KCL-MTH-98-36
hep-th/9808169
August 1998

Logarithmic Conformal Field Theory & Seiberg-Witten Models

Mic$\hbar$ael A.I. Flohr*

Department of Mathematics
King's College London
The Strand
London WC2R 2LS, United Kingdom

Abstract

The periods of arbitrary abelian forms on hyperelliptic Riemann surfaces, in particular the periods of the meromorphic Seiberg-Witten differential $\lambda_{\rm SW}$, are shown to be in one-to-one correspondence with the conformal blocks of correlation functions of the rational logarithmic conformal field theory with central charge $c = c_{2,1} = -2$. The fields of this theory precisely simulate the branched double covering picture of a hyperelliptic curve, such that generic periods can be expressed in terms of certain generalised hypergeometric functions, namely the Lauricella functions of type F_D.

* homepage: http://www.mth.kcl.ac.uk/~flohr/
email: flohr@mth.kcl.ac.uk

I. Introduction

In a seminal work [1], Seiberg and Witten found the exact low-energy effective action of four-dimensional N=2 supersymmetric $SU(2)$ Yang-Mills theory. Soon, this was generalised to general simple gauge groups [2]. At the heart of the exact solution lies a certain Riemann surface, in the case of a simple, simply-laced gauge group a hyperelliptic one, which constitutes the moduli space of the Yang-Mills theory. All information, in particular the scalar modes and the prepotential, are encoded in this hyperelliptic curve and a special meromorphic differential form associated to it, the so-called Seiberg-Witten differential $\lambda_{\rm SW}$. The task of exactly solving the low-energy effective field theory is then reduced to essentially computing the periods of $\lambda_{\rm SW}$.

In this paper, we will achieve the computation of the Seiberg-Witten periods in a new way, expressing them in terms of conformal blocks of a very special conformal field theory (CFT) with central charge $c = -2$. This theory belongs to a rather new class of CFTs, which has been studied in some detail only recently [3], the so-called logarithmic conformal field theories (LCFTs). First encountered and shown to be consistent in [4], they are not just a peculiarity but merely a generalisation of ordinary two-dimensional CFTs with broad and growing applications [5]. As is particularly true for Seiberg-Witten models, logarithmic divergences are sometimes quite physical, and so there is an increasing interest in these logarithmic conformal field theories. The relevance of LCFT in the Seiberg-Witten context has first been observed in [12].

Furthermore, this application illuminates the geometry behind logarithmic CFT. It is well known that vertex operators on worldsheet CFTs in string theory describe the equivalent of Feynman graphs with outer legs by simulating their effect on a Riemann surface as punctures. Now, in the new setting of moduli spaces of low-energy effective field theories, *pairs of* vertex operators describe the insertion of additional handles to a Riemann surface, simulating the resulting branch cuts. So, in much the same way as a smooth but infinitely long stretched tube attached to an otherwise closed worldsheet, standing for an external state, is replaced by a puncture with an appropriate vertex operator, so a smooth additional handle, standing for an intersecting 4-brane on the 5-brane worldvolume in the type IIA picture of low-energy effective field theories, is replaced by branch cuts with appropriate vertex operators at its endpoints. Hence, operator product expansions (OPEs) of such vertex operators simulating branch points, poles etc. on the curve represented as a branched covering $Z : \Sigma \to \mathbb{CP}^1$ provide an intuitive way of understanding what happens when, for instance, intersecting 4-branes run into each other or shrink to zero size.

This letter is organised as follows: In section II we briefly discuss the hyperelliptic curves and the Seiberg-Witten differential in the form relevant to our approach. Section III recapitulates the construction of 1-differentials on hyperelliptic curves in terms of vertex operators, emphasising why this leads to a logarithmic CFT. Then we compute the Seiberg-Witten periods in terms of conformal blocks in section IV, expressing them in terms of certain special functions. We conclude this last section with a brief discussion and outlook.

II. Seiberg-Witten Solutions of Supersymmetric Four-Dimensional Yang-Mills Theories

Of particular interest for the exact Seiberg-Witten low-energy effective field theory solutions of supersymmetric Yang-Mills theories is the understanding of the moduli space of vacua, which in many cases turns out to be a hyperelliptic Riemann surface. The BPS spectrum of such a model is entirely determined by the periods of a special meromorphic 1-differential on this Riemann surface, the Seiberg-Witten differential $\lambda_{\rm SW}$. Let α_i, β^j denote a canonical basis of the homology of the Riemann surface, $\alpha_i \cap \beta^j = \delta_i^{\ j}$, then the scalar modes are simply given as $a_i = \oint_{\alpha_i} \lambda_{\rm SW}$, $a_D^j = \oint_{\beta^j} \lambda_{\rm SW}$. They carry electric and magnetic charges respectively, and the mass of a BPS state with charges $(\mathbf{q}, \mathbf{g})$ is then given as $m_{(\mathbf{q},\mathbf{g})} \sim |q^i a_i + g_j a_D^j|$, momentarily neglecting possible residue terms in case of the presence of hypermultiplets.

A general hyperelliptic Riemann surface can be described in terms of two variables Z, w in polynomial form

$$w^2 + 2A(Z)w + B(Z) = 0 \tag{2.1}$$

with $A(Z), B(Z) \in \mathbb{C}[Z]$. After a simple coordinate transformation in $y = w + A(Z)$, this takes on the more familiar form $y^2 = A(Z)^2 - B(Z)$. But we might also write the hyperelliptic curve in terms of a rational map. Dividing (2.1) by $A(Z)^2$ and putting $\tilde{w} = w/A(Z) + 1$, we arrive at the representation

$$(1-\tilde{w})(1+\tilde{w}) = \frac{B(Z)}{A(Z)^2}\,. \tag{2.2}$$

This form is very appropriate in the frame of Seiberg-Witten models, since the Seiberg-Witten differential can be read off directly: The rational map $R(Z) = B(Z)/A(Z)^2$ is singular at the zeroes of $B(Z)$ and $A(Z)$, and is degenerate whenever its Wronskian $W(R) \equiv (\partial_Z A(Z)^2)B(Z) - A(Z)^2(\partial_Z B(Z))$ vanishes. This is precisely the information encoded in $\lambda_{\rm SW}$ which for arbitrary hyperelliptic curves, given by a rational map $R(Z)$, can be expressed as

$$\lambda_{\rm SW} = \frac{Z}{2\pi i}\,\mathrm{d}(\log\frac{1-\tilde{w}}{1+\tilde{w}}) = \frac{1}{2\pi i}\mathrm{d}(\log R(Z))\frac{Z}{\tilde{w}} = \frac{1}{2\pi i}\frac{W(A(Z)^2, B(Z))}{A(Z)B(Z)}\frac{Z\,\mathrm{d}Z}{y}\,. \tag{2.3}$$

