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## Logarithmic Extensions and New Indecomposable Structures of Two Dimensional Conformal Ghost Systems with Additional $\mathbb{Z}_n$ Symmetry


von

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**Zusammenfassung**

In dieser Diplomarbeit wurde das Verhalten von Geistsystemen als zweidimensionalen konformen Feldtheorien unter additiven Modifikationen des Energie-Impuls-Tensors untersucht. Die Moden dieser Modifikationsfelder sollen in der Einhüllenden der Geistmodenalgebra enthalten sein. Diese Bedingung ist erfüllbar, wenn die Geistsysteme auf riemannschen Flächen, dargestellt als verzweigte Überlagerungen der riemannschen Sphäre, definiert werden. Der unmodifizierte Energie-Impuls-Tensor agiert inkonsistent auf den Zuständen: Aus seiner Aktion auf den Zuständen lässt sich ein Widerspruch zur Assoziativität der zugrundegelegten Operator Produkt Algebra konstruieren. Im Gegensatz dazu weist der (erweiterte) Zustandsraum des $\mathbb{Z}_n$-getwisteten Geistsystems mit zentraler Ladung $c = -2$ unter der Aktion des modifizierten Energie-Impuls-Tensors unabhängig von der Ramifikationszahl $n$ die gesuchten Jordanzellen vom Rang 2 auf.

Ähnliche Modifikationen für konformen Spin $\lambda > 1$ sind möglich: Ein allgemeinerer Ansatz, der die oben genannte Modifikation für $c = -2$ als Spezialfall enthält, wurde gefunden. Konsistenzbedingungen für den Modeninhalt dieses Ansatzes und der funktionalen Abhängigkeit von anderen Parametern konnten explizit angegeben werden. Für alle Modifikationsfelder, die ausschließlich aus $b$-Moden konstruiert sind, wurde gezeigt, daß Jordanzellen für $L_0$ in allen fermionischen Geistsystemen, außer denen zu $c = -2$, ausgeschlossen sind. Nichtsdestotrotz existieren unzerlegbare Strukturen bezüglich der anderen Virasorogeneratoren. So konnten auch quasiprimäre Zustände konstruiert werden, deren korrespondierende Felder inhomogene Ward-Identitäten bezüglich des Translationsoperators erfüllen müssen. Solche unzerlegbaren Strukturen wurden meines Wissens bisher nicht beschrieben.

**Abstract**

In this thesis, I examined the behaviour of ghost systems as two dimensional conformal field theories under additive modifications of the energy-momentum tensor. The corresponding modification fields are required to have modes in the universal enveloping algebra of the algebra of the ghost modes. This condition can be satisfied, if the ghost systems in consideration are defined on a Riemann surface, represented as a ramified covering of the Riemann sphere. The unmodified energy-momentum tensor acts inconsistently on the space of states: Its action on an (enlarged) space of states contradicts the associativity of the underlying operator product algebra. Contrary to this, the modified energy-momentum tensor exhibits the desired rank-2 Jordan cells on the space of states of a $\mathbb{Z}_n$ twisted ghost system with central charge $c = -2$.

Similar modifications of the energy-momentum tensor are possible for higher spin $\lambda$: I have found a more general ansatz containing the above-mentioned as a special case. Consistency conditions on the mode content of this ansatz and on the functional dependence on the other parameters are given explicitly. It could be shown for all modifications which are required to be decomposed by $b$-modes only that Jordan cells are not possible for fermionic ghost systems except the one at $c = -2$. Nevertheless, indecomposable structures with respect to other Virasoro generatorsdo exist. I have been able to construct quasi-primary states, which have to satisfy inhomogeneous Ward identities with respect to the translation operator. To my knowledge, such indecomposable structures have not been described in the framework of logarithmic conformal field theory.

# I Introduction

Conformal field theory (CFT) in two dimensions has attracted a great deal of attention from both mathematicians and physicists since the seminal paper by Belavin, Polyakov and Zamolodchikov in 1984 [BPZ84] revealed that in principle all correlation functions have a power series expansion with the so-called conformal blocks as coefficients. They showed that a certain class of so-called minimal models comprises only finitely many of those blocks. Each of these models is therefore determined completely by a finite set of numbers and therefore is exactly solvable.

In conventional CFTs, the local coordinate dependence of correlators is restricted to a formal power series. However, in many cases, logarithmic divergencies arise in the context of conformal symmetry, e.g. in the fractional quantum Hall effect, percolation, self-avoiding walks, two dimensional dense polymers, disorder, string theory and AdS/CFT-correspondence, the abelian sandpile model, gravitational dressing and others. This was first considered to be a problem, but these logarithmic CFTs (LCFTs) helped to solve long standing puzzles not well-described by CFTs. In 1993 Gurarie investigated the $c = -2$ model and proved that it could not do without logarithmic divergencies, highlighting the necessity for a generalisation of CFTs.

Different aspects of LCFTs had been observed as early as 1987 by Knizhnik in [Kni87] and also by Saleur and Rozansky in [Sal92b], [Sal92a] and [RS93]. Saleur and Rozansky noted indecomposable structures in e.g. a WZW model of the superextension of GL$(1,1)$. For a series of models, called ghost systems, Knizhnik showed that insertions of so-called twist fields into correlators on the Riemann sphere are sufficient to mimik any correlator on a $\mathbb{Z}_n$ symmetric Riemann surface in [Kni87]. Here, the appearance of logarithms in a CFT was noticed for the first time. But it seems that neither Knizhnik nor anybody else paid very much regard to this. Knizhnik blamed the appearance of logarithms, in analogy to the free boson case, on a diverging radius of compactification. Gurarie then published his paper [Gur93] about a special model of this series, showing that the logarithms were an integral part of the model, if one takes into account the twist fields. It received a boost of attention after [Flo94] and [Flo96] were published. In the former, Flohr classified all rational conformal field theories with effective central charge $c_{eff} = 1$, including the non-unitary ones. He found theories for which modular invariant expressions could not be obtained as bilinear combinations of the characters only, but for which additional functions had to be introduced which lacked an interpretation as characters. These models contain the $c = -2$ model and proved to be logarithmic. By now, LCFTs have become a part of mathematical physics and an important tool in particle and condensed matter physics. For reviews, see [FlR03]. There, an extensive list of references can be found as well. Although the ghost systems are the first examples of CFTs with logarithmic behaviour, they are still not completely understood. The analysis of Knizhnik suggested that only irreducible representations exist. Gurarie, however, showed that indecomposable representations are present. Furthermore, Knizhnik's bosonisation formulæ of the twist fields fail to show the logarithmic divergencies which have been seen by Gurarie, if the insertion points come close to each other.

In this thesis, I try to reconcile some of the statements of [Kni87] with results of [Gur93]. In the second chapter, I review some facts about conformal field theory, Riemann surfaces and ghost systems. Then, I give a rather detailed overview over Gurarie's and Knizhnik's respective papers, and explain where they might have conflicting implications. I indicate a way to circumvent this problem for ghosts in presence of $\mathbb{Z}_2$ twists, suggested by Fjelstad et al. in [FFH+02]. In chapter III.1, I suggest a deformation[1] ansatz to do the same for all $\mathbb{Z}_n$ twisted $c = -2$ models, without using 'foreign' new modes. In fact, simple ghosts on $n$-sheeted coverings of the Riemann sphere have been proven to allow for a natural and straightforward generalisation of the ansatz of Fjelstad et al. Following that, I prove this ansatz to be applicable and derive constraints on its parameters. Lastly, I calculate the action on the space of groundstates explicitly. Naturally, it is interesting to ask what happens in the case of higher conformal spin $\lambda$. This is investigated in chapter IV, using a general ansatz containing the one of the preceeding chapter. I derive constraints on the parameters and the properties of the unknown modes. To be able to calculate the action on the groundstates explicitly, I further restrict the ansatz for the unknown modes to a linear combination of known modes. This turns out to be disadvantageous. In particular, in some cases $L_0$ fails to measure the actual conformal weight. Therefore, in chapter V, I investigate a quadrilinear ansatz which proves to be well-defined and to lead to an indecomposable action on the space of states. I also show that Jordan cells are ruled out for any additive deformation, built out of $b$-modes, for which the new Virasoro modes measure the conformal weight correctly. From that result I infer that the logarithmic conformal Ward identities connected to the 'standard Jordan cell' cannot be valid in all cases. In chapter VI, I discuss some possible implications of my results. I continue with a summary and end with an outlook on further research opportunities. The appendix contains the derivation of a deformation with adjustable lower nilpotence index.

[1] Refering to my investigations, I will use the terms 'deformation', 'improvement' and 'modification' interchangeably. Here, the term deformation is *not* used in the sense of an OPA deformation, which will be denoted explicitly.

# II Background and Motivation

## II.1 Some Facts and Notations

### II.1.1 Conformal Field Theory

As a quantum field theory, a conformal field theory (CFT) is characterised by its space of states and the set of all its correlation functions, or $n$-point- functions. In CFT, the correlators of primary, i.e. conformally covariant, fields determine the set of all observable quantities. (As will be reviewed, this concept has to be generalised somewhat in logarithmic conformal field theory.) By conformal invariance, this set can be recovered the set of two-, three- and four-point-functionsi of primary fields. Furthermore the shape of two- and three-point-functions is completely determined and four-point-functions have to satisfy dramatic constraints. Associativity of the operator product algebra (OPA) or, equivalently, crossing symmetry of the four-point-functions, imply a set of consistency or 'bootstrap' equations. Although these are often too complex to be solved, this is possible within the so-called minimal models. These models are exactly solvable in this way.

A paedagogical overview of CFT is beyond the scope of this thesis. In the following, some features essential for the understanding of the rest of this thesis will be explained exemplarily. I will give a description by equal time commutators. The OPA approach to CFT is very useful in general, but it is not explicitly needed in this thesis. In order to make this thesis more self-contained, we add some further comments to the appendix. The reader unfamiliar with this topic might wish to consult [Sch96], [Gab00], [GG00], [Sch97], [DFMS97], [Gin88] for an elaborate introduction. Locally scale invariant quantum field theories are called conformal field theories. A condition for scale invariance is the tracelessness of the energy-momentum tensor. This condition is also sufficient to render the theory invariant under all coordinate transformations $z^\mu \longrightarrow x^\mu(z)$ which leave the metric tensor invariant up to a factor:

$$g_{\mu\nu} \longrightarrow \frac{\partial z^\tau}{\partial x^\mu}\frac{\partial z^\sigma}{\partial x^\nu} = f(z) g_{\mu\nu}.$$

These transformations comprise, for obvious reasons, the conformal, i.e. angle-preserving, group. The corresponding global symmetry algebra is isomorphic to $\mathfrak{so}(p+1, q+1)$ for $d = p+q$ dimensional space with signature $((+1)^q, (-1)^p)$. Local conformal symmetry in two dimensions implies an infinite number of conserved charges:

$$Q_\varepsilon = \oint \frac{\mathrm{d}z}{2\pi i} \varepsilon(z) T(z)$$

with $\varepsilon(z)$ an arbitrary function having a Laurent expansion in some vicinity of the origin, and $T(z)$ being the energy-momentum tensor, i.e. the generator of conformal transformations of the metric. One can reconstruct the conserved charges from the modes of the energy-momentum tensor, obtained by Laurent expansion

$$L_n = \oint \frac{\mathrm{d}z}{2\pi i} \frac{1}{z^{n+1}} T(z) \quad \rightsquigarrow \quad T(z) = \sum_n L_n z^{-n-2}.$$

These modes satisfy the Virasoro algebra

$$\underbrace{\left[L_q, L_m\right] = (q-m)L_{q+m}}_{\text{de Witt algebra}} + \frac{\hat{c}}{2}\binom{q+1}{3}\delta_{q+m,0} \tag{1}$$

$$\left[L_q, \hat{c}\right] = 0 \qquad \forall q \in \mathbb{Z} \qquad \hat{c} \text{ central extension.}$$

This is the central extension of the algebra of vector fields on the circle, *Diff*($\mathbb{S}^1$). The central charge or central extension commutes (therefore the name) with all other generators of the algebra, and therefore can be represented by a number $c$. The central extension is an element of the second cohomology group of the de Witt algebra. Since central extended representations can be used instead of projective ones, (which are needed, because Hilbert spaces used in quantum mechanics are projective,) the two cocycle represents an explicit scale in the space of states. In fact, the central charge $\hat{c}$ implies that the energy-momentum tensor does not respond conformally covariantly to transformations generated by it, and therefore $\hat{c}$ introduces some soft breaking of scale invariance. In string theory, the overall central charge has to vanish, because of that. Because of (1), the energy-momentum, or stress-energy tensor is also called Virasoro field. Often it is useful to compactify the theory in consideration on the torus. Therefore, if necessary, a Wick rotation is performed first, to obtain a euclidean theory,

$$\tau, \sigma \longrightarrow \tau_+, \tau_-, \quad \tau_\pm = \tau \pm i\sigma.$$

The following conformal transformation can then be used to work on the complex plane and use complex analysis:

$$\tau_+ \longrightarrow z = e^{2\pi i \tau_+}, \quad \tau_- \longrightarrow \bar{z} = e^{2\pi i \tau_-}.$$

This transformation maps infinite past to the origin and infinite future to the 'radius at infinity'. This is called radial quantisation. (These coordinates were used already in the Laurent expansion of the energy-momentum tensor.) Tracelessness ($T^\mu_\mu = 0$) and conservation ($\partial^\mu T_{\mu\nu} = 0$) of the energy-momentum tensor imply the equations

$$T_{z\bar{z}} = 0 \quad \partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0 \quad \partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{\bar{z}z} = 0,$$

from which follows

$$0 = \partial_{\bar{z}} T_{zz} \qquad 0 = \partial_z T_{\bar{z}\bar{z}}.$$

This means that the only two independent components of the Virasoro field are holomorphic and anti-holomorphic fields, respectively. Therefore, the two Virasoro algebras have to commute. The full symmetry is $\mathfrak{Vir} \otimes \overline{\mathfrak{Vir}}$ or bigger, but it can always be factorised into a holomorphic and an anti-holomorphic part. This is also the reason why fields and derived quantities can be written in a factorised form, where the holomorphic and the anti-holomorphic part are independent of each other, e.g. $\langle T(z)T(w)\rangle \propto \frac{c}{(z-w)^4}$, but $\left\langle T(z)\bar{T}(\bar{w})\right\rangle = 0$. After the full correlator is evaluated, nevertheless, a real cut has to be taken by requiring a fixed relationship, e.g. $z = \bar{z}$. Like the stress-energy tensor, all meromorphic fields $\phi(z, \bar{z}) \in \mathcal{V}^{h,\bar{h}}[[z, z^{-1}]],\ h, \bar{h} \in \frac{\mathbb{Z}}{2}$ can be expanded into Laurent series

$$\phi(z, \bar{z}) = \phi(z)\bar{\phi}(\bar{z}) = \sum_{n,n' \in \mathbb{Z} + \left(\frac{1}{2}\right)^A} \phi_n \bar{\phi}_{n'} z^{-n-h} (\bar{z})^{-n'-\bar{h}},$$

where $A$ is 0 or 1 depending on the boundary conditions. The modes $\phi_n, \bar{\phi}_{n'}$ are obtained by contour-integration around the fields:

$$\phi_n = \oint_w \frac{\mathrm{d}z}{2\pi i} \frac{\phi(z)}{(z-w)^{n+1}}, \qquad \bar{\phi}_{n'} = \oint_{\bar{w}} \frac{\mathrm{d}\bar{z}}{2\pi i} \frac{\bar{\phi}(\bar{z})}{(\bar{z}-\bar{w})^{n'+1}}.$$

The equal-time commutators are defined as

$$\left[\phi_1(z), \phi_2(w)\right]^{grad}_{|z|=|w|} = \lim_{\delta \to 0} \left(\phi_1(z)\phi_2(w)_{|z|=|w|+\delta} - (-1)^F \phi_2(w)\phi_1(z)\right)_{|z|=|w|-\delta}$$

The commutators can be expressed in terms of modes by contour integrals:

$$\begin{aligned} &\left[A_f, B_g\right]_{e.t.} \\ &= \oint_{C_1} \frac{\mathrm{d}z}{2\pi i} f(z)A(z) \oint_{C_2} \frac{\mathrm{d}w}{2\pi i} g(w)B(w) - \oint_{C_2} \frac{\mathrm{d}w}{2\pi i} g(w)B(w) \oint_{C_1} \frac{\mathrm{d}z}{2\pi i} f(z)A(z) \\ &= \oint_0 \frac{\mathrm{d}w}{2\pi i} g(w) \oint_w \frac{\mathrm{d}z}{2\pi i} f(z)A(z)B(w). \end{aligned}$$

By choosing $f(z), g(w)$ to be (inverse) monomials, one obtains the commutators of the modes. By virtue of the field state correspondence, the grading carries over to the space of fields $\mathcal{V}$, i.e. it decomposes into so-called conformal families. A conformal family is the collection of a primary field and all its Virasoro descendants, obtained by applying lexicographically ordered words in $L_n$-modes, $n < 0$, to them.

A theory is called minimal (or rational) if it comprises only finitely many conformal families. Then the space of states decomposes into finitely many irreducible (and possibly, indecomposable) representations of the symmetry algebra. The term minimal is often used to denote that the theory is rational with respect to the Virasoro algebra alone rather than with respect to a larger algebra. Because of that, all possible observables of rational theories are determined by a finite amount of numbers. In [FQS84, FSQ86], Friedan, Qiu and Shenker showed that for central charge $c < 1$ there is a discrete set of parameters $h_{p,q}, c_{p,q}$ for which unitary theories are not excluded. Goddard, Kent and Olive constructed such theories with their famous coset construction and thereby proved their existence [GKO86, GKO85].

In CFT, the notion of primary and quasi-primary fields is of particular importance. A field $\psi(w) = \sum_n \psi_n w^{-n-h_\psi}$ is called primary, if the subsequent relations hold for all $m \in \mathbb{Z}$. The sets of relations in the second and the third line are equivalent to the first line. For quasi-primary fields, the second and the third set of relations is valid for $m \in \{-1, 0, 1\}$.

$$T(z)\psi(w) = \frac{h_\psi \psi(w)}{(z-w)^2} - \frac{\partial\psi(w)}{(z-w)} \tag{2}$$

$$\left[L_m, \psi(w)\right] = w^m(w\partial_w + (m+1)h_\psi)\psi(w) \tag{3}$$

$$\left[L_m, \psi_n\right] = (m(h-1)-n)\psi_{m+n} \qquad \forall n \in \mathbb{Z}.$$

This implies the transformation properties

$$\psi(z) \to (f'(z))^h \psi(f(z))$$

for holomorphic $f$, if $\psi$ is primary, and $f \in \mathrm{SL}(2,\mathbb{C})$, [2] if $\psi$ is quasi-primary.

Applying primary fields to the vacuum $\lim_{z\to 0} \psi_h(z)\,|0\rangle$, one obtains states which satisfy

$$L_0 \left|h_\psi\right\rangle = h_\psi \left|h_\psi\right\rangle \tag{4}$$

$$L_m \left|h_\psi\right\rangle = 0 \qquad \forall\, m > 0. \tag{5}$$

These relations define the *heighest weight states*. In order to introduce a natural pairing, one further needs an antilinear involution $\omega$, such that in particular $\omega(L_n) = L_{-n}$ coincides with usual hermitian conjugation. The field-state isomorphism is a basic characteristic of all (L)CFTs. It is usually implemented by $|h\rangle = \lim_{z\to 0}\phi_h(z)\,|0\rangle$ whereas $\langle h| = \langle 0| \lim_{z\to 0}\phi_h(z) z^{2h}$. Consistent with that, $L_n\,|0\rangle = 0$ for all $n \geq -1$ and $\langle 0|\, L_n = 0$ for all $n \leq 1$. By moving a state away from $z = 0$, the other states of the conformal family are obtained in the expansion in $z$.

[2]The group of the one-to-one maps of the whole Riemann sphere onto itself is the global conformal, or Möbius, group $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 =: \mathrm{PSL}(2,\mathbb{C})$, formed by the mappings

$$z \to \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}), \qquad a,b,c,d \in \mathbb{C}, ad - cb = 1.$$

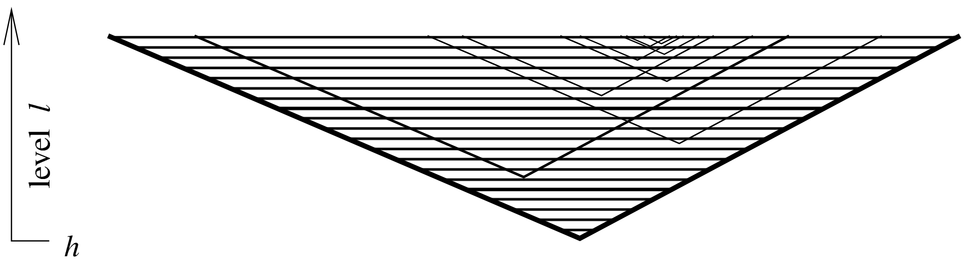



Figure 1: Modules of the Virasoro algebra. The lowest states at the edges are the so-called heighest weight states.

$\mathcal{V}^n$ is given by $\mathrm{span}_{\mathbb{C}}\left\{L_{i_r}\ldots L_{i_1}|h\rangle : \sum_r i_r = n,\ i_r \leq \ldots \leq i_1 < 0\right\}$. The number of partitions of $n$, $p(n)$, is therefore an upper bound to the dimension of the spaces $\mathcal{V}^n$, which is saturated, if there are no linear dependencies, i.e. no null vectors. A freely generated module is called Verma module $M(c,h)$, if all vectors, obtained by applying lexicographically ordered words of generators to the highest weight vector $|h\rangle$, are linearly independent as vectors. This is the generic case. But the Verma module is not necessarily irreducible, there can still be *algebraic* relations, such that certain linear combinations have the highest weight property again. These are called null vectors, because they decouple from all other vectors.

All Verma modules $M(c,h)$ are indecomposable, i.e. they cannot be written as a direct sum of two or more irreducible modules. From now on, every indecomposable module is assumed to be reducible, unless stated otherwise. In this case there exists a unique maximal proper submodule $J(c,h)$ such that $M(c,h)/J(c,h) =: V(c,h)$ is irreducible.

The submodule structure of Virasoro modules was investigated by Feigin and Fuchs [FF83, FF]. They showed three different structures to occur: There are either no, one or infinitely many null vectors. The situation is depicted in fig. II.1.1, where every cusp symbolises a singular vector with its tower of descendants. There is either only the module with the wide line style or both the wide and the middle, or all. The horizontal lines depict the level $l$, which is the difference of the conformal weight of a vector to the conformal weight of its anchestor heighest weight state. The states with non-zero level are called descendant states. The corresponding descendant fields are obtained by contour integrals of operator products with the Virasoro field

$$\phi_h^{(-n_1,\ldots,-n_l)} = \oint_{\gamma_1}\frac{\mathrm{d}w_1}{2\pi i}T(w_1)\oint_{\gamma_2}\frac{\mathrm{d}w_2}{2\pi i}T(w_2)\ldots\oint_{\gamma_l}\frac{\mathrm{d}w_l}{2\pi i}T(w_l)\phi_h(z).$$

The Virasoro field is a primary field only if $c = 0$. Otherwise, a conformal transformation leads to an additional term, the Schwartzian derivative $S(f;z)$ [3]

$$S(f;y) = \frac{\mathrm{d}^3 f(y)/\mathrm{d}y^3}{\mathrm{d}f(y)/\mathrm{d}y} - \frac{3}{2}\frac{\mathrm{d}^2 f(y)/\mathrm{d}y^2}{\mathrm{d}f(y)/\mathrm{d}y}, \qquad T(z) = \left(\frac{\mathrm{d}z}{\mathrm{d}y}\right)^{-2}\left(T(y) - \frac{c}{12}S(z;y)\right). \quad (6)$$

Eq. (3) implies the following Ward identities to hold for the chiral correlator of quasi-primary fields $G(z_1,\ldots,z_n) = \left\langle \phi_{h_1}(z_1)\ldots\phi_{h_n}(z_n)\right\rangle$

$$\begin{aligned} L_{-1}G(z_1,\ldots,z_n) &= \sum_i \partial_i G(z_1,\ldots,z_n) = 0 \\ L_0 G(z_1,\ldots,z_n) &= \sum_i (z_i\partial_i + h_i)G(z_1,\ldots,z_n) = 0 \\ L_1 G(z_1,\ldots,z_n) &= \sum_i (z_i^2\partial_i + 2z_ih_i)G(z_1,\ldots,z_n) = 0. \end{aligned} \quad (7)$$

This completely determines the one-, two- and three-point-functions of quasi-primary fields to read

$$\begin{aligned} \langle\phi_i(z)\rangle &= 0 \\ \left\langle\phi_i(z_i)\phi_j(z_j)\right\rangle &= \frac{C_{ij}}{(z_i - z_j)^{2h_i}}\delta_{ij} \\ \left\langle\phi_i(z_i)\phi_j(z_j)\phi_k(z_k)\right\rangle &= \frac{C_{ijk}}{z_{ij}^{h_i-h_j-h_k} z_{jk}^{h_j-h_k-h_i} z_{ki}^{h_i-h_k-h_j}}. \end{aligned}$$

The general shape of the correlator of quasi-primary fields is fixed to

$$\left\langle\phi_{h_1}(z_1)\ldots\phi_{h_n}(z_n)\right\rangle = F(x_1,\ldots,x_{n-3})\prod_{i>j}(z_i - z_j)^{\mu_{ij}} \quad (8)$$

where

$$\mu_{ij} = \mu_{ji}, \quad \sum_{i\neq j}\mu_{ij} = -2h_i$$

and

$$x_k = \frac{(z_1 - z_k)(z_{n-1} - z_n)}{(z_k - z_n)(z_1 - z_{n-1})}, \quad k = 2,\ldots,n-2.$$

[3] This is the unique weight-2 object with the properties

$$\begin{aligned} S(f;z) = 0 &\Leftrightarrow f(z) = \tfrac{az+b}{cz+d} \\ S\left(\tfrac{af+b}{cf+d};z\right) &= S(f;z) \\ S(f;z) &= (\partial_z g)^2 S(f;g) + S(g;z). \end{aligned}$$

Examples of CFT's are the ghost systems, which will be introduced and investigated in the following sections. The free boson is another famous example. Not only is it widely used to describe bosons, but also in order to formulate theories in a free field construction. This is possible in two space time dimensions because of the bosonisation theorem [KR87, Sto].

