is not truncated when applied to any vector, leading to an element of the algebraic completion of the space of states. All this suggests that another expansion for $|w| > |z|$ must be found in the logarithmic case.

Expansion for $|w| > |z|$, first attempt. Another possibility is to accept the equation $\log(z-w) = \log(-w+z)$ and then expand at $z = 0$ as in the cases of $T(z)T(w)$ and $T(z)\widetilde{\Omega}(w)$. In the framework of vertex operator algebras, this equation may be interpreted as $\log(-w+z) = \iota_{wz}(\log(z-w))$, generalizing the expansion map ι to a logarithmic setting. Of course, one immediately sees that it is the quantity $\log(-w)$ and not $\log w$ that appears. From an analytic point of view, both quantities are equal modulo an integer and nonvanishing multiple of $\mathrm{i}\pi$ depending on the choice of the branch of the logarithm. But if one would for example choose $\log(-w) = \log w + \mathrm{i}\pi$, this would introduce constant terms into the commutator which would destroy the necessary symmetry. In correlators of the full conformal field theory (and not just its chiral part) logarithms only appear in the form $\log|w|^2$, and this might be taken as a hint that the identification of $\log w$ and $\log(-w)$ is acceptable in some sense.[2]
For the moment I will disregard this problem and investigate how far one can get with the identification of $\log w$ and $\log(-w)$. But if this mistake is made, one can just as well make the mistake $\log(z-w) = \log(w-z)$ such that the unwanted quantity $\log(-w)$ will not appear by the definition of $\log(w-z)$. So one can calculate

$$\begin{aligned}
&-(\log(w-z))^2 - 2\log(w-z)\widetilde{\Omega}(w)\\
=&-\left(\log w - \sum_{i\geq 1}\frac{1}{i}z^i w^{-i}\right)^2 - 2\left(\log w - \sum_{i\geq 1}\frac{1}{i}z^i w^{-i}\right)\widetilde{\Omega}(w)\\
=&-(\log w)^2 + 2\log w\sum_{i\geq 1}\frac{1}{i}z^i w^{-i} - \sum_{m\geq 2}\sum_{i=1}^{m-1}\frac{2}{mi}z^m w^{-m}\\
&\quad -\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}}2\widetilde{\Omega}_{n,b}w^{-n}(\log w)^{b+1} + \sum_{i\geq 1}\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}}\frac{2}{i}\widetilde{\Omega}_{n,b}z^i w^{-i-n}(\log w)^b\\
=&-(\log w)^2 - \sum_{m\leq -1}\frac{2}{m}z^{-m}w^m\log w + \sum_{m\leq -2}\sum_{i=1}^{-m-1}\frac{2}{mi}z^{-m}w^m\\
&\quad -\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}}2\widetilde{\Omega}_{n,b}w^{-n}(\log w)^{b+1} - \sum_{m\leq -1}\sum_{n\in\mathbb{Z}}\sum_{b\in\mathbb{N}}\frac{2}{m}\widetilde{\Omega}_{m+n,b}z^{-m}w^{-n}(\log w)^b\ . \qquad (4.33)
\end{aligned}$$

Now comparing (4.31) and (4.33) yields an expression for the commutators in the domain

[2]But on the other hand, this identification is senseless in the existing rigorous formalism of vertex operator algebras, see for example the discussion related to (3.21) in [HLZ]. There, the choice $\zeta = (2r+1)\pi\mathrm{i}$ with $r \in \mathbb{Z}$ corresponds exactly to the map $y \mapsto -x$ in this context. This is the algebraic equivalent to the analytic language of the choice of the r-th branch of the logarithm.

$(m,a) \in (\mathbb{Z}_{\leq 0} \times \{0\}) \sqcup (\mathbb{Z}_- \times \mathbb{Z}_+)$:

$$\begin{aligned}\left[\widetilde{\Omega}_{n,b}, \widetilde{\Omega}_{m,a}\right] \stackrel{?}{=} & -\delta_{a,0}\delta_{b,2}\delta_{m,0}\delta_{n,0} - \delta_{a,0}\delta_{b,1}\delta_{m+n,0}\left(1-\delta_{m,0}\right)\frac{2}{m} \\ & + \delta_{a,0}\delta_{b,0}\delta_{m+n,0}(1-\delta_{m,-1})(1-\delta_{m,0})\sum_{i=1}^{-m-1}\frac{2}{mi} \\ & - \delta_{a,0}(1-\delta_{b,0})\delta_{m,0}2\widetilde{\Omega}_{n,b-1} - \delta_{a,0}(1-\delta_{m,0})\frac{2}{m}\widetilde{\Omega}_{m+n,b} \ . \end{aligned} \tag{4.34}$$

