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Four-Point Functions in Logarithmic Conformal Field Theories

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Abstract

The generic structure of 4-point functions of fields residing in indecomposable representations of arbitrary rank is given. The used algorithm is described and we present all results for Jordan-rank $r = 2$ and $r = 3$ where we make use of permutation symmetry and use a graphical representation for the results. A number of remaining degrees of freedom which can show up in the correlator are discussed in detail. Finally we present the results for two-logarithmic fields for arbitrary Jordan-rank.

Contents

1 Introduction and formulation of the problem

During the last few years, logarithmic conformal field theory (LCFT) has been established as a well-defined variety of conformal field theories in two dimensions. The defining feature of a LCFT is the occurrence of indecomposable representations which, in turn, may lead to logarithmically diverging correlation functions.

The concept of LCFTs was considered in its own right first by Gurarie [10]. Since then an enormous amount of work was done to understand LCFTs and to link LCFTs to other fields in physics, see for example the reviews [6, 7] and references therein. The number of topics (logarithmic) conformal field theories might play a role in is still growing, e. g., there are suggestions about links between Stochastic Löwner evolutions (SLEs) and (L)CFTs. A nice review on SLE is [2], and a possible relation to LCFT is discussed in [20].

Logarithmic conformal field theories are a generalization of conformal field theories (CFTs) in the sense that CFTs are LCFTs of Jordan-rank one. Since the works of Belavin, Polyakov and Zamolodchikov in 1984 [1] a powerful machinery of tools, algorithms and definitions has been developed, which nowadays is indispensable for analyzing conformal field theories. These definitions and techniques include characters, null vectors, operator product expansions (OPEs) and correlation functions, to name only a few. With the rise of LCFTs the demand for porting and generalizing these tools to LCFTs became an important endeavor. Today, porting of definitions and techniques from CFTs to LCFTs is almost finished, cf. [5, 17] and references therein. Nevertheless there exist still some areas which are not well-understood, such as modular properties of characters and partition functions.

In the course of the paper we want to discuss the generic form of four-point correlation functions, which is fixed by global conformal invariance, in the case of LCFTs. The solution for this problem in case of CFTs is well-known, but in case of LCFTs only incomplete results exist so far. An example where four-point correlators play a role in, are Abelian sandpile models which can be described by a $c = -2$ LCFT, e. g. [18, 13].

In the case of ordinary conformal field theory (CFT) it is known that every correlation function has to fulfill the so called global conformal Ward identities (GCWI) as a consequence of invariance under global conformal transformations:

$$L_q \left\langle \Psi_{(h_1)}(z_1) \ldots \Psi_{(h_n)}(z_n) \right\rangle = 0, \quad q = -1, 0, 1 \,, \tag{1}$$

where $\Psi_{(h)}(z)$ is a primary field and hence

$$L_q \left\langle \Psi_{(h_1)}(z_1) \ldots \Psi_{(h_n)}(z_n) \right\rangle = \sum_{i=1}^{n} z_i^q \left[z_i \partial_i + (q+1) h_i \right] \langle \ldots \rangle \,. \tag{2}$$

When considering logarithmic conformal field theories, primary fields appear together with so-called logarithmic partner fields which, in the simplest case form indecomposable representations in the form of Jordan cells. Then, these equations have to be slightly altered, cf. [4], by adding an additional term to the GCWI, leading to the *generalized global conformal Ward identities*:

$$\sum_{i=1}^{n} z_i^q \left[z_i\partial_i + (q+1)(h_i+\delta_i)\right] \left\langle \Psi_{(h_1,k_1)}(z_1)\ldots\Psi_{(h_n,k_n)}(z_n)\right\rangle = 0 \ , \qquad (3)$$

where $\Psi_{(h_i,k_i)}(z_i)$ denotes a logarithmic field of Jordan-level k_i respectively a primary field in case $k_i = 0$. The operator δ_i acts on these logarithmic fields by reducing the Jordan-level of the field by 1 respectively annihilating the field in case it is a primary one: $\delta_i \Psi_{(h_i,k_i)} = \Psi_{(h_i,k_i-1)}$ for $k_i > 1$ and $\delta_i \Psi_{(h_i,k_i=0)} = 0$ otherwise (field being a primary). Note that in the above equation the additional operator δ_i vanishes for $q = -1$ meaning that the LCFT version exactly matches the CFT version for this value of q. The additional operator δ_i makes it much harder to find the generic form of the correlators, because it renders the differential equations inhomogeneous, i. e., the solution will depend on solutions of lower Jordan-level. It is this additional term δ_i that makes solving the equations a lot harder compared to the CFT case.