It is this local form of the Seiberg-Witten differential which serves as a metric $\mathrm{d}s^2 = |\lambda_{\rm SW}|^2$ on the Riemann surface, and it is this local form which arises as the tension of self-dual strings coming from 3-branes in type II string theory compactifications on Calabi-Yau threefolds.*

Let us, for the sake of simplicity, concentrate on N=2 $SU(N_c)$ Yang-Mills theory with N_f massive hypermultiplets. Then, the hyperelliptic curve $y^2 = A(x)^2 - B(x)$ takes the form

$$y^2 = \left(x^{N_c} - \sum_{k=2}^{N_c} s_k x^{N_c-k}\right)^2 - \Lambda^{2N_c-N_f}\prod_{i=1}^{N_f}(x-m_i) = \prod_{j=1}^{2N_c}(x-e_j)\,, \tag{2.4}$$

where we have absorbed any dependency of $A(x) = \prod_{k=1}^{N_c}(x-\tilde{a}_k)$ on the m_i, which is the case for $N_f > N_c$, in a redefinition of the classical expectation values $\tilde{a}_k$ or s_k respectively. Then, the Seiberg-Witten differential reads

$$\lambda_{\rm SW}(SU(N_c)) = \frac{1}{2\pi i}\frac{\prod_{l=0}^{N_c+N_f-1}(x-z_l)}{\prod_{j=1}^{2N_c}\sqrt{x-e_j}\prod_{i=1}^{N_f}(x-m_i)}\,\mathrm{d}x\,, \tag{2.5}$$

where the z_l denote the zeroes of $2A(x)'B(x) - A(x)B(x)'$, and $z_0 = 0$. As a result, the total order of the general Seiberg-Witten form (2.3) vanishes, $(1 + N_c + N_f - 1)\cdot(1) + (2N_c)\cdot(-\frac{1}{2}) + (N_f)\cdot(-1) = 0$ implying that $\lambda_{\rm SW}$ has a double pole at infinity. We note that the periods of the Seiberg-Witten form are hence contour integrals with paths encircling pairs (e_i, e_j) and with an integral kernel of the form

$$\lambda_{\rm SW} \sim \prod_i (x-x_i)^{r_i}\,, \qquad \sum_i r_i = 0\,, \qquad r_i \in \{0, \pm\tfrac{1}{2}, \pm 1\}\,, \tag{2.6}$$

where the branch points e_i are a subset of the singular points x_i of the integral kernel.

III. The $c = -2$ Logarithmic CFT and 1-Differentials

The idea to represent general j-differentials ($j \in \mathbb{Z}/2$ due to locality) in terms of fields of a CFT is actually not new. We will follow here the approach put forward by Knizhnik [7], restricted to the case of interest, $j = 1$ and hyperelliptic curves, i.e. all branch points have ramification number two. As we will demonstrate, this CFT approach to the theory of Riemann surfaces naturally leads to a logarithmic CFT.

In the case of hyperelliptic curves, j-differentials are constructed by two pairs of anticommuting fields $\phi^{(j),\ell}, \phi^{(1-j),\ell}$ of spin $j, 1-j$ respectively, one pair for each sheet of the Riemann surface Σ represented as a

*This form is equivalent to the one for integrable Toda systems with spectral curve $z+1/z+r(t) = z+1/z+2A(t)/\sqrt{B(t)} = 0$, where $\lambda_{\rm SW} = t\,\mathrm{d}(\log z)$ is nothing other than the Hamilton-Jacobi function of the underlying integrable hierarchy. However, the price paid for this very simple form of $\lambda_{\rm SW}$ is that $r(t)$ is now only a fractional rational map.

branched covering of $\mathbb{CP}^1$, where the sheets are enumerated by $\ell = 0, 1$. We denote the covering map by Z. The point is that such fields behave as differentials of weight j under conformal transformations,

$$\phi^{(j),\ell}(Z', \bar{Z}')\left(\frac{\mathrm{d}Z'}{\mathrm{d}Z}\right)^j = \phi^{(j),\ell}(Z, \bar{Z})\,. \tag{3.1}$$

We assume that the operator product expansion (OPE) be normalised as

$$\phi^{(j),\ell}(Z')\phi^{(1-j),\ell}(Z) \simeq \mathbb{I}\,(Z'-Z)^{-1} + \text{regular terms} \tag{3.2}$$

with $\mathbb{I}$ denoting the identity operator. On each sheet, we have an action

$$S^{(\ell)} = \int \phi^{(j),\ell}\,\bar{\partial}\phi^{(1-j),\ell}\,\mathrm{d}^2Z = \int \phi^{(1),\ell}\,\bar{\partial}\phi^{(0),\ell}\,, \tag{3.3}$$

where integration runs over the Riemann surface Σ, and a stress energy tensor which takes the form

$$T^{(\ell)} = -j\phi^{(j),\ell}\,\partial\phi^{(1-j),\ell} + (j-1)\phi^{(1-j),\ell}\,\partial\phi^{(j),\ell} = -\phi^{(1),\ell}\,\partial\phi^{(0),\ell} \tag{3.4}$$

giving rise to a central extension $c = c_j \equiv -2(6j^2 - 6j + 1)$, i.e. in our case $c = c_1 = -2$.

Let now a genus g hyperelliptic curve be given as $y^2 = \prod_{k=1}^{2g+2}(Z - e_k)$ such that infinity would not be a branch point. At each branch point e_k, we can locally invert this to $Z(y) \sim e_k + y^2$ such that in its vicinity $y(Z) \sim (z - e_k)^{1/2}$. Let us denote the operation of moving a point around e_k by $\hat{\pi}_{e_k}$. It acts on the j-differentials with the boundary conditions

$$\hat{\pi}_{e_k}\phi^{(j),\ell}(Z) = (-)^{2j}\phi^{(j),\ell+1 \bmod 2}(Z) \tag{3.5}$$

in the vicinity of e_k. Since the $\mathbb{Z}_2$ symmetry of Σ is global, we can diagonalize $\hat{\pi}$ globally by choosing a new basis via a discrete Fourier transform,

$$\phi^{(j)}_k = \phi^{(j),0} + (-)^{j-k}\phi^{(j),1}\,, \tag{3.6}$$

with $k = 0, 1$, such that $\hat{\pi}_a\phi^{(j)}_k = (-)^{k-j}\phi^{(j)}_k$ for a any branch point. We can now define chiral currents $J_k = {:}\phi^{(j)}_k\phi^{(1-j)}_k{:}$, $\bar{\partial}J_k = 0$, which are single valued functions near a. It follows then that a branch point a carries charges $q_k = \frac{1}{2}(j-k) = \frac{1}{2}(1-k)$ with respect to these currents.

