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Pure Gauge $SU(2)$ Seiberg-Witten Theory and Modular Forms

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Abstract

We identify the spectral curve of pure gauge $SU(2)$ Seiberg-Witten theory with the Weierstrass curve $\mathbb{C}/L \ni z \mapsto (1, \wp(z), \wp(z)')$ and thereby obtain explicitely a modular form from which the moduli space parameter u and lattice parameters a, a_D can be derived in terms of modular respectively theta functions. We further discuss its relationship with the $c = -2$ triplet model conformal field theory.

1 Introduction

The low energy effective $\mathcal{N} = 2$ SYM theory is mathematically and physically extremely rich and a crosspoint for string theory, topological field theory, Riemann surfaces, algebraic topology and other interesting topics [14, 17, 20, 21].

In this paper we draw our attention mainly to the correspondence between pure gauge $SU(2)$ SW-theory and the Weierstrass formulation of elliptic curves. In particular we identify the spectral curve with the Weierstrass curve and thereby obtain some means for writing the parameter of the moduli space and the lattice parameters a and a_D in terms of theta functions, as was originally proposed in [18]. The basic motivation for doing this is that theta functions may point to characteristic functions which have a meaning as amplitudes in conformal field theory. Indeed, we find that the modular form from which we derive a and a_D can be expressed by characteristic functions of the $c = -2$ triplet model. The latter is a nice tool in order to describe the Weierstrass curve and its developing parameter.

The organization of the paper is as follows. First we review how the torus, the spectral curve for pure gauge $SU(2)$ SW-theory, is related to the charge lattice. Then we identify the corresponding algebraic curve with the Weierstrass elliptic curve. In the last chapters we explain how we derived the modular form mentioned above and then discuss the relation to the $c = -2$ triplet model. A conncetion of SW-theory to loagrithmic conformal field theory has often been speculated, see for example [5, 8, 9]. We argue that indeed the modular forms in SW-theory can only be expressed in terms of characters or torus amplitudes of a conformal field theory, if this theory is logarithmic.

2 From Pure Gauge $SU(2)$ SW-Theory to the Torus

In pure gauge SW-theory, the spectrum of stable, charged particles is given by points on a scale dependent lattice spanned by the vacuum expectation values $a(u)$ of the scalar field ϕ and its dual $a_D(u)$: $L = (a_D,\ a)^T$. The parameter u denotes a point in the moduli space $\mathcal{M} := \mathbb{CP}^1 \setminus \{\Lambda^2, -\Lambda^2, \infty\}$ of the theory, where Λ usually gets related to the scale Λ_{QCD}, $\Lambda \in \mathbb{R}^+$ and $L : \mathcal{M} \to E$ can be interpreted as a section of some flat holomorphic Γ-bundle E, $\Gamma \subset SL(2, \mathbb{Z})$. The motivation to consider such a lattice comes from the BPS-respectively central charge formula

$$Z = n_e\ a + n_m\ a_D, \tag{2.1}$$

where n_e and n_m are the quantum numbers for the electric and magnetic charges. $\Gamma \subset SL(2, \mathbb{Z})$ is the subgroup, acting on L, which leaves Z invariant. It is determined by the local symmetries of the theory which are due to the possible monodromies of L around the singularities of $\mathcal{M}$ and the global $\mathbb{Z}_2$ symmetry $u \mapsto -u$ on $\mathcal{M}$.

Geometrically the charge lattice has an interpretation as the cycles of the spectral curve of SW-theory. In particular there exists a mapping λ_{SW} of $\mathcal{M}$ to the space of meromorphic one-forms, called the SW-form, encoding the information of L and satisfying

$$\begin{aligned} &\lambda_{SW}(u): \ L(u) \rightarrow \ H_1(\mathcal{E}(u),\mathbb{Z}), \\ &(a_D(u), \ a(u)) = (\oint_B \lambda_{SW}(u), \ \oint_A \lambda_{SW}(u)) \quad A, B \in \ H_1(\mathcal{E}(u),\mathbb{Z}). \end{aligned} \tag{2.2}$$

One now can obtain the spectral curve as a Jacobian variety from the periods

$$(\Pi_D(u), \ \Pi(u)) = (\frac{\mathrm{d}}{\mathrm{d}u} a_D, \ \frac{\mathrm{d}}{\mathrm{d}u} a) = (\oint_B \omega(u), \ \oint_A \omega(u)) \quad A, B \in \ H_1(\mathcal{E}(u),\mathbb{Z}). \tag{2.3}$$