### II.1.2 Riemann Surfaces

A complex curve, i.e. a complex one dimensional connected analytic manifold, is called a Riemann surface. They prove most useful in physics: In string theory, the notion of Riemann surfaces of genus $g$ arises in a natural manner as worldsheets swept out by strings. The integral over all complex structures (or the moduli space, which is the space of admissible parameters) plays the rôle of the sum over all Feynmann graphs of order $g$. Apart from string theory, Riemann surfaces have applications in solid state physics. One example is the simulation of borders or defects, which act like branch cuts or points. Furthermore, hyperelliptic Riemann surfaces emerge as moduli spaces of Seiberg-Witten theory, if the gauge group is simple and simply-laced. In this thesis, I will only investigate algebraic curves with $\mathbb{Z}_n$ symmetry.[4] This means that all branch points $a_i$ on the surface have the same ramification number, called $n$ from now on. Every $\mathbb{Z}_n$ symmetric Riemann surface can be parametrized by numbers $a_i \in \mathbb{C}; l_i, n \in \mathbb{N}$, where $i = 1, \ldots, L$, and can be expressed as the graph

$$\Gamma = \left\{ (y, z) :\ y^n(z) = \prod_{i=1}^{L} (z - a_i)^{L_i} \right\}.$$

Then $g = (n-1)(L-1)/2$ is the genus of the Riemann surface. If any $L_i \neq 1$, the curve is called *singular*. $\mathbb{Z}_n$ symmetry implies that the monodromy around all branch points is simultaneously diagonalisable. The monodromy group acts on meromorphic[5] fields via the mappings

$$\hat{\pi}_{a_i} : \mathcal{V}^{(h,\bar{h})}[\![z, z^{-1}]\!] \to \mathcal{V}^{(h,\bar{h})}[\![z, z^{-1}]\!] \tag{9}$$

$$\varphi(z) \mapsto \varphi'(z) = \varphi(e^{2\pi i l}(z - a_i) + a_i) \qquad l \in \mathbb{Z}, \tag{10}$$

[4]The moduli space of Riemann surfaces is $3(g-1) + \delta_{g,1}$ dimensional. The moduli space of hyperelliptic Riemann surfaces of genus $g$ is $2g - 1$ dimensional, such that there is a mismatch for $g > 2$. Sadly, in string theory, one has to sum over all smooth complex curves, and for higher $g$ not all Riemann surfaces are at the least $\mathbb{Z}_n$ symmetric. This is one of the reasons why string amplitudes were calculated up to two loops only until now. Seiberg-Witten theory with non simply-laced gauge groups implies surfaces which do not satisfy this condition, neither.

[5]In the framework of conformal field theory, 'meromorphic' fields are allowed to have conformal weights $h, \bar{h} \in \mathbb{Z}/2$.

and therefore forms a representation of the fundamental group. I will denote the covering map by

$$z : \Gamma \to \mathbb{CP}^1$$

and choose local coordinates such that in the vicinity of a branch point $a$

$$z(y) = a + y^n.$$

The $n$ sheets of the Riemann surface, defined by the inverse of the covering map

$$y(z) = (z - a)^{\frac{1}{n}},$$

will be counted by $0, \ldots, n - 1$.

### II.1.3 Ghosts and Zero Modes

Ghosts are fields which exhibit the wrong statistics. Fermionic ghosts are characterised by an integer spin but anticommute, and vice versa. Ghost systems arise in different physical applications. In a two dimensional setting, there is a series of systems, which comprise pairs of fields $b, c$ with the conformal weights $\lambda, 1 - \lambda$ respectively, such that the natural pairing $\oint b(z)(\mathrm{d}z)^\lambda c(z)(\mathrm{d}z)^{1-\lambda}$ is conformally invariant. These systems can be defined for all half-integer $\lambda$, but this thesis only deals with anti-commuting fields with integer spin $\lambda$. Historically, ghosts were first used to cast away the Fadeev-Popov-determinant which makes the difference between the path integral over all field configurations and the one where integration is restricted to the physical field configurations. This gauge fixing does not spoil manifest Lorentz invariance. Modern covariant quantisation of the bosonic string e.g. makes use of the 'reparametrisation ghosts' with $\lambda = 2$; there, the gauge group is $\mathit{Diff}(\Sigma)$. In superstring theory its superpartners with conformal weight $\frac{3}{2}$ are also used. For investigation, though, it is conventional to treat the conformal weight as a parameter.

The volume of the gauge group can be integrated out yielding a functional integral with a Jacobian factor. This Jacobian can be cancelled by the rules of Grassmann integration that a determinant of an arbitrary matrix can be expressed by a Berezinian integral over the exponential of two Grassmannians times the matrix,

$$\det M = \prod_{i=1}^{N} \int \mathrm{d}b_i \mathrm{d}c_i e^{b_i M_{ij} c_j},$$

with the Grassmann variables normalised such that $\int \mathrm{d}b_i b_i = \int \mathrm{d}c_i c_i = 1$.

This trick has to be refined to calculate the determinant of an inverse propagator such as the Laplacian [Fad99], [Pok]

$$\int \prod_\alpha \mathrm{d}b_\alpha \mathrm{d}\bar{b}_\alpha \mathrm{d}c_\alpha \mathrm{d}\bar{c}_\alpha e^{-S_\lambda(b,c)} = \prod_\alpha \lambda_\alpha^2 = \det{}' \Delta^-_{1-\lambda}.$$

The second equality involves a $\zeta$-function-regularisation. This integrand is interpreted as the Lagrangian of the Grassmann fields. In our two dimensional setting, the above requirements imply the action

$$S_\lambda = \frac{1}{2\pi}\int_\Sigma dz \wedge d\bar{z}\,\sqrt{\det g_{z\bar{z}}}\,\left(b\nabla^z_{1-\lambda}c + \bar{b}\nabla^{\bar{z}}_{1-\lambda}\bar{c}\right) \tag{11}$$

with $\Sigma$ an arbitrary Riemann surface. I will mostly restrict myself to describe ghost systems on the Riemann sphere. In this case, the equations of motion,

$$\bar{\partial}c = \bar{\partial}b = \partial\bar{c} = \partial\bar{b} = 0,$$

$$\bar{\partial}(b(z)c(w)) = 2\pi\delta^2(z-w,\bar{z}-\bar{w}),$$

yield the propagators

$$b(z)c(w) = \frac{1}{(z-w)} + reg(z-w) \qquad \bar{b}(z)\bar{c}(w) = \frac{1}{(\bar{z}-\bar{w})} + reg(\bar{z}-\bar{w}) \tag{12}$$

$$c(z)c(w) = reg(z-w) \sim 0 \qquad b(z)b(w) \sim 0. \tag{13}$$

Analogous conditions apply for the right-moving chiral halves. We will not refer to them further unless necessary. Since I will work with the algebra of modes in the following, the commutators equivalent to the above OPEs are displayed below:

$$b(z) = \sum_{l\in\mathbb{Z}} b_l z^{-l-\lambda} \qquad c(z) = \sum_{l\in\mathbb{Z}} c_l z^{-l+\lambda-1}$$

$$\{b_l, c_n\} = \delta_{n+l,0}.$$

There exists a complication, though, namely that the Laplacian has eigenvectors to the eigenvalue zero, the *zero modes*. These have to be included explicitly into the path integral. This is denoted by the 'primed' determinant. Equivalently, they could be excluded from integration, because by definition they do not contribute to the exponential. Because zero modes play a crucial rôle in my thesis, and presumably in all LCFTs, some important facts about them will be summarised below. Further information is provided in the appendix.

Zero modes are annihilators to 'both sides' of the correlator, whereas their conjugate modes turn out to be creators to both sides. For the $bc$ systems, the naive calculation for correlators on the complex plane $\langle 0|1|0\rangle = \langle 0|\{b,c\}|0\rangle = \langle 0|bc+cb|0\rangle = 0$ obviously shows that the vacuum is orthogonal to itself; hence one needs a non-trivial out-state to get a non-trivial result [GSW], [AGBM$^+$87], [AGMN86]. For genus zero surfaces, therefore $c$-modes have to be included into the path integral to obtain a nonzero result. More precisely, for the $(\lambda, 1-\lambda)$ systems the modes $b_i, i = 1-\lambda,\ldots,\lambda-1$, are the zero modes such that their conjugates $c_{-i}, i = 1-\lambda,\ldots,\lambda-1$ all have to be included into a vacuum expectation value to make it non-zero. For genus $g \neq 1$, one has to insert $Q = (2\lambda - 1)(g - 1)$ $b$-modes into the path integral. Here, negative numbers imply that $c$-modes have to be inserted, in order to exclude this number of $b$-modes.

Ghost systems can be realised by a free field construction. The bosonised versions of the important fields read

$$\begin{aligned}
b^{(k)}(z) &= :e^{i\varphi^{(k)}}:(z) \qquad c^{(k)}(z) = :e^{-i\varphi^{(k)}}:(z) && (14)\\
j^{(k)}(z) &= i(\partial\varphi^{(k)})(z) && (15)\\
T^{(k)}(z) &= -\lambda:b^{(k)}\partial c^{(k)}:(z) + (1-\lambda):c^{(k)}\partial b^{(k)}:(z) && (16)\\
&= \frac{1}{2}:j^{(k)}j^{(k)}:(z) + (\frac{1}{2}-\lambda)\partial j^{(k)}, && (17)\\
S_Q &= -\frac{1}{2\pi}\int d^2z\,\sqrt{\det g_{\mu\nu}}\left(\partial_\mu\varphi\partial^\mu\varphi - \frac{i}{2}QR_g\varphi\right) \qquad Q = 2\lambda - 1. && (18)
\end{aligned}$$

where $g_{\mu\nu}$ and the afore mentioned $g_{z\bar{z}}$ refer to the induced metric, and $\phi(z)$ is a $c = 1$ bosonic field. This action is defined for all half-integer $\lambda$, where $\lambda$ and $1 - \lambda$ is the conformal spin of $b$ and $c$, respectively. These are ghost systems for $\lambda \in \mathbb{Z}$ only. Otherwise, they are just fermions, e.g. the only system with non-anomalous current are Dirac fermions with $\lambda = 1 - \lambda = \frac{1}{2}$.[6] More recently, various applications were found for $\lambda = 1$ in condensed matter physics. This system has amazing features and has been investigated very thoroughly by now. These 'simple ghosts' are used to describe self-avoiding walks, the fractional quantum hall effect, percolation, the abelian sandpile etc. Furthermore, they are used to bosonise the supersymmetry partners of the reparametrisation ghosts.

By eq. (12), ghosts are nothing but formally holomorphic differentials, i.e. they are holomorphic on the punctured disc, see e.g. [FK80], [d'H99], and therefore exist on any Riemann surface. Below space-time dimension four, the spin-statistic-theorem applies indeed, but does not yield constraints, because the Lorentz algebra in these dimensions is too small. Instead, below space-time dimension three, the bosonisation theorem is valid. Thus, in two dimensions, ghosts are allowed to be physical fields. An example is the abelian sandpile, described by a logarithmic model of central charge $c = -2$, see below (section II.2.2).

### II.1.4 Ghost Systems on Ramified Coverings of the Riemann Sphere

Considering ghost systems on non-trivial Riemann surfaces, one can make one's life easier. Below, I will present the material following the original article [Kni87]. On each of the $n$ sheets of such a surface $\Gamma$, consider a ghost system of arbitrary (half-)

[6] There is also a system of commuting variables with half-integer spin, which enjoys the same action. These are the superconformal, or bosonic, ghosts. They will not be examined in this thesis.

integral spin $\lambda$. I try to follow the conventions used in large parts of the literature, when I denote the ghosts by the letters b and c, instead of calling them $f, \phi$ as in the original paper [Kni87]. To make distinction easier, the typesetting varies when I change the monodromy basis. The usual fields are set in inclined roman, whereas the diagonalised basis is denoted by italics. Furthermore, I will denote the conformal spin by $\lambda$ instead of '$j$', favored in mathematics and parts of physics literature, in order to avoid confusion with the ghost current.

To diagonalise the monodromy, let us introduce a new basis of fields

$$b^{(k)} = \sum_{l=0}^{n-1} e^{-2\pi i l q_k} \mathrm{b}^{(l)}, \qquad c^{(k)} = \sum_{l=0}^{n-1} e^{2\pi i l q_k} \mathrm{c}^{(l)}, \tag{19}$$

$$q_k := \frac{(k + \lambda(1-n))}{n}. \tag{20}$$

instead of the old $\mathrm{b}^{(l)}, \mathrm{c}^{(l)}$ .

This also renders the currents

$$j^{(k)} = :b^{(k)}c^{(k)}: = :\mathrm{b}^{(k)}\mathrm{c}^{(k)}:$$

single-valued functions in the vicinity of the branch points. The monodromy operation $\hat{\pi}_a$ becomes $\hat{\pi}_a b^{(l)} = e^{2\pi i q_l} b^{(l)}$ and $\hat{\pi}_a c^{(l)} = e^{-2\pi i q_l} c^{(l)}$ on the $l$th diagonalised sheet. In the new basis of fields, the OPE reads

$$b^{(k)}(z')c^{(m)}(z) = \delta_{k,m}(z'-z)^{-1} + :b^{(k)}c^{(m)}:(z) + reg(z'-z). \tag{21}$$

By comparison of two different expressions connected to the OPE (21) of $b^{(k)}(z')$ and $c^{(m)}(z)$ in the vicinity of a branch point, one concludes that $q_k$ is the charge of the branch point with respect to the current on the $k$th diagonalised sheet [Kni87]. This branch point can hence be represented by an insertion of a primary field with charge $q_k$. More precisely, one can rewrite eq. (21) in terms of the old basis $\mathrm{b}^{(l)}, \mathrm{c}^{(r)}$ and perform a conformal transformation to the single-valued coordinate $y$. This gives

$$\mathrm{b}^{(k)}(z')\mathrm{c}^{(m)}(z) = \delta_{k,m}(z'-z)^{-1}\left(\frac{y'}{y}\right)^{k+\lambda(1-n)}.$$

Taylor expanding the second factor at $z' = z$, one arrives at the expression

$$\delta_{k,m}\left((z'-z)^{-1} + \frac{k+\lambda(1-n)}{n}(z-a)^{-1}\right) + reg(z'-z).$$

This expression is compared by powers of $(z - z')$ to the original one, eq. (21). Therefore, (20) are the charges with respect to the ghost currents.

These results lead [Kni87] to the following conclusion: Even if ghost theories also comprise twists, they can still be represented by a free field construction eq. (17). In this formalism, the bosonised version of the twist would read

$$V_{\boldsymbol{q}}(a) = :e^{i\sum_k q_k \varphi_k}:(a) =: :e^{i\boldsymbol{q}\boldsymbol{\varphi}}:(a). \tag{22}$$

He substantiates his conclusion (22) by the following two arguments:

1. He derives the conformal weights of the twist fields by general considerations and shows this to be equal to the weights of the bosonised twist fields (22) implied by their charges (20).

   First, one notices that $T(y)$ has to be a holomorphic field with respect to the single-valued coordinate $y$. Therefore, it is regular in a vicinity of $y = 0$. Because the transformation law of $T$ deviates from that of a primary field by the Schwartzian derivative (6), $T(z)$ aquires an additional second order pole in the vicinity of the branch point with respect to the covering map $z$.

   Assuming a different point of view, one interprets the fields living on the Riemann surface as living on the Riemann sphere with additional insertions of those twist fields. The expression $T(z)$ 'in the vicinity of a branch point $a$' therefore will be replaced by the OPE

   $$T(z)V_{\boldsymbol{q}}(a) = \frac{h_{V_{\boldsymbol{q}}} V_{\boldsymbol{q}}(a)}{(z-a)^2} + O(z-a)^{-1}. \tag{23}$$

   Comparison of powers of $(z-a)$ yields that $h_{V_q}$ has to be the coefficient of the second order pole of $T(z)$ which reads

   $$h = \frac{nc_\lambda}{24}\left(1 - \frac{1}{n^2}\right).$$

   the OPE of the Virasoro field $T(y)$ with a primary field at some coordinate $a$, evaluated at $y$, merely has a second order pole in $(y-a)$.

   By virtue of eq. (17) and eq. (20) the conformal weight $h$ can be expressed in terms of the charges through

   $$h = \sum_k h_k = \sum_k (\frac{1}{2}q_k^2 + (\lambda - \frac{1}{2})q_k), \tag{24}$$

   which leads to the expression as the second order pole of $T(z)$ in the vicinity of a branch point.

2. Furthermore, he calculates the OPE of $b^{(l)}$ and $c^{(k)}$ and a twist field in the bosonic language. The outcome of this is consistent with the fact that the branch point has charge (20) with respect to the current.

By his first consideration, the charges are fixed up to a twofold ambiguity, too. Because the conformal weight of a free field exponential is quadratic in its charge, $q_k$ and $2Q - q_k = q_k^*$ lead to the same conformal weight, where $c = 1 - 24Q^2$.

If $Q = q_k$, of course, both solutions will be equal. But this is no proof that (22) is the only possibility, indeed, it can be shown that in the case of equality, there is a further primary field of the same conformal weight, which cannot be bosonised as Knizhnik suggested. Knizhnik does not comment on the reasons from which he concludes the primarity of the twist fields. It rather seems to be an assumption than a conclusion. A possible line of argument would be that twists should introduce superselection sectors in the space of states. Those are generated by highest weight states, which correspond to primary fields.

In the following, I will speak of the $n$ 'sheets' of a Riemann surface without regard to whether the fields are from the original or the diagonalised basis, or whether the CFTs on different sheets are represented by twisted sectors.

## II.2 Some Challenges of c = −2 Systems

### II.2.1 $\mathbb{Z}_2$ Symmetry

There is no strict definition of LCFTs in the literature yet. Thus, one may only characterise it by displaying theories, which proved to be well-defined and inherit indecomposable representations (while being quasirational with respect to some algebra). In the next sections, I will review the $c = -2$ system, because it is the best-known theory in both, the ghost systems and the LCFTs. It is furthermore used as an example for what goes on and what one might expect for

1. the generalisation of $\mathbb{Z}_2$ symmetry to $\mathbb{Z}_n$ symmetry
2. the transition from spin $\lambda = 1$ to $\lambda > 1$.

The $c = -2$ model on hyperelliptic surfaces can be interpreted as the first minimal model. The central charge $c = -2$ can be obtained by setting $p = 2, q = 1$ in the formula for central charges

$$c = c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$$

in minimal models. But then, the Kač table

$$\left\{ h_{r,s} = \frac{(pr-qs)^2 - (p-q)^2}{4pq} : 0 \leq r < p, 0 \leq s < q \right\}$$

is empty, as in every $c_{p,q}$ model with $p$ or $q$ equal to 1. This corresponds to the fact that the cohomology of the fermionic screening $b_0$, i.e. the space one usually regards as the space of physical states, is trivial.

It was noted in [Kau91], [Flo96], [Kau95] that one can make sense of augmented versions of these theories, if one allows fields corresponding to the entries in the boundary of the Kač table to be part of the theory. It turns out that one can still find a finite set of representations which close under fusion. In fact, one has to drop the condition of coprimarity of $p$ and $q$. The augmented Kač table is formally given by the table of $c_{6,3}$. However, there exist infinitely many Virasoro primary fields in this theory. Therefore, in the literature, opinions diverge whether the $c_{2,1}$ model should be termed 'minimal'. This is different from theories which do not have an empty Kač table, where fields from the border are required to decouple from the bulk of the Kač table. The enlargement necessarily renders these latter theories non-unitary.

Now, the enlarged Kač table of the $c_{2,1}$ model comprises five fields. One of these, the admissible representation for $h = -\frac{1}{8}$, (eq. (24)) corresponds to the twist field on hyperelliptic surfaces. It is customary to call non-bosonised twist fields $\mu$. These fields will be labeled interchangeably with the corresponding inverse ramification number $\frac{1}{n}$, or with their charges. Gurarie showed $\mu_{\frac{1}{2}}(z)$ to exhibit logarithmic divergencies in OPEs with itself [Gur93]. This is possible only if indecomposable representations are contained in the fusion product of the participating states. The enlargement of the Kač table therefore yields fields which are termed 'prelogarithmic', to express that they are primaries, but lead to indecomposable representations in their OPEs. Section II.2.2 will discuss these indecomposable representations.

It was elucidated by [Kau00], [Flo96], [Kau95] that the $c_{2,1}$ model describes ghosts on hyperelliptic surfaces. I will analyse this relation in section II.2.3.

### II.2.2 Indecomposable Representations

Let me now briefly explain, following [Gur93], why the presence of the twist field gives rise to indecomposable representations and thus, to a LCFT. Consider a correlator of four twist fields

$$\left\langle \mu_{\frac{1}{2}}(z_1)\mu_{\frac{1}{2}}(z_2)\mu_{\frac{1}{2}}(z_3)\mu_{\frac{1}{2}}(z_4)\right\rangle = \big((z_1-z_3)(z_2-z_4)x(1-x)\big)^{\frac{1}{4}}F(x), \tag{25}$$

with $x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$ being the anharmonic ratio. The fact that the representation corresponding to $\mu_{\frac{1}{2}}(z)$ possesses a null vector at level 2 obtained by applying $(L_{-2} - 2L_{-1}^2)$ translates into the differential equation

$$x(1-x)\frac{\mathrm{d}^2F(x)}{\mathrm{d}x^2} + (1-2x)\frac{\mathrm{d}F(x)}{\mathrm{d}x} - \frac{1}{4}F(x) = 0.$$

Because the indical equation of this hypergeometric differential equation has degenerate roots, both solutions have to have equal asymptotics. This leads to the conclusion that the two solutions have equal conformal weights. Nevertheless, they are not identical. Instead, one has a logarithmic divergency. The solutions $F$ and $G$ are given by

$$F(x) = C_1 G(x) + C_2 G(1-x) \tag{26}$$

$$\text{with } G(x) = \int_{\phi=0}^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - x\sin^2(\phi)}} = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x) \quad |x| < 1 \tag{27}$$

$$G(1-x) = G(x)\log(x) + H(x), \qquad G(x), H(x) \text{ regular at } x = 0. \tag{28}$$

By eq. (28) it is clear that the first solution becomes logarithmically divergent when approaching 1 from below, thus any solution will be logarithmically divergent somewhere on the Riemann sphere. The idea is to interpret this behaviour to be due to a new kind of field [Gur93].