In order to do explicit calculations we bosonize with two analytic scalar fields φ_k, $k = 0, 1$, normalised in the usual way $\langle\varphi_k(z)\varphi_l(z')\rangle = -\delta_{kl}\log(z - z')$. Clearly, we have $\phi^{(j)}_k = {:}\exp(-i\varphi_k){:}$, $\phi^{(1-j)}_{1-k} = {:}\exp(+i\varphi_k){:}$, $J_k = i\partial\varphi_k$, and $T_k = \frac{1}{2}{:}J_kJ_k{:} + \frac{1}{2}\partial J_k$. Hence, we have a Coulomb gas CFT with background charge $2\alpha_0 = 1$. We define general vertex operators with charge $\mathbf{q} = (q_0, q_1)$ as $V_{\mathbf{q}}(a) = {:}\exp(i\,\mathbf{q}\cdot\boldsymbol{\varphi}(a)){:}$ which have conformal scaling dimensions $h(\mathbf{q}) = h_0 + h_1$ with $h_k = \frac{1}{2}(q_k^2 - q_k)$. Note that branch points are trivial objects in the $k = 1$ sector such that it suffices to only consider the $k = 0$ sector from now on.

Of course, correlators in free field realization of CFT are only non-zero, if they satisfy the charge neutrality condition. For example, the only non-vanishing two-point functions are $\langle V_{2\alpha_0-q}(z)V_q(z')\rangle = A(z - z')^{-2h(q)}$, where A is arbitrarily fixed by normalisation of the fields. However, a careful analysis shows [8] that the branch point representing vertex operator does not have a conjugate field as we expect it. The charge of a branch point is $q = \alpha_0 = 1/2$ such that $2\alpha_0 - q = q$, i.e. the branch point vertex operator appears to be self-conjugate. However, this is not true, $\langle V_{1/2}(z)V_{1/2}(z')\rangle = 0$. It turns out that the correct partner of this field is $\Lambda_{1/2} = \partial_q V_q\big|_{q=\alpha_0} = i\varphi V_{1/2}$, such that $\langle\Lambda_{1/2}(z)V_{1/2}(z')\rangle = B(z - z')^{1/4}$ and $\langle\Lambda_{1/2}(z)\Lambda_{1/2}(z')\rangle = (C - 2B\log(z - z'))(z - z')^{1/4}$. The constants A, B, C are now no longer entirely free. $SL(2,\mathbb{C})$ invariance of the two-point functions requires that $A = 0$, $B = \langle 2i\varphi V_{2\alpha_0}\rangle \equiv 1$, $C = 0$. Although this field $\Lambda_{1/2}$ is a proper primary field with respect to the stress energy tensor, it will cause logarithmic terms in the OPE with other primary fields. It will also give rise to other fields of this form, $\Lambda_q = (\partial_q h(q))^{-1}\partial_q V_q = \frac{i}{q-a_0}\varphi V_q$ which are the logarithmic partners to the primary fields V_{1-q}. Note that the latter definition of Λ_q is only valid for $q \neq \alpha_0 = \frac{1}{2}$. The conformal Ward identities forced us to put $\langle V_1\rangle = \langle V_0\rangle = \langle\mathbb{I}\rangle = 0$, while

$\langle\Lambda_1\rangle = 1$. This might seem strange but can be seen to be quite natural in a realization of this CFT by a pair of anticommuting scalar fields with manifest $SL(2,\mathbb{C})$ invariance, where the path integrals vanish unless zero modes are inserted [9]. In fact, the naive definition $\det\bar\partial_{(j)} = \int \mathcal{D}\phi^{(j),\ell}\mathcal{D}\phi^{(1-j),\ell}\exp(S^{(\ell)}) = 0$, due to $n_j - n_{1-j} = (2j-1)(g-1)$ zero modes of $\bar\partial$-holomorphic j- and $(1-j)$-differentials on a genus g Riemann surface.

To summarise, the $c=-2$ CFT of 1-differentials inevitably becomes logarithmic when we add to its field content the branch point representing vertex operator $V_{1/2}$. The reason is its proper conjugate field $\Lambda_{1/2}$, needed to cancel off the $n_{1-j}=1$ scalar zero mode. Hence, reducing an arbitrary correlation function with vertex operators V_q and logarithmic partners Λ_q ultimately will result in picking out only such nested OPEs, which lead to the only non-vanishing one-point functions $\langle\Lambda_q\rangle$. For example, the logarithmic partner of the identity, Λ_1, has the OPE $\Lambda_1(z)\Lambda_1(z') = \mathbb{I} - 2\log(z-z')\Lambda_1(z') + \ldots$ Hence, $\langle\Lambda_1(z)\Lambda_1(z')\rangle = -2\log(z-z')\langle\Lambda_1\rangle = -2\log(z-z')$.

From the above follows that we can replace the operator for a branch point by $\mu(a) = V_{1/2}(a) + \Lambda_{1/2}(a)$. We will work with reduced correlators

$$\left\langle\!\!\left\langle \prod_i \Phi_{q_i}(z_i) \right\rangle\!\!\right\rangle \equiv \prod_{k<l}(z_k - z_l)^{-q_k q_l} \left\langle \prod_i \Phi_{q_i}(z_i) \right\rangle \tag{3.7}$$

where the canonical free part has been divided off, $\Phi = V, \Lambda$. The reduced correlator is thus equal to the screening charge integrals still necessary to ensure charge neutrality. Under conformal transformations $z \mapsto M(z)$, a correlator transforms with weights $\left(\partial_z M(z)|_{z=z_i}\right)^{h(q_i)}$ for each field $\Phi_{q_i}(z_i)$. For the reduced correlators, the exponent simply has to be replaced by $-q_i/2$.