Hence, to every point $u \in \mathcal{M}$ we attach a torus $\mathcal{E}(u)$ with periods $(\Pi_D(u), \ \Pi(u))$ and developing mapping

$$\tau_{SW}(u) = \frac{\Pi_D(u)}{\Pi(u)} = \frac{\mathrm{d}a_D(u)}{\mathrm{d}a(u)}, \tag{2.4}$$

which is an element of the complex upper half plane $\mathbb{H}$ and coincides with the coupling constant of SW-theory. Since the local physical information is preserved under the symmetry Γ, which has an action on L and on τ_{SW} via (2.4), the moduli space of the theory in the perspective of the torus thus can be obtained as the quotient of the uniformization space $\mathbb{H}$ with Γ:

$$\tilde{\mathcal{M}} = \mathbb{H}/\Gamma. \tag{2.5}$$

The relations as described above can be summarized as follows:

$$\begin{array}{ccc} L & \stackrel{\lambda_{SW}}{\rightarrow} & H_1(\mathcal{E},\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathcal{M} & \stackrel{\tau_{SW}}{\rightarrow} & \tilde{\mathcal{M}} \end{array} \tag{2.6}$$

τ_{SW} is a bi-holomorphic mapping and the singularities of $\mathcal{M}$ correspond to the vertices in $\tilde{\mathcal{M}}$. The action of Γ on $H_1(\mathcal{E},\mathbb{Z})$ is promoted by λ_{SW} and can be interpreted as a change of basis which of course is a mapping between equivalent tori.

3 The spectral curve in terms of τ_{SW}

In this section we will relate the torus $\mathcal{E}(u)$, i.e. the spectral curve of $SU(2)$ SW theory to the Weierstrass standard form and thus change our perspective from the moduli space $\mathcal{M}$, parametrized by u, to the moduli space $\tilde{\mathcal{M}}$, parametrized by τ_{SW} or from holomorphic sections to modular forms. This has been done in several publications such as [18], [13] and [1]. However, the aim of this section is twofold, on the one hand we define all quantities we have used for the succeeding sections and on the other hand we focus on the relation of the uniformization parameter τ to τ_{SW}. The latter is important as it makes a first connection of the $c = -2$ triplet model with $SU(2)$ SW theory, that has already been observed in [10], even more explicit.

A torus $\mathcal{E}(u)$ has a description as an algebraic curve[1]

$$\mathcal{E}(u): \quad y^2 = (x^2 - u)^2 - \Lambda^4, \tag{3.1}$$

where we use the charge lattice normalization $L(u) = (a_D(u),\ \frac{1}{2}a(u))^T$ corresponding to the coupling $2\tau_{SW}$ and the monodromy group $\Gamma = \Gamma^0(4)$ (c.f. Appendix A). Uniformization relates $\mathcal{E}(u)$ to the family of standard elliptic curves $\mathcal{E}(\tau)$, described via the Weierstrass function $\mathbb{C}/L \ni z \mapsto (1, \wp(z), \wp'(z)) \in \mathbb{CP}^1$ and parametrized by some developing parameter τ in the complex upper half plane $\mathbb{H}$:

$$\begin{aligned} \mathcal{E}(\tau): \quad \left(\frac{\mathrm{d}}{\mathrm{d}z}\wp(z)\right)^2 &= 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau) \\ &=: 4\Pi_{i=1}^3(\wp(z) - e_i), \quad z \in H,\ \Im(\tau) > 0. \end{aligned} \tag{3.2}$$

The curve (3.1) is identified with (3.2) by first converting it to the Weierstrass standard form and then matching $J(u) = J(\tau)$, where (c.f. Appendix A)

$$J(\tau) := \frac{4}{27}\frac{(\lambda^2(\tau) - \lambda(\tau) + 1)^3}{\lambda^2(\tau)(\lambda(\tau) - 1)^2}. \tag{3.3}$$

Here we use a general parameter τ that will be related to τ_{SW}, soon. For one of the positive roots of u^2 this identification yields:

$$u(\tau) = \frac{\Lambda^2}{2}\frac{\lambda(\tau) + 1}{\sqrt{\lambda(\tau)}}. \tag{3.4}$$

