

Conformal field theory approach to bulk wave functions in the fractional quantum Hall effect

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We propose to describe bulk wave functions of fractional quantum Hall states in terms of correlators of non-unitary b/c -spin systems. These yield a promising conformal field theory analogon of the composite fermion picture of Jain. Fractional statistics is described by twist fields which naturally appear in the b/c -spin systems. We provide a geometrical interpretation of our approach in which bulk wave functions are seen as holomorphic functions over a ramified covering of the complex plane, where the ramification precisely resembles the fractional statistics of the quasi-particle excitations in terms of branch points on the complex plane. To extend Jain's main series, we use the concept of composite fermions pairing to spin singlets, which enjoys a natural description in terms of the particular $c = -2$ b/c -spin system as known from the Haldane-Rezayi state. In this way we derive conformal field theory proposals for lowest Landau level bulk wave functions for more general filling fractions. We obtain a natural classification of the experimentally confirmed filling fractions which does not contain prominent unobserved fillings. Furthermore, our scheme fits together with classifications in terms of K -matrices of effective multilayer theories leading to striking restrictions of these coupling matrices.

I. INTRODUCTION

The fractional quantum Hall effect (FQHE) is one of the most fascinating and striking phenomena in condensed matter physics.¹ Certain numbers, the filling fractions $\nu \in \mathbb{Q}$, can be observed with an extremely high precision in terms of the Hall conductivity $\sigma_H = \nu$ in natural units. These numbers are independent of many physical details such as the geometry of the sample, its purity, the temperature – at least within large bounds. The enigmatic and fascinating aspect of this phenomenon is that only a certain set of these fractional numbers ν can be observed in experiments: despite ongoing attempts in varying the purity (or disorder), the external magnetic field and various other parameters, the set of observed fractions has not changed considerably over the last few years.²⁻⁵

It was realized quite early that the FQHE shows all signs of universality and large scale behavior.^{6,7} Independence of the geometrical details of the probe and its size hints towards an effective purely topological field theory description. Indeed, since the quantum Hall effect is essentially a (2+1)-dimensional problem, the effective theory is regarded to be dominated by the topological Chern-Simons term $a \wedge da$ instead of the Maxwell term $\text{tr}F^2$. Some good reviews on the theory of the FQHE are.⁸⁻¹¹

However, one is ultimately interested in a microscopic description of the FQHE. One may start with the task of finding eigenstates of an exact microscopic Hamiltonian. This can be done numerically for small numbers of electrons. The great achievement of Laughlin was to realize how a many-particle wave function looks like if it is to respect a few common sense symmetry constraints:¹²

$$\Psi_L(z_n) = \prod_{1 \leq i < j \leq N} (z_i - z_j)^{2p+1} \exp\left(-\frac{1}{4} \sum_{1 \leq i \leq N} |z_i|^2\right). \quad (1)$$

We now know that Laughlin's wave functions are extremely good approximations to the true ground states, and they are exact solutions for Hamiltonians with certain short-range electron-electron interactions. They describe fractional quan-

tum Hall states (FQH states) with filling $\nu = 1/(2p + 1)$, $p \in \mathbb{Z}_+$. Soon after, various so-called hierarchical schemes were developed yielding ground state wave functions for other rational filling factors.¹³⁻¹⁷ The important point to note here is that the ground state eigenfunctions are time-independent up to a trivial global phase. Thus, one might view them as solutions of a (2+0)-dimensional problem. This is, more or less, the main idea behind all attempts to describe the bulk wave functions in terms of conformal field theory (CFT) correlators.

The Laughlin wave functions describe special incompressible quantum states of the electrons, so-called quantum droplets. Incompressibility is connected to the existence of energy gapless excitations on the border of the quantum state.^{6,7,18-23} The latter can successfully be described in terms of CFTs with current algebras as chiral symmetries. Furthermore, there is an exact equivalence between the (2+1)-dimensional Chern-Simons theory in the bulk and the (1+1)-dimensional conformal field theory on the boundary describing the edge excitations.²⁴ We note here that, naturally, such CFTs have to be unitary, since they describe the time evolution of spatially one-dimensional waves propagating on S^1 .

However, Laughlin's bulk wave functions in a static (2+0)-dimensional setting show a striking resemblance to correlation functions of a free Euclidean CFT put on the (compactified) complex plane. This resemblance has motivated quite a number of works trying to find a CFT description of bulk wave functions in the FQHE.²⁵⁻²⁸ Most approaches assumed from the beginning that these "bulk" theories are unitary. We stress here that this assumption is void, since the bulk wave functions one typically wants to represent are time-independent eigenfunctions. Moreover, most approaches represented the bulk wave functions in terms of building blocks belonging to classes of CFTs with continuous parameters, *e.g.*, the Gaussian $c = 1$ CFTs. The immanent problem with these approaches is that there is no principle selecting the wave functions for experimentally observed filling fractions. Therefore, almost all approaches so far easily accommodate arbitrary rational filling factors. On the other hand, it is not entirely sur-

prising that the bulk wave function should have something to do with CFT. As mentioned above, the observable quantities of the quantum Hall system are largely independent of the precise form and size of the sample. Thus, the normalized charge distributions of the electrons should be invariant under scaling (up to an exponential factor) and area preserving changes of the shape of the sample. The first symmetry is linked to conformal invariance, the latter to the $\mathcal{W}_{1+\infty}$ -algebra.^{29,30} In fact, it is known that in the two-dimensional case global scaling invariance implies full conformal invariance under certain benign circumstances.

Interestingly, there exists a particularly enigmatic FQH state with $\nu = 5/2$. This is one of the very few states with an even denominator filling. Of course, attempts have been made to describe proposed bulk wave functions for this state with the help of CFT correlators.^{21,25,31,32} In this case, however, it turned out that this can only be done if the CFT in question has central charge $c = -2$. Thus, for this FQH state we necessarily have to use a non-unitary theory. On the other hand, this CFT is well known, it is the b/c -spin system of two anti-commuting fields with spins one and zero, respectively. Therefore, following the proposal of Haldane and Rezayi it naturally yields precisely the object to be expected in this FQH state, namely spin singlet states of paired electrons. In addition, the $c = -2$ CFT contains a \mathbb{Z}_2 -twisted sector created by a primary field μ of conformal scaling dimension $h_\mu = -1/8$, which accurately describes the effect of single flux quanta piercing the quantum droplet. Thus, this theory successfully characterizes the ground state and its physically expected excitations with the correct fractional statistics, and only these. The Haldane-Rezayi state is motivated geometrically and proposes spin singlet electron states to be the fundamental objects responsible for the even denominator plateau. However, there are strong experimental indications that the $\nu = 5/2$ FQH state is spin polarized which were investigated numerically by Morf.³³

Nonetheless, the present paper starts from the geometrical viewpoint of the Haldane-Rezayi FQH state via a non-unitary spin-system CFT to revisit the question, how FQH state bulk wave functions can be represented in terms of CFTs. In contrast to other approaches we will drop the assumption that these CFTs should be unitary because there is no physical reason for it. This enables us to concentrate on a different class of CFTs, namely the b/c -spin systems of two anti-commuting fields of spins j and $(1-j)$, respectively. Locality forces $j \in \mathbb{Z}/2$ such that we confine ourselves to a discrete series of CFTs. It will turn out that our ansatz not only naturally explains all experimentally observed filling fractions, but, in addition, does not predict new unobserved series.

Besides these convenient features our approach yields a beautiful geometrical picture for the CFTs we use to represent the bulk wave functions. Additionally, we find correlations of spin j (or spin $1-j$) composite fermions with flux quanta of precisely the fractional statistics which are theoretically predicted from first principles. These statistics, say $1/m$, manifest themselves naturally in the presence of \mathbb{Z}_m -twists which in turn have the geometrical meaning of replacing the complex plane by an m -fold ramified covering of itself. Thus, the

bulk wave functions finally are recast in a language of complex analysis, *i.e.*, j - or $(1-j)$ -differentials on \mathbb{Z}_m -symmetric Riemann surfaces.

Most of the observed filling fractions $\nu \in \mathbb{Q}$ have an odd denominator, which comes from the basic fact that the elementary entities in the quantum Hall system are fermions. It turns out that unpaired fermions correspond to spin systems with spin j half-integer (remember that the paired electrons singlets in the Haldane-Rezayi state were described by an integer-spin system). An essential part of our paper is that we will propose a new hierarchical scheme in which filling fractions can be derived from others by means of forming more and more paired singlets. Besides Jain's principle series, this yields further series precisely catching all confirmed filling fractions. Unobserved filling fractions are no problem within our scheme, since they all lie at the far end of our series or are characterized by series of higher order. In contrast to this, most other hierarchical schemes predict certain unobserved fractions, since prominent experimentally confirmed ones can only be realized at a certain order k within the hierarchy while others obtained at smaller orders of the hierarchy do not show up in experiments. The problem is the lack of a physical reason why the corresponding low order FQH state does not exist, but the higher order FQH state derived from it in the hierarchy. Thus, we believe that our scheme provides a natural explanation for the completeness of the set of experimentally accessible filling fractions which does not run into this problem.

Our paper proceeds as follows: To be as self-contained as possible we collect the essential formulae and concepts of CFT in section two. We are not very general here, since we only concentrate on those facts which are relevant for the special CFTs, *i.e.*, the spin systems, that we will use throughout the remainder of the paper. The reader unfamiliar with CFT might consult the following books.^{34,35}

In section three, we briefly review the basic idea of Laughlin leading to his seminal trial wave functions. Furthermore, we present the appropriate generalization of these within the picture of Jain which allows to describe a large class of FQH states in terms of an effective integer quantum Hall effect (IQHE) of effective elementary particles, the composite fermions (CFs). We favor this idea, since our CFT ansatz contains fields which can naturally be identified with such composite fermions. Moreover, Jain's picture has the advantage to realize most of the prominently observed filling fractions within the first level of its hierarchical scheme.