We can now express an arbitrary abelian differential on the hyperelliptic curve $\Sigma : y^2 = \prod_{k=1}^{2g+2}(Z-e_k) = \prod_{k=1}^{g+1}(Z-e_k^-)(Z-e_k^+)$ in terms of fields of the $c=-2$ LCFT. In fact, with the above notations

$$\omega = \frac{\prod_{i=1}^{M}(Z-z_i)}{\prod_{k=1}^{2g+2}\sqrt{Z-e_k}\prod_{j=1}^{N}(Z-p_j)}\,\mathrm{d}Z = \prod_{i=1}^{M} V_{-1}(z_i)\prod_{k=1}^{2g+2}\mu(e_k)\prod_{j=1}^{N} V_1(p_j)\,\phi_0^{(1)}(Z)\,. \tag{3.8}$$

It is then clear that a contour integral along a closed path γ defines a conformal block

$$\oint_\gamma \omega = \left\langle\!\!\left\langle V_Q(\infty)\prod_{i=1}^{M} V_{-1}(z_i)\prod_{k=1}^{2g+2}\mu(e_k)\prod_{j=1}^{N} V_1(p_j)\right\rangle\!\!\right\rangle_{(\gamma)}\,, \tag{3.9}$$

where $Q = 2 - \sum q_i = 1 + M - N - g$ is the charge of a pole at infinity such that charge neutrality is ensured by insertion of only one screening charge $Q_- = \oint J_-$ with $J_- \equiv \phi_0^{(1)}$ being the 1-differential (note that $2\alpha_0 = 1$ and that $\phi_0^{(1)} \sim V_{-1}$ changes the charge by -1). We now choose (part of)* the basis of conformal blocks to coincide with the canonical homology basis of cycles, i.e. $\gamma \in \{\alpha_i, \beta^i\}_{1\le i\le g}$ which can be choosen as $\alpha_i = C_{(e_i^-, e_i^+)}$, $\beta^i = C_{(e_i^+, e_{g+1}^-)}$. Here, $C_{(a,b)}$ denotes a closed path encircling a, b.

IV. Periods of the Seiberg-Witten Differential

Let us start with a warm up by calculating the periods of the only holomorphic one-form for the torus, i.e. for gauge group $SU(2)$. The torus in question is given by $y^2 = (x^2-u)^2 - \Lambda^4$ with the four branch points $e_1 = \sqrt{u-\Lambda^2}$, $e_2 = -\sqrt{u+\Lambda^2}$, $e_3 = -\sqrt{u-\Lambda^2}$, $e_4 = \sqrt{u+\Lambda^2}$. The standard periods of the holomorphic form $\mathrm{d}x/y$ are easily computed (where the normalization has been fixed to be in accordance with the asymptotic behavior of a and a_D in the weak coupling region). With $\xi = 1/M(e_4) = \frac{(e_1-e_4)(e_3-e_2)}{(e_2-e_1)(e_4-e_3)}$

*Further singular points z_i, p_j of ω can either be multiplied out to yield a sum of smaller integral kernels, or simply contribute residual terms. But we can treat them on equal footing with the branch points e_k in the CFT picture by analytic continuation of correlation functions with $q_i \notin \mathbb{Z}/2$ to these particular values.

the inverse crossing ratio, $\xi = (u - \sqrt{u^2 - \Lambda^4})/(u + \sqrt{u^2 - \Lambda^4})$, we have

$$\begin{aligned}
\pi_1 = \frac{\partial a}{\partial u} &= \frac{\sqrt{2}}{2\pi}\int_{e_2}^{e_3}\frac{\mathrm{d}x}{y} = \frac{\sqrt{2}}{2\pi}\langle\!\langle \mu(e_1)\mu(e_2)\mu(e_3)\mu(e_4)\rangle\!\rangle_{(e_2,e_3)} \\
&= \frac{\sqrt{2}}{2\pi}(e_3-e_2)^{-\frac{1}{2}}(e_4-e_1)^{-\frac{1}{2}}\langle\!\langle \mu(\infty)\mu(1)\mu(0)\mu(M(e_4))\rangle\!\rangle_{(0,1)} \\
&= \frac{\sqrt{2}}{2}(e_2-e_1)^{-\frac{1}{2}}(e_4-e_3)^{-\frac{1}{2}}{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\xi)\,, \qquad (4.1)
\end{aligned}$$

The other period is obtained in complete analogy by exchanging e_2 with e_1, yielding

$$\pi_2 = \frac{\partial a_D}{\partial u} = \frac{\sqrt{2}}{2\pi}\int_{e_1}^{e_3}\frac{\mathrm{d}x}{y} = \frac{\sqrt{2}}{2}(e_1-e_2)^{-\frac{1}{2}}(e_4-e_3)^{-\frac{1}{2}}{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;1-\xi)\,. \qquad (4.2)$$

Here and in the following, (generalized) hypergeometric functions with arguments such as $1-\xi$ are understood as expansions around $1-\xi$ and should be analytically continued to a region around ξ. This will result in the desired logarithmic divergencies. For example, with the usual Frobenius process we find (the factor $\pi = \Gamma(\frac{1}{2})^2$ stems from the formula for analytic continuation of hypergeometric functions)

$$\begin{aligned}
\pi\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;1-\xi) &= {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\xi)\log(\xi) + \sum_{n=0}^{\infty}\left(\frac{\partial}{\partial\varepsilon}\frac{(\frac{1}{2}+\varepsilon)_n(\frac{1}{2}+\varepsilon)_n}{(1+\varepsilon)_n(1+\varepsilon)_n}\right)\Big|_{\varepsilon=0}\xi^n \qquad (4.3)\\
&= {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\xi)\log(\xi) + \partial_\varepsilon\,{}_3F_2(1,\tfrac{1}{2}+\varepsilon,\tfrac{1}{2}+\varepsilon;1+\varepsilon,1+\varepsilon;\xi)\big|_{\varepsilon=0}\,.
\end{aligned}$$