It is possible to invert this equation in order to derive an expression for λ in terms of u but another equivalent and physically inspired way is to identify λ with the inverse crossing ratio of the branch points $\bar{e}_1 := \sqrt{u - \Lambda^2},\ \bar{e}_2 := -\sqrt{u + \Lambda^2},\ \bar{e}_3 := -\sqrt{u - \Lambda^2}$ and $\bar{e}_4 := \sqrt{u + \Lambda^2}$ of the curve $\mathcal{E}(u)$

$$\lambda = \xi := \frac{(\bar{e}_1 - \bar{e}_4)(\bar{e}_3 - \bar{e}_2)}{(\bar{e}_2 - \bar{e}_1)(\bar{e}_4 - \bar{e}_3)} \tag{3.5}$$

in the $c = -2$ triplet model on the sphere, as it is done in [9]. This is possible due to relation (A.4) which canonically appears in the $c = -2$ triplet model as a quotient of

[1]There exists another, physically equivalent formulation of the torus related to the one above by an isogeny [21], [15].

certain correlation function (c.f. (5.2)). We will comment on that more extensively in chapter 5. Notice that thereby we relate the triplet model to the Weierstrass formulation but if $\tau_{SW} = \tau$ this also means a connection to SW theory. Simplifying (3.5) yields

$$\lambda(\alpha) = \frac{\sqrt{\frac{\alpha}{\alpha-1}} - 1}{\sqrt{\frac{\alpha}{\alpha-1}} + 1} \quad , \quad \alpha = \frac{u^2}{\Lambda^4}. \tag{3.6}$$

In order to identify τ and τ_{SW} use the fact that by means of uniformization the structure of the algebraic curve (3.1) is encoded in a hypergeometric differential equation for the charge lattice parameters a and a_D (f.i. [3], [14], [17]):

$$\hat{\mathcal{L}}\ s(u) := \left((\Lambda^4 - u^2)\frac{\mathrm{d}^2}{\mathrm{d}u^2} - \frac{1}{4} \right)\ s(u) = 0. \tag{3.7}$$

The two solutions are [3]

$$\begin{aligned} a(u) &= \sqrt{\frac{u-\Lambda^2}{2}}\ {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; \frac{2\Lambda^2}{\Lambda^2 - u}\right), \\ a_D(u) &= -\frac{\Lambda^2 + u}{2\Lambda}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; \frac{\Lambda^2 + u}{2\Lambda^2}\right). \end{aligned} \tag{3.8}$$

Now using (2.4) and (A.4) yields $\tau = \tau_{SW}$.

Expressions of the scalar modes in terms of hypergeometric functions are, of course, not new. However, we provide these here in order to make the paper more self contained.

4 Modular Forms

In order to get in contact with cfts or string theory it is suggestive to express the main functions above as modular forms (c.f. Appendix A). To be more specific, in our case this is important as some of the characters of the triplet model have a characteristic inhomogeneity with respect to their modular weight. We will find that the main character of this section (4.4) can be expressed in terms of characters of a cft that has to have the same properties and hence is the $c = -2$ triplet theory.

The moduli space parameter u can be written in terms of the Dedekind eta function from (3.4):

$$u(\tau) = \frac{\Lambda^2}{8}\ \left(\left(\frac{\eta(\frac{\tau}{4})}{\eta(\tau)} \right)^8 + 8 \right). \tag{4.1}$$

This is the key for deriving all other quantities as functions of τ. In [18] Nahm pointed out that the combination $c(\tau) := \tau a - a_D$ transforms like a modular form of weight -1.

Another promising feature of this combination is that given $\tau_{SW} = \tau$ one yields a by differentiating c by τ

$$\frac{\mathrm{d}c}{\mathrm{d}\tau} = a + \frac{\mathrm{d}a_D}{\mathrm{d}\tau}\left(\tau - \frac{\mathrm{d}a_D}{\mathrm{d}a}\right) = a. \tag{4.2}$$

Hence a and a_D should be given in terms of modular functions, likewise.