Together with the commutator in (4.30) this gives for arbitrary m, n, a and b:

$$\begin{aligned}\left[\widetilde{\Omega}_{m,a}, \widetilde{\Omega}_{n,b}\right] \stackrel{?}{=} & \left(\delta_{a,0}\delta_{b,2} - \delta_{a,2}\delta_{b,0}\right)\delta_{m,0}\delta_{n,0} + \left(\delta_{a,1}\delta_{b,0} + \delta_{a,0}\delta_{b,1}\right)(1-\delta_{m,0})\delta_{m+n,0}\frac{2}{m} \\ & - \left(\sum_{i=1}^{m-1}\frac{1}{i} + \sum_{i=1}^{-m-1}\frac{1}{i}\right)\delta_{a,0}\delta_{b,0}\delta_{m+n,0}\frac{2}{m} \\ & - \delta_{m,0}\left(\delta_{a,1}2\widetilde{\Omega}_{n,b} - \delta_{a,0}(1-\delta_{b,0})2\widetilde{\Omega}_{n,b-1}\right) + \delta_{a,0}(1-\delta_{m,0})\frac{2}{m}\widetilde{\Omega}_{m+n,b} \ . \end{aligned} \tag{4.35}$$

One may convince oneself that the terms in the first three lines on the right-hand side of (4.35) are indeed antisymmetric with respect to the permutation $(m,a) \leftrightarrow (n,b)$. In a manner of speaking, these terms are the "correct generalization" of the restricted commutator in (4.30). In contrast to this, the remaining two terms are *not* antisymmetric with respect to the permutation $(m,a) \leftrightarrow (n,b)$.

The following makes the problem with the above expansion even clearer: since there are also nontrivial $(\log z)$-powers in the second term in equation (4.31), but there are no such $(\log z)$-powers in equation (4.33) due to the formal expansion at $z = 0$, a comparison shows that

$$\text{for all } (n,b) \in \mathbb{Z} \times \mathbb{N} : \ \left[\widetilde{\Omega}_{n,b}, \widetilde{\Omega}_{m,a}\right] = 0 \text{ if } (m,a) \in \mathbb{Z}_{\leq -1} \times \mathbb{Z}_+ \ . \tag{4.36}$$

Now let $(m,a) = (0,1)$. Then according to (4.30) or (4.35) and because of $2\widetilde{\Omega}_{n,b} \neq 0$, the commutator $[\widetilde{\Omega}_{m,a}, \widetilde{\Omega}_{n,b}]$ does *not* vanish for $(n,b) \in \mathbb{Z}_{\leq -1} \times \mathbb{Z}_+$, in contradiction to (4.36). As the proof of (4.30) for $(m,a) \in (\mathbb{Z}_+ \times \{0\}) \sqcup (\mathbb{N} \times \mathbb{Z}_+)$ does not seem to be problematic, the reason must be the wrong expansion of $\log(z-w)$ for $|w| > |z|$: $\log w$ is not equal to $\log(-w)$ and also for $|w| > |z|$ or $(m,a) \in (\mathbb{Z}_{\leq 0} \times \{0\}) \sqcup (\mathbb{Z}_- \times \mathbb{Z}_+)$, the expansion of the right-hand side of the operator product expansion (4.27) must contain the quantity $\log z$. An incorrect expansion leads to contradictions like (4.36).

Expansion for $|w| > |z|$, second attempt. The last paragraph makes it clear that the first attempt lacks sufficiently many $(\log z)$-terms. As the first three terms in (4.35) seem to be correct and as the expansion $\log(-w+z) = \iota_{wz}(\log(z-w))$ also seems sensible, now only the term $2\log(z-w)\widetilde{\Omega}(w)$ from the operator product expansion (4.27) shall be expanded differently. To do this one may first note that

$$2\log(w-z)\widetilde{\Omega}(w) = 2\log(w-z)\sum_{i\in\mathbb{N}}\frac{1}{i!}(w-z)^i\partial^i\widetilde{\Omega}(z) \sim 2\log(w-z)\widetilde{\Omega}(z) \ .$$

Motivated by this, now $\widetilde{\Omega}(w)$ is simply replaced by $\widetilde{\Omega}(z)$ in the operator product expansion in the case of $|w| > |z|$. Thus,

$$\begin{aligned}
2\log(w-z)\widetilde{\Omega}(z) &= \sum_{m\in\mathbb{Z}}\sum_{a\in\mathbb{N}} 2\widetilde{\Omega}_{m,a} z^{-m} (\log z)^a \log w - \sum_{i\geq 1}\sum_{m\in\mathbb{Z}}\sum_{a\in\mathbb{N}} \frac{2}{i}\widetilde{\Omega}_{m,a} z^{i-m} w^{-i} (\log z)^a \\
&= \sum_{m\in\mathbb{Z}}\sum_{a\in\mathbb{N}} 2\widetilde{\Omega}_{m,a} z^{-m} (\log z)^a \log w \\
&\quad - \sum_{m\in\mathbb{Z}}\sum_{n\geq 1}\sum_{a\in\mathbb{N}} \frac{2}{n}\widetilde{\Omega}_{m+n,a} z^{-m} w^{-n} (\log z)^a \ .
\end{aligned} \tag{4.37}$$

If this is compared to (4.31) without paying attention to the domains of the sums, one finds that now the relevant terms

$$\delta_{b,1}\delta_{n,0}\widetilde{\Omega}_{m,a} - \delta_{b,0}(1-\delta_{n,0})\frac{2}{n}\widetilde{\Omega}_{m+n,a}$$

appear in the commutator $[\widetilde{\Omega}_{n,b}, \widetilde{\Omega}_{m,a}]$ – and these are exactly those terms that by adding them would antisymmetrize the last two terms in the commutator (4.30). Unfortunately, the domains of the sums in (4.31) and (4.37) do not agree, and this leads to inconsistencies as before.