These results are, of course, well known. Less known might be the fact that for the case without hypermultiplets, $N_f = 0$, we can express the periods of the Seiberg-Witten form by the Lauricella function $F_D^{(3)}$. In fact,

$$\begin{aligned}
a(u) &= \frac{\sqrt{2}}{2\pi}\int_{e_2}^{e_3}\frac{4x^2\,\mathrm{d}x}{y} = \frac{2\sqrt{2}}{\pi}\langle\!\langle V_2(\infty)\mu(e_1)\mu(e_2)\mu(e_3)\mu(e_4)V_{-2}(0)\rangle\!\rangle_{(e_2,e_3)} \\
&= \frac{2\sqrt{2}}{\pi}\frac{e_1^2}{(e_3-e_2)^{\frac{1}{2}}(e_4-e_1)^{\frac{1}{2}}}\langle\!\langle \mu(\infty)\mu(1)\mu(0)\mu(M(e_4))V_{-2}(M(0))V_2(M(\infty))\rangle\!\rangle_{(0,1)} \\
&= 2\sqrt{2}\frac{e_3^2}{(e_4-e_3)^{\frac{1}{2}}(e_2-e_1)^{\frac{1}{2}}}F_D^{(3)}(\tfrac{1}{2},\tfrac{1}{2},-2,2,1;\xi,\eta,\varpi)\,, \qquad (4.4)
\end{aligned}$$

with the second inverse cross ratio $\eta = 1/M(0) = \frac{e_1(e_2-e_3)}{(e_1-e_2)e_3}$, and $\varpi = 1/M(\infty) = \frac{e_2-e_3}{e_2-e_1}$ the inverse of the image of the double pole at infinity. The Lauricella D-type functions are generalized hypergeometric functions in several variables, given as power series (with $(a)_n = \Gamma(a+n)/\Gamma(a)$ the Pochhammer symbol)

$$\begin{aligned}
&F_D^{(n)}(a,b_1,b_2,\ldots,b_n,c;x_1,x_2,\ldots,x_n) = \\
&\quad\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a)_{m_1+m_2+\ldots+m_n}(b_1)_{m_1}(b_2)_{m_2}\ldots(b_n)_{m_n}}{(c)_{m_1+m_2+\ldots+m_n}(1)_{m_1}(1)_{m_2}\ldots(1)_{m_n}}x_1^{m_1}x_2^{m_2}\ldots x_n^{m_n}\,, \qquad (4.5)
\end{aligned}$$

whenever $|x_1|,|x_2|,\ldots,|x_n| < 1$. Its integral representation has the form of a CFT screening integral, $\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}F_D^{(n)}(a,b_1,\ldots,b_n,c;x_1,\ldots,x_n) = \int_0^1 u^{a-1}(1-u)^{c-a-1}\prod_{i=1}^n(1-ux_i)^{-b_i}\,\mathrm{d}u$. For $n=1$, it reduces to the ordinary Gauss hypergeometric function ${}_2F_1(a,b_1;c;x_1)$, and for $n=2$, it is nothing else than the Appell function $F_1(a;b_1,b_2;c;x_1,x_2)$. A great deal of information on these functions may be found for example in the book [10] by Exton. An important fact is that $F_D^{(n)}$ satisfies the following system of partial differential equations of second order:

$$\left[(1-x_j)\sum_{k=1}^{n}x_k\frac{\partial^2}{\partial x_k\partial x_j} + (c-(a+b_j+1)x_j)\frac{\partial}{\partial x_j} - b_j\sum_{\substack{k=1\\k\neq j}}^{n}x_k\frac{\partial}{\partial x_k} - ab_j\right]F = 0\,, \qquad (4.6)$$

where $j = 1, \ldots, n$. Interestingly, this remains true even in the case that massive hypermultiplets are present ($N_f > 0$), while the Picard-Fuchs equations now are of third order. However, the price paid is an artifically enlarged number of variables. Furthermore, we easily can write down differential equations of second and third order for each field in the correlator which is proportional to $F_D^{(n)}$, depending on whether the field is degenerate of level two, e.g. $\mu = \Psi_{1,2}$, $V_{-1} = \Psi_{2,1}$, or three as $V_1 = \Psi_{1,3}$ (where we consider the $c = -2$ CFT as the degenerate model with $c = c_{2,1}$) according to [11]. We extensively exploit the special properties of these functions in our forthcoming paper [6].

Again, we may obtain the dual period by exchanging e_2 with e_1, yielding

$$a_D(u) = 2\sqrt{2}\frac{e_3^2}{(e_4 - e_3)^{\frac{1}{2}}(e_1 - e_2)^{\frac{1}{2}}} F_D^{(3)}(\tfrac{1}{2}, \tfrac{1}{2}, -2, 2, 1; 1 - \xi, 1 - \eta, 1 - \varpi)\,. \tag{4.7}$$

The two periods given above are by construction the $a_{(\alpha)}$ and $a_{(\beta)}$ periods respectively. It is worth noting that the dependency on three variables is superficial, since all cross ratios are solely functions in the four branch points. Indeed, we have $\xi = \varpi^2$, $\eta = -\varpi$. The inverse crossing ratios have the nice property that they tend to zero for $|u| \gg 1$, e.g. $\xi \sim (\frac{1}{2}\frac{\Lambda^2}{u})^2 + O(u^{-4})$. Hence, the overall asymptotics of $a(u)$ and $a_D(u)$ is entirely determined by the prefactors, which are $a(u) \sim \frac{2\sqrt{2}e_3^2}{\sqrt{e_4 - e_3}\sqrt{e_2 - e_1}} \sim \sqrt{2u} + O(u^{-\frac{1}{2}})$ and $a_D(u) \sim \frac{\sqrt{2}e_3^2}{\pi\sqrt{e_4 - e_3}\sqrt{e_1 - e_2}} \log(\xi) \sim \frac{i}{\pi}\sqrt{2u}\log(u) + O(u^{-\frac{1}{2}}\log(u))$. Expanding $a(u)$ as a power series in $1/u$ yields the familiar result

$$\begin{aligned} a(u) &= \sqrt{2u}\left[1 - \frac{1}{16}\frac{\Lambda^4}{u^2} - \frac{15}{1024}\frac{\Lambda^8}{u^4} - \frac{105}{16384}\frac{\Lambda^{12}}{u^6} - \frac{15015}{4194304}\frac{\Lambda^{16}}{u^8} + O(u^{-10})\right] \\ &= \sqrt{2}\sqrt{u + \Lambda^2}\, {}_2F_1(-\tfrac{1}{2}, \tfrac{1}{2}, 1; \frac{2\Lambda^2}{u + \Lambda^2})\,. \end{aligned} \tag{4.8}$$