By inserting $u(\tau)$ into (3.8), a and a_D are expressed as functions of $\tau = \tau_{SW}$. In order to write them as modular functions it suffices to consider a and a_D in some region of $\mathcal{M}$, they can then be analytically continued to the whole of $\mathcal{M}$. Hence, we expand a and a_D around $u = \infty$ and then insert $u(\tau)$ in order to find the q expansions. The u expansion of a at infinity is straightforward whereas for a_D one has to use [4]

$$\begin{gathered}\frac{\Gamma(a)}{\Gamma(c)}\ {}_2F_1(a,a;c;z) = \frac{(-z)^{-a}}{\Gamma(c-a)}\sum_{n=0}^{\infty}\frac{(a)_n(1-c+a)_n}{n!^2}\ z^{-n}\left(\log(-z)+h_n\right), \\ h_n = 2\psi(1+n)-\psi(a+n)-\psi(c-a-n).\end{gathered} \tag{4.3}$$

Using inspiration from Nahm's paper and maple we were able to match (analytically) the q expansion of a with the q expansion of the derivative of some function c by τ, where

$$c(\tau) := \frac{\mathrm{i}\Lambda}{\pi}\left(\frac{\theta_{02}(\tau)-\theta_{22}(\tau)}{\eta^3(\tau)}\right) \tag{4.4}$$

in accordance with Nahm and, indeed,

$$a(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}c(\tau). \tag{4.5}$$

What relates expression (4.4) and, it being the central element generating a and a_D, the whole story to the $c = -2$ triplet model is its modular weight. The characters of ordinary rational cft's are homogeneous functions of modular weight zero, however $c(\tau)$ has weight -1. Thus, using $\chi_{-\frac{1}{8}}$ and $\chi_{\frac{3}{8}}$ (c.f. Appendix A) one finds

$$c(\tau) = \frac{\mathrm{i}\Lambda}{\pi}\left(\frac{\chi_{-\frac{1}{8}}-\chi_{\frac{3}{8}}}{\eta^2}\right) \tag{4.6}$$

and the interesting factor of η^2 in the denominator. This, however, can only be expressed in terms of characters of the $c = -2$ triplet model as only there characters appear with inhomogeneous modular weight, and which are of the form $\frac{\theta_{s.t.}}{\eta} + k\eta^2$ (c.f. Appendix A).

When expressing a_D by means of c and a we found a relation which differed from Nahm's given above, namely

$$a_D(\tau) + 2a(\tau) = \tau a(\tau) - c(\tau). \tag{4.7}$$

Anyway, we can use the $\mathbb{Z}_2$ symmetry $u \mapsto -u$ as it has an effect on a_D. Indeed, when going to $-u$ we calculated in exactly the same way as above

$$a_D(\tau) = \tau a(\tau) - c(\tau), \tag{4.8}$$

and obviously

$$u \mapsto -u \quad \Rightarrow \quad a_D \mapsto a_D - 2a, \tag{4.9}$$

in accordance with, for example, [17]. Further, with these relations and also with the help of the series expansions obtained we succeeded in checking, again

$$\tau_{SW} = \frac{\mathrm{d}a_D}{\mathrm{d}a} = \tau. \tag{4.10}$$

5 Correspondence to the $c = -2$ Triplet Model

As we discovered in the section before, equation (4.4) gives a strong hint that SW theory is related to the $c = -2$ triplet theory. We will now give some more relations concerning the main quantities of pure gauge $SU(2)$ SW theory. Thereby all important parameters are obtained as amplitudes and characters of the triplet model.

First of all, it is possible to write $c(\tau)$ completely in terms of characters of the $c = -2$ triplet theory (on $\mathbb{CP}^1$):

$$c(\tau) = \frac{\mathrm{i}\Lambda}{\pi} \left(\frac{\chi_{-\frac{1}{8}}(\tau) - \chi_{\frac{3}{8}}(\tau)}{\chi_1(\tau) - \chi_0(\tau)} \right). \tag{5.1}$$

We consider this very simple expression for $c(\tau)$ in terms of the characters of the triplet theory as the main result of the paper. We once more stress that it is not possible to express the modular form $c(\tau)$ of weight -1 in terms of characters or torus amplitudes of any ordinary (rational) conformal field theory as these necessarily have to be modular forms of weight zero. Only within logarithmic conformal field theory do torus amplitudes arise, which have non-vanishing modular weight, see [11].

There is some geometric interpretation of this behind it which is strongly suggested in [9] and [16]. The involved characters in the numerator belong to the twistfield μ of conformal weight $h_\mu = -\frac{1}{8}$ and its excited partner σ with weight $h_\sigma = \frac{3}{8}$. Now, the μ field can be thought of as representing some branch point on $\mathbb{CP}^1$ and one may think of the torus as being constructed as a double cover of $\mathbb{CP}^1$ with four fields μ producing two branch cuts.