Section four is the core of our paper. More general Laughlin type trial wave functions in the lowest Landau level (LLL) projection are obtained from multilayer states which are derived from Chern-Simons theory. In this scheme, all essential information can be understood to be encoded in a certain matrix K describing the coupling of the layers, *i.e.*, of different quantum fluids. It respects many general principles, such as topological order. Evident physical properties lead to severe constraints on these K -matrices, which will be seen to coincide nicely with the constraints we find for our spin system CFTs. Step by step we develop the b/c -spin approach in terms of beginning with the simplest case of the principal main se-

ries of Jain's hierarchy. All other confirmed filling fractions are consecutively obtained by pairing of CFs to spin singlet states. This can be done to a lower or higher degree resulting in our novel hierarchical scheme being independent of the pure Chern-Simons formalism. By this, we do not have to make use of the principle of particle-hole duality that is not suitable to form a hierarchy of states by order of stability. Furthermore, our pairing scheme, which is represented by tensoring the spin CFTs with additional spin-singlet b/c -spin systems of central charge $c = -2$, puts severe constraints on the possible form of the bare K -matrix, restricting it essentially to block form. After developing our approach to the point that all observed filling fractions are obtained and certain prominent rational numbers, which were never experimentally confirmed, are ruled out within our approach, we finally provide some predictions for future experiments.

In the concluding fifth section, we summarize our results and try to put them into context. We also mention unsolved problems and some directions for possible research in the future. The appendix contains some sketchy remarks, that the space of states of our non-unitary theories appropriately coincides with the space of states of the (1+1)-dimensional theories describing the edge excitations.

II. CONFORMAL FIELD THEORY

During the last decades conformal field theory (CFT) became one of the most powerful tools of modern theoretical physics.³⁶ Surely, one of the most important impulses came from statistical mechanics: CFT is well-known for its applicability to statistical systems at criticality. At a continuous phase transition the correlation length diverges and the system becomes scale invariant. In two dimensions this usually implies conformal invariance of the system. If the corresponding CFT is identified and found to be rational we can derive the partition function and the problem is solved in a very elegant and effective way. Apart from that there are lots of phenomena, for instance bosonization, in solid state physics involving CFT even if it is not apparent at first sight. Often when geometrical or topological aspects arise CFT is close at hand and allows to derive global properties without detailed knowledge of microscopic structures.

We want to stress that this paper deals with bulk CFTs in 2+0 dimensions. Therefore, it does not make sense to argue about unitarity, time evolution and similar aspects. Of course, the corresponding edge theory has to be unitary, but this implies no crucial restriction for the bulk part.

The theories used in our picture are the b/c -spin systems that were analyzed in detail by Knizhnik.^{37,38} They are described by the action

$$S = \int d^2z b(z)\bar{\partial}c(z) + h.c. \quad (2)$$

Here, $b(z)$ and $c(z)$ are anti-commuting conformal fields of weight $j \in \mathbb{Z}/2$, and $1 - j$ respectively, where z is a coordinate in the complex plane. Mathematically spoken the fields $b(z)$ and $c(z)$ describe j - and $1 - j$ -differentials. Therefore,

they are directly related to the cohomology of the topological space they live on. Furthermore, these theories are chiral CFTs so we can treat the holomorphic part independently. This nicely coincides with the fact that FQH states considered in the lowest Landau level (LLL) are described by holomorphic wave functions.

By variation of the action via path integral we get the equations of motion:

$$\begin{aligned} (\bar{\partial}c(z))b(z') &= (\bar{\partial}b(z))c(z') = \delta^2(z - z', \bar{z} - \bar{z}') \\ \text{and } \bar{\partial}b(z) &= \bar{\partial}c(z) = 0. \end{aligned} \quad (3)$$

In classical terms we would expect

$$(\bar{\partial}c(z))b(z') = (\bar{\partial}b(z))c(z') = 0. \quad (4)$$

Thus, the normal-ordered product of the two fields in order to satisfy (4) reads:

$$:b(z)c(z'):= b(z)c(z') - \frac{1}{z - z'}. \quad (5)$$

In 2d CFT a product of local chiral operators can be expanded in an operator valued Laurent series with meromorphic functions as coefficients. In the evaluation of correlators these so-called operator product expansions (OPEs) play an important role. The OPEs of the two fields $b(z)$ and $c(z')$ can be read off directly from (5):

$$b(z)c(z') \sim \frac{1}{z - z'} \quad , \quad c(z)b(z') \sim \frac{1}{z - z'}. \quad (6)$$

Here, ' \sim ' denotes 'equivalent up to regular terms'. These regular terms vanish if evaluated in a correlator.

The energy-momentum tensor $T(z)$ of the theory can be derived by varying the action S with respect to the induced metric. This yields

$$T(z) = (1 - j):(\partial b(z))c(z): - j:b(z)(\partial c(z)): . \quad (7)$$

In principle there are just a few facts we have to know about a general CFT: the central charge c and the set of conformal weights $\{h_i\}$ of its fields are two of them. They can be derived by OPEs involving the energy-momentum tensor:

$$T(z)b(w) \sim \frac{j}{(z - w)^2}b(w) + \frac{1}{z - w}\partial_w b(w) \quad , \quad (8)$$

$$T(z)c(w) \sim \frac{1 - j}{(z - w)^2}c(w) + \frac{1}{z - w}\partial_w c(w) \quad , \quad (9)$$

$$\begin{aligned} T(z)T(w) &\sim \frac{\frac{1}{2}(-12j^2 + 12j - 2)}{(z - w)^4} + \frac{2}{(z - w)^2}T(w) \\ &+ \frac{1}{z - w}\partial_w T(w) \quad . \end{aligned} \quad (10)$$

Equations (8) and (9) can be understood as the definition of a primary conformal field, the numerator of the first term of the OPE yields its conformal weight h . The third OPE contains a so-called anomalous term that is not proportional to the field itself or its derivatives. This term is due to the existence of a

central extension of the algebra of conformal symmetries. In fact, in all CFTs the OPE of $T(z)$ with itself reads

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) . \quad (11)$$

We find

$$c_{b/c\text{-spin}} = -2(6j^2 - 6j + 1) . \quad (12)$$

For $j \neq \frac{1}{2}$ the central charge is negative (as $j \in \mathbb{Z}/2$). Therefore, the b/c -spin systems used in our scheme are non-unitary. We will briefly discuss this issue and how our approach fits together with the unitary edge theories in the appendix. Furthermore, there exists an additional symmetry of the action. Under the simultaneous transformation

$$b(z) \rightarrow b(z) \exp(i\alpha) \quad \text{and} \quad c(z) \rightarrow c(z) \exp(-i\alpha) \quad (13)$$

the action remains unchanged. The corresponding conserved spin current $j(z)$ reads:

$$j(z) = -:b(z)c(z): \quad (14)$$

with its conserved charge

$$Q_{(i\alpha),j} = \frac{1}{2\pi i} \oint_0 dz (i\alpha)j(z) . \quad (15)$$

To stress it again, the b/c -spin systems are directly related to the topology they live on. In our picture we are interested in Riemann surfaces (RS) with global \mathbb{Z}_n -symmetry. This means that every branch point is of order n and that all monodromy matrices can be diagonalized simultaneously. It is sufficient to do the calculation locally for a single branch point at z_0 . The results can be directly extended to m branch points.

A \mathbb{Z}_n -symmetric RS can be locally represented by a branched covering of the compactified complex plane ($\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) with the following map:

$$z : \text{RS} \rightarrow \widehat{\mathbb{C}} , \quad z(y) = z_0 + y^n . \quad (16)$$

We identify the RS locally by n sheets of $\widehat{\mathbb{C}}$ via the inverse map of (16). The b/c -spin fields living on the RS are therefore represented by an n -dimensional vector of identical copies of the b/c -fields $b^{(l)}(z)$ and $c^{(l)}(z)$ on the complex plane with boundary conditions

$$\begin{aligned} \hat{\Pi}_{z_0} b^{(l)}(z) &= b^{(l+1)}(z) , \quad l = 0, \dots, n-1 , \\ b^{(n)}(z) &= b^{(0)}(z) , \end{aligned} \quad (17)$$

where

$$\hat{\Pi}_{z_0} : (z - z_0) \rightarrow (z - z_0) \exp(2\pi i) . \quad (18)$$

For further investigation we introduce a Fourier basis

$$\begin{aligned} b_k(z) &= \sum_{l=0}^{n-1} \exp\left(\frac{-2\pi i(k+j(1-n))l}{n}\right) b^{(l)}(z) , \\ c_k(z) &= \sum_{l=0}^{n-1} \exp\left(\frac{+2\pi i(k+j(1-n))l}{n}\right) c^{(l)}(z) . \end{aligned} \quad (19)$$

This basis diagonalizes $\hat{\Pi}_{z_0}$:

$$\begin{aligned} \hat{\Pi}_{z_0} b_k(z) &= \exp\left(\frac{+2\pi i(k+j(1-n))}{n}\right) b_k(z) , \\ \hat{\Pi}_{z_0} c_k(z) &= \exp\left(\frac{-2\pi i(k+j(1-n))}{n}\right) c_k(z) . \end{aligned} \quad (20)$$

As a consequence the conserved spin current (14) becomes single-valued. Therefore, the corresponding charge vector α_k identified with the branch point z_0 is

$$\alpha_k = -\frac{k+j(1-n)}{n} . \quad (21)$$

Now we bosonize the theory. This means that we express the spin fields in terms of exponentials of analytic scalar bosonic fields Φ_k

$$\begin{aligned} b_k(z) &=: \exp(+i\Phi_k(z)) : , \\ c_k(z) &=: \exp(-i\Phi_k(z)) : . \end{aligned} \quad (22)$$

This yields that the branch point of the \mathbb{Z}_n -symmetric RS is related to a primary conformal field of the b/c -spin system:

$$V_{\vec{\alpha}}(z_0) = : \exp\left(-i \sum_{k=0}^{n-1} \alpha_k \Phi_k\right) : . \quad (23)$$

It is called Vertex operator and has conformal weight

$$h_{\vec{\alpha}} = \sum_{k=0}^{n-1} h_{\alpha_k} = \sum_{k=0}^{n-1} \left(\frac{1}{2}\alpha_k^2 - (j - \frac{1}{2})\alpha_k\right) . \quad (24)$$