These solutions have to be interpreted as the outcome of the OPEs of the fields involved times a suitable outstate. The conclusion is that the channel $G(z)$ has to imply $\mu_{\frac{1}{2}}(z)\mu_{\frac{1}{2}}(0) = z^{\frac{1}{4}}\mathbb{1}$, whereas the other channel, $G(z)\log(Z) + H(z)$, has to correspond to $\mu_{\frac{1}{2}}(z)\mu_{\frac{1}{2}}(0) = z^{\frac{1}{4}}\left(\mathbb{1}\log(z) + \tilde{\mathbb{1}}\right)$. From the transformation properties of primary fields (in particular, under rotations of $z$ by $\lambda$, implemented by conjugation with $e^{\lambda\mathcal{L}_0}$, of $\mu_{\frac{1}{2}}(z)\mu_{\frac{1}{2}}(0)$) it can be deduced that $\tilde{\mathbb{1}}$ cannot be the identity, but has to correspond to a state $|\tilde{0}\rangle$ which satisfies

$$L_0|\tilde{0}\rangle = |0\rangle \tag{29}$$

(As opposed to the action of the energy operator on the $\mathfrak{sl}(2,\mathbb{C})$ invariant vacuum $L_0\,|0\rangle = 0$.) Thus, the matrix representation of $L_0$ on the basic fields contains a Jordan cell spanned by the vectors $|0\rangle$ and $|\tilde{0}\rangle$ with eigenvalue zero. This is an example of an indecomposable representation. One can easily infer from this property that such non-trivial vacuum structure like that of the ghost systems is mandatory, if the vacuum resides in a non-trivial Jordan cell with respect to $L_0$, because of

$$0 = \left\langle L_0 0|\tilde{0}\right\rangle = \left\langle 0|L_0|\tilde{0}\right\rangle = \langle 0|0\rangle\,.$$

This new kind of field $\widetilde{\mathbb{1}}$ leads to unusual OPEs. One has to admit powers of logarithms as well

$$A(z)B(0) = z^{h_C - h_A - h_B}\sum_r \log^r(z)\,(C_{m-r} + \ldots) \text{ with } 0 \le r < m. \tag{30}$$

Here $m$ denotes the rank of the Jordan cell in which the $C_{m-r}$ reside. This result for the OPE was generalised to other LCFTs and fixed in its form in [Flo02]. In fact, associativity of the OPA and the generic form of the two- and three-point functions for an arbitrary rank LCFT lead to the restriction that the possible modification of the OPE can only yield powers of logarithms in addition to its standard form, independently of the precise form of the underlying indecomposable structure. If one assumes that rank-$m(h_i)$ Jordan cells are spanned by highest weight states $\Psi_{(h_i;0)}$ and their at least quasi-primary logarithmic partners $\Psi_{(h_i;k_i)}, \quad 0 < k_i < m(h_i)$, whereas states corresponding to prelogarithmic fields do not reside in Jordan cells, one can constrain the shape of logarithmic OPE's further. With the so-called zero mode content bounds on the set of indices over which to sum can be derived, and the prefactors can be fixed in terms of two- and three-point-functions. It is important to note that scale invariance is not broken by these modifications of the OPE, because the logarithms only appear dependent on scale invariant quantities, as the crossing ratios. An LCFT is not fixed by two-, three- and four-point-functions of primary fields only. In ordinary CFT, this was derived from the fact that any correlator of descendant fields can be rewritten in terms of correlators of their anchestors. But in an LCFT there exist fields which are neither primary nor descendant fields, usually, these are the log-partners to some primary fields. But in fact, LCFTs should be fixed by two, three and four-point-functions which involve the log-partners instead of their corresponding primaries.

A detailed exposition of Guraries analytic approach can also be found in [Gab03]. In this paper, Gaberdiel furthermore establishes a more formal algebraic approach to LCFT using the co-multiplication formula [GK96].

**A Further Ghost System at $c = -2$: Zamolodchikov Ghosts.** The most thoroughly investigated model is the one with highest central charge, $c = -2$, as it seems to have some additional features which simplify investigation. Besides the above mentioned pair of Grassmann fields with weight $(\lambda, 1 - \lambda)$, which seems to be the most natural choice for describing dense polymers and to bosonize the SUSY partners of the $c = -26$ ghosts, the $c = -2$ system naturally admits two further realisations by 'symplectic fermions'. The ghosts of these theories are obtained from the above-mentioned by field redefinitions such that both ghosts have equal conformal weights. Because the other realisations helped to get a deeper insight, I will give a short overview. The name symplectic stems from the fact that the OPA of the pairs of new fields can equally be described by an OPA of one field, the 'fermion', (a two-tuple consisting of the aforementioned fields) with a symplectic product. One such symplectic ghost system of weight $(0, 0)$ [Gur93] was invented, but not published by Zamolodchikov, I will refer to it as Zamolodchikov ghosts, or, more conventional $\theta\bar{\theta}$ system. The Zamolodchikov ghosts enjoy the mode expansion

$$\theta(z) := \partial^{-1}b(z) = \xi^+ + \theta_0^+ \log(z) + \sum_{n\neq 0} \theta_n^+ z^{-n}, \tag{31}$$

$$\bar{\theta}(z) := c(z) + \theta_0^- \log(z) = \xi^- + \theta_0^- \log(z) + \sum_{n\neq 0} \theta_n^- z^{-n}. \tag{32}$$

The action is simply

$$S = \int \mathrm{d}^2 z \partial\theta\partial\bar{\theta} \tag{33}$$

Therefore the Virasoro field reads

$$T_{\theta\bar{\theta}} = \partial\theta\partial\bar{\theta} = \sum_n \sum_l \frac{(n-l)}{l} \theta_l^+ \theta_{n-l}^- z^{-n-2}.$$

Note that the field $\bar{\theta}(z)$ is neither the chiral partner of $\theta(z)$ nor the complex conjugate of it. [7]

Comparison yields that the mode expansions of $T_{bc}$ and $T_{\theta\bar{\theta}}$ differ by the term $\theta_0^- \sum_n \frac{\theta_n^+}{n} z^{-n-2}$. This stems from the fact that $\partial\theta\partial\bar{\theta} = b\partial c + b\partial(\theta_0^- \log(z))$. Considering this system with $\mathbb{Z}_2$-twists, thus putting them on a hyperelliptic surface, allows for an identification of the new zero modes with the old ones from the other sheet and vice versa [Floa]. The calculation of this will not be given here, because it is a special case of the result in section III.1.

Great success has been made by describing the abelian sandpile within this model [Rue02],[MR01],[PR04a],[PR04b],[PR04c]. It describes the surface of a sandpile in a simplified discretised manner. The field $\mu$ is used to simulate borders.

Instead of the above mentioned procedure of formally integrating $b$ to $\theta$, one can put both fields on equal footing differentiating $c$ to a spin-1 field. One obtains a system with weights $(1, 1)$, which has been considered by Kausch [Kau00].

**A Deformation .** We have seen that certain conformal fields, namely the twist fields in the $c = -2$ ghost system, necessarily give rise to indecomposable representations. One may now ask the inverse question, whether one can consistently extend a CFT by indecomposable representations. This has first been considered by Fjelstad et al. in [FFH+02]. In [FFH+02] a deformation technique is developed by which the $\widehat{\mathfrak{sl}}(2, \mathbb{C})$ WZW models and the $c_{2,q}$ models, which include the simple ghosts or symplectic fermions, are enlarged to logarithmic CFTs. The space of states is enlarged by taking the tensor product with a finite dimensional vector space K. (See appendix for

[7] There are other naming conventions, e.g. barred quantities instead of the superscript '−', and the superscript '+' omitted.

details.) The Virasoro modes get improved by an operator $\beta \in \mathrm{End}_+\mathrm{K}$, which is the subset of annihilators of a chosen vacuum of K, times a recognised mode. As an application they examined, among others, the simple ghost system. Their ansatz amounts to a deformation of the energy-momentum tensor by a deformation field

$$U(z) = \beta b(z). \tag{34}$$

In this case, the annihilation zero mode $b_0$ corresponds to the fermionic screening, which they call $\oint E$. The space of states gets enlarged by the creators with respect to their chosen vacuum, $\alpha \in \mathrm{End}_-\mathrm{K}$ of this new auxiliary vector space. Fields corresponding to states which are products of $c$-modes and the conjugate $\alpha \in \mathrm{End}_-\mathrm{K}$ will span Jordan cells with respect to $(L_0 + U_0)$. This construction for the special case of simple ghosts looks like the extra term in $T_{\theta\bar{\theta}}$ after identification of $\beta = \theta_0^-$. There, the logarithmic field $\widetilde{\mathbb{1}}$ corresponds to $\widetilde{\mathbb{1}} = \left|:\theta\bar{\theta}:\right\rangle$.

Consult [Flo03],[Gab03],[Kaw03],[MARS03],[RT03], also published in [FlR03], and references therein for a broader review.

### II.2.3 The Relation of the $c = -2$ LCFT to the Ghost Systems on a Ramified Covering

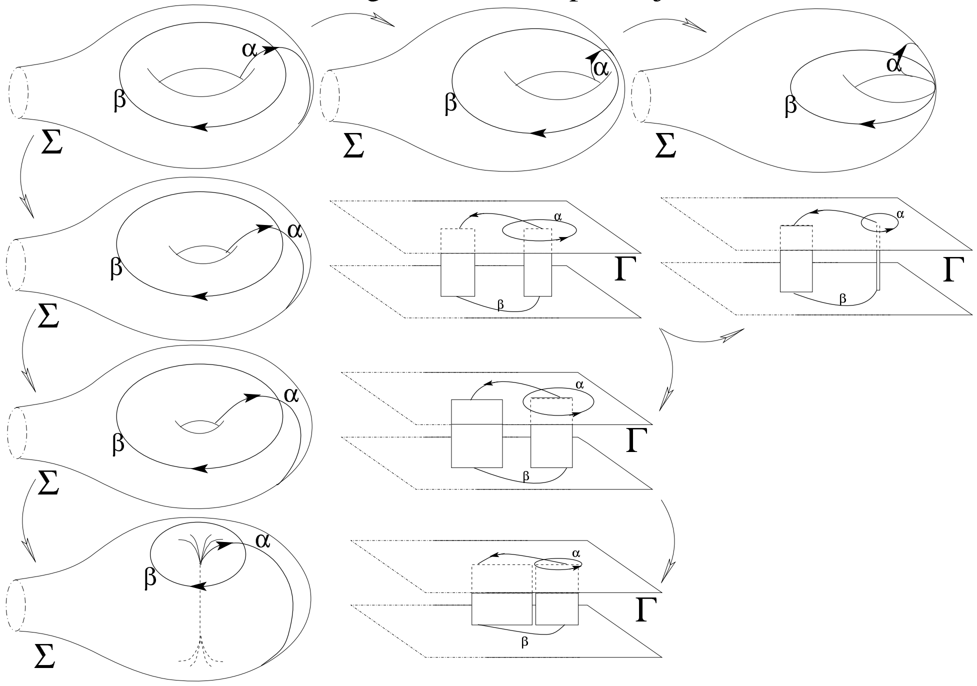
It is now apparent that the field $\mu$ with $h_\mu = \frac{1}{8}$, which in Knizhnik's approach creates a $\mathbb{Z}_2$ branch point, gives rise to a major modification of the CFT: Some of the modules become indecomposable, and, along with that, logarithmic divergencies arise in the correlators. Knizhnik did not notice the indecomposable structures directly, because though he considered them to be ordinary fields, he fixed their points of insertions in order to describe the geometry of distinct non-singular Riemann surfaces. Therefore, his demand on the complex curve to be nonsingular, i.e.

$$\Gamma = \left\{ (y, z) : \ y^n(z) = \prod_{i=1}^{M} (z - a_i), \quad M = mn \right\},$$

was perfectly admissible. This picture changes if the twist fields are allowed to propagate. The interpretation of two branch points coming close to each other, as required in the OPE of such, is that the represented Riemann surface leaves the class of surfaces of genus $\frac{N-2}{2}$ and $r$ punctures[8] to reside in the class of surfaces of genus $\frac{N-4}{2}$ and $r+1$ punctures. When branch points run into each other, the homology cycles between them get 'pinched'. Figure 2 is to be understood as a somewhat handwaving motivation why one expects something to happen if branch points join. It is not meant as an explanation or precise description of what happens.

[8] I try to stick to the convention to address distinguished points by marked points, whereas an insertion of a puncture alters the topology.

Figure 2: A sketch of what happens with a hyperelliptic Riemann surface, represented as a ramified double covering, when branch points join




It is supposed to show a 'handle' of a hyperelliptic Riemann surface Σ, which is also represented as a branched covering Γ, and the homology cycles attached to it. For the sake of simplicity, I sketch the process of 'pinching' of the latter, which happens if the OPE of twist fields is inserted. Two striking inconsistencies emerge from the results of Gurarie and Knizhnik.

1. If one calculates the energy-momentum tensor by variation of eq. (11) with respect to the metric, in the so-called twist field formalism, we find a perfectly diagonalisable action of the Virasoro modes on the space of states, which is in contradiction to the results of Gurarie. If twists are present in the theory, a modification of the Virasoro field is therefore needed to unveil the logarithmic structure. As a consequence, eq. (11) had to be modified to yield the modified Virasoro field.

2. Knizhnik's bosonisation formulæ do not mirror the facts proved by Gurarie. By calculating the OPE of two of the bosonised twist fields, one immediately sees that they, being constructed from ordinary free-field exponentials, do *not* lead to logarithms nor logarithmic partner fields

$$(z-z')^{-\boldsymbol{q}\boldsymbol{q}^*} V_{\boldsymbol{q}}(z) V_{\boldsymbol{q}^*}(z') = \frac{\mathbb{1}}{z-z'} + j(z) + T(z)(z-z') + reg(z-z'). \tag{35}$$

This requires a redefinition of the bosonisation formulæ.

This thesis is concerned with the resolution of the first inconsistency. The second inconsistency is far more difficult to tackle, since there is no mathematically well-defined bosonisation scheme of the (prelogarithmic) twist field. First attempts in this direction have been made via so-called puncture operators, see [KL98].

**Some further remarks on $\mathbb{Z}_n$ twisted theories.** In [Kau95, GK99, Kau00] it was shown that for all twists the $\mathcal{W}(2,3)$ algebra is contained in the extended symmetry algebra. However, not much is known about these extended symmetries. There are indications that the affine Lie algebra for $\mathbb{Z}_n$ symmetry is $\widehat{\mathfrak{su}(n)}$, i.e. there is an entry in the corresponding Kač table for a degenerate $c = -2$ model [Floa], which has suitable conformal weight to describe the $\mathbb{Z}_n$-twist. This would render the symmetry algebra an extension of a $\mathcal{W}(A_{n-1})$-algebra. In [Wan98b, Wan98a] the modules of $\mathcal{W}(2,3)$ were classified. There are no null vectors which would yield differential equations allowing to do a similar reasoning as Gurarie did. However, in [Kau00] a co-multiplication formula of [GK96] was exploited to derive a differential equation on twist-correlators in the $(1,1)$-symplectic fermion model with $\mathcal{W}(2,3)$ as symmetry algebra.

### II.2.4 Putting Things Together

Fjelstad et al. showed in [FFH+02] that with help of a certain operator, acting on the tensor product of the Hilbert space with an auxiliary vector space a logarithmic deformation operator can be obtained. They invented a tool to enlarge CFT's to LCFT's, i.e. irreducible representations to reducible but indecomposable representations at the cost of postulating new states in the spectra.

The required artificial enlargement of the space of states by an auxiliary vector space may seem out of place in applications. New states in the spectrum would seem somewhat ad hoc - one is forced to introduce logarithmic fields, but does not know, where the auxiliary states for the deformation should come from. Of course, Gurarie also enlarged the field content of the theory, but for a different reason: This was to maintain consistency of the operator products with his results on the shape of certain four point functions.

**Renaturation** The problem with the auxiliary vector space introduced in [FFH+02] is that it obstructs a geometric interpretation. This would be resolved if one solely applied recognised modes onto the vacuum to build the space of states. Under certain circumstances, this is indeed possible. If the theory is set on a $\mathbb{Z}_n$ symmetric ramified covering on the Riemann sphere, modes at one's disposal suffice. On any sheet, one can allow for all words in $c$-zero-modes from all other sheets to be applied on the original states, and then take the direct sum. The space of ground spaces, as a whole, is not altered at all, because zero modes have to be included in correlators in any case, because of the charge anomaly. On distinct sheets, one can introduce deformation fields in a fashion similar to that of [FFH+02], and investigate whether the space of states decomposes into indecomposable representations of the sum of the Virasoro and the deformation field. To see that this is a more natural construction, one should remember that the ghost systems do perfectly well without logarithms, in case no twist fields are present. But if they are, they introduce branch cuts – at least at the level of correlators, as discussed above. That is, the other fields of the theory naturally live on branched coverings of a Riemann sphere or plane. But then, Knizhnik's procedure allows – at least nearly – to diagonalise the monodromy, such that one ends up with $n$ copies of CFTs on $n$ sheets, such that the total theory is a tensor product of the theories on the individual sheets. Actually, these theories turn out not to be completely independent. In fact, the OPE of two branch points gives rise to logarithmic fields, for which corresponding states have to be constructed. The only consistent way to introduce these additional states without resorting to artificial add-ons is to let the zero modes 'shine through'. This thesis will make this idea explicit. However, it is worth noting that the $c = -2$ Zamolodchikov ghosts fit in this concept, since each of the $\theta$ fields contains an additional zero mode compared to the $bc$ system. As stated above, the energy-momentum tensor $T_{\theta\bar{\theta}}$ associated to these fields comprises an extra term, which looks like the deformation in [FFH+02]. To summarise: The existence of twist fields, which are used to simulate branch points, together with the properties of a well-defined OPA, imply the existence of indecomposable structures. The necessary enlargement of the space of states will be achieved, in contrast to [FFH+02], with use of additional modes, which are already present. Namely, I will use only (products of) modes of the other sheets of a ramified covering, without the need for an artificial construction.

## II.3 Some Questions Connected with $\lambda > 1$

In section III.1 et sqq. we describe new (L)CFTs connected with ghost systems in the presence of higher twists. Here, we want to display our original motivation to investigate. This paragraph can be omitted on first reading.

It seems natural to ask whether other ghost systems exhibit logarithmic divergencies and indecomposable structures as well. There are striking differences between models with $\lambda = 1$ and $\lambda > 1$, but also a lot of similarities.

To my knowledge there is no direct proof of other ghost systems being logarithmic until now. One sees by inspection of the respective formulæ of the central charges for the minimal models and the ghost systems that for $\lambda > 1$ ghost systems cannot be minimal, but at most rational, because they do not fit in the allowed set of $c_{p,q}$ values. Those ghost systems can be parametrized by [GT89] $c^j = 1 + 3\varepsilon Q^2$ with $\varepsilon = (-)^F$

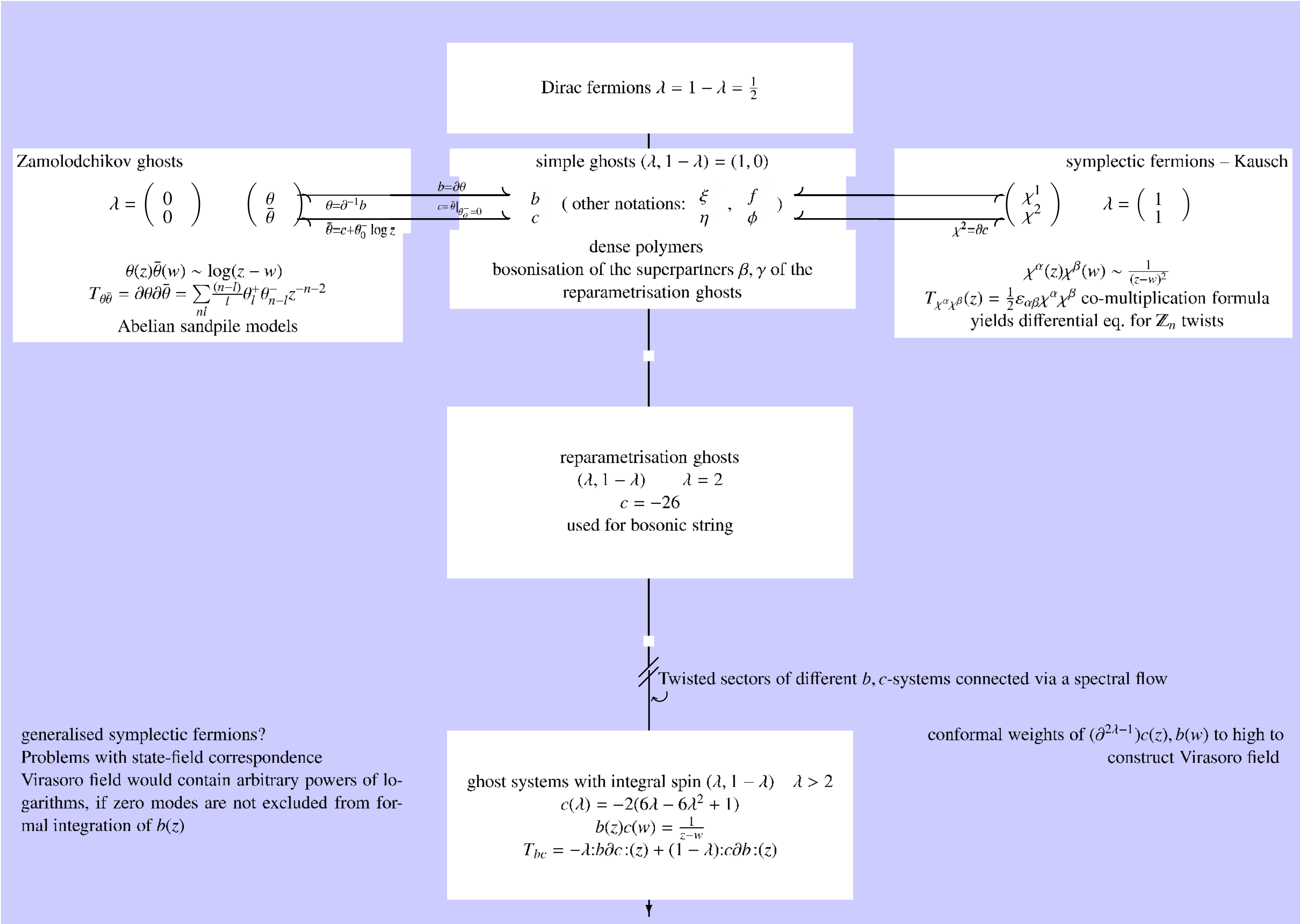
being the eigenvalue of the Witten index, and $Q$ the (integral) background charge at infinity. Virasoro minimal models are parametrized by $c_{p,q} = 13 - 6(\frac{p}{q} + \frac{q}{p})$ with integer, coprime $q, p$. For this reason the ghost systems may not be regarded as such, except for $p = q, q = 2p$ or $p = 2q$ (and $Q = 1$ or 0), if one drops the condition of coprimarity.

In [GT89] it is shown that all central charges and conformal weights correspond to rational numbers, which seems to be characteristic of rational models. However, this does not prove that with respect to a the Virasoro algebra alone, there are only finitely many primaries. To get the rationality of these structures you might have to invoke the full maximal extended symmetry algebra. It is not refered to the maximally extended algebra, and even the simplest case, $c = -2$, is not rational with respect to the Virasoro algebra alone, but only quasirational, i.e. it has a *countably infinite* set of irreducible representations (and possibly indecomposable representations). In contrast to the reasoning in [GT89] all *Virasoro algebras* realised by ghost systems fail to meet the condition [FSQ86, FQS84, BPZ84, Flob] that the effective central charge must be strictly less than unity rational models, which are rational also with respect to the Virasoro algebra alone. On the contrary, it is easy to see that the effective central charge, $c_{eff} = c - 24 \cdot h_{min}$, is exactly unity for all ghost systems.

Thus, rationality could only be achieved with respect to an extended symmetry algebra. These extended symmetry algebras are not known, and it is a conjecture that if the set of all conformal weights of a model is a subset of $\mathbb{Q}$ this implies that this theory has only finitely many respresentations. In particular, certain null vectors exist only with respect to the maximally extended algebra, but not with respect to the Virasoro algebra alone. But the maximally extended algebras of the ghost systems with $\lambda > 1$ are not known. Therefore even the lowest null vectors are unknown and so is the classification of the representations of the $\lambda > 1$ ghost systems.

However, there are some facts that indicate indirectly that indecomposable structures do exist in these models: Most strikingly, the different ghost systems are connected via a spectral flow, which is indicated by the fact that the partition function of different twisted sectors depends, up to a prefactor $z^{\alpha}$, only on the sum of the conformal spin $\lambda$ of the antighost $b$ and and the twist $\alpha$. See [EFH98] for further information on that topic.

One particular example has been investigated by Krohn and Flohr [KF03]. Via formal integrations and field redefinitions, reparametrisation ghosts in the presence of $\mathbb{Z}_2$-twists, are turned into 'generalised' symplectic fermions which resemble the Zamolodchikow ghosts. But their results seem not to work for other twists, and the calculations have to be redone separately for any model with higher spin. The approach of this thesis is somewhat different, because generalised symplectic fermions require $(2\lambda - 1)$ integrations and therefore the mode expansion of the stress-energy tensor becomes logarithmic as well. One way to circumvent this would be to try to exclude the zero modes from integration and afterwards add them with the right powers of $z$. This could cause problems with the field-state-correspondence, or at least look unnatural. Rather, in this thesis, I directly deform $T$ without the attempt to interpret this as a consequence of a redefinition of the basic fields. The investigation will be performed by pure algebraic means, no use is made of the properties of the OPE. Unlike the modus operandi in [KF03] I will not try to express the modified energy-momentum tensor by deformed fields. To obtain a logarithmic deformation, $\beta$ has to be nilpotent. I am thus interested in a realisation of $\beta$ by either known nilpotent (finite sums of) operators or nilpotent fields obtained by (concatenated) normal ordering procedures or differentation of the so-called basic fields. These fields should satisfy the field-state-correspondence. I conclude this section with a summary of the different realisations of the ghost systems. This might prove useful to get a better overview and to relate my general results to the simple and well-known example $c = -2$.