However, as we learn from [16], this geometric interpretation of branch points being represented by some analytic fields is not a speciality of the triplet theory alone but of a whole class of theories of pairs of analytic anticommuting fields of integer spins j and $1-j$. Nonetheless the triplet theory is somewhat outstanding. All these theories lead to different kinds of Coulomb gas models according to which one can calculate different four-point functions. The main point is that the four-point function for the "geometric" field μ in the $c=-2$ triplet theory leads to a hypergeometric ode of type[2] $(\frac{1}{2},\frac{1}{2},1)$ and hence to the projective solution (A.4), formally

$$\tau = \mathrm{i}\,\frac{\langle\mu(\infty)\mu(1)\mu(\xi)\mu(0)\rangle_2}{\langle\mu(\infty)\mu(1)\mu(\xi)\mu(0)\rangle_1}, \tag{5.2}$$

where the index labels the different conformal blocks respectively homology cycles, that are, as explained, in one to one correspondence to the periods, and the crossing ratio ξ matches the modular function λ, as in (3.5). Explicitly, they read $\langle\mu\mu\mu\mu\rangle_k = [\xi(1-\xi)]^{1/4}F_k(\xi)$ with $F_1(\xi) = {}_2F_1(\frac{1}{2},\frac{1}{2};1;\xi)$ and $F_2(\xi) = F_1(1-\xi)$. In this respect the triplet theory is a natural tool in order to express the family of Weierstrass curves $\mathcal{E}(\tau)$ by means of four point functions of the twistfield μ.

The relation to SW theory is then introduced indirectly via the isomorphism $\mathcal{E}(u) \simeq \mathcal{E}(\tau)$ under which $\tau_{SW} = \tau$ and the $c=-2$ model is thus natural for the Weierstrass formulation with developing mapping (A.4) rather than for the SW moduli space in general. This is mainly due to the fact that the $c=-2$ theory does not know about the restrictions from the monodromy incorporated in the SW model, however it may serve as a nice tool in order to describe the geometric properties of general tori in the Weierstrass formulation. For example one can express the SW-form

$$\lambda_{\mathrm{SW}} = \frac{\mathrm{i}}{\sqrt{2}\pi}\frac{x^2}{y(x)}\mathrm{d}x \tag{5.3}$$

and the derived lattice paramters a and a_D in terms of quotients of conformal blocks of the $c=-2$ triplet model [9], where one uses the algebraic curve $y(x) = \prod_{i=1}^4 (x-\bar{e}_i)$ defined by the four branch points $\bar{e}_i$ as in (3.5). In particular (2.2) has an expression as six point functions on the sphere [8] or in another interpretation two point functions on the torus, represented by the twistfields:

$$\begin{aligned} a_i(u) &= \frac{\langle V_2(\infty)\mu(\bar{e}_1)\mu(\bar{e}_2)\mu(\bar{e}_3)\mu(\bar{e}_4)V_{-2}(0)\rangle_i}{\langle V_1(\infty)\mu(\bar{e}_1)\mu(\bar{e}_2)\mu(\bar{e}_3)\mu(\bar{e}_4)V_{-2}(0)\rangle} \\ &= \frac{(\mathrm{i})^i}{\sqrt{2}\pi}\frac{\bar{e}_3^2}{\sqrt{(\bar{e}_4-\bar{e}_3)(\bar{e}_2-\bar{e}_1)}}\frac{\langle V_2(\varpi)\mu(\infty)\mu(1)\mu(\xi)\mu(0)V_{-2}(\eta)\rangle_i}{\langle V_1(\varpi)\mu(\infty)\mu(1)\mu(\xi)\mu(0)V_{-2}(\eta)\rangle}, \end{aligned} \tag{5.4}$$

[2] c.f. [12].

that is

$$
\begin{aligned}
a(u) &= \mathrm{i}\frac{\Lambda^2(1+\varpi^2)^2}{\sqrt{\varpi}\left(\sqrt{(\varpi+1)^2\Lambda^2}+\sqrt{(\varpi-1)^2\Lambda^2}\right)}F_D^{(3)}(\tfrac{1}{2},\{\tfrac{1}{2},2,-2\},1;\varpi^2,-\varpi,\varpi),\\
a_D(u) &= \frac{\Lambda^2(1+\varpi^2)^2}{\sqrt{\varpi}\left(\sqrt{(\varpi+1)^2\Lambda^2}+\sqrt{(\varpi-1)^2\Lambda^2}\right)}F_D^{(3)}(\tfrac{1}{2},\{\tfrac{1}{2},2,-2\},1;1-\varpi^2,1+\varpi,1-\varpi).
\end{aligned}
\tag{5.5}
$$