The strength of the CFT picture becomes apparent when asymptotic regions of the moduli space are to be explored. Then, OPE and fusion rules provide easy and suggestive tools. For example, the asymptotics of $a(u)$ and $a_D(u)$ follow directly from the OPE of the field μ as discussed in the preceeding section. The logarithmic partners of primary fields appear precisely, if the contour of the screening charge integration gets pinched between the two fields whose OPE is inserted. Thus, the choice of contour together with the choice of internal channels (due to inserted OPEs) determines which term of the OPE $\mu(z)\mu(0) \sim z^{1/4}(V_1(0) + \Lambda_1(0) - 2\log(z)V_1(0) + \ldots)$ is picked. So, when expanded in ξ, both periods, $a(u)$ and $a_D(u)$ have asymptotics according to inserting the OPEs $\mu(e_2)\mu(e_3)$ and $\mu(e_1)\mu(e_4)$. Keeping in mind (3.7) when inserting an OPE, we find with $e_{ij} = e_i - e_j$

$$\begin{aligned} a(u) &\sim [e_{12}e_{13}e_{42}e_{43}]^{-1/4}\,\frac{e_1e_2}{e_3e_4}\,[e_{34}\,\langle\!\langle V_2(\infty)V_1(e_3)V_1(e_4)V_{-2}(0)\rangle\!\rangle + \ldots] \\ &\sim [e_{12}e_{13}e_{42}e_{43}]^{-1/4}\,\frac{e_1e_2e_4}{e_3}\,[\langle\!\langle V_2(\infty)\Lambda_1(e_4)V_{-2}(0)\rangle\!\rangle + \ldots] \\ &\sim \sqrt{2u} + \ldots\,, \end{aligned} \tag{4.9}$$

where the three-point functions evaluate trivially. In a similar fashion, we obtain

$$\begin{aligned} a_D(u) &\sim \frac{1}{i\pi}\,[e_{12}e_{13}e_{42}e_{43}]^{-1/4}\,\frac{e_1e_2}{e_3e_4}\,[e_{34}\,\langle\!\langle V_2(\infty)\Lambda_1(e_3)\Lambda_1(e_4)V_{-2}(0)\rangle\!\rangle + \ldots] \\ &\sim \frac{1}{i\pi}\,[e_{12}e_{13}e_{42}e_{43}]^{-1/4}\,\frac{e_1e_2e_4}{e_3}\,[-2\log(e_4 - e_3)\,\langle\!\langle V_2(\infty)\Lambda_1(e_3)V_{-2}(0)\rangle\!\rangle + \ldots] \\ &\sim \frac{i}{\pi}\sqrt{2u}\,[\log(u) + 2\log(2) + \ldots]\,. \end{aligned} \tag{4.10}$$

Of course, other internal channels can be considered. In particular, we may insert the OPE for $|e_1 - e_3| \ll 1$ to get the behavior of the periods for the case $u \longrightarrow \Lambda^2$. In fact, $a_D(u)$ and $a(u)$ exchange their rôle since now the monopole becomes massless. Put differently, duality in Seiberg-Witten models cooks down to crossing symmetry in our $c=-2$ LCFT. The leading term can be read off from $a_D(u)$ above (the OPE factors turn out to be the same upto a braiding phase) to be proportional to $i(u-\Lambda^2)/\sqrt{2\Lambda^2}$. The relative normalization of the logarithmic operator Λ_1 with respect to its primary partner is fixed by considering $a_D(u)$ as the analytic continuation of $a(u)$ via crossing symmetry yielding a factor of $(i\pi)^{-1}$.

There is one further BPS state which can become massless, since there is one further zero of the discriminant

$$\Delta(y^2(x)) = (\det\bar\partial_{(j=\frac12)})^8 = \left(\left\langle\prod_{i=1}^{2g+2} V_{1/2}(e_i)\right\rangle_{c=1}\right)^8 = \prod_{j<k}(e_j-e_k)^2\,, \tag{4.11}$$

namely $e_2 \longrightarrow e_4$. This is a dyonic state with charge $(q,g)=(-2,1)$, meaning that both, the α cycles as well as the β cycle, get pinched in this limit. It follows that both, $a(u)$ as well as $a_D(u)$, will receive logarithmic corrections when $u\longrightarrow -\Lambda^2$, which is well known to be the case.

Within the CFT picture, higher gauge groups as well as additional (massive) flavours are treated in the same way. Hence, we obtain for the $SU(2)$ case with $N_f<4$ hypermultiplets, after simple algebra in the numerator,

$$\begin{aligned}\lambda_{\rm SW} &= \frac{1}{2\pi i}\frac{x\,{\rm d}x}{y\prod_{k=1}^{N_f}(x-m_k)}\left(4x\prod_{k=1}^{N_f}(x-m_k)-(x-\sqrt{u})(x+\sqrt{u})\sum_{k=1}^{N_f}\prod_{l\neq k}(x-m_l)\right)\\ &= \frac{{\rm d}x}{2\pi i}\left((4-N_f)\frac{x^2}{y}+N_f\frac{u}{y}-\sum_{k=1}^{N_f}m_k\left(\frac{x^2}{y(x-m_k)}-\frac{u}{y(x-m_k)}\right)\right)\,,\end{aligned} \tag{4.12}$$

such that we immediately can express the periods of the Seiberg-Witten form in 4-point and 5-point functions. Using $\frac{x^2}{y(x-m_k)}=\frac{x+m_k}{y}+\frac{m_k^2}{y(x-m_k)}$ to rewrite the last term, we obtain

$$\begin{aligned}\oint\lambda_{\rm SW} = \frac{1}{2\pi i}\Bigg(&(4-N_f)\,\langle\!\langle V_2(\infty)\mu(e_1)\mu(e_2)\mu(e_3)\mu(e_4)V_{-2}(0)\rangle\!\rangle\ +uN_f\,\langle\!\langle\mu(e_1)\mu(e_2)\mu(e_3)\mu(e_4)\rangle\!\rangle \\ &-\sum_{k=1}^{N_f}m_k\left[\langle\!\langle V_1(\infty)\mu(e_1)\mu(e_2)\mu(e_3)\mu(e_4)V_{-1}(-m_k)\rangle\!\rangle-(u-m_k^2)\,\langle\!\langle V_{-1}(\infty)\mu(e_1)\mu(e_2)\mu(e_3)\mu(e_4)V_1(m_k)\rangle\!\rangle\right]\Bigg)\end{aligned} \tag{4.13}$$

as the CFT expression. We recover hence the well know result that for all $m_k=0$ the scalar modes have roughly the same form as in the $N_f=0$ case. The above results in the following expression ($x(\cdot)=1/M(\cdot)$ denote the inverse crossing ratios)