As these fields are of central importance in our context, let us look a bit more carefully at them. They are primary conformal fields arising in the most general case from the CFT of the free boson with an embedded background charge and are defined by

$$V_{\ell}(z) = : \exp(i\ell\Phi(z)) : . \quad (25)$$

Here, $\Phi(z)$ is a free scalar bosonic field with conformal weight $h = 0$. The OPE reads

$$\Phi(z)\Phi(w) \sim -\ln(z-w) . \quad (26)$$

The energy-momentum tensor is given by

$$T(z) = -\frac{1}{2} : \partial_z \Phi(z) \partial_z \Phi(z) : + i\alpha_0 \partial^2 \Phi(z) , \quad (27)$$

where α_0 is a background charge placed at infinity. This leads to the following OPEs:

$$\begin{aligned} T(z)T(w) &\sim \frac{\frac{1}{2}(1-12\alpha_0^2)}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) \\ &\quad + \frac{1}{z-w}\partial_w T(w) , \end{aligned} \quad (28)$$

$$\begin{aligned} (z)\partial\Phi(w) &\sim \frac{2i\alpha_0}{(z-w)^3} + \frac{1}{(z-w)^2}\partial_w\Phi(w) \\ &\quad + \frac{1}{z-w}\partial_w^2\Phi(w) . \end{aligned} \quad (29)$$

The background charge is derived from (28) and (12)

$$\alpha_0 = \frac{1}{2} - j \quad (30)$$

in order to bosonize the theory correctly. Furthermore, it follows from (29) that $\partial\Phi$ is not a primary conformal field unless the background charge vanishes. In fact, even Φ itself is not primary, as it is expected from (26). Due to the logarithmic term in its OPE the Vertex operator is the remaining candidate for a primary field. In fact,

$$T(z)V_\ell(w) \sim \frac{\ell^2/2 - \alpha_0\ell}{(z-w)^2} V_\ell(w) + \frac{1}{z-w} \partial_w V_\ell(w) \quad (31)$$

proves that $V_k(z)$ is primary with conformal weight $h = \ell^2/2 + (j - 1/2)\ell$. This is indeed the result of (24).

Having a closer look at (21) we immediately find that the charge vector of the Vertex operator is dominated by the \mathbb{Z}_n -symmetry of the RS. The spin j simply provides an offset which is just visible in the conformal weight of the fields since the phase is determined by $\alpha_k \bmod 1$. In addition we have to distinguish between two different types of fields. First, there are twist fields that contain the full information of the branch point. Therefore, the charge vector $\vec{\alpha}$ has to keep track of analytic continuation. For example, given a \mathbb{Z}_3 -symmetric RS and $j = 3/2$, the charge vector is derived as

$$\vec{\alpha}_{n=3,j=3/2} = (1, 2/3, 1/3) . \quad (32)$$

Secondly, there are projective fields. Their charge components are identical as if we simply had an n -fold copy of $\hat{\mathbb{C}}$. This yields charge vectors $\vec{\alpha}^p$ with

$$\vec{\alpha}_1^p = \dots = \vec{\alpha}_n^p \in \left\{ 0, \frac{1}{n}, \dots, 1 \right\} . \quad (33)$$

We stress once more the important role of the charge vectors $\vec{\alpha}$. Besides local chiral fields, whose charge vectors have integer valued components only, we include fractional ones (33). The effect of the corresponding vertex operators is to precisely simulate the action of a branch point of ramification number n . This is exactly the effect we expect from fractional statistics of quasi-particles. Thus, we incorporate the statistics into a geometrical setting, where the complex plane is replaced by an n -fold ramified covering of itself, created by flux quanta piercing it.

Naturally, we expect to find the projective fields in order to describe FQH states in the lowest Landau level (LLL) projection correctly. Since the bosons Φ_k are free fields, the correlators of their Vertex operators read

$$\langle \Omega | V_{\vec{\alpha}_1}(z_1) \dots V_{\vec{\alpha}_n}(z_n) | 0 \rangle = \prod_{i < j}^n (z_i - z_j)^{\vec{\alpha}_i \cdot \vec{\alpha}_j} , \quad (34)$$

where $\langle \Omega |$ is an out-state connected to the background charge at infinity.

The set of equations (33) and (34) including their geometric features is all we need to derive the LLL projected FQH bulk wave functions for filling fractions $0 \leq \nu \leq 1$.

III. LAUGHLIN STATES

To begin the analysis of the FQHE that was first discovered by Tsui, Störmer and Gossard¹ it is natural to start with the Laughlin states. Their wave functions are given by

$$\Psi_L(z_j) = \mathcal{N} \prod_{k < l}^n (z_k - z_l)^{2p+1} \exp\left(-\frac{1}{4} \sum_i^n |z_i|^2\right), \quad (35)$$

where $p \in \mathbb{N}$, $z_i = x_i + iy_i$ is the position of the i -th electron in unified complex coordinates and \mathcal{N} is a normalization factor.

These wave functions describe a uniform incompressible quantum fluid of electrons in the LLL widely separated from each other that obey phase correlations as if carrying $2p$ flux quanta of the magnetic field. They are completely anti-symmetric, correspond to filling fractions $\nu = \frac{1}{2p+1}$ and were conceived by Laughlin¹² as the variational ground state wave functions for the model Hamiltonian

$$\mathcal{H} = \sum_k^n \left[\frac{1}{2m} \left(\frac{\hbar}{i} \nabla_k - \frac{e}{c} \vec{A}(\vec{r}_k) \right)^2 + V_{\text{bg}}(\vec{r}_k) \right] + \sum_{k < l}^n \frac{e^2}{|\vec{r}_k - \vec{r}_l|} . \quad (36)$$

Here, V_{bg} is a potential of a background charge distribution that neutralizes the electrons' Coulomb repulsion. This guarantees that the system is stable. The vector potential is taken in the symmetric gauge

$$\vec{A}(\vec{r}) = \frac{B}{2} (-y, x, 0) . \quad (37)$$

We stress that electron-electron interaction is a crucial necessity for the FQHE. In contrast to the IQHE, which can be understood as a one-particle effect involving disorder, the fractional regime is found to be a strongly correlated system (SCS).

Furthermore, the modulus squared of the wave function is equivalent to the Boltzmann distribution of a 2d one-component plasma. This yields further information with respect to the thermodynamic limit,

$$|\Psi|^2 = \exp(-\beta\Phi), \quad (38)$$

where $\beta = 1/(2p + 1)$ and

$$\Phi = -2(2p + 1)^2 \sum_{k < l}^n \ln |z_k - z_l| + \frac{2p + 1}{2} \sum_k^n |z_k|^2 . \quad (39)$$

Hence, for small p the system is a liquid rather than a Wigner crystal.

Another important property is the incompressibility of the Laughlin states. This leads to the existence of plateaus in the Hall conductance. The Laughlin ground state can be extended with respect to quasi-hole excitations by introducing a simple polynomial factor

$$\Psi_{\text{exc.}} = \mathcal{N}(\zeta_i) \prod_{k,l} (z_k - \zeta_l) \prod_{r < s} (z_r - z_s)^{2p+1} \times \exp\left(-\frac{1}{4} \sum_i |z_i|^2\right) . \quad (40)$$

Here, the ζ_i denote the positions of the quasi-hole excitations. With respect to (39) the excited states, in contrast to the ground states, have a non-uniform charge distribution. In comparison with the 2d plasma one can calculate a charge deficit of $\frac{e}{2p+1}$ at the point ζ_i , which means that the quasi-holes are fractionally charged.

Thus, in order to analyze their statistics more carefully, we derive the Berry connection, first stated by Arovas *et al.*,³⁹ from the normalization factor (a detailed comment on the derivation is provided in chapter 2 of Stone's book¹⁰):

$$\Psi_{\text{exc.}} = \mathcal{N} \prod_{k,l} (z_k - \zeta_l) \prod_{r < s} (z_r - z_s)^{2p+1} (\zeta_r - \zeta_s)^{\frac{1}{2p+1}} \times \exp \left[-\frac{1}{4} \sum_i \left(|z_i|^2 + \frac{1}{2p+1} |\zeta_i|^2 \right) \right]. \quad (41)$$

Therefore, the quasi-particles obey fractional statistics and the non-holomorphic part in the wave function describing quasi-particle interactions gives rise to the complex geometry the Laughlin states are built on. This geometrical features are directly embedded in the b/c -spin systems. Given a filling fraction $\nu = 1/(2p+1)$ we identify a \mathbb{Z}_{2p+1} -symmetric projective field with the electron e^- and another one with the flux quantum Φ , respectively.

The charge vectors are related to the statistics, thus $(2p+1)$ -dimensional and take the form

$$\vec{\alpha}_{e^-} = \left(1, \dots, 1 \right), \quad \vec{\alpha}_{\Phi} = \left(\frac{1}{2p+1}, \dots, \frac{1}{2p+1} \right). \quad (42)$$

The correlators (34) yield the correct wave functions (35) and (41) up to the exponential factor:

$$\Psi_L = \langle \Omega | V_{\vec{\alpha}_{e^-}}(z_1) \dots V_{\vec{\alpha}_{e^-}}(z_n) | 0 \rangle = \prod_{i < l}^n (z_i - z_l)^{2p+1}, \quad (43)$$

$$\Psi_{\text{exc}} = \langle \Omega | \prod_{i=1}^n V_{\vec{\alpha}_{e^-}}(z_i) \prod_{j=1}^k V_{\vec{\alpha}_{\Phi}}(\zeta_j) | 0 \rangle = \prod_{r,s}^{n,k} (z_r - \zeta_s) \prod_{i < l}^n (z_i - z_l)^{2p+1} \prod_{p < q}^k (\zeta_p - \zeta_q)^{\frac{1}{2p+1}} \quad (44)$$

We have to make a comment here: In our approach, the CFT always lives on a ramified covering of the compactified complex plane, *i.e.*, on the Riemann sphere. On the other hand, the FQH system lives on a certain chunk of the plane, the sample. Thus, in a correct treatment, wave functions of the FQH system must be elements of a suitable test space. It turns out that this is the Bargmann space.⁴⁰ The elements of the Bargmann space for N complex variables are of the form

$$\psi(\{z\}) = p(z_1, \dots, z_N) \prod_{i=1}^N \exp(-c_i |z_i|^2).$$

There are further restrictions on the constants c_i and on the multi-variate polynomial $p(\{z\})$ whenever the function

$\psi(\{z\})$ is symmetric or anti-symmetric under certain permutations of its arguments. The only effect of the exponential factor is to guarantee a sufficient fast decay of the modulus squared of the wave function if one or more of its arguments become large. It can be shown rigorously that this factor is absent if the FQH problem is considered in a different setting, *i.e.*, on a sphere pierced by the field of a magnetic monopole positioned in its centre. This idea was first stated by Haldane.¹⁵ Since this is a compact space, so is the support of the wave function. When computing bulk wave functions in terms of CFT correlators, we automatically move to this latter setting on the compact sphere. Thus, it is natural to expect that the CFT picture reproduces the bulk wave functions on the sphere and not on the plane. However, for completeness, we mention that it is possible to reproduce the exponential factors within the CFT picture by explicitly including a homogeneous background charge distribution confining the support of the wave function as it was shown by Moore and Read.^{25,41}

We can deduce that the \mathbb{Z}_n -symmetry of the RS the spin fields live on has a one-to-one correspondence with the statistics and charges of the (quasi-)particles in the Laughlin states, *e.g.*, the $(2p+1)$ -dimensional charge vectors (42) yield the wave functions (44), and $n = 2p+1$. Furthermore, the scalar products of the charge vectors determine the particles' interaction, *i.e.*, order of zeros in the polynomial terms of the wave functions. We stress that in spite of the electron with elementary charge obeying simple fermionic statistics the field's nature has a geometric background in terms of the topology of the RS. This will become more apparent in states of higher order.