Dirac fermions $\lambda = 1-\lambda = \frac{1}{2}$
Zamolodchikov ghosts
$\lambda = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} \theta \\ \bar\theta \end{pmatrix}$
$\theta = \partial^{-1} b$
$\bar\theta = c + \theta_0^- \log z$
$b = \partial\theta$
$c = \bar\theta|_{\theta_0^- = 0}$
$\theta(z)\bar\theta(w) \sim \log(z-w)$
$T_{\theta\bar\theta} = \partial\theta\partial\bar\theta = \sum_{nl} \frac{(n-l)}{l}\theta_l^+\theta_{n-l}^- z^{-n-2}$
Abelian sandpile models
simple ghosts $(\lambda, 1-\lambda) = (1,0)$
$b$, $c$ ( other notations: $\xi$, $\eta$ , $f$, $\phi$ )
dense polymers
bosonisation of the superpartners $\beta, \gamma$ of the reparametrisation ghosts
symplectic fermions – Kausch
$\begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix}$ $\lambda = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\chi^2 = \partial c$
$\chi^\alpha(z)\chi^\beta(w) \sim \frac{1}{(z-w)^2}$
$T_{\chi^\alpha\chi^\beta}(z) = \frac{1}{2}\varepsilon_{\alpha\beta}\chi^\alpha\chi^\beta$ co-multiplication formula yields differential eq. for $\mathbb{Z}_n$ twists
reparametrisation ghosts
$(\lambda, 1-\lambda)$ $\lambda = 2$
$c = -26$
used for bosonic string
Twisted sectors of different $b, c$-systems connected via a spectral flow
generalised symplectic fermions?
Problems with state-field correspondence
Virasoro field would contain arbitrary powers of logarithms, if zero modes are not excluded from formal integration of $b(z)$
ghost systems with integral spin $(\lambda, 1-\lambda)$ $\lambda > 2$
$c(\lambda) = -2(6\lambda - 6\lambda^2 + 1)$
$b(z)c(w) = \frac{1}{z-w}$
$T_{bc} = -\lambda{:}b\partial c{:}(z) + (1-\lambda){:}c\partial b{:}(z)$
conformal weights of $(\partial^{2\lambda-1})c(z), b(w)$ to high to construct Virasoro field

# III Simple ghosts and arbitrary twists

## III.1 Possible Deformations for $\mathbb{Z}_n$ Symmetric Surfaces

In the last sections I reviewed how simple ghosts on a hyperelliptic surface can be made a logarithmic theory. I also stated that the additional zero modes can be chosen to be 'primal' modes from the other sheet. This section is devoted to survey if and how this identification can be generalised for the $\mathbb{Z}_n$ symmetric case. To that end, I derive constraints on recognised states which render the auxiliary vector space dispensable. For the beginning, I will stick to an ansatz similar to the one proposed in [FFH+02] for $c = -2$, as discussed in II.2.2, to check whether the original Hilbert space suffices for $\mathbb{Z}_n$ symmetric Riemann surfaces as well. As done in [FFH+02],[KF03] I will add an 'improvement term' $U^{(k)}(z)$ to the Virasoro field $T^{(k)}(z)$ (16) on the individual sheets. The modes of this improvement term are denoted by [9]

$$U_m^{(k)} := :\beta^{(k)} b_m^{(k)}:$$

$\widetilde{L}_m^{(k)} = L_m^{(k)} + U_m^{(k)}$ refers to the new Virasoro modes.

Besides two mixed terms, a commutator of two 'improved' modes leads to a commutator of two deformation terms, which has to vanish in most cases, because the action of the deformation field should be nilpotent, and a commutator of Virasoro modes. Recall that the latter yields the famous Virasoro algebra

$$\left[L_q^{(k)}, L_m^{(r)}\right] = \left((q-m)L_{q+m}^{(k)} + \frac{c}{2}\binom{q+1}{3}\delta_{q+m,0}\right)\delta_{k,r}.$$

In the following, I investigate for which choices of $\beta$ this algebraic structure remains

[9] In the following, fermionic normal ordering is implicitly understood for the Virasoro field and its deformation. If $A, B, C, D$ are fermionic $[AB, CD] = A\{B, C\}D - \{C, A\}BD + CA\{B, D\} - C\{D, A\}B$. Interchanging $C$ and $D$ gives the same result with an overall minus sign, so normal ordering can be neglected inside commutators *up to normal ordering constants*. (The same holds, of course, for $A$ and $B$ because of the antisymmetry of the commutator). Another way to view this is to recall that a normal ordered product of *two* operators differs from the formal product only by certain commutators of modes. If both operators are contained in an algebra which only allows for central terms, this merely yields c-numbers. A commutator of such formal products is the same as a commutator of the normal ordered products plus central terms. Therefore, normal ordering is irrelevant for the derivation of the constraints – under my assumptions, it could in the worst case lead to constants, which have to vanish.

the same.

$$
\begin{aligned}
&[L_q^{(k)} + U_q^{(k)}, L_m^{(r)} + U_m^{(r)}] = [\widetilde{L}_q^{(k)}, \widetilde{L}_m^{(r)}] \\
&\overset{!}{=} ((q-m)(L_{q+m}^{(k)} + U_{q+m}^{(k)}) + \frac{c}{2}\binom{q+1}{3}\delta_{q+m,0})\delta_{rk} \\
&= [L_q^{(k)}, L_m^{(r)}] \\
&+\sum_l (q-l)\left(b_l^{(k)}\left\{c_{q-l}^{(k)},\beta^{(r)}\right\}b_m^{(r)} - \left\{\beta^{(r)}, b_l^{(k)}\right\}c_{q-l}^{(k)}b_m^{(r)} + \beta^{(r)}b_l^{(k)}\delta_{q+m,l}\delta_{k,r}\right) && \Big\}\,[L_q^{(k)}, U_m^{(r)}] \\
&-\sum_t (m-t)\left(b_t^{(r)}\left\{c_{m-t}^{(r)},\beta^{(k)}\right\}b_q^{(k)} - \left\{\beta^{(k)}, b_t^{(r)}\right\}c_{m-t}^{(r)}b_q^{(k)} + \beta^{(k)}b_t^{(r)}\delta_{m+q,t}\delta_{r,k}\right) && \Big\}\,[L_m^{(r)}, U_q^{(k)}] \\
&+\beta^{(r)}\left\{b_m^{(r)},\beta^{(k)}\right\}b_q^{(k)} - \left\{\beta^{(k)},\beta^{(r)}\right\}b_m^{(r)}b_q^{(k)} - \beta^{(k)}\left\{b_q^{(k)},\beta^{(r)}\right\}b_m^{(r)} && \Big\}\,[U_q^{(k)}, U_m^{(r)}]
\end{aligned}
$$

With abbreviations

$$
\begin{aligned}
\Gamma_{r,k} &:= \left\{\beta^{(r)},\beta^{(k)}\right\} \\
\gamma_{r,l|k} &:= \left\{c_l^{(k)},\beta^{(r)}\right\} \\
\alpha_{r,q|k} &:= \left\{b_q^{(r)},\beta^{(k)}\right\}
\end{aligned}
$$

the condition reads

$$
\begin{aligned}
(q-m)U_{q+m}^{(r)}\delta_{rk} &= \sum_l (q-l)\left(-\alpha_{k,l|r}c_{q-l}^{(k)}b_m^{(r)} - b_m^{(r)}\gamma_{k,q-l|r}b_l^{(k)} + \beta^{(r)}b_l^{(k)}\delta_{r,k}\delta_{m+q,l}\right) \\
&-\sum_t (m-t)\left(-\alpha_{r,t|k}c_{m-t}^{(r)}b_q^{(k)} - b_q^{(k)}\gamma_{r,m-t|k}b_t^{(r)} + \beta^{(k)}b_t^{(r)}\delta_{k,r}\delta_{m+q,t}\right) \\
&+\beta^{(r)}\alpha_{r,m|k}b_q^{(k)} - \Gamma_{k,r}b_m^{(r)}b_q^{(k)} - \beta^{(k)}\alpha_{k,q|r}b_m^{(r)}. \qquad (36)
\end{aligned}
$$

At this step I have made the assumption that $\beta$ is not a product of modes, or at least, not a product that contains $c$-modes. For *any* $\beta$ that satisfies the following equations for all $m, q \in \mathbb{Z}$,

$$\sum_l \left((m-l)\gamma_{r,m-l|k}b_q^{(k)}b_l^{(r)} + (q-l)\gamma_{k,q-l|r}b_l^{(k)}b_m^{(r)}\right) = \Gamma_{k,r}b_m^{(r)}b_q^{(k)}, \qquad (37)$$

$$\sum_l \left((m-l)\alpha_{r,l|k}c_{m-l}^{(r)}b_q^{(k)} - (q-l)\alpha_{k,l|r}c_{q-l}^{(k)}b_m^{(r)}\right) = 0, \qquad (38)$$

$$\beta^{(r)}\alpha_{r,m|k}b_q^{(k)} - \beta^{(k)}\alpha_{k,q|r}b_m^{(r)} = 0, \qquad (39)$$

the modified Virasoro modes still generate a Virasoro algebra:

$$\sum_l \left((q-l)\beta^{(r)}b_l^{(k)}\delta_{rk}\delta_{m+q,l} - (m-l)\beta^{(k)}b_l^{(r)}\delta_{kr}\delta_{q+m,l}\right) = (m-q)\beta^{(r)}b_{m+q}^{(r)}\delta_{rk}. \qquad (40)$$

Unless $l = 0$, eq. (37) involves a vanishing $\gamma_{r,l|k}$, because each term contributes a unique mode content. If, on the other hand, $l = 0$, all terms have the same mode content. In this case, evidently, the prefactor vanishes as well, which implies that $\Gamma_{k,r}$ is zero. Therefore $\beta$ could be a linear combination of zero modes from different sheets. Eq.s (38) and (39) imply that all individual terms have to vanish separately due to their diverse mode-content. Only $b_0$-modes can contribute to a linear combination as an ansatz for $\beta$. Of course, monomials in $b_0$-modes from different sheets are also possible. These will be scrutinized further in section IV.1. With such a linear ansatz, the rhs of eq. (37) vanishes due to the fact that all $b_0^{(k)}$'s anticommute with each other. Also, the lhs vanishes because of the coefficient being proportional to the second entry of $\gamma_{k,t|r}$ which is proportional to $\delta_{t,0}$. Eq.s (38), (39) are fulfilled trivially without further constaints. Eq. (40) is fulfilled independently of any ansatz for $\beta$.

## III.2 The Action of the Deformed Virasoro Modes on the State Space

The action of the new Virasoro modes on the various ground states may result in additional conditions if the matrix representation of $\widetilde{L}_0^{tot}$ is to contain Jordan cells. I denote the ground states as $\prod_{k=0}^{n-1}(c_0^{(k)})^{N_k}|0\rangle$ with $N_k \in \{0, 1\}$. The ordering prescription is that $c^{(k)}$ resides on the left of $c^{(l)}$ if $0 \leq k < l \leq n-1$. The preceeding consideration showed that $\beta^{(k)}$ has to meet the same commutation relation as a $b$-mode from another sheet. I therefore impose a linear combination of those as an ansatz for $\beta^{(k)}$

$$\beta^{(k)} = \sum_{s=0}^{n-1} M_{sk} b_0^{(s)}.$$

With this, the deformation $U_m^{(k)} = \beta^{(k)} b_m^{(k)}$ acts as

$$\begin{aligned} U_0^{(k)} \prod_{l=0}^{n-1}(c_0^{(l)})^{N_l}|0\rangle &= \sum_{s=0}^{n-1} M_{sk} b_0^{(s)} b_0^{(k)} \prod_{l=0}^{n-1}(c_0^{(l)})^{N_l}|0\rangle \\ &= -\sum_{s=k+1}^{n-1} M_{sk} (-)^{\sum_{l'=k}^{s} N_{l'}} \delta_{N_k,1}\delta_{N_s,1} \prod_{\substack{l=0 \\ k\neq l\neq s}} (c_0^{(l)})^{N_l}|0\rangle \\ &\quad + \sum_{s=0}^{k} M_{sk} (-)^{\sum_{l'=k}^{s} N_{l'}} \delta_{N_k,1}\delta_{N_s,1} \prod_{\substack{l=0 \\ k\neq l\neq s}} (c_0^{(l)})^{N_l}|0\rangle \end{aligned} \tag{41}$$

on arbitrary states matching the afore mentioned conventions. In the last step I 're-arranged' signs by raising the summation index in the exponential to avoid a space-consuming case discrimination. The summation is understood to give the same result if upper and lower bound are interchanged. Demanding the existence of Jordan cells also for $\widetilde{L}_0^{tot}$ yields

$$0 \neq U_0^{tot} \prod_{l=0}^{n-1} (c_0^{(l)})^{N_l} |0\rangle = -\sum_{k=0}^{n-1} \sum_{s=k+1}^{n-1} M_{sk} (-)^{\sum_{l'=k}^{s} N_{l'}} \delta_{N_k,1} \delta_{N_s,1} \prod_{\substack{l=0 \\ k \neq l \neq s}} (c_0^{(l)})^{N_l} |0\rangle \quad (42)$$
$$+ \sum_{k=0}^{n-1} \sum_{s=0}^{k} M_{sk} (-)^{\sum_{l'=s}^{k} N_{l'}} \delta_{N_s,1} \delta_{N_k,1} \prod_{\substack{l=0 \\ s \neq l \neq k}} (c_0^{(l)})^{N_l} |0\rangle .$$

By relabeling of dummy indices in the second term one finds

$$0 \neq -\sum_{\substack{k=0 \\ s=k+1}}^{n-1} M_{sk} (-)^{\sum_{l'=k}^{s} N_{l'}} \delta_{N_k,1} \delta_{N_s,1} \prod_{\substack{l=0 \\ k \neq l \neq s}} (c_0^{(l)})^{N_l} |0\rangle \quad (43)$$
$$+ \sum_{\substack{k=0 \\ s=k+1}}^{n-1} M_{ks} (-)^{\sum_{l'=k}^{s} N_{l'}} \delta_{N_k,1} \delta_{N_s,1} \prod_{\substack{l=0 \\ k \neq l \neq s}} (c_0^{(l)})^{N_l} |0\rangle$$

and finally

$$0 \neq \sum_{\substack{k=0 \\ s=k+1}}^{n-1} (M_{ks} - M_{sk}) (-)^{\sum_{l'=k}^{s} N_{l'}} \delta_{N_k,1} \delta_{N_s,1} \prod_{\substack{l=0 \\ k \neq l \neq s}} (c_0^{(l)})^{N_l} |0\rangle .$$

Thus, if one denotes $\mathcal{P}_{rs} := \text{span}\{|\psi\rangle = \prod_{l=0}^{n-1} (c_0^{(l)})^{N_l} |0\rangle : N_k,\ N_s \neq 0\}$ it follows that

$$\boxed{M_{sk} \neq M_{ks} \ \Rightarrow \widetilde{L}_0^{tot} |\psi\rangle \notin \mathcal{P}_{rs}.} \quad (44)$$

This assures rank-two Jordan cells in the action of suitable fields, regardless of the number of sheets. All groundstates are highest weight states with respect to the deformed Virasoro modes, because only $c_0$-modes contribute. Therefore $\widetilde{L}_n^{(s)} \prod_{l=0}^{n-1} (c_0^{(l)})^{N_l} |0\rangle = 0$ for all $n > 0$. This shows that the extension of the energy-momentum tensor leads to logarithmic divergencies in OPEs. The results above are in perfect concordance with the results of [Kau00]. Considering weight $(1, 1)$ symplectic fermions Kausch could prove that for all ramification numbers, the functional dependence of the four-point function $\left\langle \mu_{\frac{1}{r}}(\infty) \mu_{\frac{1}{s}}(1) \mu_{\frac{1}{t}}(x) \mu_{\frac{1}{u}}(0) \right\rangle$ on the crossing ratio aquires at most one logarithm as a factor. This exactly happens if either $\frac{1}{r} + \frac{1}{s} \in \mathbb{N}$ or $\frac{1}{r} + \frac{1}{t} \in \mathbb{N}$ or $\frac{1}{r} + \frac{1}{u} \in \mathbb{N}$, the other sums are then fixed by charge balance. This just expresses the fact that those twists are dual to each other.

# IV Deformations for $\lambda > 1$

Elated by the possibility to realise a logarithmic extension of the simple ghost system on $\mathbb{Z}_n$ symmetric Riemann surfaces, one could try a similar procedure to see if other ghost system CFT's representations can be enlarged alike.

Apart from the differences between models with $\lambda = 1$ and $\lambda > 1$, a comment on the word "alike" is in order: Fjelstad et al. remark that only if their operator "$E$" is a primary screening current, no logarithms show up in the deformed fields. This, in particular, should be the only possibility for the deformed Virasoro field not to contain logarithms, i. e. to have a well-defined Laurent expansion. (An attempt to deform $T$ with $R(w)\log(w)$ can only be consistent with the property of $T$ to be a primary field – except for the conformal anomaly – if $R$ contains logarithms as well. A consistent construction, if possible, looks unnatural and therefore will not be pursued in the present paper). Fjelstad et al. corroborate that their construction works with nilpotent bosonic operators as well. Despite that, they consider it for their purposes appropriate to restrict to fermionic screenings. (On that account, they had to restrict their examination to the $c_{p,1}$- and the $c_{2,q}$-models, $q$ odd.)

For the simple ghost system, the $b$-field may serve as the primary fermionic screening current and permits the construction discussed above. But in the other models no fermionic primary screening current is available, because of $b$ and $c$ being $\lambda$- and $(1-\lambda)$-differentials, respectively. $N$-fold integration and division by powers of $z$ do not only lower the conformal weight, they also convert the $\lambda$-differentials into non-primary fields. However, the $c = -2(6\lambda^2 - 6\lambda + 1)$, $\lambda > 1$ models lack primary screening currents at all: For $\lambda > -\frac{1}{2}$ the current $j(w) = :bc:(w) \equiv :\mathscr{b}\mathscr{c}:(w)$ is anomalous

$$T(z)j(w) \sim \frac{1-2\lambda}{(z-w)^3} + \frac{1}{(z-w)^2}j(w) + \frac{1}{z-w}(\partial j)(w).$$

An attempt to deform $T$ by modes of the original space of states without use of logarithms is *not* along the lines of [FFH+02]. Of course, one could look for a more general setting allowing for a deformation without logarithms (in $T$) and auxiliary states. Instead, the important point is that it is not known whether ghost systems with $j > 1$ are genuinely subtheories of logarithmic ones. I will examine ghost systems with $\lambda > 1$ the way I did for $c = -2$, believing they admit deformations of $T$ which act constistenly on the space of states, because the setup is similar to that of the case before.

## IV.1 Derivation of the Constraints

Below, I investigate if a more general deformation of the Virasoro modes will leave the commutation relations unchanged. Proceeding as in section III.1, I use the following ansatz

$$U_y^{(s)} := \sum_{l\in\mathbb{Z}} P(\lambda, y, l, s)\beta_l^{(s)} b_{y-l}^{(s)}$$

which results in the requirement

$$(x-y)U_{x+y}^{(s)} := (x-y)\sum_l P(\lambda, x+y, l, s)\beta_l^{(s)} b_{x+y-l}^{(s)}\delta_{r,s} \tag{45}$$

$$\begin{aligned}
&= \sum_k (\lambda x - k) b_k^{(r)} \left\{ c_{x-k}^{(r)}, \sum_l P(\lambda, y, l, s)\beta_l^{(s)} \right\} b_{y-l}^{(s)} && \clubsuit\\
&\quad - \left\{ \sum_l P(\lambda, y, l, s)\beta_l^{(s)}, \sum_k (\lambda x - k) b_k^{(r)} \right\} c_{x-k}^{(r)} b_{y-l}^{(s)} && \diamondsuit\\
&\quad + \sum_l P(\lambda, y, l, s)\beta_l^{(s)} (\lambda x - x + l - y) b_{y+x-l}^{(r)} \delta_{r,s} && \spadesuit\\
&- \Bigg( \sum_k (\lambda y - k) b_k^{(s)} \left\{ c_{y-k}^{(s)}, \sum_m P(\lambda, x, m, r)\beta_m^{(r)} \right\} b_{x-m}^{(r)} && \clubsuit\\
&\quad - \left\{ \sum_m P(\lambda, x, m, r)\beta_m^{(r)}, \sum_k (\lambda y - k) b_k^{(s)} \right\} c_{y-k}^{(s)} b_{x-m}^{(r)} && \diamondsuit\\
&\quad + \sum_m P(\lambda, x, m, r)\beta_m^{(r)} (\lambda y - y + m - x) b_{x+y-m}^{(r)} \delta_{r,s} \Bigg) && \spadesuit\\
&+ \Bigg( \sum_m \sum_l P(\lambda, x, m, r) P(\lambda, y, l, s) \beta_m^{(r)} \left\{ b_{x-m}^{(r)}, \beta_l^{(s)} \right\} b_{y-l}^{(s)} && \heartsuit\\
&\quad - \sum_l \sum_m P(\lambda, y, l, s) P(\lambda, x, m, r) \left\{ \beta_l^{(s)}, \beta_m^{(r)} \right\} b_{x-m}^{(r)} b_{y-l}^{(s)} && \clubsuit\\
&- \sum_l \sum_m P(\lambda, y, l, s) P(\lambda, x, m, r) \beta_l^{(s)} \left\{ b_{y-l}^{(s)}, \beta_m^{(r)} \right\} b_{x-m}^{(r)} \Bigg). && \heartsuit
\end{aligned}$$

One obviously has to collect terms with the same total mode-content, i.e. the total number of modes in products, first. The total mode content is the same for commutators of two deformation modes and commutators of Virasoro modes with deformation modes, only if the ansatz is a linear combination. Then terms in eq. (45) denoted by different suits contain different numbers of $b$'s and $c$'s (because the commutators give c-numbers), and therefore can be treated separately. All graded commutators of pure products in $b$-modes obviously vanish.[10] Furthermore, each proper polynomial ansatz

[10]Constant central terms are excluded, because these would alter the central charge – but the central charges of ghost systems are fixed to $c = -2(6\lambda^2 - 6\lambda + 1)$ by the properties of the OPE. For deformations,

can be treatet degree-wise, because only normal ordering could intermix terms of different total mode content. With linear ansatzes or $b$-mode-monomials,[11] this amounts to a set of four equations

$$\left\{(\spadesuit),\ (\clubsuit),\ (\diamondsuit),\ (\heartsuit)\right\},$$

the implications of which will be discussed below.
The terms marked by $\spadesuit$ result in the subsequent constraint

$$\begin{aligned}&(x-y)\cdot U^{(r)}_{x+y} \\ &= (x-y)\cdot\sum_k P(\lambda, x+y, k, s)\beta^{(r)}_k b^{(r)}_{x+y-k}\delta_{r,s} \\ &\overset{!}{=} \sum_l P(\lambda, y, l, s)\beta^{(s)}_l(\lambda x - x + l - y) b^{(s)}_{y+x-l}\delta_{r,s} \\ &- \sum_m P(\lambda, x, m, r)\beta^{(r)}_m(\lambda y - y + m - x) b^{(r)}_{x+y-m}\delta_{r,s}.\end{aligned} \tag{$46_\spadesuit$}$$

Comparing coefficients yields

$$\begin{aligned}&P(\lambda, y, l, s)(\lambda x - x + l - y) - P(\lambda, x, l, s)(\lambda y - y + l - x) \\ &= (x-y)P(\lambda, x+y, l, s) \qquad \forall l, x, y.\end{aligned} \tag{$47_\spadesuit$}$$

This recurrence relation can be turned into an explicit one. By setting $y = 0$ one obtains

$$P(\lambda, 0, l, s)(\lambda x - x + l) - P(\lambda, x, l, s)(l - x) = x\cdot P(\lambda, x, l, s).$$

The recurrence relation can thus be solved, provided $P(\lambda, x, l, s)$ satisfies the condition

$$\boxed{P(\lambda, 0, l, s)\cdot((\lambda-1)\cdot x + l) = l\cdot P(\lambda, x, l, s).} \tag{$\spadesuit$}$$

One finds that the solution factorises into one part for $l$, $\lambda$, $s$ fixed, and an up to now unspecified part which could be a functional depending on $l$, $\lambda$, $s$. As a first difference to the simple ghost system, the Virasoro algebra condition fixes the general form of solutions. In the former case, one had to get rid of several terms, but nothing had to be adjusted beyond it. Clearly, for $\lambda > 1$ the solution reads

$$P(\lambda, x, l, s) = A_{\lambda,l,s}\cdot((\lambda-1)\cdot x + l), \tag{$\spadesuit\spadesuit$}$$

but for $\lambda = 1$ *any* dependence on $l$ is allowed. In particular, the solution may be chosen to be *independent* of $l$. It is thus possible to deform $L_0$ by a term containing only

which contain $c$-modes as well, eq. (45) would only be correct to the highest degree in modes, because I neglected normal ordering.