$$\begin{aligned}\oint\lambda_{\rm SW} &= \Bigg(\frac{(4-N_f)e_3^2}{(e_4-e_3)^{\frac12}(e_2-e_1)^{\frac12}}F_D^{(3)}(\tfrac12,\tfrac12,-2,2,1;x(e_4),x(0),x(\infty))\\ &+\ \frac{uN_f}{(e_2-e_1)^{\frac12}(e_4-e_3)^{\frac12}}\,{}_2F_1(\tfrac12,\tfrac12;1;x(e_4))\\ &-\ \sum_{k=1}^{N_f}\frac{m_k(e_3+m_k)}{(e_2-e_1)^{\frac12}(e_4-e_3)^{\frac12}}F_D^{(3)}(\tfrac12,\tfrac12,-1,1,1;x(e_4),x(-m_k),x(\infty))\\ &+\ \sum_{k=1}^{N_f}\frac{m_k(u-m_k^2)}{(e_2-e_1)^{\frac12}(e_4-e_3)^{\frac12}(e_3-m_k)}F_D^{(3)}(\tfrac12,\tfrac12,1,-1,1;x(e_4),x(m_k),x(\infty))\Bigg)\,.\end{aligned} \tag{4.14}$$

Since the $F_D^{(3)}$ Lauricella functions have a negative integer as one of the numerator parameters, they can be expanded as polynomials in F_1 Appell functions, i.e. 5-point functions via

$$F_D^{(3)}(a;b,b',b'';c;x,y,z)=\sum_{m=0}^{\infty}\frac{(a)_m(b')_m y^m}{(1)_m(c)_m}\,F_1(a+m;b,b'';c+m;x,z)\,, \tag{4.15}$$

since this expansion truncates for $b' \in \mathbb{Z}_-$. Of course, we could have expressed this from the beginning by only one correlation function proportional to $F_D^{(2N_f+3)}$ of $2N_f+3$ variables, as indicated in (3.9), which is to be contrasted with the approach taken in [12].

As one further example, we consider $SU(3)$ without hypermultiplets, where $R(Z) = \Lambda^6/(Z^3-uZ+v)^2$ such that the resulting hyperelliptic curve has six branch points e_i and its metric $|\lambda_{\rm SW}|^2$ possesses three zeroes z_j. We get

$$\begin{aligned}\oint_\gamma \lambda_{\rm SW} &= 2\left\langle\!\!\left\langle V_2(\infty)\mu(e_1)\ldots\mu(e_6)V_{-1}(-\sqrt{u/3})V_{-1}(0)V_{-1}(\sqrt{u/3})\right\rangle\!\!\right\rangle_{(\gamma)} \\ &= \prod_{i=1}^{3}(\partial_{e_i}M(e_i))^{\frac14}\prod_{i=4}^{6}\left(\frac{\partial_{e_i}M(e_i)}{M(e_i)^2}\right)^{\frac14}\prod_{j=1}^{3}\left(\frac{\partial_{z_j}M(z_j)}{M(z_j)^2}\right)^{-\frac12}\lim_{z\to\infty}\left(\frac{z^2\partial_z M(z)}{M(z)^2}\right) \\ &\times\ F_D^{(7)}(\tfrac12,\tfrac12,\tfrac12,\tfrac12,-1,-1,-1,2,1;x(e_4),x(e_5),x(e_6),x(0),x(-\sqrt{u/3}),x(\sqrt{u/3}),x(\infty))\,,\end{aligned} \tag{4.16}$$

whith the last equality valid for $\gamma = \alpha_1 \equiv C(e_2,e_3)$. This Lauricella D-system for seven variables provides the complete set of all periods. There exist more compact expressions in the literature for this case, where the Appell function F_4 is involved [13]. However, presenting the solution in this way is more transparent, if we view the moduli space of low-energy effective field theory as created from string- or M-theory, e.g. as intersecting NS-5 and D-4 branes. Then, the branch points e_i and mass poles m_k are the directly given data – they denote the endpoints of the intersections. It remains to interpret the zeroes of the Seiberg-Witten form within the brane picture, since they appear on equal footing with the other singular points in our CFT approach. Moreover, this approach suggests that BPS states from geodesic integration paths [14] joining two zeroes of $\lambda_{\rm SW}$ can be described in much the same way as the more familiar BPS states connected to the periods. The zeroes of $\lambda_{\rm SW}$ correspond to branching points in the fibration of Calabi-Yau threefold compactifications of type II string theory, and the corresponding BPS states are related to 2-branes ending on the 5-brane worldvolume $\mathbb{R}^4\times\Sigma$.

Expressing the Seiberg-Witten periods in terms of correlation functions reveals a further complication in exploring the moduli space of low-energy effective field theories. These periods depend only on the moduli s_k and perhaps masses m_l. So, for the $SU(3)$ example above, the periods really depend only on two variables, u, v. However, $\lambda_{\rm SW}$ in its factorized form naturally leads to a 10-point function! The complete set of solutions of the associated Lauricella $F_D^{(7)}$ system which covers all of $\mathbb{C}^7$ is actually quite large, and exceeds by far the set of periods obtainable from simple paths enclosing two of the singular points (Pochhamer paths).

The reason behind all this enrichment is buried in the fact that we are dealing with a Riemann surface together with an associated metric $\lambda_{\rm SW}$. A detailed analysis of all these features relies on a deeper knowledge of the analytic properties of Lauricella functions and will be carried out in our forthcoming paper [6].

Acknowledgment: I have enjoyed many fruitful discussions with and comments from Ralph Blumenhagen, Wolfgang Eholzer, Matthias Gaberdiel, Amihay Hanani, Andreas Honecker, Horst Kausch, Neil Lambert, Werner Nahm, Gerard Watts, and Peter West. This research is funded by the EU TMR network number FMRX-CT96-0012.