Beyond Laughlin

As already pointed out the FQHE is a strongly correlated system (SCS). In such systems interactions dominate the physics and long range effects take place. Well known examples are superconductivity and the Hubbard model which can be described in terms of effective theories. The common feature of these theories is the demand for the existence of effective particles in the system, *e.g.*, Cooper pairs (superconductivity) or spinons and holons (Hubbard model). Concerning the FQHE one widely accepted effective theory with direct correspondence to experimental facts was developed by Jain.^{16,42,43} He explained the fractional effect by introducing the composite fermion (CF) model. A CF consists of one electron with a number of pairs of flux quanta of the magnetic field attached to it. Jain showed that the FQHE is an effective IQHE for the CFs and proposed sequences of states to appear in a certain order. These are found in agreement with experimental data.

With respect to trial wave functions the attachment of p pairs of flux quanta is conducted by multiplying the IQHE wave function Ψ_I (filling fraction ν_I) with a polynomial Jastrow factor

$$\Psi_{\text{CF}} = \prod_{i < j}^N (z_i - z_j)^{2p} \Psi_I. \quad (45)$$

The filling fraction of the CF state is then derived as

$$\nu_{\text{CF}} = \frac{\nu_1}{2p\nu_1 + 1}. \quad (46)$$

Here, ν_1 corresponds to the IQH state Ψ_1 . This procedure neither destroys the correlations of the system nor the incompressibility of the state. Laughlin's wave functions with filling fraction $\nu = 1/(2p + 1)$ are the simplest examples of this scheme. We start from a $\nu = 1$ IQH state Ψ_1 and attach p pairs of flux quanta:

$$\Psi_L = \prod_{i < j}^N (z_i - z_j)^{2p} \underbrace{\prod_{i < j}^N (z_i - z_j)}_{\Psi_1} \exp\left(-\frac{1}{4} \sum_i |z_i|^2\right) \quad (47)$$

In principle, it is possible to get any rational number as filling factor by applying Jain's construction repeatedly. This forms the hierarchical scheme of Jain. Thus, instead of starting with an IQH state, one starts with a FQH state obtained from Jain's construction, and forms new CFs out of the old ones by attaching additional pairs of flux quanta. The new filling fraction is obtained via (46) by replacing ν_1 by ν_{CF} to obtain a new filling ν'_{CF} . In this way, arbitrarily continued fractions of the form

$$\nu = [2p_1, \dots, 2p_n, \nu_1] = \frac{1}{2p_1 + \frac{1}{2p_2 + \frac{1}{\ddots + \frac{1}{2p_n + \frac{1}{\nu_1}}}}} \quad (48)$$

can be constructed, and thus arbitrary positive rational numbers $\nu < 1$. However, this hierarchical scheme shares with all the other hierarchical schemes that it soon produces way too many unobserved filling fractions. Moreover, it is crucial to invoke the principle of particle-hole duality in order to get some of the experimentally confirmed filling fractions within the first few levels of the hierarchy. Unfortunately, the set of all experimentally observed FQH states can not be characterized by particle-hole duality very well since related states do not necessarily have to share physical features, e.g., size of the energy gap and width of the plateau. Thus, if a hierarchical scheme is to represent the experimental order of stability and to describe sets and series of FQH states in a proper way, particle-hole duality should not be taken into account.

IV. MULTILAYER STATES AND CFT APPROACH

Not relying on particle-hole duality as an argument and without predicting unobserved fractions, we derive sequences of all FQH states ($0 \leq \nu \leq 1$) observed up to now in agreement with experimental data²⁻⁵ with a very few exceptions. Furthermore, a unifying scheme for the construction of bulk wave functions in terms of CFT correlators is provided. To arrive at these wave functions we have to generalize Jain's composite fermion (CF) approach to multilayer states. One way to provide this is to start from an effective field theory.

It is well-known that QED in (2+1) dimensions consists of a Maxwell part and a topological Chern-Simons term. It is true that the latter is neglectable compared to the first one in many cases, but it was rigorously shown that it dominates the FQH regime.⁷ Therefore, the FQH system can be described in terms of an effective Chern-Simons theory. It turns out that a FQH system can consist of several quantum fluids which may be coupled to each other. Each fluid i is described in the effective field theory by a vector potential a_i^μ in addition to the external field A^μ with couplings κ_i . For completeness, we provide the general form of the Lagrangian:

$$\mathcal{L} = -\frac{1}{4\pi} a_{i\mu} K_{ij} \epsilon^{\mu\nu\lambda} \partial_\nu a_{j\lambda} - \frac{e}{2\pi} \kappa_i A_\mu \epsilon^{\mu\nu\lambda} \partial_\nu a_{i\lambda} + \dots, \quad (49)$$

where we left out possible other terms such as the contribution of the quasi-hole current. To introduce this concept of multilayer states we follow the original theory of Wen.^{44,45} The complete Lagrangian contains various couplings and sources which are irrelevant for our purposes. The only important conclusion in our context is that the internal structure of a so-called m -layer FQH state is encoded in the invertible $m \times m$ matrix K_{ij} describing the couplings of different layers or quantum fluids with each other. This matrix encodes various information of the FQH state, e.g., the filling fraction, topological order, ground state degeneracy and the structure of corresponding trial wave functions. As a result, for a spin-polarized electron system K_{ij} can be written in compact form and has to satisfy the following conditions (represented in the symmetric electron basis of Chern-Simons theory):

$$K_{ij} = \begin{cases} \text{odd integer} & i = j \\ \text{integer} & i \neq j \end{cases}. \quad (50)$$

The filling fraction is

$$\nu_K = \sum_{i,j}^m K_{ij}^{-1}. \quad (51)$$

In addition, the trial wave functions can be read off directly:

$$\Psi_K = \prod_{i < j}^N \prod_{\mu}^m (z_i^{(\mu)} - z_j^{(\mu)})^{K_{\mu\mu}} \prod_{i,j}^N \prod_{\mu < \lambda}^m (z_i^{(\mu)} - z_j^{(\lambda)})^{K_{\mu\lambda}} \quad (52)$$

Jain's Main Series

To follow Jain's approach (45) we start from a double-layer IQH state Ψ_1 with two filled Landau levels (LLs):

$$K_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \nu = 2. \quad (53)$$

These two layers do not interact. Attaching p pairs of flux quanta to each electron yields

$$K_{ij} = \begin{pmatrix} 2p+1 & 2p \\ 2p & 2p+1 \end{pmatrix}, \quad \nu = \frac{2}{4p+1}. \quad (54)$$

The flux quanta introduce interactions between different layers. Two filled LLs of CFs correspond to a LLL FQH state. Generalized to m layers we obtain

$$K_{ij} = \begin{pmatrix} 2p+1 & 2p & \cdots & \cdots & 2p \\ & 2p & 2p+1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 2p+1 & 2p \\ & 2p & \cdots & \cdots & 2p & 2p+1 \end{pmatrix}, \quad (55)$$

which corresponds to $\nu_p = m/(2mp+1)$. This implies the following sequences of filling fractions:

$$\begin{aligned} \nu_1 &= \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}, \frac{7}{15}, \frac{8}{17}, \frac{9}{19}, \frac{10}{21}, \dots \\ \nu_2 &= \frac{1}{5}, \frac{2}{9}, \frac{3}{13}, \frac{4}{17}, \frac{5}{21}, \frac{6}{25}, \dots \\ \nu_3 &= \frac{1}{7}, \frac{2}{13}, \frac{3}{19}, \dots \\ \nu_4 &= \frac{1}{9}, \frac{2}{17}, \dots \\ &\vdots \end{aligned} \quad (56)$$

These are limited by the Wigner crystal regime for $\nu \rightarrow 0$ depending on the quality of the sample. Therefore, the series for $p \geq 5$ were still not observed. On the other hand we have a cutoff if m , the number of LLs of CFs building the state, is increased. In terms of an effective IQHE this corresponds to the classical limit $B_{\text{eff}} \rightarrow 0$.

The trial wave functions (52) are LLL projections of the true FQH states. To do the projection properly CFs of different LLs labelled by (μ) have to be distinguished. The resulting wave function is anti-symmetric only within each LL, anti-symmetrization over different LLs is unphysical and would yield a vanishing Ψ_K in most cases.

The complete set of states for the sequences (56) is included in the b/c -spin system approach, (quasi-)particles, their charges and statistics are described in terms of \mathbb{Z}_{2mp+1} -symmetric projective fields. As before, p labels the number of pairs of flux quanta attached to the electron and m is the number of filled CF LLs. Each layer $\mu \in \{1, \dots, m\}$ is connected with a $(2mp+1)$ -dimensional charge vector:

$$\vec{\alpha}_i^{(\mu)} = \begin{cases} 1 & 1 \leq i \leq 2p \\ 1 & i = 2mp + 2 - \mu \\ 0 & \text{otherwise} \end{cases}. \quad (57)$$

This yields

$$\vec{\alpha}_i^{(\mu)} \cdot \vec{\alpha}_i^{(\lambda)} = 2p + \delta_{\mu, \lambda}. \quad (58)$$

Naively one might have expected a $(2p+1)m$ -dimensional charge vector for an m -layer state. However, this would mean that the flux quanta were independent for each layer.