[11] For powers in $b$-modes, the last three terms have higher total mode content and therefore have to vanish separately, and, of course they do.

zero modes of conformal weight 0, i.e. $A_{1,l,s} = B_s \delta_{l,0}$. For $\mathbb{Z}_2$ twists, this yields the deformation suggested in [FFH⁺02], which partly characterised my initial position. If $\lambda > 1$, terms, which contain only zero modes of conformal weight 0, do not contribute to $L_0$ due to the additional factor of $l = 0$.

As long as $\lambda > 1$, eq. ($\spadesuit$) reveals that the deformation terms have to have exactly the same form as the original Virasoro modes, with the $c$-modes replaced by $\beta$'s. One might be tempted to arrange them in the same manner as the modes constituting the original Virasoro modes. This, however, leads to a recurrence relation which does not decouple and thus seems not to be solvable in an acceptable amount of time and effort. Instead, I deliberately ordered them the other way around – and got a relation which is easily solved.

Next, the $\clubsuit$ terms are collected.

$$\begin{aligned} &\sum_{k,l} P(\lambda, y, l, s)(\lambda x - k) \left\{c^{(r)}_{x-k}, \beta^{(s)}_l\right\} b^{(r)}_k b^{(s)}_{y-l} \\ +&\sum_{k,m} P(\lambda, x, m, r)(\lambda y - k) \left\{c^{(s)}_{y-k}, \beta^{(r)}_m\right\} b^{(r)}_{x-m} b^{(s)}_k \\ =&\sum_{m,l} P(\lambda, y, l, s) P(\lambda, x, m, r) \left\{\beta^{(s)}_l, \beta^{(r)}_m\right\} b^{(r)}_{x-m} b^{(s)}_{y-l} \end{aligned} \tag{$48_\clubsuit$}$$

Again, this equation can only be fulfilled mode-wise. After imposing eq. ($\overset{\spadesuit}{\spadesuit}$), it becomes clear that $\left\{c^{(r)}_{x-k}, \beta^{(s)}_{(\lambda-1)y}\right\}$ does not contribute to the above sum, because $\beta^{(s)}_{(\lambda-1)y}$ does not contribute to $U^{(s)}_y$ either. Nevertheless this mode will contribute to any other $\left\{c^{(r)}_{x'-k'=x-k}, U^{(s)}_{y'}\right\}$, $x'$, $y'$, $k'$, chosen so that $A_{\lambda,y',l'} \neq 0 \neq (\lambda x' - k')$, and therefore its anti-commutators are fixed. Hence, eq. ($48_\clubsuit$) simplifies to

$$\begin{aligned} &P(\lambda, y, l, s)(\lambda x - x + m) \left\{c^{(r)}_m, \beta^{(s)}_l\right\} \\ +&P(\lambda, x, m, r)(\lambda y - y + l) \left\{c^{(s)}_l, \beta^{(r)}_m\right\} \\ =&\ P(\lambda, y, l, s) P(\lambda, x, m, r) \left\{\beta^{(s)}_l, \beta^{(r)}_m\right\} . \end{aligned} \tag{$49_\clubsuit$}$$

Imposing the solution ($\overset{\spadesuit}{\spadesuit}$) yields

$$\boxed{A_{\lambda,l,s} \left\{c^{(r)}_m, \beta^{(s)}_l\right\} + A_{\lambda,m,r} \left\{c^{(s)}_l, \beta^{(r)}_m\right\} = A_{\lambda,l,s} A_{\lambda,m,r} \left\{\beta^{(s)}_l, \beta^{(r)}_m\right\} .} \tag{$\clubsuit$}$$

In contrast to their predecessor, eq. ($\clubsuit$), the equations which arise from the $\diamondsuit$, $\heartsuit$ terms can have nontrivial solutions only on the same sheet (or perhaps in the sum of all the individual sheet-theories). This is because the $\clubsuit$ terms are the only ones which merge two $b$-modes on different sheets rather than two different modes on different sheets.

The $\diamond$ terms read altogether

$$\sum_{m,k} P(\lambda, x, m, r)(\lambda y - k) \left\{\beta_m^{(r)}, b_k^{(s)}\right\} c_{y-k}^{(s)} b_{x-m}^{(r)}$$
$$= \sum_{k,l} P(\lambda, y, l, s)(\lambda x - k) \left\{\beta_l^{(s)}, b_k^{(r)}\right\} c_{x-k}^{(r)} b_{y-l}^{(s)} \quad . \tag{$50_\diamond$}$$

Setting on the rhs

$$l = y - x + m', \qquad k = x - y + k'$$

gives

$$\sum_{m,k} P(\lambda, x, m, r)(\lambda y - k) \left\{\beta_m^{(r)}, b_k^{(s)}\right\} c_{y-k}^{(s)} b_{x-m}^{(r)} \tag{$51_\diamond$}$$
$$= \sum_{m',k'} P(\lambda, y, y - x + m', s)(\lambda x - x + y - k') \left\{\beta_{y-x+m'}^{(s)}, b_{x-y+k'}^{(r)}\right\} c_{y-k'}^{(r)} b_{x-m'}^{(s)}.$$

Imposing eq. ($\spadesuit$), comparison of the modes involved yields

$$A_{\lambda,m,r}((\lambda - 1)x + m)(\lambda y - k) \left\{\beta_m^{(r)}, b_k^{(s)}\right\} \tag{$52_\diamond$}$$
$$= A_{\lambda,y-x+m,s}((\lambda - 1)y + y - x + m)(\lambda x - x + y - k) \left\{\beta_{y-x+m}^{(s)}, b_{x-y+k}^{(r)}\right\} .$$

Obviously, this can only be fulfilled if the coefficients are zero for $r \neq s$ or if one drops the condition that the different Virasoro algebras commute, i.e. only looks at the total theory. The first possibility amounts to $\left\{\beta_m^{(r)}, b_k^{(s)}\right\} = 0$ except for $r = s$. Summation over all sheet-labels in order to obtain a weaker condition on the other hand yields

$$A_{\lambda,m,r}((\lambda - 1)x + m)(\lambda y - k) \left\{\beta_m^{(r)}, b_k^{(s)}\right\} \tag{$\diamondsuit$}$$
$$= A_{\lambda,y-x+m,s}((\lambda - 1)y + y - x + m)(\lambda x - x + y - k) \left\{\beta_{y-x+m}^{(r)}, b_{x-y+k}^{(s)}\right\} ,$$

where the dummy indices have been relabelled on the rhs. The equation had to be fulfilled for all integer values of $x$ and $y$. Setting $y - x =: a$ and assuming both sides to be non-zero numbers one infers

$$\frac{A_{\lambda,a+m,s}((\lambda - 1)y + a + m)(\lambda x + a - k)}{A_{\lambda,m,r}((\lambda - 1)x + m)(\lambda y - k)} \stackrel{!}{=} \frac{\left\{\beta_m^{(r)}, b_k^{(s)}\right\}}{\left\{\beta_{a+m}^{(r)}, b_{-a+k}^{(s)}\right\}}. \tag{$53_\diamond$}$$

The rhs does not depend separately on $x$ and $y$, but only on the difference $a$, whereas the lhs is a rational function of both, which contradicts the assumption. Even if only the total theory is required to be well-defined, $\{\beta_m^{(r)}, b_k^{(s)}\}$ has to be zero, if $\beta$ is realised as a state corresponding to a free field. Collecting the ♡'s gives

$$\sum_{m,l} P(\lambda, x, m, r) P(\lambda, y, l, s) \left\{ b_{x-m}^{(r)}, \beta_l^{(s)} \right\} \beta_m^{(r)} b_{y-l}^{(s)}$$
$$= \sum_{m,l} P(\lambda, y, l, s) P(\lambda, x, m, r) \left\{ b_{y-l}^{(s)}, \beta_m^{(r)} \right\} \beta_l^{(s)} b_{x-m}^{(r)}. \qquad (\heartsuit)$$

Referring to this constraint, condition of the total theory to be a well-defined tensor product actually implies the individual theories to commute. A similar computation as before implies that all commutators of $\beta$ and $b$ have to vanish, if the ansatz for $\beta$ is linear. This excludes any $c$-modes as constituents of the field corresponding to $\beta$ in a linear ansatz also.

## IV.2 A Deformation by Free Fields

The preceeding section IV.1 leads to the conclusion that $\beta$, as for $c = -2$, is allowed to be a linear combination of $b$-modes. In this case the rhs of eq. (♣) vanishes.

$$A_{\lambda,l,s} \left\{ c_m^{(r)}, \beta_l^{(s)} \right\} = -A_{\lambda,m,r} \left\{ c_l^{(s)}, \beta_m^{(r)} \right\}$$

Imposing the ansatz $\beta_l^{(s)} =: \sum_{k=0}^{n-1} M_{ks} b_l^{(k)}$ gives

$$0 = \sum_{k=0}^{n-1} M_{ks} A_{\lambda,l,s} \left\{ c_m^{(r)}, b_l^{(k)} \right\} + \sum_{p=0}^{n-1} M_{pr} A_{\lambda,m,r} \left\{ c_l^{(s)}, b_m^{(p)} \right\}$$

which implies

$$\boxed{M_{rs} A_{\lambda,l,s} + M_{sr} A_{\lambda,-l,r} \stackrel{!}{=} 0.} \qquad (54_{\clubsuit})$$

If one allows $M_{rr}$ to be non-zero on the $r$th sheet, $A_{\lambda,-l,r}$ is fixed as well, $A_{\lambda,l,r} = A_{\lambda,-l,r}$. A term of the form $M_{rs} A_{\lambda,l,s}$ occurs with a prefactor of $((\lambda - 1)n + l)$ in $U_n^{(s)}$ for *every n* and some $l$ if any $L_m^{(s)}$ is to be deformed. The only possibility to deform $T^{(s)}(z)$ without changing a distinct mode, say $L_k^{(s)}$, is to demand

$$A_{\lambda,l,s} = A_{\lambda,(1-\lambda)k,s} \delta_{(1-\lambda)k,l},$$

because then $U_n^{(s)}$ reads

$$\sum_s M_{rs} A_{\lambda,(1-\lambda)k,s} ((\lambda - 1)n + (1 - \lambda)k) b_{(1-\lambda)k}^{(r)} b_{n-(1-\lambda)k}^{(s)},$$

which obviously vanishes if and only if $n = k$, provided $A_{\lambda,(1-\lambda)k,s}$ and any of the $M_{rs}$ are non-zero. Due to eq. (54$_\clubsuit$) $U^{(r)}_{-n}$ vanishes, too, but

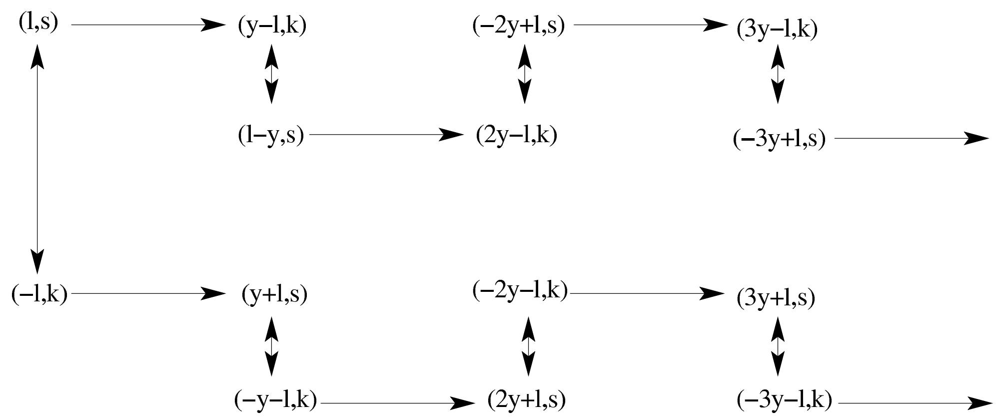
$$\sum_r M_{sr} A_{\lambda,-(1-\lambda)k,r}((\lambda-1)n-(1-\lambda)k) b^{(s)}_{-(1-\lambda)k} b^{(r)}_{n+(1-\lambda)k} = U^{(r)}_n$$

is forced to be non-zero. This implies in particular that if on any sheet, say $s$, $L^{(s)}_m$ is to be deformed, there will be another sheet, for instance called $r$, on which $L^{(r)}_{-1}$ will be deformed. Thus, deforming in this manner implies the existence of at least one sheet with a deformed translation operator. Of course, this leads to something 'new', only if its action on quasi-primary fields is altered as well. Then, the Ward identities eq. (74) would no longer be valid. This shows that some of the conditions were indeed necessary, to generalise the Ward identities in such a simple and elegant way.

Opposed to that in the *total* theory, one can choose a whole subalgebra (spanned by $L^{tot}_{\pm y}, L^{tot}_0$) of $\mathfrak{Vir}$ for one $y$ to remain undeformed. Because of this, one can even enforce 'non-logarithmic' Ward identities. Let's see how this works. There, an additional symmetry can be used to make two of the deformation modes vanish. Because of eq. (54), with a non-zero $A_{\lambda,l,k}M_{sk}$, $A_{\lambda,-l,s}M_{ks}$ will be non-zero as well. For a term in a total deformation mode with conformal weight $-y$, one can find exactly one second term with the same mode content, by inversion of the second index in $A_{\lambda,l,s}$ at $y$. This implies $A_{\lambda,l,s}M_{ks}$ and $A_{\lambda,y-l,k}M_{sk}$ to be prefactors of terms with the same mode content in $U^{tot}_y$. By fixing $A_{\lambda,l,s}M_{ks}$ for one tuple $l, s, k$ and adjusting other prefactors, depending on the $y$ chosen, recursively, in principle all terms contributing to a specific $U^{tot}_y$ could be made to vanish. This leads to a series of constraints:

$$\left.\begin{aligned} -((\lambda-1-i)y+l)A_{\lambda,l-iy,s}M_{ks} &= ((\lambda+i)y-l)A_{\lambda,(i+1)y-l,k}M_{sk} \\ -((\lambda-1-i)y-l)A_{\lambda,-l-iy,k}M_{sk} &= ((\lambda+i)y+l)A_{\lambda,(i+1)y+l,s}M_{ks} \end{aligned}\right\} \forall i \in \mathbb{N} \tag{55}$$

The logic behind this is to fix $A_{\lambda,l,s}M_{ks}$ for one tuple $(l, k, s)$ to a distinct non-zero value. In the total theory, now the values $A_{\lambda,y-l,k}M_{sk}$ could be adjusted for one $y$ and the same $(l, k, s)$ as before. Therefore, I have a non-zero deformation which would vanish in the total theory, if it were not for the extra terms fixed by (54$_\clubsuit$). These could be cancelled by further terms, as denoted in the figure below. This picture has to be understood such that the prefactors of the modes corresponding to the displayed tuples have to be adjusted to be equal if joined by horizontal arrows, whereas terms connected by vertical arrows are mutually fixed by (54$_\clubsuit$).




Generically, if these conditions are to be met, infinitely many terms have to contribute to the individual deformation modes. This extends to terms which do not contain zero modes any more and are mappings from excited states with negative ghost charge to such with positive ghost charge. However, if the 'initial' $l$ is chosen to be zero, there is only one condition in the first place. Furthermore, the sum will terminate at $(\lambda - 1)y = i$. The prefactor of the second term cannot become zero at all, so one may adjust all $A_{\lambda,(i'+1)y,k}M_{sk}$ with $|i'| < (\lambda - 1)|y|$ to meet the above conditions.

The choice of $l$ to be zero implies that the equations emanate from each other by permuting their sheet-labels

$$\left.\begin{array}{l} 0 = (\lambda - 1 - i)yA_{\lambda,-iy,s}M_{ks} + (\lambda + i)yA_{\lambda,(i+1)y,k}M_{sk} \\ 0 = (\lambda - 1 - i)yA_{\lambda,-iy,k}M_{sk} + (\lambda + i)yA_{\lambda,(i+1)y,s}M_{ks} \end{array}\right\} \forall i \in \mathbb{N}. \tag{56}$$

However, eq. ($54_\clubsuit$) then implies that this equation is still valid when the sign of $y$ is flipped. Thus, with a vanishing $U^{tot}_y$, $U^{tot}_{-y}$ also vanishes.

A similar behaviour of an improved $c = -26$ theory was noted in [KF03].

### IV.2.1 The Action on the Space of States

Imposing the constraints derived above one should explore the action of such a deformation on the space of states: I restrict myself to the action of $U_0^{(k)}$ on the various ground states which have the form

$$\prod_{r=1-\lambda}^{\lambda-1}\prod_{l_r=0}^{n-1}(c_r^{(l_r)})^{N_{r,l_r}}\,|0\rangle \qquad \text{with} \qquad N_{r,l_r} =: N_{l_r} \in \{0,\ 1\}\,.$$

Thereby, I implicitly adopt the ordering prescription that

$$\text{if} \quad (n-1)r + k_r =: a(k_r) < a(k'_{r'}), \qquad c_r^{(k_r)} \text{ resides on the left of } c_{r'}^{(k'_{r'})}. \tag{57}$$

An occupation number formalism seems to be more adequate than a tensor product formalism. This notation could prove to be more natural if, like in the $c = -2$ case, some of the zero modes "show through". At least the mode $c^{(l)}_{\lambda-1}$ with conformal weight $1-\lambda$, being the *constant* term in the Laurent expansion of the field $c$, is present on all sheets around $z = 0$, and not only on the $l$th. But at $z = \infty$, its hermitian conjugate $c^{(l)}_{1-\lambda}$ is the constant term, because the conformal mapping gives an additional factor of $z^{\pm 2\lambda}$. On the contrary, the zero modes of the pair of weight-0 'Zamolodchikov' ghosts remain the constants of the mode expansion.

As one would expect from a tensor product, even the unmodified $L^{(k)}_n$ act on these states in an illdefined way. This will be investigated in depth later in this section. However, that is why I am interested in Jordan cells which survive the summation over sheet labels. To this end, it is worth noting that in the attempt to find Jordan cells, the terms involving the conformal weights $l' \neq \pm l$ act independently even after summation over sheets labels, i. e. they yield different mode contents of the resulting states. Hence, one could as well investigate them separately, because summation can not destroy the Jordan cell structure:

$$\overset{i}{u}{}^{(k)}_m := \sum_{s=0}^{n-1} M_{sk}(((\lambda-1)m+i)A_{\lambda,i,k}b^{(s)}_i b^{(k)}_{m-i} + ((\lambda-1)m-i)A_{\lambda,-i,k}b^{(s)}_{-i} b^{(k)}_{m+i}),$$

with $U^{(k)}_m = \sum_{i=0} \overset{i}{u}{}^{(k)}_m$. The action of the 'atomic' deformation $\overset{i}{u}{}^{(k)}_0$ of $L^{(k)}_0$ on the ground states then reads

$$\overset{i}{u}{}^{(k)}_0 \prod_{r=1-\lambda}^{\lambda-1} \prod_{l_r=0}^{n-1} (c^{(l_r)}_r)^{N_{l_r}} |0\rangle$$

$$= -\sum_{s=0}^{n-1} iM_{sk}\left(A_{\lambda,i,k}b^{(k)}_{-i}b^{(s)}_i + A_{\lambda,-i,k}b^{(s)}_{-i}b^{(k)}_i\right) \prod_{r=1-\lambda}^{\lambda-1} \prod_{l_r=0}^{n-1} (c^{(l_r)}_r)^{N_{l_r}} |0\rangle$$

$$= \sum_{s=0}^{n-1} iM_{sk}\Bigg(A_{\lambda,i,k}(-)^{\sum_{r'_{l'}=s_{-i}}^{k_i} N_{r'_{l'}}} \delta_{N_{s_{-i}},1}\delta_{N_{k_i},1} \prod_{\substack{r=1-\lambda \\ s_{-i}\neq l_r\neq k_i}}^{\lambda-1} \prod_{l_r=0}^{n-1} (c^{(l_r)}_r)^{N_{l_r}} |0\rangle$$

$$+A_{\lambda,-i,k}(-)^{\sum_{r'_{l'}=s_i}^{k_{-i}} N_{r'_{l'}}} \delta_{N_{s_i},1}\delta_{N_{k_{-i}},1} \prod_{\substack{r=1-\lambda \\ s_i\neq l_r\neq k_{-i}}}^{\lambda-1} \prod_{l_r=0}^{n-1} (c^{(l_r)}_r)^{N_{l_r}} |0\rangle\Bigg). \qquad (58)$$

The overall sign in the l.h.s. of the above equation stems from the ordering with respect to increasing indices of the first term which is proportional to $A_{\lambda,i,k}$. In the r.h.s., one has to pay attention that to move one of the $b$-modes to the right of its conjugate requires

$$\sum_{r_m=0_{1-\lambda}}^{k_{-l-1}} N_{r_m} := \sum_{\substack{r,m \\ a(r_m)<a(k_{-l})}} N_{r_m}$$

permutations. E.g. a term with mode-content $b_{-l}^{(s)} b_l^{(k)}$, $l > 0$ produces the sign

$$(-1)^{\sum\limits_{r_m=0_{1-\lambda}}^{k_{-l}-1} N_{r_m} + \sum\limits_{r_m=0_{1-\lambda}}^{s_l-1} N_{r_m}} = (-1)^{\sum\limits_{r_m=k_{-l}}^{s_l-1} N_{r_m}} .$$

Because states residing in the complement of the kernel of $b_{-l}^{(s)} b_l^{(k)}$ are characterised by both $N_{k_{-l-1}}$ and $N_{s_{l-1}}$ being equal to 1, we can account for the additional minus sign mentioned above by extending the summation over occupation numbers. From eq. (58) it is evident that all resulting states appear twice, once with $A_{\lambda,i,k}$ and once with $A_{\lambda,-i,k}$. To check whether coefficients cancel each other, one relabels roughly one half of the terms: Interchanging labels in (58) for $s > k$ yields $\quad {}^{i}u_0^{(k)} \prod\limits_{r=1-\lambda}^{\lambda-1} \prod\limits_{l_r=0}^{n-1} (c_r^{(l_r)})^{N_{l_r}} |0\rangle$

$$= \sum_{\substack{s,k \\ s<k}}^{n-1} i \Big( \big(A_{\lambda,i,k} M_{sk} + A_{\lambda,-i,s} M_{ks}\big) (-)^{\sum\limits_{r'_{l'}=s_{-i}}^{k_i} N_{r'_{l'}}} \delta_{N_{s_{-i}},1} \delta_{N_{k_i},1} \prod_{r=1-\lambda}^{\lambda-1} \prod_{\substack{l_r=0 \\ s_{-i}\neq l_r \neq k_i}}^{n-1} (c_r^{(l_r)})^{N_{l_r}} |0\rangle$$

$$+ \big(A_{\lambda,i,s} M_{ks} + A_{\lambda,-i,k} M_{sk}\big) (-)^{\sum\limits_{r'_{l'}=k_{-i}}^{s_i} N_{r'_{l'}}} \delta_{N_{k_{-i}},1} \delta_{N_{s_i},1} \prod_{r=1-\lambda}^{\lambda-1} \prod_{\substack{l_r=0 \\ k_{-i}\neq l_r \neq s_i}}^{n-1} (c_r^{(l_r)})^{N_{l_r}} |0\rangle \Big)$$

$$+ i \sum_{k}^{n-1} \big(A_{\lambda,i,k} M_{kk} + A_{\lambda,-i,k} M_{kk}\big) (-)^{\sum\limits_{r'_{l'}=k_i}^{k_{-i}} N_{r'_{l'}}} \delta_{N_{k_i},1} \delta_{N_{k_{-i}},1} \prod_{r=1-\lambda}^{\lambda-1} \prod_{\substack{l_r=0 \\ k_i\neq l_r \neq k_{-i}}}^{n-1} (c_r^{(l_r)})^{N_{l_r}} |0\rangle .$$

A final step of relabeling condenses this to

$$\sum_{s,k}^{n-1} i \big(A_{\lambda,i,k} M_{sk} + A_{\lambda,-i,s} M_{ks}\big) (-)^{\sum\limits_{r'_{l'}=s_{-i}}^{k_i} N_{r'_{l'}}} \delta_{N_{s_{-i}},1} \delta_{N_{k_i},1} \prod_{r=1-\lambda}^{\lambda-1} \prod_{\substack{l_r=0 \\ s_{-i}\neq l_r \neq k_i}}^{n-1} (c_r^{(l_r)})^{N_{l_r}} |0\rangle ,$$

which yields the equivalence

$$\boxed{A_{\lambda,i,k} M_{sk} + A_{\lambda,-i,s} M_{ks} = 0 \Leftrightarrow L_0^{tot} \text{ diagonalisable} .} \qquad (59)$$

One recognises this formula as the condition for the Virasoro algebras on different sheets to commute and give a well-defined total symmetry algebra! Thus, no Jordan cells for $L_0^{tot}$ are possible via deformations with free $b$-fields from other sheets. The summation over sheet labels is necessary, because the new individual Virasoro modes do not act diagonally on the states. Only the $L_n^{tot}$ give the correct result, e.g. only $L_0^{tot}$

gives the right conformal weight. The $L_n^{(k)}$ are "blind" with respect to all modes from the $l$th sheet if $k \neq l$, so any state not containing modes of sheet $k$ behaves like the vacuum under the action of $L_n^{(k)}$.