References

[1] N. Seiberg, E. Witten, Nucl. Phys. **B431** (1994) 484-550 [hep-th/9408099]; Nucl. Phys. **B426** (1994) 19-52, Erratum ibid. **B430** (1994) 485-486 [hep-th/9407087]

[2] See for example [13] and
W. Lerche, *Introduction to Seiberg-Witten theory and its stringy origins*, Proceedings of the *Spring School and Workshop in String Theory*, ICTP, Trieste (1996) [hep-th/9611190]
A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, Phys. Lett. **B344** (1995) 169 [hep-th/9411048]
P. Argyres, A. Faraggi, Phys. Rev. Lett. **74** (1995) 3931 [hep-th/9411057]
A. Hanany, Y. Oz, Nucl. Phys. **B452** (1995) 283 [hep-th/9505074]
N. Seiberg, E. Witten, [hep-th/9607163]

N. Seiberg, Phys. Lett. **B388** 753 (1996) [hep-th/9608111]
and references therein.

[3] See for example [4, 8, 9] and
J.-S. Caux, I.I. Kogan, A. Lewis, A.M. Tsvelik, Nucl. Phys. **B289** (1997) 469 [hep-th/9606138]
M. Flohr, Int. J. Mod. Phys. **A11** (1996) 4147-4172 [hep-th/9509166]; Int. J. Mod. Phys. **A12** (1997) 1943-1958 [hep-th/9605151]; Nucl.Phys. **B514** (1998) 523-552 [hep-th/9707090]
M.R. Gaberdiel, H.G. Kausch, Nucl. Phys. **B477** (1996) 293 [hep-th/9604026]; Phys. Lett. **B386** (1996) 131 [hep-th/9606050]; *A Local Logarithmic Conformal Field Theory*, preprint DAMTP 1998-74, DTP-98/37 [hep-th/9807091]
A.M. Ghezelbash, V. Karimipour, Phys. Lett. **B402** (1997) 282-289 [hep-th/9704082]
M. Khorrami, A. Aghamohammadi, A.M. Ghezelbash, to appear in Phys. Lett. **B** [hep-th/9803071]
M. Khorrami, A. Aghamohammadi, M.R. Rahimi Tabar, Phys. Lett. **B419** (1998) 179-185 [hep-th/9711155]
N.E. Mavromatos, R.J. Szabo, Phys. Lett. **B430** (1998) 94-101 [hep-th/9803092]
M.R. Rahimi Tabar, A. Aghamohammadi, M Khorrami, Nucl. Phys. **B497** (1997) 555-566 [hep-th/9610168]
M.R. Rahimi Tabar, S. Rouhani, Phys. Lett. **B431** (1998) 85-89 [hep-th/9707060]
F. Rohsiepe, *On Reducible but Indecomposable Representations of the Virasoro Algebra*, preprint BONN-TH-96-17 [hep-th/9611160]
A. Shafiekhani, M.R. Rahimi Tabar, Int. J. Mod. Phys. **A12** (1997) 3723-3738 [hep-th/9604007]

[4] L. Rozansky, H. Saleur, Nucl. Phys **B376** (1992) 461 [hep-th/9203069]
V. Gurarie, Nucl. Phys. **B410** (1993) 535 [hep-th/9303160]

[5] See for example [9] and
A. Bilal, I.I. Kogan, Nucl. Phys. **B449** (1995) 569 [hep-th/9503209]
J.-S. Caux, *Exact Multifractality for Disordered N-Flavour Dirac Fermions in Two Dimensions*, [cond-mat/9804133]
J.-S. Caux, I.I. Kogan, A.M. Tsvelik, Nucl. Phys. **B466** (1996) 444 [hep-th/9511134]
J.-S. Caux, N. Taniguchi, A. M. Tsvelik, to appear in Nucl. Phys. **B** [cond-mat/9801055]
J. Ellis, N.E. Mavromatos, D.V. Nanopoulos, Int. J. Mod. Phys. **A12** (1997) 2639-2674 [hep-th/9605046]; M. Flohr, Mod. Phys. Lett. **A11** (1996) 55-68 [hep-th/9605152]; Nucl. Phys. **B482**[FS] (1996) 567-578 [hep-th/9606130]
A.M. Ghezelbash, M. Khorrami, A. Aghamohammadi, *Logarithmic conformal field theories and AdS correspondence*, [hep-th/9807034]
I.I. Kogan, A. Lewis, Phys. Lett. **B431** (1998) 77-84 [hep-th/9802102]
I.I. Kogan, N.E. Mavromatos, Phys. Lett. **B375** (1996) 111 [hep-th/9512210]
I.I. Kogan, N.E. Mavromatos, J.F. Wheater, Phys. Lett. **B387** (1996) 483 [hep-th/9606102]
I.I. Kogan, C. Mudry, A.M. Tsvelik, Phys. Rev. Lett. **77** (1996) 707 [cond-mat/9602163]
Z. Maassarani, D. Serban, Nucl. Phys. **B489** (1997) 603 [hep-th/9605062]
N.E. Mavromatos, R.J. Szabo, *Matrix D-brane Dynamics, Logarithmic Operators and Quantization of Noncommutative Spacetime*, preprint OUTP-98-54P [hep-th/9808124]
M.R. Rahimi Tabar, S. Rouhani, Ann. Phys. **246** (1996) 446-458 [hep-th/9503005]; Nuovo Cim. **112B** (1997) 1079-1084 [hep-th/9507166]; Europhys. Lett. **37** (1997) 447-451 [hep-th/9606143]
S. Skoulakis, S. Thomas, *Logarithmic Conformal Field Theory Solutions of Two Dimensional Magnetohydrodynamics*, preprint QMW-PH-97-41 [cond-mat/9802040]

[6] M. Flohr, *Logarithmic Conformal Field Theories & Riemann Surfaces, Part I: The Hyperelliptic Case & Seiberg-Witten Models*, to appear [hep-th/98soon]

[7] V.G. Knizhnik, Phys. Lett. **B180** (1986) 247-254; Commun. Math. Phys. **112** (1987) 567-590

[8] I.I. Kogan, A. Lewis, Nucl. Phys. **B509** (1998) 687-704 [hep-th/9705240]

[9] V. Gurarie, M. Flohr, C. Nayak, Nucl. Phys. **B498** (1997) 513-538 [cond-mat/9701212]

[10] H. Exton, *Multiple Hypergeometric Functions and Applications*, Chichester: Ellis Horwood etc. (1976)

[11] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333

[12] A. Cappelli, P. Valtancoli, L. Vergnano, Nucl. Phys. **B524** (1998) 469-501 [hep-th/9710248]

[13] A. Klemm, W. Lerche, S. Theisen, Int. J. Mod. Phys. **A11** (1996) 1929-1974 [hep-th/9505150]

[14] A. Klemm, W. Lerche, P. Mayr, C. Vafa, N. Warner, Nucl. Phys. **B477** (1996) 746-766 [hep-th/9604034]
J. Schulze, N. Warner, Nucl. Phys. **B498** (1997) 101-118 [hep-th/9702012]
A. Brandhuber, S. Stieberger, Nucl. Phys. **B488** (1997) 199-222 [hep-th/9610053]