Identifying these or, equivalently, the base spaces of the m copies of the ramified complex plane immediately leads to $(2p+1)m - (m-1) = 2mp+1$. The correlators (34) are hence derived to read

$$\begin{aligned} \Psi_{p,m}(z_i^{(\mu)}) &= \langle \Omega | \prod_{\mu}^m V_{\vec{\alpha}^{(\mu)}}(z_1^{(\mu)}) \cdots V_{\vec{\alpha}^{(\mu)}}(z_N^{(\mu)}) | 0 \rangle \\ &= \prod_{i < j}^N \prod_{\mu}^m (z_i^{(\mu)} - z_j^{(\mu)})^{2p+1} \prod_{i,j}^N \prod_{\mu < \lambda}^m (z_i^{(\mu)} - z_j^{(\lambda)})^{2p} \end{aligned} \quad (59)$$

Equation (59) generalizes the result of (44) and the basic Jain series (56) with $\nu_p = \frac{m}{2mp+1}$ are identified.

Composite Fermion Pairing

Concerning other filling fractions all known hierarchical systems^{13,15,43,46,47} invoke the principle of particle-hole duality, relating, for example the series

$$\nu_1 = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}, \frac{7}{15}, \frac{8}{17}, \frac{9}{19}, \frac{10}{21}, \dots \quad (60)$$

and

$$\nu_1^{(1)} = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \frac{6}{11}, \frac{7}{13}, \frac{8}{15}, \frac{9}{17}, \frac{10}{19}, \dots \quad (61)$$

The latter is of type $\nu_p^{(1)} = \frac{m}{2mp-1}$ and can naively be represented in terms of m -layer K -matrices

$$K_{ij} = \begin{cases} 2p-1 & i = j \\ 2p & i \neq j \end{cases}. \quad (62)$$

This does not suit our approach in its basic form: it demands the existence of charge vectors $\vec{\alpha}$ and $\vec{\beta}$ corresponding to different layers with

$$\vec{\alpha}^2 = \vec{\beta}^2 = 2p-1 \quad \text{and} \quad \vec{\alpha} \cdot \vec{\beta} = 2p \not\equiv 0. \quad (63)$$

This is not possible since it contradicts Schwarz' inequality and indicates that these 'dual' series possess completely new physical features. The analytic structure of the wave function (52) for K -matrices (62) exhibits that CFs living in the same layer repulse each other with the power of $(2p-1)$ while those of different layers repulse themselves by $2p$. This suggests the existence of an effectively attractive CF interaction within a LL, *i.e.*, pairing. This is induced by the $c = -2$ logarithmic b/c -spin system with spin $j = 1$ as it was proven for the Haldane-Rezayi state which was discussed for some time as proposal to describe the even-denominator filling fraction $\nu = 5/2$.^{21,25,31,32}

In analogy to (22) the fields $b(z)$ and $c(z)$ can be bosonized on a ramified covering of the compactified complex plane locally representing the \mathbb{Z}_n -symmetric RS in terms of Vertex operators:

$$\begin{aligned} b_{\vec{\gamma}}(z) &= : \exp(+i\vec{\gamma}\vec{\Phi}(z)) : \\ c_{\vec{\gamma}}(z) &= : \exp(-i\vec{\gamma}\vec{\Phi}(z)) : \end{aligned} \quad \gamma_k \in \{0, 1\}. \quad (64)$$

In terms of CFT the pairing effect of the CFs is described by $b(z)\partial c(z')$. The OPE

$$b_{\tilde{\gamma}}(z)\partial c_{\tilde{\gamma}}(z') \sim \frac{\tilde{\gamma}^2}{(z-z')^2} \quad (65)$$

yields the so-called Pfaffian form $\text{Pf}(z_i, z'_i)$ if the fields (65) are evaluated in a correlator:

$$\langle \Omega | \prod_{i=1}^N b_{\tilde{\gamma}}(z_i) \partial_{z'_i} c_{\tilde{\gamma}}(z'_i) | 0 \rangle = \tilde{\gamma}^2 \text{Pf}(z_i, z'_i),$$

$$\text{Pf}(z_i, z'_i) \equiv \sum_{\sigma \in S_N} \text{sgn}(\sigma) \prod_{i=1}^N \frac{1}{(z_i - z'_{\sigma(i)})^2}. \quad (66)$$

In this way, the $\nu_p^{(1)}$ series can be identified by the same fields as the basic Jain series (59) if additional inner-LL pairings are included. To find a physical and stable system we expect all CF LLs to be paired. In order to describe this in a proper way, each layer $\mu \in \{1, \dots, m\}$ possesses an m -dimensional charge vector:

$$\tilde{\gamma}_i^{(\mu)} = \delta_{\mu, i} \Rightarrow \tilde{\gamma}^{(\mu)} \cdot \tilde{\gamma}^{(\lambda)} = \delta_{\mu, \lambda}. \quad (67)$$

The CFs themselves correspond to the charge vectors (57). Thus, the wave function reads

$$\Psi_{p, m}^{(1)}(z_i^{(\mu)}) = \langle \Omega | \prod_{\mu}^m V_{\tilde{\alpha}^{(\mu)}}(z_1^{(\mu)}) \cdots V_{\tilde{\alpha}^{(\mu)}}(z_{2N}^{(\mu)}) | 0 \rangle$$

$$\times \langle \Omega | \prod_{\mu}^m \prod_{i=1}^N b_{\tilde{\gamma}^{(\mu)}}(z_i^{(\mu)}) \partial_{z_{i+1}} c_{\tilde{\gamma}^{(\mu)}}(z_{i+1}^{(\mu)}) | 0 \rangle$$

$$= \prod_{\mu}^m \text{Pf}(z_i^{(\mu)}, z_{N+i}^{(\mu)})$$

$$\times \underbrace{\prod_{i < j}^{2N} \prod_{\mu}^m (z_i^{(\mu)} - z_j^{(\mu)})^{2p+1}}_{(*)} \prod_{i, j}^{2N} \prod_{\mu < \lambda}^m (z_i^{(\mu)} - z_j^{(\lambda)})^{2p}. \quad (68)$$

We want to stress that equation (68) is in principle completely independent of the Chern-Simons approach. Nonetheless, it can be related to the K -matrix (62). Concerning trial wave functions, the Pfaffian cancels two powers of the paired CF contribution to $(*)$. Thus, paired CFs repulse each other by $(z_i^{(\mu)} - z_j^{(\mu)})^{2p-1}$ in either wave function (52) and (68). Additionally, both yield the same filling fractions

$$\nu_{p, m}^{(1)} = \frac{m}{2mp-1}. \quad (69)$$

Yet there is another subtlety. It can be shown that the procedure of attaching flux quanta as proposed in Wen's original Chern-Simons approach for spin-polarized states can not be extended to systems with non-trivial geometry,⁴⁸ e.g., the paired CF states represented by (68). As a consequence, the corresponding K -matrices can not be represented in a compact form with respect to global gauge transformations. Minimal alternative theories are suggested, e.g., for the case of

spin-polarized states,⁴⁹ but despite their beautiful and fundamental approach to FQH systems in terms of symmetries, they lack proposing of trial wave functions and at the moment can not be discussed in this context. However, the CFT approach proposed in this paper allows to face these problems from a completely different viewpoint and is thought of to imply geometrically motivated restrictions and consequences rather than to derive them from first principles.

It is worth mentioning that the bulk wave functions resulting from this CFT scheme including pairing are given by two factors in terms of correlation functions, as indicated in (68). The factor representing the pairing has a different geometrical meaning, when the vertex operators are again viewed as generating non-trivial Riemann surfaces. The pairing introduces poles into the picture of branched ramified coverings of the Riemann sphere, such that the CFT description leaves the realm of compact Riemannian geometry. Whether a connection to the above mentioned observations^{48,49} can be made, must remain open at this time.

Reconsidering (68), we identify the first order paired series:

$$\nu_1^{(1)} = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \frac{6}{11}, \frac{7}{13}, \frac{8}{15}, \frac{9}{17}, \frac{10}{19}, \dots$$

$$\nu_2^{(1)} = \frac{1}{3}, \frac{2}{7}, \frac{3}{11}, \frac{4}{15}, \frac{5}{19}, \frac{6}{23}, \dots$$

$$\nu_3^{(1)} = \frac{1}{5}, \frac{2}{11}, \frac{3}{17}, \dots$$

$$\nu_4^{(1)} = \frac{1}{7}, \frac{2}{15}, \dots$$

$$\vdots$$

$$(70)$$

This proposal can be extended in a natural way imagining that the structure of paired CF singlets is not restricted to be an inner-LL effect. Two LLs of CFs that are completely paired among each other can form a new incompressible quantum liquid and can hence interact with other blocks or single layers of paired droplets. Related to Wen's original Chern-Simons approach, we find two natural series of K -matrices $(K^{e, \circ})_{ij}$ with an even and an odd number of layers, respectively.⁶⁵

$$(K^{e, \circ})_{ij} = \begin{cases} 2p-1 & i=j \\ 2p-2 & i \neq j, \quad 2(k-1)+1 \leq i, j \leq 2k, \\ & \text{where } (1 \leq k \leq b) \\ 2p & \text{otherwise} \end{cases}. \quad (71)$$

Here, b is the number of paired 2×2 -blocks. The first series, given a $2b$ -layer FQH state, reads:

$$(K^e)_{ij} = \begin{pmatrix} 2p-1 & 2p-2 & 2p & \cdots & 2p \\ 2p-2 & 2p-1 & 2p & \ddots & \vdots \\ 2p & 2p & \ddots & 2p & 2p \\ \vdots & \ddots & 2p & 2p-1 & 2p-2 \\ 2p & \cdots & 2p & 2p-2 & 2p-1 \end{pmatrix}, \quad (72)$$

with $\nu_p^{(2)e} = 2b/(4bp - 3)$. The latter, given a $2b + 1$ -layer FQH state, has a remaining solely self-paired layer and corresponds to filling fractions

$$\nu_p^{(2)o} = \frac{2b + 3}{2p(2b + 3) - 3}. \quad (73)$$

Together, they yield the second order paired series:

$$\begin{aligned} \nu_1^{(2)e} &= \frac{4}{5}, \left(\frac{6}{9}\right), \frac{8}{13}, \frac{10}{17}, \dots & \nu_2^{(2)e} &= \frac{2}{5}, \frac{4}{13}, \dots, \\ \nu_1^{(2)o} &= \frac{5}{7}, \frac{7}{11}, \left(\frac{9}{15}\right), \dots & \nu_2^{(2)o} &= \dots \end{aligned} \quad (74)$$