If we consider the states corresponding to the fields $\Psi_{(h_i,k_i)}$, the action of δ_i leads to the following property for L_0

$$L_0\,|h;k\rangle = h\,|h;k\rangle + |h;k-1\rangle \qquad (4)$$

where additionally

$$|h;-k\rangle = 0 \quad \forall k > 0 \qquad (5)$$

holds. This shows, that the fields $\Psi_{(h_i,k_i)}$ indeed correspond to Jordan cells with respect to L_0. The representation of a LCFT with the largest Jordan cell defines the rank r of the LCFT, i. e., $k_i < r$.

The representation space is, as usual, spanned by the states $|h,k\rangle$ defined by the field-state isomorphism $|h,k\rangle := \lim_{z\to 0}\Psi_{(h,k)}\,|0\rangle$. All these states are typically assumed to be quasi-primary in the sense that $L_n\,|h,k\rangle = 0\ \forall n > 0$ and for all k. Thus, they almost behave as highest-weight states, up to the non-diagonal action of L_0. This is not true in general, because states to logarithmic partner fields may fail to be quasi-primary, i. e., $L_1\,|h,k>0\rangle \neq 0$. However, under certain assumptions, this does not affect the form of correlation functions. Furthermore, from the results for 1-, 2- and 3-point functions we can expect the vacuum representation to have the maximal Jordan-rank. No counter-examples are known up to now and thus we assume that the Jordan-rank is the same for all representations without loss of generality. The latter is justified as follows: in case some smaller Jordan-rank representation does show up, we can extend this representation by adding additional fields which we set to zero. In essence, this simply means that the general results remain valid with some of the structure constants set to zero. For further details on the precise assumptions in the case of non quasi-primary fields and on the maximal rank of the vacuum representation see [5].

While there are generic methods to determine 2- and 3-point correlation functions, e. g. see [5, 9, 11, 14, 19] and the particular elegant approach in [16], no such method exists, to our knowledge, for 4-point correlation functions. However, in [14, 15] a solution for the case of 4-point functions involving a level two null vector field is given. Any $n > 3$-point correlator of chiral fields can be reduced to 2- and 3-point functions. This is still possible for the full (non-chiral) theory. However, in order to determine the correct normalisation constants of the 2- and 3-point functions in this case, it is necessary to analyse the 4-point correlation functions. Therefore one can compute all observable quantities of a CFT–at least in principle–if one knows all 2-, 3- and, 4-point functions. Thus, this work attempts to close the remaining gap by providing the prescription to fix the generic form of 4-point correlators in the case of arbitrary rank Jordan-cells in LCFT.

While the generic form of 2- and 3-point functions is fixed up to structure constants the generic form of 4-point functions can be fixed only up to functions $F_{i_1 i_2 i_3 i_4}(x)$ of the globally conformally invariant crossing ratios x. As in the case of ordinary conformal field theory these structure functions can be computed if additional local symmetries, i. e., null vectors, exist. Indeed, such null vectors can exist in the logarithmic case [3], but the resulting differential equations are more difficult to solve because they are inhomogeneous in general [4].

In this paper we describe how the most general ansatz can be constructed and how the emerging constants can be calculated in order to find a valid ansatz for equation (3). Most of the constants can be fixed with the help of the global conformal Ward identities, but we will also encounter cases where some degrees of freedom are left. A necessary condition for these additional degrees of freedom is that all four fields in the four-point function are of logarithmic origin. The number of degrees of freedom very much depends on the form of the correlator. Furthermore we find that we have to identify some of the structure functions $F_{i_1 i_2 i_3 i_4}$ that are part of the correlator.

We then will use the discussed methods to determine all correlators for Jordan-rank $r = 2$ and $r = 3$. The results are given in a graphical representation and also we make use of permutation symmetries in order to keep the terms as short as possible. In the last section we consider the special case that only two of the four fields are logarithmic and we show how the resulting equations can be solved in this case for arbitrary Jordan-rank r.

2 Approaching the problem

In this section we describe how we simplify the initial problem and what algorithm we use to solve it for a Jordan-rank $r = 2$ and $r = 3$ theory. We also discuss the appearance of additional degrees of freedom that may show up if all four fields are of logarithmic type. For understanding of this section it might be helpful to have a glance at the next section which in detail discusses the most simple non-trivial case, that is Jordan-rank $r = 2$.