The index $i = 1,2$ labels the standard homology basis, i.e. a and a_D, respectively, and the fields $V_2(\infty)$ and $V_{-2}(0)$ correspond to the double pole at infinity and the double zero at the origin of the SW-differential (5.3). The denominator serves in order to extract the nontrivial part of the numerator [9]. That a can be written as a Lauricella function and that the three crossing ratios are related to each other was obtained in [2]. Again, ξ is related to λ via (3.5) and $\varpi^2 = \xi$, $\eta = -\varpi$, such that the periods are functions of one complex variable only.

By means of $u(\tau)$, i.e. (3.4), the expression for $a(u)$ as above can be expanded in q via the elliptic modulus $\lambda(\tau) = \varpi^2$. This q-series coincides with the one for the simple hypergeometric functions as given in (3.8). Using this form of $a(u)$, we find

$$
\begin{aligned}
a(u) &= \Lambda\left(\frac{(\varpi-1)^2}{4\varpi}\right)^{\frac{1}{2}}{}_2F_1\left(\frac{1}{2},-\frac{1}{2};1;\frac{-4\varpi}{(1-\varpi)^2}\right)\\
&= \Lambda(-\xi')^{-1/2}{}_2F_1\left(\frac{1}{2},-\frac{1}{2};1;\xi'\right)\\
&= \frac{\mathrm{i}\Lambda}{\sqrt{2}\pi}\frac{\langle\sigma(\infty)\sigma(1)\mu(\xi')\mu(0)\rangle_1}{\langle\sigma(\infty)\sigma(1)\mu(\xi')\sigma(0)\rangle},\\
a_D(u) &= -\left(1-\frac{1}{\xi'}\right){}_2F_1\left(\frac{1}{2},\frac{1}{2};2;1-\frac{1}{\xi'}\right)\\
&= \frac{\mathrm{i}\Lambda}{\sqrt{2}\pi}\frac{\langle\sigma(\infty)\sigma(1)\mu(\xi')\mu(0)\rangle_2}{\langle\sigma(\infty)\sigma(1)\mu(\xi')\sigma(0)\rangle},\\
&= \frac{-\Lambda}{\sqrt{2}\pi}\frac{\langle\sigma(\infty)\sigma(1-1/\xi')\mu(1)\mu(0)\rangle_1}{\langle\sigma(\infty)\sigma(1-1/\xi')\mu(1)\sigma(0)\rangle}.
\end{aligned}
\tag{5.6}
$$

Here, ξ' is interpreted as the inverse crossing ratio

$$
\xi' := \frac{-4\varpi}{(\varpi-1)^2}. \tag{5.7}
$$

Again, the label of the correlation functions refers to the conformal blocks. In the denominator, $\langle\sigma(\infty)\sigma(1)\mu(\xi')\sigma(0)\rangle = [\xi'(1-\xi')]^{-1/4}(C_1\xi' + C_2)$, and the block needed here is obtained with the choice $C_1 = 1, C_2 = 0$. The final expression for $a_D(u)$ is obtained by analytic continuation of the second conformal block to the region $\{\xi' : |\xi'| > 1, |1-\xi'| < 1\}$.

Performing again an expansion in q for ξ', we find explicitly the relation

$$\xi = \varpi^2 = \lambda(q) \quad \text{and} \quad \xi' = \lambda(-\sqrt{q})\,. \tag{5.8}$$

This means that the modular parameter associated to ξ' is just half of the modular parameter associated to ξ, since $-\sqrt{q} = -(\exp(\pi \mathrm{i}\tau))^{1/2} = \exp(\pi\mathrm{i}(\tau/2 \pm 1))$, and $\tau/2 \pm 1$ is equivalent under the $PSL(2,\mathbb{Z})$ action to $\tau/2$. We note that substituting $u \mapsto -u$ would have led us immediately to the result $\xi' = \lambda(\sqrt{q})$, which is the elliptic modular form for a torus with lattice parameter $\frac{1}{2}\tau$. In this respect the correlation functions in (5.6) have an interpretation as zero-point functions on a torus with $\frac{1}{2}\tau$, where the branchpoints are represented not by the μ fields alone but also by the excited fields σ.