Fractions in brackets are not coprime and also appear in other series. This indicates that these states can exist in different forms of quantum liquids. We are now able to generalize this scheme to the case of $n \times n$ blocks of paired LLs and derive the n -th order series. There exist $n - 1$ sub-series determined by the number r of remaining solely self-paired LLs, *e.g.*, $r = 0$ in the even case for second order and $r = 1$ in the odd case, respectively. Let b denote the number of fully paired blocks then the $m \times m$ -matrix $K_{p,m}^{(n)}$ with $m = bn + r$ of the n -th order paired FQH state reads:

$$\left(K_{p,m}^{(n)}\right)_{ij} = \begin{cases} 2p - 1 & i=j \\ 2p - 2 & i \neq j, (k-1)n + 1 \leq i, j \leq kn, \\ & \text{where } (1 \leq k \leq b) \\ 2p & \text{otherwise} \end{cases}. \quad (75)$$

The corresponding filling fractions are

$$\nu_{p,m}^{(n)r} = \frac{bn + r(2n - 1)}{2p(bn + r(2n - 1)) - (2n - 1)}. \quad (76)$$

By this, we deduce the third order states confirmed by experiment⁶⁶ (higher orders do not yield additional observed fractions):

$$\begin{aligned} \nu_1^{(3)0} &= \left[\frac{6}{7}\right], \frac{9}{13}, \dots & \nu_1^{(3)1} &= \frac{8}{11}, \dots & \nu_1^{(3)2} &= \dots \\ \nu_2^{(3)0} &= \frac{3}{7}, \dots & \nu_2^{(3)1} &= \dots & \nu_2^{(3)2} &= \dots \end{aligned} \quad (77)$$

Spending a closer look on (75) the question arises to what extent our access to FQH pairing is too restrictive. One could imagine more general K -matrices with band-like or even more complicated structures yielding arbitrary ν . For example, $\nu = 4/11$, a state that was recently confirmed by experiment,⁵ could be realized by

$$K_{ij} = \begin{pmatrix} 3 & 2 & 2 & 4 \\ 2 & 3 & 4 & 2 \\ 2 & 4 & 3 & 2 \\ 4 & 2 & 2 & 3 \end{pmatrix}. \quad (78)$$

This K -matrix describes a ring of two second order blocks. Remarkably, the result of a detailed analysis of equation (76) shows that certain fractions do not appear, for example $7/9$,

$10/13$, $5/13$, and $4/11$, indicating that if they exist they presumably have to be another kind of FQH fluid. This set of fractions was very recently confirmed by Pan and Störmer⁵⁰ and seemingly manifests itself due to CF-CF interactions. As exactly these fractions lie beyond the access of our scheme, the b/c -spin systems motivate a reasonable physical constraint for the Chern-Simons formalism in order to classify FQH states. We directly deduce this from the CFT picture of the fields given by (64). If we had an off-block pairing structure, there would exist a triple of

$$\begin{aligned} b_{\vec{\gamma}_1}(z_i^{(1)}) \partial c_{\vec{\gamma}_1}(z_j^{(1)}), & \quad b_{\vec{\gamma}_2}(z_i^{(2)}) \partial c_{\vec{\gamma}_2}(z_j^{(2)}), \\ b_{\vec{\gamma}_3}(z_i^{(3)}) \partial c_{\vec{\gamma}_3}(z_j^{(3)}), & \end{aligned} \quad (79)$$

with the charge vectors obeying the following set of equations:

$$\begin{aligned} \vec{\gamma}_1^2 = \vec{\gamma}_2^2 = \vec{\gamma}_3^2 = 1, & \quad \vec{\gamma}_1 \cdot \vec{\gamma}_2 = \vec{\gamma}_1 \cdot \vec{\gamma}_3 = 1, \\ \text{and } \vec{\gamma}_2 \cdot \vec{\gamma}_3 = 0. & \end{aligned} \quad (80)$$

Since their components are restricted to be either 0 or 1, we end up with a contradiction:

$$\vec{\gamma}_1 = \vec{\gamma}_2 = \vec{\gamma}_3 \quad \text{and} \quad \vec{\gamma}_2 \neq \vec{\gamma}_3 \quad \not\vdash. \quad (81)$$

As a consequence the most general K -matrix for a correct description of paired FQH states is restricted to be built from blocks:

$$\left(K_{p,m}^{b,n_b}\right)_{ij} = \begin{cases} 2p - 1 & i=j \\ 2p - 2 & i \neq j, 1 + \sum_{l=1}^{k-1} n_l \leq i, j \leq \sum_{l=1}^k n_l, \\ & \text{where } (1 \leq k \leq b) \\ 2p & \text{otherwise} \end{cases}. \quad (82)$$

Here, b denotes the number of blocks and n_b their corresponding size. Therefore, $m = \sum_{l=1}^b n_b$, if we denote singly paired layer by $n_b = 1$. We stress that the new series of filling fractions ν_p^{b,n_b} obtained from (82) are rather unlikely to be seen in experiments as their K -matrices are less symmetric than the ones given by (75). Since it is quite difficult to derive a general formula for ν_p^{b,n_b} , we simply provide the only additional fraction that may be seen in the nearer future:

$$\nu_1^{2,(3,2)} = \frac{19}{23}. \quad (83)$$

Therefore, the set of matrices (75) remains as the natural candidate to describe series of paired FQH states by order of stability. The corresponding bulk wave functions $\Psi_{p,m}^{(n)}$ of the n -th order paired FQH states can be calculated as a direct generalization of (68). Given the matrix $K_{p,m}^{(n)}$, an m -dimensional charge vector with respect to a paired block $B \in \{1, \dots, b + r\}$ (either $n \times n$ or a remaining 1×1 layer) is identified with each layer μ :

$$\vec{\gamma}_i^{(\mu)} = \delta_{B(\mu),i} \Rightarrow \vec{\gamma}^{(\mu)} \cdot \vec{\gamma}^{(\lambda)} = \delta_{B(\mu),B(\lambda)}. \quad (84)$$

Additionally, each layer k possesses a $(2mp+1)$ -dimensional charge vector for the CFs:

$$\vec{\alpha}_i^{(\mu)} = \begin{cases} 1 & 1 \leq i \leq 2p \\ 1 & i = 2mp + 2 - \mu \\ 0 & \text{otherwise} \end{cases} \quad (85)$$

$$\Rightarrow \vec{\alpha}^{(\mu)} \cdot \vec{\alpha}^{(\lambda)} = 2p + \delta_{\mu, \lambda} .$$

Let I denote the set of paired LLs, *e.g.*, $I = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$ describes a triple-layer state with $\nu_{p,3}^{(2)1} = \frac{5}{10p-3}$ where we find a 2×2 -block of the first two LLs while the third is solely self-paired. The wave functions read

$$\begin{aligned} \Psi_{p,m}^{(n)}(z_i^{(\mu)}) &= \langle \Omega | \prod_{\mu}^m V_{\vec{\alpha}^{(\mu)}}(z_1^{(\mu)}) \cdots V_{\vec{\alpha}^{(\mu)}}(z_{2N}^{(\mu)}) | 0 \rangle \\ &\times \langle \Omega | \prod_{(\mu, \lambda) \in I} \prod_{i=1}^N b_{\vec{\gamma}^{(\mu)}}(z_i^{(\mu)}) \partial_{z_{i+1}} c_{\vec{\gamma}^{(\lambda)}}(z_{i+1}^{(\lambda)}) | 0 \rangle \\ &= \prod_{(\mu, \lambda) \in I} \text{Pf}(z_i^{(\mu)}, z_{N+i}^{(\lambda)}) \\ &\times \underbrace{\prod_{i < j}^{2N} \prod_{\mu}^m (z_i^{(\mu)} - z_j^{(\mu)})^{2p+1} \prod_{i, j}^m \prod_{\mu < \varrho} (z_i^{(\mu)} - z_j^{(\varrho)})^{2p}}_{\Psi_{p,m}(z_i^{(\mu)})} , \quad (86) \end{aligned}$$

where $\Psi_{p,m}(z_i^{(\mu)})$ is the bulk wave function of the basic Jain series (59).

Combining equations (56), (70), (74), and (77) we find the complete set of experimentally confirmed filling fractions by order of stability except for $\nu = 4/11$, which is presumably a non-Abelian FQH state falling outside our approach, $\nu = 7/9$, $\nu = 10/13$, and $\nu = 5/13$. We find a natural cutoff if either the number of participating CF LLs m increases or $\nu \rightarrow 0$. Series of more complicated CFs (larger p) are less developed, complete pairings ($r = 0$) are favored and each series precisely keeps track of the stability of the FQH states found in experiments whereas no unobserved fraction is predicted.

We want to make another comment on the fact that the $\nu = 7/9$ state is established that weak. If we naively assumed the series

$$\nu = \frac{k}{2k-5} = \frac{6}{7}, \frac{7}{9}, \frac{8}{11}, \frac{9}{13}, \dots ,$$

we would consider $\nu = 7/9$ to be more likely to appear than $\nu = 8/11$. Furthermore, it can not be argued that $7/9$ is dominated by the $\nu = 1$ plateau since $\nu = 4/5$ exists. This seems rather unusual or even exceptional but is precisely predicted by our approach since $7/9$ is assumed to belong to another class of quantum Hall fluids. Therefore, the series in figure 1 simply indicate where new fractions given by (76) will show up. By our hierarchical order of stability the following filling fractions are predicted if experimental circumstances are improved in the future (we just indicate fractions with denominator $d \leq 29$).