To investigate the appearance of Jordan cells for individual Virasoro modes, one could try to restrict the set of ground states to a certain subspace. For each sheet, say $k$, and each sector with total conformal weight $h^{tot}$, one could choose the states satisfying

$$h^s = h^{tot}\delta_{k,s}.$$

Here, I defined the conformal weights $h^{tot}, h^s$ as the eigenvalues of $L_0^{tot}$ and $L_0^s$, respectively,

$$h^{(s)}\psi := L_0^{(s)}\psi, \qquad h^{tot}\psi := L_0^{tot}\psi.$$

This subspace is assumed to consist of all states satisfying the condition that the conformal weights of their constituent modes from the sheets $k,\ k \neq s$, add up to zero *independently*. The space of states is obtained as usual by taking the direct sum over all sheets of lexicographically ordered free-field-descendants of distinct sheets. On these states, the modified Virasoro modes are considered to act diagonally. Nevertheless, one runs into problems considering the action of the $\widetilde{L}_0^{(k)}$ even on the ground states. It embeds the subset with $h^s = h^{tot}\delta_{k,s}$ into its complement in the $h^s$ sector of $\mathcal{F}$. This means that applying the $\widetilde{L}_0^{(k)}$ once more gives a contribution from the unmodified $L_0^{(k)}$. As one can infer from

$$[L_0^{(k)}, U_0^{(k)}] = \sum_l lP(\lambda, 0, l, k)\beta_l^{(k)} b_{-l}^{(k)},$$

the action of $L_0^{(k)}$ after $U_0^{(k)}$ on any log-partner of ground states will have contributions to the total conformal weight which do not add up to zero separetely on individual sheets, because $\beta_l^{(k)}$ is decomposed by modes from other sheets only. Consider for example the action of the deformation on the ground state $\left|c^0_{1-\lambda}c^1_{1-\lambda}c^0_{\lambda-1}c^1_{\lambda-1}\right\rangle$

$$L_0^{(0)}\Big(L_0^{(0)} - \sum_{\substack{s=0\\ i\in\mathbb{N}}}^{n-1} iM_{s0}\big(A_{\lambda,i,0}b_{-i}^{(0)}b_i^{(s)} + A_{\lambda,-i,0}b_{-i}^{(s)}b_i^{(0)}\big)\Big)\left|c^0_{1-\lambda}c^1_{1-\lambda}c^0_{\lambda-1}c^1_{\lambda-1}\right\rangle$$

$$= L_0^{(0)}\Bigg((1-\lambda)\underbrace{M_{00}\big(A_{\lambda,\lambda-1,0} + A_{\lambda,1-\lambda,0}\big)}_{=\,0,\ \text{because of }(54\clubsuit)}b^{(0)}_{1-\lambda}b^{(0)}_{\lambda-1}\left|c^0_{1-\lambda}c^1_{1-\lambda}c^0_{\lambda-1}c^1_{\lambda-1}\right\rangle$$

$$+(1-\lambda)M_{10}\big(A_{\lambda,\lambda-1,0}\left|c^0_{1-\lambda}c^1_{\lambda-1}\right\rangle + A_{\lambda,1-\lambda,0}\left|c^1_{1-\lambda}c^0_{\lambda-1}\right\rangle\big)\Bigg)$$

$$= (1-\lambda)^2 M_{10}\Big(A_{\lambda,\lambda-1,0}\left|c^0_{1-\lambda}c^1_{\lambda-1}\right\rangle - A_{\lambda,1-\lambda,0}\left|c^1_{1-\lambda}c^0_{\lambda-1}\right\rangle\Big)$$

One sees, the first state has conformal weight $1-\lambda$ with respect to $L_0^{(k)}$, whereas the second has conformal weight $\lambda - 1$. It obviously would not help to apply $(L_0^{(k)})^2$ on this ground state, because the additional deformation mode just leads to additional terms proportional to the vacuum.

This behaviour of the new Virasoro modes is not surprising, because the $k'$th conformal weight of $\widetilde{L}_n^{(k)}$ itself is not $-n$, but a sum of terms with $k'$th conformal weight $-n, -l-n, l-n,$ if $k = k'$ and of $0, l, -l,$ if $k \neq k'$. One can trace back this behaviour to the fact that the ansatz for $U_y^{(s)}$ is an unnatural choice as well, since its sheet-wise conformal weight is unequal to its total conformal weight. It is therefore manifest that the definition of $\widetilde{L}_0^{(k)}$ is problematic, the $\widetilde{L}_n^{(s)}$ cease to act diagonally.

This is a further difference to the simple ghost system: There, the individual $L_0^{(k)}$ would act in a well-defined way on all groundstates, because the conformal weights of the inserted $c$-modes are zero. For $\lambda > 1$, however, exclusive insertions of $c_0$-modes cannot give rise to Jordan cells, since $L_0$ does not contain suitable pairs of $b_0$-modes.

The action of the deformed Virasoro modes yields descendants and states with ghost charge diminished by two. That suggests an indecomposable structure, but due to the problems with the action of the Virasoro modes in general, this will not be substantiated further, however, some remarks can be found in [KF03]. An improvement term, which yields well-defined energy operators $L_0^{(k)}$ with $k \in \{0, \ldots, n-1\}$, is described in the next sections, where it is also shown to have an indecomposable structure.

One can infer from the calculation below that even on hyperelliptic Riemann surfaces, such an indecomposable structure is not forced to have filtration length two [Roh96], if there exist two indices $i, i', i \neq i'$, for which $A_{\lambda,i,k} \neq 0 \neq A_{\lambda,i',k}$ on some sheet $k$. If there is a deformation, such two indices exist at least, namely $i, -i$. As before, I only consider a product of atomic deformations, to obtain the full, one has to sum over $i$ and $i'$. But of course, it suffices to set $i'=-i$.

$$\overset{i}{u}{}_m^{(k)} \overset{i'}{u}{}_r^{(k)} :=$$

$$\sum_{s=0}^{n-1} \big( M_{sk} \left( ((\lambda-1)m + i) A_{\lambda,i,k} \; b_i^{(s)} b_{m-i}^{(k)} + ((\lambda-1)m - i) A_{\lambda,-i,k} \; b_{-i}^{(s)} b_{m+i}^{(k)} \right) \big)$$
$$\times \sum_{s'=0}^{n-1} \big( M_{s'k} \left( ((\lambda-1)r + i') A_{\lambda,i',k} b_{i'}^{(s')} b_{r-i'}^{(k)} + ((\lambda-1)r - i') A_{\lambda,-i',k} b_{-i'}^{(s')} b_{r+i'}^{(k)} \right) \big)$$

$$
\begin{aligned}
= -\sum_{s,s'=0}^{n-1} M_{sk}M_{s'k}\Big(&((\lambda-1)m+i)((\lambda-1)r+i')A_{\lambda,i,k}\ \ A_{\lambda,i',k}b_i^{(s)}b_{i'}^{(s')}b_{m-i}^{(k)}b_{r-i'}^{(k)}\\
&+((\lambda-1)m+i)((\lambda-1)r-i')A_{\lambda,i,k}\ \ A_{\lambda,-i',k}b_i^{(s)}b_{-i'}^{(s')}b_{m-i}^{(k)}b_{r+i'}^{(k)}\\
&+((\lambda-1)m-i)((\lambda-1)r+i')A_{\lambda,-i,k}\ \ A_{\lambda,i',k}b_{-i}^{(s)}b_{i'}^{(s')}b_{m+i}^{(k)}b_{r-i'}^{(k)}\\
&+((\lambda-1)m-i)((\lambda-1)r-i')A_{\lambda,-i,k}A_{\lambda,-i',k}b_{-i}^{(s)}b_{-i'}^{(s')}b_{m+i}^{(k)}b_{r+i'}^{(k)}\Big)
\end{aligned}
$$

Even if one fixes $m = r$ and $i' = -i$, this product is not forced to be zero, because at least in the generic case for $m$ and $r$, the different terms are linearly independent. It seems pretty obvious that one can find deformations with $(U_m^{(k)})^3 \neq 0$ , if $s'$ is not constrained to be $s$ or $k$.

The deformation of reparametrisation ghosts on hyperelliptic surfaces found in [KF03],[FK03] is not an example of this type of deformation, although it is constructed similarly. But this model is somewhat special, because the different Virasoro algebras do not commute with each other. The identification of the additional zero modes was made 'cross-over', such that a certain zero mode would act as an annihilator on one, and a creator on the other sheet. It seems impossible to generalise their choice to higher twists.

# V A Quadrilinear Deformation

Looking for Jordan cells of well-defined $L_0$'s, I try to mimic the properties of the deformation introduced in II.2.2 for simple ghosts. Recall that on each sheet there was a pair of zero modes of the Zamolodchikov ghosts. Being the constant terms in the Laurent expansion, they were interpreted to be present everywhere on the Riemann sphere without the branch points, i.e. independent of the actual coordinate patch. On the contrary, zero modes from other ghost systems can be the constants of a Laurent expansion only locally in the vicinity of a branch point. Inversion of the coordinates and transition between in- and out-states exchanges the sign of the conformal weight of the mode which is actually the constant zero mode. For the sake of simplicity, I will restrict myself to an algebraic point of view in terms of modes. The realisation of this deformation in terms of fields is left to future work. I redefine $U_y^{(s)}$ to be

$$U_y^{(s)} := \sum_l P(\lambda, y, l, s)A_0^{(s)}b_l^{(s)}b_{y-l}^{(s)}$$

demanding $A_0^{(s)}$ to be composed out of modes whose sheetwise conformal weight adds up to zero *by pairs*, which, of course, implies $\left[L_0^{(k)}, A_0^{(s)}\right] = 0$ for all $k, s$. Furthermore, I stipulate $\left[b_n^{(s)}, A_0^{(s)}\right] = \left[c_n^{(s)}, A_0^{(s)}\right] = \left[L_n^{(s)}, A_0^{(s)}\right] = 0$ for all $s, n$. This ansatz should be understood as assembled from two composite bosonic operators, though it can be written in the same form as the deformation before. Next to the $A_0^{(s)}$ there is the operator $\sum_l b_l^{(s)} b_{n-l}^{(s)}$ with modes from the same sheet, determining the total conformal weight. With these choices the commutator of the deformed Virasoro modes differs by

$$\left[L_x^{(r)}, U_y^{(s)}\right] - \left[L_y^{(s)}, U_x^{(r)}\right] \tag{60}$$

$$= \sum_l P(\lambda, y, l, s) \left[L_x^{(r)}, A_0^{(s)} b_l^{(s)} b_{y-l}^{(s)}\right] - \sum_t P(\lambda, x, t, r) \left[L_y^{(s)}, A_0^{(r)} b_t^{(s)} b_{y-t}^{(s)}\right]$$

$$= \sum_l P(\lambda, y, l, s) \left(\left[L_x^{(r)}, A_0^{(s)}\right] b_l^{(s)} b_{y-l}^{(s)} (1 - \delta_{s,r})\right. \tag{60a)}$$

$$\left. + ((\lambda - 1)x - l) A_0^{(s)} b_{x+l}^{(s)} b_{y-l}^{(s)} \delta_{s,r} + ((\lambda - 1)x - y + l) A_0^{(s)} b_l^{(s)} b_{x+y-l}^{(s)} \delta_{s,r}\right)$$

$$- \sum_t P(\lambda, x, t, r) \left(\left[L_y^{(s)}, A_0^{(r)}\right] b_t^{(r)} b_{x-t}^{(r)} (1 - \delta_{s,r})\right. \tag{60b)}$$

$$\left. + ((\lambda - 1)y - t) A_0^{(r)} b_{y+t}^{(r)} b_{x-t}^{(r)} \delta_{s,r} + ((\lambda - 1)y - x + t) A_0^{(r)} b_t^{(r)} b_{x+y-t}^{(r)} \delta_{s,r}\right)$$

from the original one. On may explore the cases $r = s$ and $r \neq s$ separately, of course. On the same ($r = s$) sheet, this gives the conditions

$$\sum_l P(\lambda, y, l, s) \left(((\lambda - 1)x - l) A_0^{(s)} b_{x+l}^{(s)} b_{y-l}^{(s)} + ((\lambda - 1)x - y + l) A_0^{(s)} b_l^{(s)} b_{x+y-l}^{(s)}\right)$$

$$- \sum_l P(\lambda, x, l, s) \left(((\lambda - 1)y - l) A_0^{(s)} b_{y+l}^{(s)} b_{x-l}^{(s)} + ((\lambda - 1)y - x + l) A_0^{(s)} b_l^{(s)} b_{x+y-l}^{(s)}\right)$$

$$\stackrel{!}{=} \sum_l (x - y) P(\lambda, x + y, l, s) A_0^{(s)} b_l^{(s)} b_{x+y-l}^{(s)}. \tag{61}$$

Because every term on the r.h.s. inside the braces of (61) looks promising, we examine the subset of solutions given below by (62) and (63)

$$\sum_l P(\lambda, y, l, s)((\lambda - 1)x - l) A_0^{(s)} b_{x+l}^{(s)} b_{y-l}^{(s)}$$
$$\stackrel{!}{=} \sum_l P(\lambda, x, l, s)((\lambda - 1)y - l) A_0^{(s)} b_{y+l}^{(s)} b_{x-l}^{(s)} \tag{62}$$

and

$$P(\lambda, y, l, s)((\lambda - 1)x - y + l) A_0^{(s)} b_l^{(s)} b_{x+y-l}^{(s)}$$
$$- P(\lambda, x, l, s)((\lambda - 1)y - x + l) A_0^{(s)} b_l^{(s)} b_{x+y-l}^{(s)}$$
$$\stackrel{!}{=} (x - y) P(\lambda, x + y, l, s) A_0^{(s)} b_l^{(s)} b_{x+y-l}^{(s)}. \tag{63}$$

The second equation yields (♠) with solution (♣). Collecting terms contributing to the same combinations of modes in (62) gives

$$\sum_l (P(\lambda, y, l, s)((\lambda - 1)x - l) + P(\lambda, x, -l, s)((\lambda - 1)y + l)) A_0^{(s)} b_{x+l}^{(s)} b_{y-l}^{(s)} \overset{!}{=} 0.$$

This is a special case of ($49_\clubsuit$) and gives

$$\begin{aligned} P(\lambda, y, l, s) &= A_{\lambda,l,s}((\lambda - 1)y + l) \text{ which is } (\spadesuit) \\ \text{where } &\boxed{A_{\lambda,l,s} = -A_{\lambda,-l,s}.} \end{aligned} \tag{64}$$

Alternatively one could have shifted $l$ by $-x$ in the second, and by $-y$ in the fourth term. This leads to a single, more complicated recurrence relation, which will be displayed in the appendix A. One should note that $l \to x + y - l$ yields the same mode content. Due to the increased symmetry eq. (64) implies that there is no deformation $L_0^{(s)}$ for any $s$. But one can show with an more general reasoning that there is no solution to the complicated recurrence relation allowing for a deformed $L_0^{(s)}$. This will be the topic of section V.2. To analyse the remaining terms of (60 a, b) which combine modes from different sheets I impose the ansatz

$$A_0^{(s)} = \sum_{i,p} B_i^{ps} b_{-i}^{(p)} b_i^{(p)}.$$

This gives

$$\begin{aligned} &\sum_l P(\lambda, y, l, s) \left[L_x^{(r)}, A_0^{(s)}\right] b_l^{(s)} b_{y-l}^{(s)} = \sum_t P(\lambda, x, t, r) \left[L_y^{(s)}, A_0^{(r)}\right] b_t^{(r)} b_{x-t}^{(r)} \qquad (65) \\ \Leftrightarrow &\sum_{l,i} P(\lambda, y, l, s) B_i^{rs} \left(((\lambda - 1)x + i) b_{x-i}^{(r)} b_i^{(r)} + ((\lambda - 1)x - i) b_{-i}^{(r)} b_{x+i}^{(r)}\right) b_l^{(s)} b_{y-l}^{(s)} \\ = &\sum_{t,i} P(\lambda, x, t, r) B_i^{sr} \left(((\lambda - 1)y + i) b_{y-i}^{(s)} b_i^{(s)} + ((\lambda - 1)y - i) b_{-i}^{(s)} b_{y+i}^{(s)}\right) b_t^{(r)} b_{x-t}^{(r)}. \end{aligned}$$

Assorting terms by mode-content leads to

$$\begin{aligned} &\sum_{l,t} ((\lambda - 1)x + t) P(\lambda, y, l, s)(B_t^{rs} - B_{-t}^{rs}) b_t^{(r)} b_{x-t}^{(r)} b_l^{(s)} b_{y-l}^{(s)} \\ = &\sum_{l,t} ((\lambda - 1)y + l) P(\lambda, x, t, r)(B_l^{sr} - B_{-l}^{sr}) b_t^{(r)} b_{x-t}^{(r)} b_l^{(s)} b_{y-l}^{(s)}. \end{aligned} \tag{66}$$

The requisite relabelings can be performed because $P(\lambda, x, 0, r) = 0$ for all $x, r$ by (64). If

$$0 = B_l^{sr} - B_{-l}^{sr} := M_l^{sr},$$

the deformation vanishes as well. Plugging in (♠) and (62) reveals the requirements

$$\boxed{A_{\lambda,l,s} M_t^{rs} = A_{\lambda,t,r} M_l^{sr}} \tag{67}$$

$\forall l, t \in \{0, \cdots, \lambda - 1\},\ r, s \in \{0, \ldots, n-1\}$. If the $A_{\lambda,l,s}$ are chosen to be the same on all sheets, $M_t^{rs}$ is a symmetric matrix for all $t > 0$. Furthermore, if any $M_t^{rs}, A_{\lambda,t,r}$ are non-zero, then $A_{\lambda,l,s} = 0 \Leftrightarrow M_l^{sr} = 0$. As before, this implies that some translation operator gets affected by the deformation. The impact of this on the Ward identities will be investigated in the next section. The above, in particular eq. (67) implies that we have $\frac{n^2(j-1)^2+n(j-1)}{2}$ independent deformation 'directions'. (We can choose one independent coefficient for any pair of tuples $(r, s, t, l) \neq (s, r, l, t),\ r, s \in \{0, \ldots, n-1\},\ l, t \in \{1, \ldots, \lambda - 1\}$ and one for each $(r, s, t, l) = (s, r, l, t)$.) At first sight, it seems that this time, one could even adjust these coefficients such that the global conformal group remains unaltered on the level of individual sheets. But eq. (64) excludes to choose $l = 0$ as a starting point. The set of indices $l$ to be summed over then had to be infinite. This had as a consequence that summands in the deformation term altered the ghost charge of excited states, while vanishing on their anchestor ground states. Above, I excluded such implicitly by restricting to $A_{\lambda,l,s} = 0$ if $|l| \geq \lambda$.

## V.1 The Action on the Space of States

The action of the $\widetilde{L}_n^{(k)}$ on the ground states is as follows:

$$U_n^{(s)} \prod_{r=1-\lambda}^{\lambda-1} \prod_{k_r=0}^{n-1} (c_r^{(k_r)})^{N_{k_r}} |0\rangle = \sum_{i,p,l} B_i^{ps} A_{\lambda,l,s}((\lambda-1)n + l) \times \tag{68}$$

$$\times (-)^{\left\{ {}_{i}{}^{p}{}_{n}{}^{s}{}_{l} \right\}} \delta_{N_{p_i},1} \delta_{N_{p_{-i}},1} \delta_{N_{s_{-l}},1} \delta_{N_{s_{l-n}},1} \prod_{\substack{r=1-\lambda \\ k_r \notin \{p_i, p_{-i}, s_{-l}, s_{l-n}\}}}^{\lambda-1} \prod_{k_r=0}^{n-1} (c_r^{(k_r)})^{N_{r,k}} |0\rangle$$

with $\left\{ {}_{i}{}^{p}{}_{n}{}^{k}{}_{l} \right\}$ counting the number of permutations necessary to group mutually conjugate modes next to each other, $b$'s to the left. I introduced this sign as a shorthand notation for a lenghthy expression leading to an irrelevant sign. To assure oneself that it really only leads to an overall sign, I display its precise definition and the calcutation leading to the above stated

$$\left\{ {}_{i}{}^{p}{}_{n}{}^{k}{}_{l} \right\} := \left( \sum_{\substack{k',r': \\ a(k'_{r'})=a(0_{1-\lambda})}}^{a(s_{l-n-1})} + \sum_{\substack{k',r': \\ a(k'_{r'})=a(0_{1-\lambda})}}^{a(s_{-l-1})} + \sum_{\substack{k',r': \\ a(k'_{r'})=a(0_{1-\lambda})}}^{a(o_{-i-1})} + \sum_{\substack{k',r': \\ a(k'_{r'})=a(0_{1-\lambda})}}^{a(o_{i-1})} \right) N_{k',r'} + \frac{1-\mathrm{sign}(\sigma_{p,k,i,l,n})\cdot 1}{2}$$

with

$$\sigma_{p,k,i,l,n} \in \mathfrak{S}_4 : \sigma_{p,k,i,l,n}(a(p_{-i})) < \sigma_{p,k,i,l,n}(a(p_i)) < \sigma_{p,k,i,l,n}(a(s_l)) < \sigma_{p,k,i,l,n}(a(s_{n-l})).$$

It is useful to note $\left\{{}^{p\ k}_{i\ \ l}{}_{0}\right\} = \left\{{}^{k\ p}_{l\ \ i}{}_{0}\right\}$, because $(\sigma_{p,k,i,l,n})$ odd $\Leftrightarrow$ $(\sigma_{k,p,l,i,n})$ odd. But if $i \to -i$ or $l \to -l$, $\left\{{}^{p\ k}_{i\ \ l}{}_{0}\right\}$ changes by one.

Surprisingly, $L_0^{(s)}$ remains undeformed on any sheet $s$. This is because of the requirement (62) and its consequence (64). The ansatz for $U_0^{(s)}$ inherits an additional symmetry: Terms in $U_0^{(s)}$ with identical mode contents are obtained by $i \to -i$ or $l \to -l$, whereas for $n \neq 0$ it is only $i \to -i$ . This implies

$$\begin{aligned} & U_0^{(s)} \prod_{r=1-\lambda}^{\lambda-1} \prod_{k_r=0}^{n-1} (c_r^{(k_r)})^{N_{k_r}} |0\rangle \\ = & \sum_{i,p,l>0} \left( B_i^{ps} A_{\lambda,l,s}(l)(-)^{\left\{{}^{p\ s}_{i\ \ l}{}_{0}\right\}} + B_i^{ps} A_{\lambda,-l,s}(-l)(-)^{\left\{{}^{p\ s}_{i\ \ -l}{}_{0}\right\}} \right) \times \\ & \times \delta_{N_{p_i},1} \delta_{N_{p_{-i}},1} \delta_{N_{s_l},1} \delta_{N_{s_{-l}},1} \prod_{\substack{r=1-\lambda \\ k_r \notin \{p_i, p_{-i}, s_l, s_{-l}\}}}^{\lambda-1} \prod_{k_r=0}^{n-1} (c_r^{(k_r)})^{N_{r,k}} |0\rangle \overset{(62),(64)}{=} 0 \end{aligned} \quad (69)$$

Thus, Jordan cells are *not* possible for this deformation. This is true more generally:

## V.2 Curiosities at $\lambda > 1$

In the following, I display some general results on additive redefinitions of the Virasoro field. However, this section does *not* deal with OPA deformations. Both, the bilinear and the quadrilinear deformation have in common a diagonalisable $L_0^{tot}$, but in the former case the full theory is the only well-defined. The quadrilinear deformation being well-defined even on individual sheets in turn implies $L_0$ to be diagonalizable even on individual sheets. This is a general feature of the higher spin ghost systems, which can be seen from the Virasoro constraints for a particular choice of modes:

$$[L_x^{(r)}, U_0^{(s)}] - [L_0^{(s)}, U_x^{(r)}] \overset{!}{=} x \cdot U_x^{(s)} \delta_{r,s}. \quad (70)$$

$U_x^{(r)}$ is restricted to be composed out of $b$-modes such that

$$h^{(t)}_{U_x^{(r)}} = -x\delta_{r,t}$$

to render the action on at least a restricted space of states welldefined, i.e. one imposes the condition that $\widetilde{L^{(r)}}_0$ has to act as the *energy operator*. This enables us to calculate $[L_0^{(s)}, U_x^{(r)}] = -x \cdot U_x^{(r)} \delta_{r,s}$. This obviously yields

$$[L_x^{(r)}, U_0^{(s)}] \overset{!}{=} 0 \ \ \forall x \text{ if } r = s.$$

In my framework, this set of equations generically requires $U_0^{(s)}$ to be zero. Exemptions are

- deformations by $b$-zero-modes of conformal weight zero for $\lambda = 1$, because
$$[L_x^{(s)}, b_0^{(r)}] = ((0)x - 0)b_x^{(r)}\delta_{r,s} = 0,$$
which characterises, as expected, the well known $c = -2$ LCFT deformation.[12]

It is impossible to achieve the same within other ghost systems than $c = -2$, using monomials in $b$-modes of finite degree. An attempt to write something down like this yields the condition that all modes from a distinct sheet have to be present in the product, if any modes are present. This is not a product of finitely many modes, and obviously not well-defined: Any permutation of modes times its signature is an identity, but the sum over all integers is not absolutely convergent.