To conclude the discussion, we remark that the interpretation of the denominator of equation (5.1) is much more involved. That $c(\tau)$ can be expressed solely in terms of characteristic functions of the triplet model can be read as a strong hint that there really is a relation, though up to know we obtained all quantities in a non-constructive way. It would be interesting to look at SW with matter and other gauge groups in order to find out if still the triplet model appears. For completeness, we give the SW periods a and a_D expressed in terms of characters of the $c = -2$ triplet model:

$$\begin{aligned}
a(\tau) &= \Lambda\frac{\theta_3(\theta_2^4-\theta_3^4)\partial_q\theta_2}{\theta_2^2(\theta_2\partial_q\theta_3-\theta_3\partial_q\theta_2)}\\
&= \Lambda\frac{\eta^2(\chi_{-\frac{1}{8}}+\chi_{\frac{3}{8}})\left(16(\chi_0+\chi_1)^4-(\chi_{-\frac{1}{8}}+\chi_{\frac{3}{8}})^4\right)\partial_q(\eta(\chi_0+\chi_1))}{4(\chi_0+\chi_1)^2\left((\chi_0+\chi_1)\partial_q(\eta(\chi_{-\frac{1}{8}}+\chi_{\frac{3}{8}}))-(\chi_{-\frac{1}{8}}+\chi_{\frac{3}{8}})\partial_q(\eta(\chi_0+\chi_1))\right)}\\
&= \Lambda\frac{(\chi_0-\chi_1)(\chi_{-\frac{1}{8}}+\chi_{\frac{3}{8}})^2\left(16(\chi_0+\chi_1)^4-(\chi_{-\frac{1}{8}}+\chi_{\frac{3}{8}})^4\right)}{4(\chi_0+\chi_1)^2}\\
&\quad\times\frac{\partial_q((\chi_0-\chi_1)(\chi_0+\chi_1)^2)}{\left((\chi_0+\chi_1)^2\partial_q((\chi_0-\chi_1)(\chi_{-\frac{1}{8}}+\chi_{\frac{3}{8}})^2)-(\chi_{-\frac{1}{8}}+\chi_{\frac{3}{8}})^2\partial_q((\chi_0-\chi_1)(\chi_0+\chi_1)^2)\right)}\,,\\
a_D(\tau) &= (\tau-2)a(\tau)-c(\tau)\,,
\end{aligned} \tag{5.9}$$

where $c(\tau)$ is given in eq. (5.1) and the theta as well as the characteristic functions are all taken at $\sqrt{q}$, i.e. at the value $\frac{1}{2}\tau$. Note that this corresponds to ξ' above, i.e. half of the modular parameter associated to the original formulation in ξ. To arrive at these formulæ, one essentially uses the facts that, firstly, $a(u)$ is given by

$$a(k) = \frac{2\Lambda}{\pi}\frac{E(k)}{k}\,, \tag{5.10}$$

with $k^2 = \lambda(\sqrt{q})$ the elliptic modulus and that, secondly, the complete elliptic integral of the second kind can be expressed as

$$\frac{E(k)}{k} = (1-k^2)\left(\frac{K(k)}{k} + \frac{\mathrm{d}K(k)}{\mathrm{d}k}\right) \tag{5.11}$$

in terms of the complete elliptic integral of the first kind. Then, the result follows by simply plugging in $K(k) = \frac{1}{2}\pi\theta_3^2(\sqrt{q})$ and $k = \sqrt{\lambda(\sqrt{q})} = \theta_2^2(\sqrt{q})/\theta_3^2(\sqrt{q})$. With the help of the appendix, the expression in Jacobi theta functions can then easily be rewritten in terms of characters of the $c=-2$ triplet model.