TABLE I: Expected Hall fractions

p	ν_p	$\nu_p^{(1)}$	$\nu_p^{(2)}$	$\nu_p^{(3)}$	$\nu_p^{(4)}$
1	$\frac{11}{23}$	$\frac{11}{21}$	$\frac{11}{19}, \frac{12}{21}$	$\frac{11}{17}, \frac{12}{19}, \frac{13}{21}$	$\frac{8}{9}, \frac{11}{15}, \frac{18}{29}$
2	$\frac{7}{29}$	$\frac{7}{27}$	$\frac{5}{17}, \frac{6}{21}$	$\frac{6}{19}, \frac{8}{27}$	$\frac{8}{25}$
3	$\frac{4}{25}$	$\frac{4}{23}$	$\frac{4}{21}, \frac{5}{27}$		
4	$\frac{3}{25}$	$\frac{3}{23}$	$\frac{4}{29}$		

Quasi-Particle Excitations

One of the most striking results in the study of the FQHE was the discovery of quasi-particles with fractional charges and statistics.¹² Experimentally it has been proven very difficult to measure them (even for the Laughlin states) and a lot of effort is spent to analyze them in more detail. The two sets of wave functions (59) and (86) describe the electron ground state for a given filling fraction ν . As already shown for the Laughlin series the geometric features of excitations responsible for statistics and charges are directly embedded in the b/c -spin systems and are related to the \mathbb{Z}_m -symmetry of the RS the fields live on, *i.e.*, the dimension of the CF charge vectors (85). However, a basic quasi-particle excitation of an m -layer state has to be considered more carefully. First of all, we would like to have trivial statistics of a quasi-particle with respect to the CFs. Thus, we should expect $\vec{\alpha}_{\Phi} \cdot \vec{\alpha}_{\text{CF}}^{(\mu)} = 1$, where $\vec{\alpha}_{\text{CF}}^{(\mu)}$ is the charge vector of the CF in the μ -th layer as given by (57). The naive solution

$$\vec{\alpha}_{\Phi} = \left(\frac{1}{2mp+1}, \dots, \frac{1}{2mp+1} \right), \quad (\vec{\alpha}_{\Phi})^2 = \frac{1}{2mp+1} \quad (87)$$

yields the value $\vec{\alpha}_{\Phi} \cdot \vec{\alpha}_{\text{CF}}^{(\mu)} = \frac{2p+1}{2pm+1}$, which is not an integer for $m > 1$. A simple generalized solution exists, namely

$$\vec{\alpha}_{\Phi, i} = \frac{1}{2pm+1} \begin{cases} m & 1 \leq i \leq 2p \\ 1 & 2pm+2-m \leq i \leq 2pm+1 \\ 0 & \text{otherwise} \end{cases} , \quad (88)$$

which coincides with (42) for $m = 1$. By this, we obtain the desired result for all layers. Furthermore,

$$\vec{\alpha}_{\Phi} \cdot \vec{\alpha}_{\Phi} = \frac{1}{(2pm+1)^2} (2pm^2 + m) = \frac{m}{2pm+1} , \quad (89)$$

which yields the correct quasi-particle statistics for an m -layer state since each layer contributes $1/(2pm+1)$.

Therefore, the quasi-particle excitations of the wave functions $\Psi_{p,m}$ and $\Psi_{p,m}^{(n)}$ are predicted to carry a phase $\Theta \sim \pi/(2mp+1)$ and have the charge $q \sim e/(2mp+1)$ as it is shown in Table II. Since several filling fractions, *e.g.*, $2/5$, belong to more than one series and, thus, exist in different

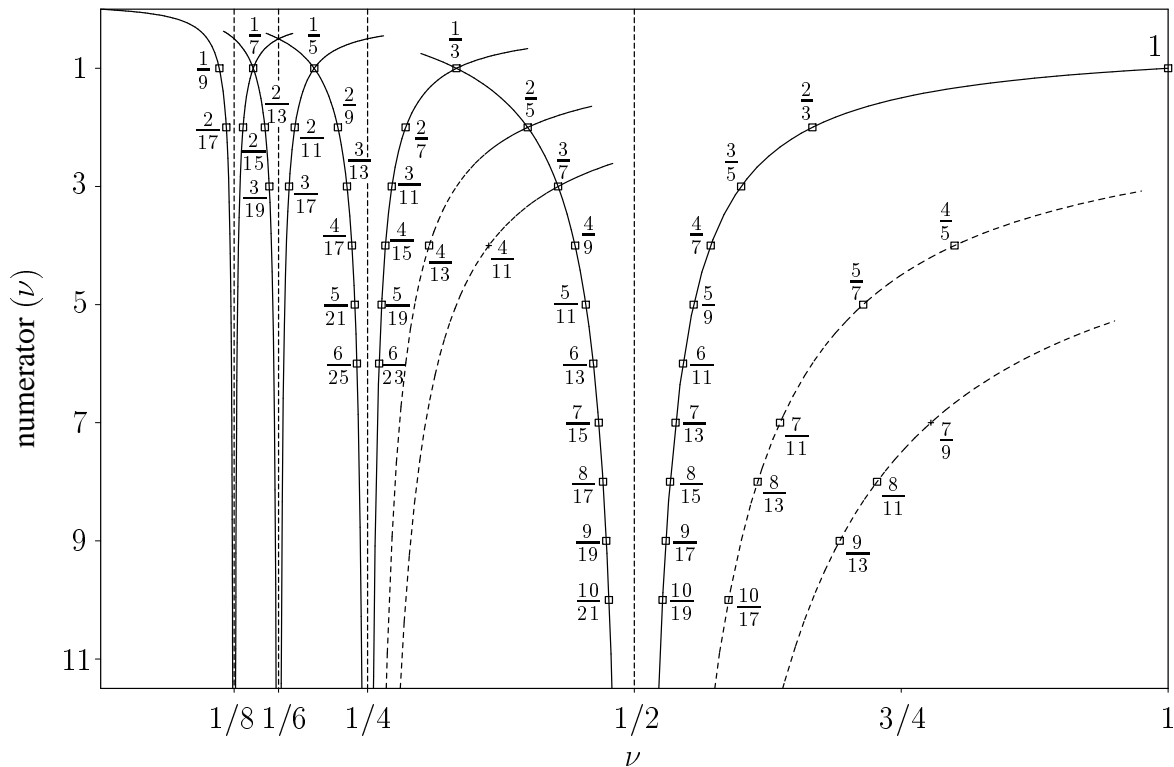


FIG. 1: Observed Hall fractions in the interval $0 \leq \nu \leq 1$. Established fractions are labelled by '□'. The symbol '+' denotes cases that exceed our scheme. The basic Jain series ν_p approximate $1/2p$ from below, the corresponding first order paired series $\nu_p^{(1)}$ from above (both marked by continuous lines) as well as the higher order series $\nu_p^{(n)}$ (marked by dashed lines).

TABLE II: Quasi-particle statistics for confirmed FQH states

Θ	ν	Θ	ν
$\frac{\pi}{3}$	$\frac{1}{3}$	$\frac{\pi}{15}$	$\frac{7}{15}, \frac{7}{13}, \left(\frac{9}{15}\right)$
$\frac{\pi}{5}$	$\frac{1}{5}, \frac{2}{5}, \frac{2}{3}$	$\frac{\pi}{17}$	$\frac{2}{17}, \frac{2}{15}, \frac{4}{17}, \frac{4}{15}, \frac{4}{13}, \frac{4}{9}, \frac{8}{17}, \frac{8}{15}, \frac{8}{13}$
$\frac{\pi}{7}$	$\frac{1}{7}, \frac{3}{7}, \frac{3}{5}, \frac{5}{7}$	$\frac{\pi}{19}$	$\frac{3}{19}, \frac{3}{17}, \frac{3}{13}, \frac{9}{19}, \frac{9}{17}, \frac{9}{13}$
$\frac{\pi}{9}$	$\frac{1}{9}, \frac{2}{9}, \frac{2}{7}, \frac{2}{5}, \frac{4}{9}, \frac{4}{7}, \frac{4}{5}, \frac{8}{11}$	$\frac{\pi}{21}$	$\frac{5}{21}, \frac{5}{19}, \frac{5}{11}, \frac{10}{21}, \frac{10}{19}, \frac{10}{17}$
$\frac{\pi}{11}$	$\frac{5}{11}, \frac{5}{9}, \frac{7}{11}$	$\frac{\pi}{23}$	
$\frac{\pi}{13}$	$\frac{2}{13}, \frac{2}{11}, \frac{3}{13}, \frac{3}{11}, \frac{3}{7}, \frac{6}{13}, \frac{6}{11}, \left(\frac{6}{9}\right)$	$\frac{\pi}{25}$	$\frac{6}{25}, \frac{6}{23}$

forms of quantum liquids, we also find various types of quasi-particles. Direct experimental observations are still difficult, and — as far as we know — good indications solely exist for the Laughlin series. Thus, the correct identification of the quasi-particle within the spectrum of our CFT must remain

open. We finally note that our choice (57) for the charge vectors of the CF and (88) for the quasi-particles is not unique, although physically motivated, particularly simple and symmetric. The ambiguity is not disturbing since most other solutions are related to ours by a change of basis within the tensor

product of the CFTs. The advantage of our approach is that the CFTs themselves are confined to a discrete series leaving not much room for arbitrariness.

V. SUMMARY AND OUTLOOK

The success of the analysis of the Haldane-Rezayi state via $c = -2$ spin systems^{32,51} stimulated our approach. With a few general and physically motivated assumptions restricting to a discrete set of CFTs we were able to construct a hierarchical scheme that precisely keeps track of experimental results. Developing these features in a natural and simple way, we consecutively derived, with a few exceptions, the complete set of filling fractions by order of stability in the FQH regime of $0 \leq \nu \leq 1$ without predicting fractions not confirmed by experiment.

More precisely, we constructed CFTs yielding geometrical descriptions of FQH states. Since odd-denominator fillings refer to fermionic statistics, the natural choice are $(j, 1 - j)$ b/c -spin systems with j half-integer. Moreover, the statistics of the flux quanta, as suggested by Jain's composite fermion picture, are now more general such that we are led to consider Riemann surfaces with global \mathbb{Z}_n -symmetry. Representing these surfaces as n -fold ramified covering of the complex plane, the effect of a flux quantum is geometrically the same as a branch point. The CFT correlators are then sections of certain vector bundles. The bulk ground state wave function is given by a correlator of vertex operators whose twist numbers are purely fermionic resembling the quantum numbers of a composite fermion. With these ingredients we obtained bulk wave functions for the principal main series $\nu = \frac{m}{2pm+1}$. It turned out that our choice of CFTs has not only a direct geometric interpretation, but furthermore puts severe constraints on possible FQH states. The description of the FQHE via an effective Chern-Simons theory leads to a classification of FQH states in terms of the so-called K -matrices. Our approach rules out many K -matrices corresponding to Wen's original theory, since the corresponding bulk wave functions can not be written in factorized form in terms of CFT correlators.