A similar reasoning forbids Jordan cells for $L_0^{tot}$, even if they are present on individual sheets due to the relinquishment of a consistent action of the individual $L_0^{(s)}$. Because it is to be required to act as the energy operator, $L_0^{tot}$ has to count the correct total conformal weight. For that reason, all $U_x^{(r)}$ have $-x$ as total conformal weight. The constraint for the total Virasoro algebra reads

$$\sum_{r,s}[L_x^{(r)}, U_0^{(s)}] - \sum_{r,s}[L_0^{(s)}, U_x^{(r)}] \stackrel{!}{=} x\sum_s U_x^{(s)}. \tag{71}$$

This again requires $[L_x^{tot}, U_0^{tot}] = 0$. I will now make use of the primarity of $b$ and $c$,

$$[L_y^{(s)}, b_x^{(r)}] = ((\lambda - 1)y - x)b_{y+x}^{(r)}\delta_{r,s} \qquad [L_y^{(s)}, c_x^{(r)}] = -(\lambda y + x)c_{y+x}^{(r)}\delta_{r,s}, \tag{72}$$

with respect to the *old* Virasoro modes. (Note that the improvement terms do not need to respect primarity, in particular, of the $c$'s.) If there are nontrivial contributions from $[L_x^{(r)}, U_0^{tot}]$, which cancel in the sum over sheet-labels, the contributions from different sheets have to cancel each other in $U_0^{tot}$ as well. Consider:

$$\begin{aligned}[L_0^{(t)}, [L_x^{(r)}, U_0^{tot}]] &= -[L_x^{(r)}, [U_0^{tot}, L_0^{(t)}]] - [U_0^{tot}, [L_0^{(t)}, L_x^{(r)}]] \\ &\stackrel{(45)}{=} -[L_x^{(r)}, [L_0^{tot}, U_0^{(t)}]] - [U_0^{tot}, (-x)\delta_{r,t}L_x^{(r)}]]\end{aligned}$$

In the first line, I used the Jacobi identity. Because $L_0^{tot}$ counts the total conformal weight of $U_0^{(t)}$, which at least must be required to be well-defined, the first term vanishes. This implies that $[L_x^{(r)}, U_0^{tot}]$ has conformal weight $(-x)\delta_{r,t}$ on the $t$th sheet. Therefore $[L_x^{tot}, U_0^{tot}] = 0$, if and only if for all $u$

$$[L_x^{(u)}, U_0^{tot}] = 0,$$

[12]This also implies that we cannot find other additive deformations of the Virasoro field which have Jordan cells with rank greater than two, because the deformation of $L_0^{(s)}$ then at least has to contain $b_0^{(s)}$ as an overall factor. Otherwise the individual Virasoro algebras are either trivially deformed, or do not commute.

because $[L_x^{(r)}, U_0^{tot}]$ has different quantum numbers (namely $h^{(t)}_{[L_x^{(u)},U_0^{tot}]}$) for $u \neq r$.

Apart from the exceptions made above for $c = -2$, $U_0^{tot}$ has to vanish in my framework. Thus, if a deformation, which is decomposed by $b$-modes only, is required to yield an energy operator, it cannot have Jordan cells.

A further rationale shows that for deformations also containing $c$-modes, at least a weakened condition holds:

$$\sum_{r,s}[L_x^{(r)}, U_0^{(s)}] - \sum_{r,s}[L_0^{(s)}, U_x^{(r)}] + \sum_{r,s}[U_x^{(r)}, U_0^{(s)}] \stackrel{!}{=} x\sum_{s} U_x^{(s)}$$

$$\Leftrightarrow$$

$$\sum_{r,s}[\widetilde{L}_x^{(r)}, U_0^{(s)}] = [\widetilde{L}_x^{tot}, U_0^{tot}] = 0 \qquad \sum_{r,s}[U_x^{(r)}, U_0^{(s)}] = 0$$

In section IV.1 it was already shown that an ansatz containing $c$-modes cannot be linear. Because of that, the second equation in the second line holds true as well, because the terms in the first line have unequal mode contents. The above exceptions are not present in this case, because the $c$-modes transform according to the second part of (72). This implies in particular that for any $s$ the $U_0^{(s)}$ have to reside in the center of the new Virasoro algebra on the $s$th sheet.

Nevertheless, because the maximally extended algebra is not known, I cannot decide whether $U_0^{(s)}$ has to commute with its generators. Therefore I do *not* conclude that Jordan cells are forbidden in a larger algebra, although there can be no Jordan cells with respect to $L_0^{tot}$, as long as it is required to measure the conformal weight correctly.

### V.2.1 Surprising Indecomposable Structures

Although there are no Jordan cells possible for the energy operators, there indeed are indecomposable structures. The action of the deformed Virasoro creation modes on $\mathfrak{Vir}$-primary, free-field-descendants with conformal weight zero yields Virasoro descendants with the original ghost charge plus a sum of Virasoro primaries of lesser charge. All terms are of the same level. Because the symmetry with respect to the lower indices, which excluded Jordan cells eq. (69) exists only for $n = 0$, summation over sheet-labels does not affect this indecomposable structure.
This becomes obvious by considering that there are states $\left|c_{-l}^{(s)}c_{l-n}^{(s)}c_{-i}^{(r)}c_i^{(r)}C^{(\hat{s})}\right\rangle$, with $r \neq s$, $l, i$ suitable for the choice of the parameters $A_{\lambda,l,s}$, which get mapped to $\left|C^{(\hat{s})}\right\rangle$ by the action of $U_n^{(s)}$. In particular, $\left|C^{(\hat{s})}\right\rangle$, which is thought to be an arbitrary product of $c$-modes from the other sheets, can be chosen to be the SL$(2,\mathbb{C})$ invariant (total) vacuum. Therefore, there is one module on every sheet, which contains the irreducible vacuum module as a submodule. The vacuum module is unaltered by the transition to

the deformed Virasoro algebra, because the whole vacuum module is invariant under $U_n^{(r)}$ for all $r$. With respect to the old Virasoro modes, all ground states with equal ghost charge are contained in the same conformal family. With respect to the new Virasoro modes, all ground states with the equal ghost charge modulo four are contained in the one indecomposable module, but only the vacuum vector is a heighest weight vector. The vectors with $s$th weights $j^{(s)}\lambda - \frac{j^{(s)}(j^{(s)}+1)}{2}$, $j^{(s)} < \lambda - 1$ are the quasi-primary states with ghost charge $j^{(s)}$ on which the indecomposable representations are built. This shown more explicit in the subsequent section.

Remark: The modified Virasoro-modes do *not* commute with the current-zero-mode, instead

$$[J_0^{(t)}, U_n^{(k)}] = 2U_n^{(k)}\delta_{k,t} + 2\sum_i B_i^{lt} A_{\lambda,l,k}((\lambda-1)n+l)b_{-i}^{(t)}b_i^{(t)}b_l^{(k)}b_{n-l}^{(k)}(1-\delta_{k,l})$$

as long as modes from the $l$th sheet contribute to $U_n^{(k)}(z)$, i.e. as long as for some $i$, $B_i^{tk}$ is non-zero. In contrast to an LCFT with the usual Jordan cells, the Hilbert space in my deformed theories with $\lambda > 1$ is graded with respect to the energy operators. However, in contrast to usual CFT, the representations of the ghost charge do not coincide with the modules of the deformed Virasoro modes in the setting described before.

## V.3 Logarithmic Ward Identities Revisited?

One may now ask whether there are highest weight states or states corresponding to quasi-primary fields except for the vacuum. With respect to the total old Virasoro algebra, only the $\mathfrak{sl}(2,\mathbb{C})$-invariant vacuum is a highest weight state. Every state, which satisfies the condition that

$$N_{l_s} = 1 \Rightarrow N_{(l+1)_s} = 1 \tag{73}$$

for any of its occupation numbers, is quasi-primary with respect to the individual old Virasoro modes $\mathfrak{Vir}_{old}^{(s)}$.

To investigate which states $|\psi\rangle$ are quasi-primary with respect to the new Virasoro modes $\widetilde{L}_1|\psi\rangle = 0$, it suffices to look for states, satisfying eq. (73), which remain invariant under the deformation mode $U_1^{(s)}$. Because the deformation lowers the mode content, both the Virasoro and the deformation term with conformal weight $-1$ have to vanish separately on these states. Needless to say that I do not refer to the case where the global conformal group is chosen to be undeformed. Evidently $U_1^{(s)}$ vanishes on states which have the form $\left|\psi^{(s)}\right\rangle|\widehat{0^{(s)}}\rangle$, i.e. where only modes from the present sheet contribute. Apart from that it suffices to consider only the part $\left|\psi^{(s)}\right\rangle$ on the actual sheet $s$. By eq. (68) it follows that for all $s$ the action of $U_1^{(s)}$ vanishes on states that do not simultaneously contain $c_{-l}^{(s)}$ and $c_{l-1}^{(s)}$.

This leads to the conclusion that states which contain only modes with conformal weights $\leq 0$ on the significant sheet, and that satisfy the condition eq. (73), are quasi-primary with respect to the new Virasoro modes.

Now consider $U_{-1}^{(s)}$. By the same reasoning as before, it will vanish on all states which do not simultaneously contain $c_{-l}^{(s)}$ and $c_{l+1}^{(s)}$. But, obviously, there is a gap, the product of modes $c_0^{(s)}c_1^{(s)}$ vanishes under the action of $U_1$, but it could give a non-zero result under the action of $U_{-1}$, even in suitable $\mathfrak{Vir}_{old}$ quasi-primary combinations with other modes. On the contrary, there are no inhomogeneities possible in the Ward identity corresponding to $\widetilde{\mathcal{L}}_0$, and there could exist quasi-primary fields satisfying homogeneous differential equations derived from $\widetilde{\mathcal{L}}_{-1}$, which then necessarily satisfy homogeneous differential equation with respect to $\widetilde{\mathcal{L}}_1$. The inverse of the last statement is *not* true. There *are* quasi-primary fields, i.e. which enjoy homogeneous Ward identity involving $\widetilde{\mathcal{L}}_1$ but have inhomogeneities with $\widetilde{\mathcal{L}}_{-1}$. An example is a field corresponding to $\left|\varphi^{\widehat{(s)}}\right\rangle \otimes \left|c_0^{(s)}c_1^{(s)}\right\rangle$, with $\left|\varphi^{\widehat{(s)}}\right\rangle$ containing at least one pair $c_{-i}^{(r)}c_i^{(r)}$ $r \neq s, A_{\lambda,l,s}M_i^{rs} \neq 0$ for at least one $l$.

To my knowledge, it was conventional to assume LCFTs to contain only representations which comprise 'standard' Jordan cells, i.e. there is one primary field and $r-1$ quasi-primary 'logarithmic partner field', such that

$$L_0 \left|h_\psi\right\rangle = h_\psi \left|h_\psi\right\rangle + (1-\delta_{k,0}) \left|h; k-1\right\rangle \qquad k \in \{0, \ldots, r-1\}.$$

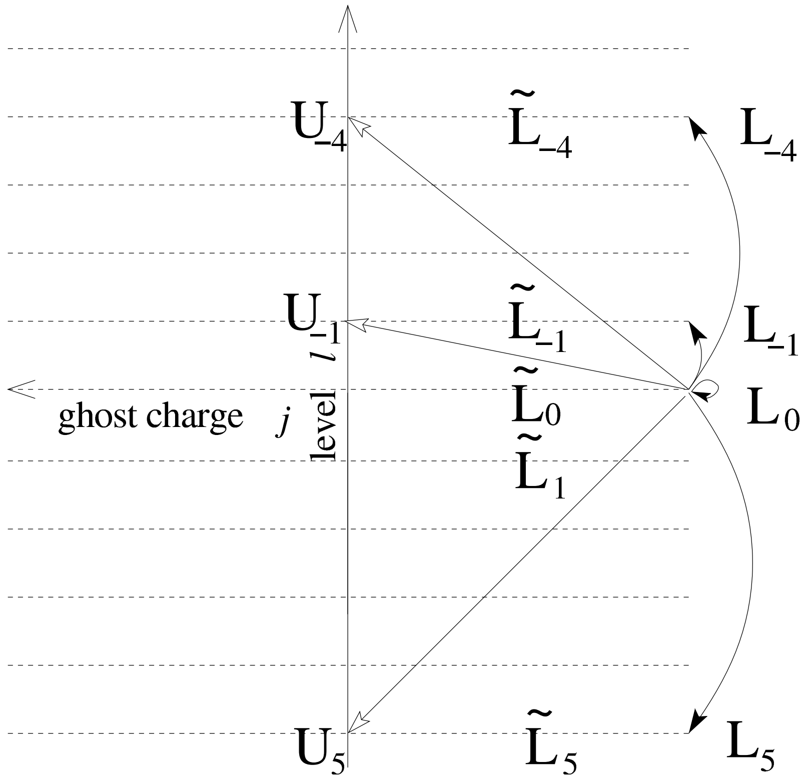
Here, $r$ denotes the rank of the Jordan cell. The Ward identities for logarithmic partner fields containing the operators $\mathcal{L}_0^{log}, \mathcal{L}_1^{log}$ become inhomogenous in this case

$$\begin{aligned} L_{-1}G(z_1,\ldots,z_n) &= \sum_i \partial_i G(z_1,\ldots,z_n) = 0 \\ L_0 G(z_1,\ldots,z_n) &= \sum_i (z_i\partial_i + h_i + \hat{\delta}_{h_i})G(z_1,\ldots,z_n) = 0 \\ L_1 G(z_1,\ldots,z_n) &= \sum_i (z_i^2\partial_i + 2z_i[h_i + \hat{\delta}_{h_i}])G(z_1,\ldots,z_n) = 0, \end{aligned} \tag{74}$$

where $G(z_1,\ldots,z_n) = \left\langle \Psi_{h_1,k_1}(z_1)\ldots\Psi_{h_n,k_n}(z_n)\right\rangle$ and $\hat{\delta}_{h_i}\Psi_{(h_j;k_j)} = \delta_{ij}\Psi_{(h_j;k_j-1)}$, $\hat{\delta}_{h_i}\Psi_{(h_j;0)} = 0$.

These logarithmic Ward identities had to be modified further for the models studied in this and the preceeding section. This observation sets us in the position to see an effect of the extraordinary structure of the 'logarithmically' extended ghost theories for $1 < \lambda \in \mathbb{N}$. To summarise, the indecomposable structures one encounters here have the property that $L_0$ is unaltered (and thus blind with respect to the indecomposable structure), and suitable states are annihilated by $\widetilde{L}_1$, but not by $\widetilde{L}_{-1}$ at the same time. In particular, the indecomposable structure becomes visible in the action of $\widetilde{L}_{-1}$, or $U_{-1}$, respectively. This is a completely different and new sort of indecomposable representation (to LCFT), since the ordinary Jordan-cell-type representations are distinguished by the feature that $L_{-1}$ remains undeformed.

Figure 3: Action of some of the quadrilinear Virasoro modes on a quasiprimary state.



# VI Discussion

## VI.1 Remarks on Hermiticity

The combination of $b$-modes contained in the deformation of simple ghosts is antihermitian $\left(b_0^{(s)}b_n^{(r)}\right)^\dagger = b_{-n}^{(r)}b_0^{(s)} = -b_0^{(s)}b_{-n}^{(r)}$. This resembles the deformation by free fields for $\lambda > 1$, but for these, the prefactor $P(\lambda, y, l, s)$ is mandatory, which on the one hand excludes deformation terms by modes with conformal weight only, and on the other hand, allows to chose the hermiticity properties of the deformation

$$U_y^\dagger = \sum_l P(j, y, l, s)(:\beta_l^s b_{y-l}^s :)^\dagger = \sum_l P(j, y, l, s) : b_{y-l}^{s\dagger}\beta_l^{s\dagger} :$$

$$= -\sum_l P(j, y, -l, s) : \beta_l^s b_{-y-l}^s := \sum_l A(j, -l, s)((j-1)(-y) + l) : \beta_l b_{-y-l} :$$

The deformation is antihermitian if $A(j, l, s) = -A(j, -l, s)$. (54$_\clubsuit$) then implies that $M_{ks} = -M_{sk}$ and analogously for hermitian deformations. Eq. (54) reveals that it can be chosen such that the deformation is either hermitian or antihermitian. As usual in CFT, $(L_n^{(s)})^\dagger = L_{-n}^{(s)}$ and $j_n^{(s)\dagger} = -j_{-n}^{(s)}$. It is still possible that with respect to a (modified) ghost current, states organise into more common indecomposable structures. In particular, due to the anti-hermicity of $j(z)$ opposed to the hermicity of $T(z)$, $j_0$ might possibly admit a logarithmic deformation. Some WZW models are already known to have *currents* exhibiting Jordan cells, the Virasoro fields built out of these currents nevertheless acts diagonalisable. However, an investigation of whether ghost systems admit for similar structures is left to future work. One could be tempted to infer that the deformation $U_0^{(s)}$ has to be antihermitian to render the matrix representation of $L_0^{(s)}$ non-diagonalisable. This contradicts the observation that Jordan cells arise in the matrix representation of $L_0^{(s)}$ regardless of its hermiticity properties. This contradiction is easily resolved by recalling that the vacuum has zero norm, because the pairing is non-trivial. Because the current is anomalous, any non-zero correlation function has to contain $n(2\lambda - 1)$ creation zero modes in excess. The appropriate outstate for the $\mathfrak{sl}(2, \mathbb{C})$-invariant vacuum has to comprise these. Therefore, hermiticity, and its converse, becomes a completely formal concept, which cannot be used to predict indecomposable structures at all. The quadrilinear deformation has a distinct hermicity property

$$A_0^{(s)\dagger} = \sum_{i,p} B_i^{ps^*} b_i^{(p)\dagger} b_{-i}^{(p)\dagger} = \sum_{i,p} B_i^{ps^*} b_{-i}^{(p)} b_i^{(p)} = A_0^{(s)}$$

$$(U_y^{(s)})^\dagger := \sum_l P(\lambda, y, l, s)^* b_{y-l}^{(s)\dagger} b_l^{(s)\dagger} A_0^{(s)\dagger} = \sum_l P(\lambda, y, l, s) b_{l-y}^{(s)} b_{-l}^{(s)} A_0^{(s)}$$

$$= \sum_l - A_{\lambda,-l,s}(\lambda - 1)y - l) b_l^{(s)} b_{-y-l}^{(s)} A_0^{(s)} = A_{\lambda,-l,s}(\lambda - 1)((-y) + l) A_0^{(s)} b_l^{(s)} b_{-y-l}^{(s)}$$

$$= -U_{-y}^{(s)}.$$

## VI.2 Interpretation of the Results of section V.2

One could interpret the vanishing of the deformation of $L_0$ and the modification of the action of $L_{-1}$ on some quasi-primary states as follows: In theories compactified on the torus, the generators of the global conformal group still act as differential operators, $\mathcal{L}_i, i \in \{-1, 0, 1\}$ corresponding to translations, scalings and special conformal transformations, on quasi-primary fields. But the transformation to the new coordinates for radial quantisation maps scalings, translations and special conformal transformations not separately onto themselves, but only the whole group. A rotation in the old coordinates, generated by $L_0$, corresponds to a translation in the new and $\mathcal{L}_1$ generates rotations instead. It is worth noting in this context that for ghost theories with integral spin $\lambda > 1$, we only know applications – namely, the bosonic string – where the worldsheet is naturally compactified on a torus (in light-cone coordinates), in contrast to condensed matter applications of $c = -2$, which naturally live on the complex plane.

One could speculate that indeed these theories may have applications only in cases, where in some sense the indecomposable structure manifests itself in the action on the space of states of the generator of rotations, whereas the translation operator acts diagonalisable.

# VII Summary

In section III.1, I constructed a logarithmic extension of the $c = -2$ model for arbitrary twists by adding 'improvement' terms $U_m^{(k)} := :\beta^{(k)} b_m^{(k)}:$ to the Virasoro modes as a straightforward generalisation of the construction in [FFH+02]. I required the improvement modes to reside in the universal covering of the ghost-mode algebra. I found this to be possible if $\beta$ satisfies $\{\beta^{(k)}, \beta^{(m)}\} = \{\beta^{(k)}, b_l^{(m)}\} = \{\beta^{(k)}, c_r^{(m)}\} = 0$ for all $k, m, \in \{0, \dots, n-1\}$, $\quad l \in \mathbb{Z}$, and, if $k \neq m$, for all $r \in \mathbb{Z} \setminus \{0\}$. The upper index in braces counts the sheets of a ramified covering on the Riemann sphere. I explicitly investigated the action on the space of states of the ansatz

$$\beta^{(k)} := \sum_{s=0}^{n-1} M_{sk} b_0^{(s)}.$$

The action of the modified energy operators $\widetilde{L}_0^{(k)}$, $k \in \{0, \ldots, n-1\}$, on the space of ground states shows the desired rank 2 Jordan cells.

I showed that the simple ghost system is the only fermionic ghost system that allows for this construction. Rank 2 Jordan cells could furthermore be shown to be possible if and only if the ansatz for $\beta$ is a sum of products of weight-zero $b$-modes from different sheets.

In section IV, I developed a general construction to extend arbitrary fermionic ghost system CFTs to contain indecomposable representations. The deformation

$$U_y^{(s)} := \sum_{l \in \mathbb{Z}} P(\lambda, y, l, s) \beta_l^{(s)} b_{y-l}^{(s)}$$

was shown to be subject to the following consistency conditions ( $A_{\lambda,l,s} := P(\lambda, 0, l, s)$):

$$P(\lambda, 0, l, s) \cdot ((\lambda - 1) \cdot x + l) = l \cdot P(\lambda, x, l, s) \qquad (\spadesuit)$$

$$A_{\lambda,l,s} \left\{ c_m^{(r)}, \beta_l^{(s)} \right\} + A_{\lambda,m,r} \left\{ c_l^{(s)}, \beta_m^{(r)} \right\} = \begin{cases} A_{\lambda,l,s} A_{\lambda,m,r} \left\{ \beta_l^{(s)}, \beta_m^{(r)} \right\} & \beta \text{ linear in } b, c \\ 0 & \text{otherwise} \end{cases} \qquad (\clubsuit)$$

$$\left\{ \beta_l^{(s)}, b_m^{(k)} \right\} = 0$$

For the more explicit ansatz $\beta_l^{(k)} := \sum_{k=0}^{n-1} M_{ks} b_l^{(k)}$ the second set of consistency conditions reduces to

$$M_{rs} A_{\lambda,l,s} + M_{sr} A_{\lambda,-l,r} \stackrel{!}{=} 0 \qquad (54)$$

But the individual $\widetilde{L_0^{(k)}}$ do not measure the conformal weight of the states correctly, hence they cannot be interpreted as energy operators on te single sheets. Only the sum $\widetilde{L_0^{tot}}$ can as the energy operator for the tensor product of the theories on the distinct sheets. By eq. (54), $\widetilde{L_0^{tot}}$ is diagonalisable. Above that, this deformation admits a whole subalgebra $\widetilde{L}_0^{(tot)}, \widetilde{L}_{\pm y}^{(tot)}$ of the Virasoro algebra to be left undeformed.

In order to find improvement terms on the individual sheets with a well-defined action on the states, I scrutinised a further ansatz consisting of composite bosonic operators in section V

$$U_y^{(s)} := \sum_l P(\lambda, y, l, s) A_0^{(s)} b_l^{(s)} b_{y-l}^{(s)}.$$

In analogy with the $c = -2$ case I demanded

$$\left[b_n^{(s)}, A_0^{(s)}\right] = \left[c_n^{(s)}, A_0^{(s)}\right] = \left[L_n^{(s)}, A_0^{(s)}\right] = 0$$

for all $s, n$. Then, the solution of (♠), i.e. (♠̂), is still a solution if in addition

$$A_{\lambda,l,s} = -A_{\lambda,-l,s} \tag{64}$$

holds true. Specifying to $A_0^{(s)} := \sum_{i>0,p} M_i^{ps} b_{-i}^{(p)} b_i^{(p)}$ I found

$$A_{\lambda,l,s} M_t^{rs} = A_{\lambda,t,r} M_l^{sr} \tag{67}$$

as sufficient condition for the ansatz to yield commuting Virasoro algebras.