It is worth noting that there is an interesting relation between the modular form $c(\tau)$ and the elliptic modulus k. To see this, let us introduce modified characters which take into account the fermionic nature of the $c=-2$ triplet model realized in terms of symplectic fermions. It turns out that the irreducible highest weight representations with conformal weights $h=3/8$ and $h=1$ are spin doublets with all states having odd fermion number, while the irreducible representations with weights $h=-1/8$ and $h=0$ are spin singlets with all states having even fermion number. It is then useful to introduce characters including Witten's index F, i.e.

$$\chi_h(z,q) = \mathrm{tr}_{\mathcal{V}_h} z^F q^{L_0 - c/24}\,. \tag{5.12}$$

Defining in addition the quantity

$$\kappa(z,\tau) = \frac{\chi_0(z,q)+\chi_1(z,q)}{\chi_{-\frac{1}{8}}(z,q)+\chi_{\frac{3}{8}}(z,q)}\,, \tag{5.13}$$

we find the remarkable result

$$c(\tau) = -\frac{\mathrm{i}\Lambda}{\pi}\frac{1}{\kappa(-1,\tau)}\,, \qquad k(\tau) = 4\kappa^2(1,\tau)\,. \tag{5.14}$$

It is worth mentioning that $\kappa(z,\tau)$ is just the quotient of the chiral partition functions of the twisted and untwisted sectors, respectively, of the $c=-2$ ghost system [6]. Therefore,

$$\begin{aligned} a(\tau) &= \frac{\Lambda}{2\pi}\left(1-16\kappa^4(1,\tau/2)\right)\left(\frac{K(4\kappa^2(1,\tau/2))}{\kappa^2(1,\tau/2)} + \frac{1}{2\kappa(1,\tau/2)}\frac{\mathrm{d}K(4\kappa^2(1,\tau/2))}{\mathrm{d}\kappa(1,\tau/2)}\right)\,, \\ a_D(\tau) &= (\tau-2)a(\tau) + \frac{\mathrm{i}\Lambda}{\pi}\frac{1}{\kappa(-1,\tau)}\,, \end{aligned} \tag{5.15}$$

Another guess is that the characteristic functions hint to topological string theory, as there is a relation between Gromov-Witten invariants and modular forms. This idea is strongly supported by [19]. Only recently there was published a paper by Huang and Klemm who indeed related the results of [19] to the topological B-model on a local CY [13] and expressed the pure gauge $SU(2)$ prepotential up to genus 6 in terms of modular forms. The expressions for a and u calculated there (which they do for the isogeneous curve) match ours but a factor of 2 in the q-expansion. In another interesting paper [1] Klemm and collaborators proved even more evidence for the suggestion above.

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A Useful Formulas and Definitions

$$
\begin{aligned}
q &= \exp(\mathrm{i}\pi\tau) \\
\eta(q) &= q^{\frac{1}{12}} \prod_{n=1}^{\infty}(1-q^{2n}), \\
\theta_2(z,q) &= 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} q^{n(n+1)} \cos((2n+1)z), \\
\theta_3(z,q) &= 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos(2nz), \\
\theta_{\lambda,k}(q) &= \sum_{-\infty}^{\infty} q^{\frac{(2kn+\lambda)^2}{2k}},
\end{aligned}
$$

The characteristic functions in the $c=-2$ triplet model are the following:

$$
\begin{aligned}
\chi_{-\frac{1}{8}}(q) &= \frac{\theta_{0,2}(q)}{\eta(q)}, \\
\chi_{\frac{3}{8}}(q) &= \frac{\theta_{2,2}(q)}{\eta(q)}, \\
\chi_0(q) &= \frac{1}{2\eta(q)}(\theta_{1,2}(q) - \eta(q)^3), \\
\chi_1(q) &= \frac{1}{2\eta(q)}(\theta_{1,2}(q) + \eta(q)^3), \\
\chi_{\mathcal{R}_{0/1}}(q) &= 2\eta(q)^{-1}\theta_{1,2}(q),
\end{aligned}
$$

where the last character appears twice [12], one for an indecomposable singlet highest weight representation $\mathcal{R}_0$ and the other for an indecomposable doublet highest weight representation $\mathcal{R}_1$.

$$
\Gamma^0(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \; : \; b = 0 \bmod n \right\} \tag{A.1}
$$

The function $\lambda(\tau)$ is the standard elliptic modular function

$$
\lambda(\tau) := \frac{e_3(\tau) - e_2(\tau)}{e_1(\tau) - e_2(\tau)}. \tag{A.2}
$$

The e_i are the branch points of the algebraic curve (3.2) and (3.3) is invariant under $\Gamma(2)$. λ can be expressed in terms of theta functions [4]

$$\lambda(\tau) = \frac{\theta_2^4(0,q)}{\theta_3^4(0,q)}. \tag{A.3}$$

A nice property of the modular function is that its inverse can be given via hypergeometric functions

$$\tau = \mathrm{i}\,\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\lambda\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\lambda\right)}. \tag{A.4}$$

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