Besides the main series of Jain, we obtain other filling fractions by one further principle. Not relying on the so-called particle-hole duality, which is not suitable to develop a hierarchy of states in correspondence with experiments, we introduce pairings of composite fermions instead. This leads to a new hierarchy of states obtained from the principal series by a growing number of pairings that are effectively described by additional CFTs, namely the already mentioned $c = -2$ spin singlet systems. The requirement that the bulk wave function can be written in terms of factorized CFT correlators demands that only pairings leading to K -matrices in block form are possible. By this, we nearly obtain all experimentally observed filling fractions. However, the great strength and predictive power we see in our approach is that it precisely avoids all filling fractions which are supposed to be formed from more complicated CF-CF interactions. Our ansatz yields a natural order of stability in perfect agreement with experimental data suggesting a clear picture of series which can be

observed up to a given maximal numerator of ν . Thus, we are able to denote the next members of these series, as indicated by Fig. 1 and Table I, which might be observed under improved experimental conditions.

The main advantage of our scheme is that it avoids arbitrariness and the concept of pairing is not exceptional as well. First of all, it precisely agrees with the natural and geometrical motivation and structure of the Haldane-Rezayi state. A nice discussion is provided by these papers.^{52,53} Moreover, pairing effects are indicated by numerical studies,^{54,55} and are in analogy to similar phenomena in other fields of condensed matter physics, such as certain exactly integrable models in the context of BCS pairing.⁵⁶ Although our proposed bulk wave functions which describe paired FQH states differ from the ones predicted by the naive K -matrix formalism, they share important asymptotic features. This stimulates the question how a valid Chern-Simons approach that covers non-Abelian states on a non-trivial geometrical setting has to look like and how trial wave functions can be derived from it. A check of our bulk wave functions should be done numerically, but is beyond the scope of this paper.

Our description in terms of b/c -spin systems seems to be sufficiently complete. It should be possible to incorporate FQH states from non-Abelian Chern-Simons theories^{57,58} as well, since we believe that the geometric principle remains unchanged. The main difference lies in the nature of the quasi-particle excitations. In our approach, non-trivial statistics is a consequence of the twists introduced by the flux quanta and is – in the LLL – always of Abelian nature since all monodromies are simultaneously diagonalized. Non-Abelian statistics is involved and can not be represented within the simple CFTs we used. However, we point out that the $c = -2$ CFT coming into play with pairing is actually a logarithmic CFT and thus includes fields with non-diagonalizable monodromy action.³² In order to understand this in more detail, we would have to work with the full twist fields, not only the projective ones. This immediately leads to further restrictions for the twist fields in order to be inserted in a correlator. If the twists are summed over all insertions they have to be trivial in all n copies of the b/c -spin system we consider. However, at this stage, the full description of quasi-particle excitations remains an unsolved problem. Another one is the correct choice of the spin system, *i.e.*, of the conformal weights $(j, 1 - j)$ of the field $b(z)$ and $c(z)$. This problem is related to the fact that our b/c -spin systems possess partition functions which are equivalent to Gaussian $c = 1$ models. Unfortunately, the partition function of a $(j, 1 - j)$ system is closely related to the partition function of any other $(j', 1 - j')$ system, in particular if $j - j' \in \mathbb{Z}$. Thus, CFT alone is not able to fix j . However, if we take the composite fermion as the basic object, we might expect that the FQH state involving composite fermions made out of electrons with p attached pairs of flux quanta should correspond to spin $j = \frac{1}{2}(2p + 1)$ fields in the CFT description. These should be elementary in the sense that the spectrum of the CFT does not contain fermionic fields with smaller spin in the non-twisted sector. Moreover, the twists related to the quasi-particle excitations should have a minimal charge of $\alpha = 1/(2pm + 1)$ for an m layer state, since this is the ex-

pected fractional statistics. The fractional charge is entirely determined by the geometry, *i.e.*, by the number of sheets in the covering of the complex plane. But the requirement that the composite fermions shall be the effective elementary particles fixes $j = \frac{1}{2}(2p+1)$ or $j = \frac{1}{2}(2p+3)$ due to the duality $j \longleftrightarrow 1-j$. A very interesting question is, whether an effective theory of transitions between different FQH states could yield a mechanism how our CFTs are mapped onto each other, *e.g.*, along the lines of the following papers.^{28,59}

Finally, we point out that our scheme should be understood as a proposal. Although we provided a stringent geometrical setting which identified our choice of CFTs, we can not connect these CFTs to the full (2+1)-dimensional bulk theory via rigorous first principles. For instance, and in contrast to the (1+1)-dimensional edge theory, there is no mathematical rigorous theorem which would allow us to invoke some sort of Chern-Simons versus CFT equivalence. Furthermore, our expressions for the bulk wave functions in terms of CFT correlators, as all existing proposals for bulk wave functions, should be understood as trial ones, since exact solutions are not known (this even applies to the Laughlin wave functions). Comparison with others obtained from numerical diagonalization of the exact Hamiltonian can only be made for a small number of electrons and not in the thermodynamic limit. On the other hand, trial wave functions such as the ones conceived by Laughlin possess many special features or symmetries, *e.g.*, topological order or incompressibility, *i.e.*, symmetry under area-preserving diffeomorphisms.

We hope that future research will reveal the physical nature of such properties such that the connection with CFT is eventually put on firmer ground and trial wave functions are more thoroughly checked or even derived from first principles.

APPENDIX A: DISCUSSION ON UNITARITY

It might seem disturbing that the CFTs proposed to describe the FQH bulk regime are non-unitary. We stress again that these CFTs are not meant to yield the bulk wave functions from a dynamical principle, nor do they provide an effective Hamiltonian. Moreover, since the relevant states are stationary eigenstates of the of a full (2+1)-dimensional system, no time evolution is involved. In this sense, the bulk theory can be reduced to a truly Euclidean one which is (2+0)-dimensional. The topological nature of the full (2+1)-d system suggests the bulk theory to be at least scale invariant. Thus, the assumption that bulk wave functions should have a CFT description is reasonable, but the requirement that these CFTs should be unitary is not necessary and does not contain any physically relevant information. The bulk CFT describes purely geometry, namely how the corresponding wave functions can be understood in terms of vector bundles over Riemann surfaces.⁶⁰ As we have argued in the main text, the fractional statistics of the quasi-particle excitations results in a multi-valuedness of the wave functions, considered as functions over the complex plane. One of the central features of our approach is to replace this setting by the geometrically more natural scheme of holomorphic functions over a rami-

fied covering of the complex plane leading to the non-unitary $(j, 1-j)$ b/c -spin systems.

However, the question of unitarity is not irrelevant. In order to be consistent, we should require that our ansatz fits together with the (1+1)-d CFTs describing edge excitations. These describe waves propagating along the one-dimensional edge of the quantum droplet and hence necessarily have to be unitary. This also follows from the strict one-to-one correspondence of (2+1)-dimensional Chern-Simons theories on a manifold M with unitary (1+1)-dimensional CFTs living on the boundary ∂M . Consistency requires that the space of states of either CFT, the edge and the bulk one, should be equivalent. In other terms, both should have the same partition functions. Fortunately, the b/c -spin systems have well-known partition functions which are indeed equivalent to those of certain $c = 1$ Gaussian models. These latter unitary CFTs are precisely the candidates for the description of the edge excitations which are most widely used.⁶⁷

To be more explicit, we consider a spin $(j, 1-j)$ b/c -spin system in some twisted sector with twist α . The full character of this system, including the ghost number, is defined as

$$\chi^{(j,\alpha)}(q, z) \equiv \text{tr}_{\mathcal{H}(\alpha)} q^{L_0^{(j,\alpha)} - \frac{c_j}{24}} z^{j_0^{(\alpha)}}, \quad (\text{A1})$$

where we have clearly indicated that the mode expansions of the Virasoro field and the ghost current depend on the twist sector. Explicitly computed, these characters read:

$$\chi^{(j,\alpha)}(q, z) = q^{\frac{1}{2}(j+\alpha)(j+\alpha+1) + \frac{1}{12}z\alpha} \times \prod_{n=1}^{\infty} (1 + zq^{n+(j+\alpha)-1})(1 + z^{-1}q^{n-(j+\alpha)}). \quad (\text{A2})$$

It is evident from this formula that the characters (almost) only depend on $(j + \alpha)$. In particular, we obtain the equivalence:

$$\chi^{(j,\alpha)}(q, z) = z^{\frac{1}{2}-j} \chi^{(\frac{1}{2}, \alpha+j-\frac{1}{2})}. \quad (\text{A3})$$

Thus, the Virasoro characters (putting $z = 1$) of the b/c -spin systems are all equivalent to characters of the complex fermion with $c = 1$ where the twist sectors α get mapped to others with $\alpha + j - \frac{1}{2}$. Thus, all sectors which are mapped in this way keep their statistics, since $j \in \mathbb{Z} + \frac{1}{2}$ and $\alpha \equiv \alpha + j - \frac{1}{2} \pmod{1}$. A more detailed analysis⁶¹⁻⁶⁴ reveals that the partition functions are indeed equivalent. This extends to the $c = -2$ spin system describing pairing.^{32,51} Therefore, the space of states of b/c -spin systems with twists $\alpha = k/m$, $k = 0, \dots, m-1$, is equivalent to the space of states of a rational $c = 1$ (\mathbb{Z}_2 orbifold) theory with radius of compactification $2R^2 = 1/m$. The careful reader should note that this equivalence holds. Although we always consider m copies of our b/c -spin systems, we work in an Abelian projection where the charges (or twists) of all copies of the fields are closely related to each other. Since they are not chosen independently, we only get one copy of the Hilbert space.

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 - ⁶⁷ There are some other proposals making use of so-called minimal $\mathcal{W}_{1+\infty}$ models or $\widehat{SU}(m)$ Kac-Moody algebras for m -layer states.^{7,17,23,45}