No expansion for $|w| > |z|$? The above two failed attempts to correctly expand for $|w| > |z|$ could motivate all the terms that are sufficient for an antisymmetric commutator $[\widetilde{\Omega}_{m,a}, \widetilde{\Omega}_{n,b}]$ (without explicitly aiming at antisymmetry), but they could not do so consistently. On the other hand, the derivation of the commutator (4.30) for $(m,a) \in (\mathbb{Z}_+ \times \{0\}) \sqcup (\mathbb{N} \times \mathbb{Z}_+)$ appears to be correct. So one may propose to argue in the following way: only one expansion (for $|z| > |w|$) has to be carried out as this already gives all the crucial information on the commutator (as in the case of $T(z)T(w)$ and $T(z)\widetilde{\Omega}(w)$). The missing terms are simply added such that antisymmetry is warranted. This suggests that the commutator should be

$$\begin{aligned}
\left[\widetilde{\Omega}_{m,a}, \widetilde{\Omega}_{n,b}\right] &\stackrel{?}{=} (\delta_{a,0}\delta_{b,2} - \delta_{a,2}\delta_{b,0})\,\delta_{m,0}\delta_{n,0} + (\delta_{a,1}\delta_{b,0} + \delta_{a,0}\delta_{b,1})\,(1-\delta_{m,0})\delta_{m+n,0}\frac{2}{m} \\
&\quad - \left(\sum_{i=1}^{m-1}\frac{1}{i} + \sum_{i=1}^{-m-1}\frac{1}{i}\right)\delta_{a,0}\delta_{b,0}\delta_{m+n,0}\frac{2}{m} - \delta_{a,1}\delta_{m,0}2\widetilde{\Omega}_{n,b} + \delta_{b,1}\delta_{n,0}2\widetilde{\Omega}_{m,a} \\
&\quad + \delta_{a,0}(1-\delta_{m,0})\frac{2}{m}\widetilde{\Omega}_{m+n,b} - \delta_{b,0}(1-\delta_{n,0})\frac{2}{n}\widetilde{\Omega}_{m+n,a}\ ,
\end{aligned} \tag{4.38}$$

where only the minimal number of new terms was added to the commutator (4.30) to secure antisymmetry; additional terms are not to be expected "because of symmetry".

So (4.38) is my best proposal for the commutator of two logarithmic $\widetilde{\Omega}$-modes. At this point it may only be viewed as well-motivated but certainly not proven as the correct expansion for $|w| > |z|$ remains unclear.

It is interesting to note that the commutation relation (4.38) cannot be the bracket for a Lie algebra spanned by the modes $\widetilde{\Omega}_{m,a}$. Indeed, explicitly calculating $[[\widetilde{\Omega}_{l,a},\widetilde{\Omega}_{m,b}],\widetilde{\Omega}_{n,c}] + [[\widetilde{\Omega}_{m,b},\widetilde{\Omega}_{n,c}],\widetilde{\Omega}_{l,a}] + [[\widetilde{\Omega}_{n,c},\widetilde{\Omega}_{l,a}],\widetilde{\Omega}_{m,b}]$ with a computer algebra system, one finds that this satisfies the ordinary Lie algebra Jacobi identity for arbitrary $l,m,n \in \mathbb{Z}$ only if the logarithmic indices a,b,c are positive integers. An analogous statement is true for the double-commutator $[[L_l,\widetilde{\Omega}_{m,b}],\widetilde{\Omega}_{n,c}]$ and its cyclic permutations, using in addition the commutation relation (4.14) for $[L_l,\widetilde{\Omega}_{m,b}]$. On the other hand, the Jacobi identity

$$\Big[\Big[L_m,L_n\Big],\widetilde{\Omega}_{l,a}\Big] + \Big[\Big[L_n,\widetilde{\Omega}_{l,a}\Big],L_m\Big] + \Big[\Big[\widetilde{\Omega}_{l,a},L_m\Big],L_n\Big] = 0$$

is satisfied for all $l,m,n \in \mathbb{Z}$ and $a \in \mathbb{N}$. This is yet another manifestation of the correctness of (4.14). But conversely, the failure of the commutator (4.38) to satisfy the Jacobi identity may not be reason enough for it to be disqualified, as not much is known on the algebras of modes in logarithmic conformal field theory. In particular, at the present stage I am not aware of a necessity for the modes to span a Lie algebra. In the general setting of logarithmic conformal field theory it might simply not be true that all modes are elements of a Lie algebra as in the case of ordinary vertex operator algebras – if at all it is possible to treat logarithmic conformal field theory at the level of vertex operator algebras and not only their modules in the first place; this remains an open question.

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Ich versichere, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die Zitate kenntlich gemacht habe.

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