Furthermore, I proved that the requirements of the deformed Virasoro-zero-mode to be an energy operator and $\beta$ to be a member of the universal enveloping algebra of the $b$-modes of the whole Riemann surface suffice to exclude Jordan cells for $L_0^{\lambda>1}$. I studied the action on an augmented space of states and showed that the action nevertheless is indecomposable. I found quasi-primary states on which $L_{-1}$ acts differently than reckoned before: Any state, whose occupation numbers satisfy

$$N_{s,-l} = 0 \quad \vee \quad N_{s,l\pm 1} = 0,$$

vanishes under the action of $U_{\pm 1}^{(s)}$

$$U_{\pm 1}^{(s)} \left|\psi^{(s)}\right\rangle |\widehat{C^{(s)}}\rangle = 0 \qquad \forall \ |\widehat{C^{(s)}}\rangle.$$

In particular, on any sheet $s$, the quasiprimary states $\prod_{i=1}^{\lambda-1} \left|c_i^{(s)}\right\rangle |\widehat{C^{(s)}}\rangle$ vanish under $U_1^{(s)}$, but not under $U_{-1}^{(s)}$. Here, $|\widehat{C^{(s)}}\rangle$ denotes an arbitrary state with insertions of $c$-zero modes from sheets other that $s$. Thus, the global conformal Ward identities have to be altered further for the general case. Hence, the specific conditions, under which the LCFT Ward identities could originally be derived, were necessary indeed.

I constructed a further deformation with higher nilpotence index which is a generalisation to the former ansatz and might prove useful in special applications. In the appendix B, a similar bosonic ansatz with

$$A_0^{(s)} := \sum_{h>0} M(s,h) \prod_{p\neq s} \left(b_{-h}^{(p)} b_h^{(p)}\right)^{N_{h,p}^{(s)}}$$

will be displayed. It leads to the requirements $A_{\lambda,h,r} M(r,l) N_{l,s}^{(r)} = A_{\lambda,l,s} M(s,h) N_{h,r}^{(s)}$.

# VIII Outlook

A very interesting question arising from above is whether the deformations found above are consistent with different extensions of the Virasoro algebra. This could be the maximally extended $\mathcal{W}$-algebra, or a superextension. It is unclear whether for $\lambda > 1$ there could be extended algebras which admit Jordan cells for the zero modes of other generators.

Furthermore, it remains to be investigated whether the indecomposable structures found for higher spin $\lambda$, which lack Jordan cells, lead to logarithmic singularities in correlators. A possibility to verify this would be to find a modified current $\widetilde{j}^{(k)}(z)$ which is consistent with $\widetilde{T}^{(k)}(z) = \frac{1}{2}\!:\widetilde{j}^{(k)}\widetilde{j}^{(k)}\!:(z) + (\frac{1}{2} - \lambda)\partial\widetilde{j}^{(k)}(z)$. If one were able to show that $\widetilde{j}_0$ had Jordan cells, the theory would clearly exhibit logarithmic divergencies. Also, if the zero mode of this current was to commute with all Virasoro modes, $\widetilde{j}_0$ would respect the $\widetilde{L}_0$-grading of the Hilbert space. This would imply that $L_0$ maps states to sums of states which reside in different superselection sectors. This also would be a proof for the theory to be logarithmic. Further examination of the anomaly of the current $j$ and, if found, the deformed current $\widetilde{j}_0$ seems to be necessary.

Another interesting question would be to explore the antiperiodic sector, and, in particular, the Ramond sector of the fermionic 'ghost' systems with half-integer spin $\lambda$. It is as yet unclear whether the twist remains primary with respect to the modified Virasoro modes. Because it has no mode decomposition except in the bosonic language, which proved not to be trustworthy by [Gur93], one would have to use the twisted Borcherd identity of [EFH98] to decide that. Particularly exciting would be a bosonisation scheme of the twists which is compatible with the OPE.

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# A The Long Recurrence Relation

To check whether something went wrong or whether there is a further obvious solution, I redo the calculation with the "big" recurrence relation, collecting the coefficients of distinct combinations of modes:

$$\begin{aligned}
&\{P(\lambda,m,(l-n),k)(\lambda n-l) &&-P(\lambda,m,(m-l),k)((\lambda-1)n-m+l)\\
&+P(\lambda,m,l,k)((\lambda-1)n-m+l) &&-P(\lambda,m,(n+m-l),k)((\lambda-1)n-l+n)\\
&-P(\lambda,n,(l-m),k)(\lambda m-l) &&+P(\lambda,n,(n-l),k)((\lambda-1)m-n+l)\\
&-P(\lambda,n,l,k)((\lambda-1)m-n+l) &&+P(\lambda,n,(n+m-l),k)((\lambda-1)m-l+m)\}\\
&\stackrel{!}{=}(n-m)\{P(\lambda,(n+m),l,k) &&-P(\lambda,(n+m),(n+m-l),k)\}
\end{aligned}$$

My solution from the short recurrence relation obviously solves it.

$$\begin{aligned}
&\big\{A_{\lambda,(l-n),k}((\lambda-1)m+l-n)(\lambda n-l) - A_{\lambda,(m-l),k}(\lambda m-l)((\lambda-1)n-m+l)\\
&+A_{\lambda,l,k}((\lambda-1)m+l)((\lambda-1)n-m+l) - A_{\lambda,(n+m-l),k}(\lambda m+n-l)(\lambda n-l)\\
&-A_{\lambda,(l-m),k}((\lambda-1)n+l-m)(\lambda m-l) + A_{\lambda,(n-l),k}(\lambda n-l)((\lambda-1)m-n+l)\\
&-A_{\lambda,l,k}((\lambda-1)n+l)((\lambda-1)m-n+l) + A_{\lambda,(n+m-l),k}(\lambda n+m-l)(\lambda m-l)\big\}\\
&= \big\{A_{\lambda,(l-n),k}((\lambda-1)m+l-n)(\lambda n-l) + A_{\lambda,(l-m),k}(\lambda m-l)((\lambda-1)n-m+l)\\
&+A_{\lambda,l,k}((\lambda-1)m+l)((\lambda-1)n-m+l) - A_{\lambda,(n+m-l),k}(\lambda m+n-l)(\lambda n-l)\\
&-A_{\lambda,(l-m),k}((\lambda-1)n+l-m)(\lambda m-l) - A_{\lambda,(l-n),k}(\lambda n-l)((\lambda-1)m-n+l)\\
&-A_{\lambda,l,k}((\lambda-1)n+l)((\lambda-1)m-n+l) + A_{\lambda,(n+m-l),k}(\lambda n+m-l)(\lambda m-l)\big\}\\
&\stackrel{!}{=} (n-m)\big\{A_{\lambda,l,k}((\lambda-1)(n+m)+l) - A_{\lambda,(n+m-l),k}(\lambda(n+m)-l)\big\}
\end{aligned}$$

# B A Bosonic Deformation with Lower Nilpotence Index

The deformations discussed above have in common that their nilpotence indices vary with the number of sheets on which the Virasoro field is being deformed. If one is interested in deformations which are nilpotent of a certain lower order one should increase the zero mode content with positive ghost charge of the deformation term. This can be done with a product rather than a sum ansatz: Because commutators with Virasoro modes act derivatively on products, the constraints happen to be quite similar to those obtained from the quadrilinear deformation. The calculations of section V for equal sheet-labels $r = s$ remain valid, but I redo the cases $r \neq s$ by setting $\widetilde{A}_0^{(s)}$ to be $\widetilde{A}_0^{(s)} = M(s) \prod_{\substack{i>0,p \\ p\neq s}} (b_{-i}^{(p)} b_i^{(p)})^{N_{i,p}^{(s)}}$. This gives

$$
\begin{aligned}
0 &= \sum_l P(\lambda, y, l, s) \left[L_x^{(r)}, \widetilde{A}_0^{(s)}\right] b_l^{(s)} b_{y-l}^{(s)} - \sum_t P(\lambda, x, t, r) \left[L_y^{(s)}, \widetilde{A}_0^{(r)}\right] b_t^{(r)} b_{x-t}^{(r)} \qquad (75) \\
&= \sum_{a>0,l} P(\lambda, y, l, s) M(s) (((\lambda-1)x + a) b_{x-a}^{(r)} b_a^{(r)} + ((\lambda-1)x - a) b_{-a}^{(r)} b_{a+x}^{(r)}) \cdot N_{a,r}^{(s)} b_l^{(s)} b_{y-l}^{(s)} \\
&\qquad \prod_{\substack{i>0,p, \\ p\neq s}} ((\widehat{b_{-a}^{(r)} b_a^{(r)}}) \cdot N_{a,r}^{(s)} (b_{-i}^{(p)} b_i^{(p)}))^{N_{i,p}^{(s)}} \\
&- \sum_{b>0,t} P(\lambda, x, t, r) M(r) (((\lambda-1)y + b) b_{y-b}^{(r)} b_b^{(r)} + ((\lambda-1)y - b) b_{-b}^{(s)} b_{b+y}^{(s)}) \cdot N_{b,s}^{(r)} b_t^{(r)} b_{x-t}^{(r)} \\
&\qquad \prod_{\substack{i>0,p, \\ p\neq r}} ((\widehat{b_{-b}^{(s)} b_b^{(s)}}) \cdot N_{b,s}^{(r)} (b_{-i}^{(p)} b_i^{(p)}))^{N_{i,p}^{(r)}},
\end{aligned}
$$

Where the hat denotes that the modes have been removed by the action of the Virasoro modes. The second and fourth line have to be prefactors of the same products, which have to cancel each other. The third and the fifth line are the same, if from every sheet $r$ at most one pair of modes with the equal conformal weights, say $h$ and $-h$, contributes to the product, i.e. if $N_{i,p}^{(r)}$ has the form $N_{i,p}^{(r)} = \delta_{i,h}(1 - \delta_{p,r})$. Because of that, it seems to be more feasible to call $\widetilde{A}_0^{(s)} := A_0^{(s)}(h)$ and redefine $A_0^{(s)}$ to be

$$
A_0^{(s)} := \sum_{h>0} M(s, h) \prod_{p\neq s} \left(b_{-h}^{(p)} b_h^{(p)}\right)^{N_{h,p}^{(s)}}.
$$

By relabeling one obtains

$$\sum_{h>0,l} P(\lambda, y, l, s) M(s, h)(((\lambda - 1)x + h) b_{x-h}^{(r)} b_h^{(r)} + ((\lambda - 1)x - h) b_{-h}^{(r)} b_{h+x}^{(r)}) N_{h,r}^{(s)} b_l^{(s)} b_{y-l}^{(s)}$$

$$\prod_{\substack{p,\\ p\neq s}} ((\widehat{b_{-h}^{(r)} b_h^{(r)}})^{N_{h,r}^{(s)}} (b_{-h}^{(p)} b_h^{(p)}))^{N_{h,p}^{(s)}}) \tag{76}$$

$$-\sum_{h>0,l} P(\lambda, x, l, r) M(r, h)(((\lambda - 1)y + h) b_{y-h}^{(r)} b_h^{(r)} + ((\lambda - 1)y - h) b_{-h}^{(s)} b_{h+y}^{(s)}) N_{h,s}^{(r)} b_l^{(r)} b_{x-l}^{(r)}$$

$$\prod_{\substack{p,\\ p\neq r}} ((\widehat{b_{-h}^{(s)} b_h^{(s)}})^{N_{h,s}^{(r)}} (b_{-h}^{(p)} b_h^{(p)}))^{N_{h,p}^{(r)}})$$

After the domain of summation has been split, the coefficients, i.e. the first and third line above, read:

$$\begin{aligned}
0 = &\sum_{h>0,l>0} P(\lambda, y, l, s) M(s, h)(((\lambda - 1)x + h) b_{x-h}^{(r)} b_h^{(r)}) \cdot N_{h,r}^{(s)} b_l^{(s)} b_{y-l}^{(s)} && \triangle\\
&- \sum_{h>0,l>0} P(\lambda, x, l, r) M(r, h)(((\lambda - 1)y + h) b_{y-h}^{(r)} b_h^{(r)}) \cdot N_{h,s}^{(r)} b_l^{(r)} b_{x-l}^{(r)} && \triangle\\
&+ \sum_{h>0,l>0} P(\lambda, y, l, s) M(s, h)(((\lambda - 1)x - h) b_{-h}^{(r)} b_{h+x}^{(r)}) \cdot N_{h,r}^{(s)} b_l^{(s)} b_{y-l}^{(s)} && \square\\
&- \sum_{h>0,l>0} P(\lambda, x, l, r) M(r, h)(((\lambda - 1)y - h) b_{-h}^{(s)} b_{h+y}^{(s)}) \cdot N_{h,s}^{(r)} b_l^{(r)} b_{x-l}^{(r)} && \circ\\
&- \sum_{h>0,l<0} P(\lambda, y, l, s) M(s, h)(((\lambda - 1)x + h) b_{x-h}^{(r)} b_h^{(r)}) \cdot N_{h,r}^{(s)} b_{y-l}^{(s)} b_l^{(s)} && \circ\\
&+ \sum_{h>0,l<0} P(\lambda, x, l, r) M(r, h)(((\lambda - 1)y + h) b_{y-h}^{(r)} b_h^{(r)}) \cdot N_{h,s}^{(r)} b_{x-l}^{(r)} b_l^{(r)} && \square\\
&- \sum_{h>0,l<0} P(\lambda, y, l, s) M(s, h)(((\lambda - 1)x - h) b_{-h}^{(r)} b_{h+x}^{(r)}) \cdot N_{h,r}^{(s)} b_{y-l}^{(s)} b_l^{(s)} && \triangledown\\
&+ \sum_{h>0,l<0} P(\lambda, x, l, r) M(r, h)(((\lambda - 1)y - h) b_{-h}^{(s)} b_{h+y}^{(s)}) \cdot N_{h,s}^{(r)} b_{x-l}^{(r)} b_l^{(r)} && \triangledown
\end{aligned}$$

Equal tokens indicate that different terms comprise the same mode-contents. Because of eq. (64) one does not need to include terms with $l = 0$ into the sum in order to make the above equation valid. For all $h > 0$ and all $l \neq 0$ as well as $(x, y) \in \mathbb{Z}^2$ and $(r, s) \in [0, \ldots, n-1]^2$ it leads e.g.[13] to the requirements

$$\begin{aligned} &- \sum_{h>0,l>0} P(\lambda, x, l, r) M(r, h)(((\lambda-1)y - h) b^{(s)}_{-h} b^{(s)}_{y+h}) \cdot N^{(r)}_{h,s} b^{(r)}_{l} b^{(r)}_{x-l} \\ &= \sum_{h>0,l>0} P(\lambda, y, -l, s) M(s, h)(((\lambda-1)x + h) b^{(r)}_{x-h} b^{(r)}_{h}) \cdot N^{(s)}_{h,r} b^{(s)}_{y+l} b^{(s)}_{-l}, \end{aligned}$$

thus

$$\begin{aligned} &-P(\lambda, x, h, r) M(r, l) N^{(r)}_{l,s}(((\lambda-1)y - l) \\ &= P(\lambda, y, -l, s) M(s, h) N^{(s)}_{h,r}(((\lambda-1)x + h), \end{aligned}$$

which implies

$$A_{\lambda,h,r} M(r, l) N^{(r)}_{l,s} = -A_{\lambda,-l,s} M(s, h) N^{(s)}_{h,r}.$$

By eq. (64) this becomes, finally

$$\boxed{A_{\lambda,h,r} M(r, l) N^{(r)}_{l,s} = A_{\lambda,l,s} M(s, h) N^{(s)}_{h,r}.} \tag{77}$$

Although the calculations are nearly identical to the case before, there is an important difference: The nilpotence index can be lowered without the need to restrict to a certain number of different contributing zero modes. This could be interesting if one does not want quadratic contributions of the deformation in normal ordered products of the new Virasoro modes, e.g. if one tries to keep generators of an extension to a $\mathcal{W}$-algebra.

Of course, linear combinations of terms characterised by $\sum_p N^{(s)}_{h,p} = m$ are possible as well, if the above condition are met for every power in $b$-modes.

# C Some Words about the Notion of Zero Modes

Having specified a positive definite metric, tensors on $\Sigma$ can be written as section of a line bundle $K^{(m,n)}$. Then, the covariant derivative acts as

$$\nabla \phi = \nabla^{(m)}_{\bar{y}} \phi + \nabla^{(m)}_{y} \phi$$

on sections $\phi \in K^{(m,n)}$. $\nabla^{(m)}_{\bar{y}}$ is the Cauchy-Riemann operator. Sections in $K^{(m,n)}$ can be identified with those of $K^{(m-n,o)}$. The Laplacians then read

$$\Delta^{+}_{(m)} = -2\nabla^{y}_{(m+1)} \nabla^{(m)}_{y} \qquad \Delta^{-}_{(m)} = -2\nabla^{(m-1)}_{y} \nabla^{y}_{(m)}. \tag{78}$$

[13]There are four sets of equations, which lead to identical results.

with

$$\begin{aligned} \nabla^y_{(m)} : K^m &\to K^{m-1} \qquad \nabla^y_{(m)}\phi = (g_{y\bar{y}})^{-1}\partial_{\bar{y}}\phi \otimes (\mathrm{d}y)^{-1} \\ \nabla^{(m)}_y : K^m &\to K^{m+1} \qquad \nabla^{(m)}_y\phi = (g_{y\bar{y}})^{m}\partial_{\bar{y}}(g_{y\bar{y}})^{-m}\phi \otimes \mathrm{d}y. \end{aligned}$$

The Riemann-Roch-Atiyah-Singer theorem states

$$\dim \mathrm{Ker}\nabla^{(m)}_y - \dim \mathrm{Ker}\nabla^y_{(m+1)} = (m + \frac{1}{2})\chi(\Gamma) = (2m - 1)(1 - g).$$

That implies that at least the number on the r.h.s (the index of the Cauchy-Riemann operator) of zero modes have to be included into the path integral. Only for $g = 1$ one has to include all $b$- and $c$-zero modes, in the other cases, one has to include exactly the number the index dictates.

# D Vertex Operators and Vertex Operator Algebras

I want to note here that a meromorphic CFT fulfills the axioms of a vertex operator algebra (VOA). For meromorphic CFTs it provides a very powerful and condensed formulation. To my knowledge a similarly stringent formulation for non-meromorphic or even logarithmic settings is a field of active research.

A vertex operator algebra $\mathfrak{V}$ is determined by the following data [FLM]:

1. a $\mathbb{Z}$-graded Vector space $\mathcal{V} = \bigoplus_{n\in\mathbb{Z}} \mathcal{V}^n$, $\mathcal{V}^n$ finite dimensional and homogeneous
2. Two distinct elements $|0\rangle \in \mathcal{V}^0$, $L_{-2}|0\rangle \in \mathcal{V}^2$
3. A distinguished endomorphism $L_{-1} \in \mathrm{End}(\mathcal{V})$ with $L_{-1}|0\rangle = 0$.
4. A field-state-correspondence: A linear mapping $Y(\cdot, z) : \mathcal{V} \to \mathrm{End}(\mathcal{V})[[z, z^{-1}]]$, $z$ being a formal variable.

satisfying the following relations

1. $Y(|0\rangle, z) = \mathbb{1}_{\mathcal{V}}$ as well as $Y(\phi, z)|0\rangle \in \phi + z\mathcal{V}[[z]]\ \forall\phi \in \mathcal{V}$.
2. $Y(L_{-2}|0\rangle, z) = T(z) = \sum_n L_n z^{-n-2}$ The Virasoro field has a mode expansion, the modes of which satisfy the Virasoro algebra

$$\left[L_q, L_m\right] = (q - m)L_{q+m} + \frac{\hat{c}}{2}\binom{q+1}{3}\delta_{q+m,0}$$

$$\left[L_q, \hat{c}\right] = 0\ \forall q \in \mathbb{Z}$$

3. The field-state correspondence is compatible with the grading: $\phi_{-n}$, which is a mode of the expansion $Y(\phi, z) = \sum_n \phi_n z^{-n-h_\phi}$, is a mapping from $\mathcal{V}^m$ to $\mathcal{V}^{m+n}$.

4. $\forall \phi \in \mathcal{V} : Y(L_{-1}\phi, z) = \partial Y(\phi, z)$.

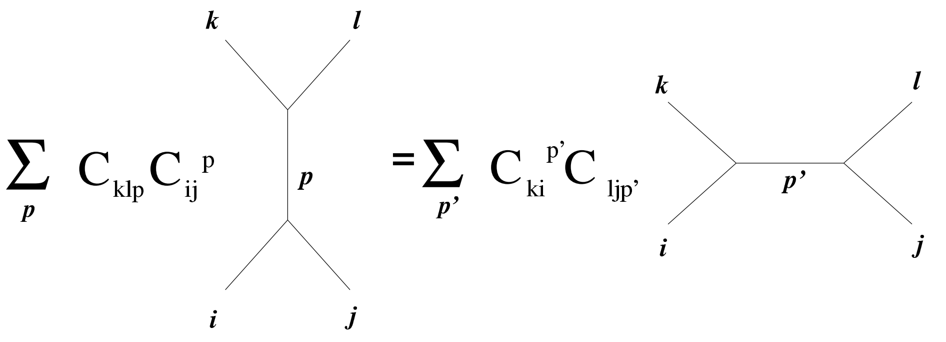
5. Vectors of $\mathcal{V}$ enjoy a locality property: $\forall \phi, \psi \in \mathcal{V}\ \exists N_{\phi\psi} \in \mathbb{N}$ : $(z-w)^{N_{\phi\psi}} \left[Y(\phi, z), Y(\psi, w)\right] = 0$.

Wilson proved that a theory can be characterised equivalently by fixing equal-time commutators or by specifying ann operator product algebra (OPA). In the latter case, one fixes a (finite) basis of so-called 'simple' fields and demands that all fields of the theory are contained in the vector space, which is spanned by them and all words in coordinate derivatives and normalordered products applied to them. This space of fields is assumed to close under a certain product, the operator product expansion (OPE), which is specified by other requirements of the theory, such as associativity, locality, Lorentz or conformal covariance . . . The OPA formulation is most appropriate and convenient in two dimensional CFT, because the additional symmetry causes the OPE of simple fields to close within themselves. For that reason, two- and three-point-functions could in principle be calculated exactly by the so-called bootstrap approach.

$$\phi_i(z)\phi_j(y) = \sum_k C_{ij}^k(z-y)\phi_k(y)$$

and the associativity

$$(\phi_i(z)\phi_j(y))\phi_k(x) = \phi_i(z)(\phi_j(y)\phi_k(x)) =$$
$$\sum_{p',l} C_{ij}^{p'}(z-y)C_{lkp'}(y-x)\phi_k(x) = \sum_{p,l} C_{ijp}(z-x)C_{lkp}(y-x)\phi_k(x)$$




Here, I briefly review the approach of Fjelstad et al., because some of their concepts proved very useful to our approach and its motivation.

It was already mentioned that in a first step, they enlarged the space of states by taking the tensor product with a finite dimensional vector space K. Thereby, the Virasoro modes get improved by an operator $\beta \in$ End_K times a recognised mode. Then, the whole OPA is deformed by applying a superderivation on it. Under certain circumstances, this is an isomorphism of OPAs, and the improvement modes of the Virasoro field are obtained by expressing it in terms of the deformed basic fields. They deform Operator Product Algebras, i.e. graded (super-) vector spaces with a collection of bilinear operations

$$[\cdot,\cdot]_n : \mathcal{V}\times\mathcal{V} \longrightarrow \mathcal{V} \qquad \forall n \in \mathbb{Z}. \tag{79}$$

which respect grading and associativity. They define a mapping

$$\begin{aligned} \Delta\cdot:\ & \mathcal{V}\times\mathcal{V} \longrightarrow \widetilde{\mathcal{V}} \otimes_{\mathbb{C}} \mathbb{C}[[z,z^{-1}]][\log(z)] \\ & (E,A) \mapsto \Delta_E A = [E,A]_1 \log(z) + \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}\frac{[E,A]_{n+1}}{z^n}, \end{aligned}$$

treating $\log(z)$ as a new formal variable. To fix a field in the first factor renders the map a superderivation on the second. They introduce an auxiliary vector space K and take the tensor product with the old space of states $\mathcal{V}\otimes$ K, such that for a distinct state $\omega \in$ K, $\Omega_{\mathcal{V}} \otimes \omega = \Omega$ is the vacuum of the enlarged stace of states. They define $\mathrm{End}_+(\mathrm{K}) = \{\beta : \beta\omega = 0\}$ and $\mathrm{End}(\mathrm{K}) = \mathrm{End}_+(\mathrm{K}) + \mathrm{End}_-(\mathrm{K})$. Then they consider the mapping

$$A \mapsto \exp(\beta\Delta_E)A \qquad \beta \in \mathrm{End}K$$

They observe that if either $E$ or $\beta$ are nilpotent, e.g. $E,\beta$ fermionic, the mapping is an isomorphism of OPA's (and the exponential becomes a finite sum). They observe that if and only if $E$ is chosen to be a (fermionic) screening current, no logarithms occur in the deformed fields. In this case, $\oint E$ is the differential in a complex. Because the pristine and the deformed OPA are isomorphic under the afore-mentioned circumstances, deformed composite fields (like the Virasoro field) are obtained by composition of the corresponding deformed basic fields, i.e. the deformation commutes with expanding operator products. The choice of the annihilation mode to be nilpotent renders the exponential a polynomial, in their case, a linear one. By this OPA deformation the deformed Virasoro field $T(z) + :\beta\frac{b(z)}{z}:$ is obtained as usual in CFT, but is composed out of the deformed basic fields.

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Ich versichere, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die Zitate kenntlich gemacht habe.

Bonn, Januar 2005