2.1 Simplification

As motivated in the introductory section it is sufficient to consider four-point functions and this is what we will do in the following. We also noted in the introduction that equation (3) is equal to the global conformal Ward identity in CFT for $q = -1$, meaning that the ansatz has to be translation invariant. Thus the ansatz depends on $z_{ij} := z_i - z_j$. Thus the ansatz has the following form

$$\begin{aligned}\left\langle \Psi_{(h_1,k_1)}(z_1)\ldots\Psi_{(h_4,k_4)}(z_4)\right\rangle &= \prod_{i<j} z_{ij}^{\mu_{ij}}\, F\begin{bmatrix} h_1,k_1 & h_2,k_2 \\ h_3,k_3 & h_4,k_4\end{bmatrix}(z_{12},z_{13},z_{14},z_{23},z_{24},z_{34}) \\ &=: \prod_{i<j} z_{ij}^{\mu_{ij}}\, F_{k_1k_2k_3k_4}(\log(z_{12}),\ldots,\log(z_{34}),x)\ ,\end{aligned} \tag{6}$$

where the exponents $\mu_{ij} = \mu_{ji}$ have to satisfy the conditions

$$\sum_{j\neq i} \mu_{ij} = -2h_i\ . \tag{7}$$

Note that the structure function F does not necessarily depend only on the crossing ratio x, since it will contain all the logarithms. The factor $\prod_{i<j} z_{ij}^{\mu_i}$ exists to counter the h_i terms on the left hand side of equation (3) and therefore we can without loss of generality set all conformal weights to zero, $h_i = 0$. Note that the full correlator of course depends on the conformal weights. The point here is that the global symmetries are not sufficient to fix the complete correlator, but they are strong enough to fix the generic form and this form has no dependence on h_i, except through the prefactor involving the exponents μ_{ij}. Therefore, we can simplify the resulting formulas by omitting the trivial direct dependency on the conformal weights. If we set all conformal weights to zero then (3) becomes

$$\sum_{i=1}^{n} z_i^q\,[z_i\partial_i + (q+1)\delta_i]\,\langle k_1k_2k_3k_4\rangle = 0\ , \tag{8}$$

where we write k_i instead of the much longer form $\Psi_{(h_i,k_i)}(z_i)$. The remaining two equations for $q = 0, 1$ have a δ_i term acting on the correlator and thus lowering the sum of the Jordan-levels by one. Because of calculating the expressions recursively we can assume the predecessors $\delta_i\langle\ldots\rangle$ to be known. This leads to the final form

$$O_0\langle\ldots\rangle := \sum_{i=1}^{4} z_i\partial_i\langle\ldots\rangle = -\sum_i \delta_i\langle\ldots\rangle \tag{9}$$

$$O_1\langle\ldots\rangle := \sum_{i=1}^{4} z_i^2\partial_i\langle\ldots\rangle = -2\sum_i z_i\delta_i\langle\ldots\rangle\ , \tag{10}$$

where the correlators depend on the difference z_{ij} only. Though looking simple for given predecessors $\delta_i\langle\ldots\rangle$ at first glance, it is not easy to find an ansatz for the correlator at all. Moreover we will learn that in some cases the result is not unique. We sometimes use the sloppy term "integrating" the predecessors $\delta_i \langle \ldots \rangle$ as a shortage for finding an ansatz that fulfills the above equations.

The starting point for the recursion is given by

$$\langle k_1 k_2 k_3 k_4 \rangle = F_0(x) \quad \text{for} \quad \sum_i k_i = r-1 \quad \text{respectively} \tag{11}$$

$$\langle k_1 k_2 k_3 k_4 \rangle = 0 \quad \text{for} \quad \sum_i k_i < r-1 \tag{12}$$

where x is the anharmonic ratio of the four points, $x = \frac{z_{12}z_{34}}{z_{14}z_{32}}$. In essence this means that a correlation function with *total Jordan-level* $K := \sum_i k_i = r-1$ behaves like a correlation function in ordinary conformal field theory, i. e., it depends on one function of the globally conformally invariant anharmonic ratio.

The reason for these initial conditions comes from the fact that the only non-vanishing one-point function in LCFT is the one of the highest level logarithmic partner of the identity, $\Psi_{(h=0,k=r-1)}$. Evaluating a correlation function amounts to contracting the inserted fields, in all possible ways, down to a one-point function. Therefore, it is only natural to expect that the total Jordan-level K of a non-vanishing correlator must at least be equal to $r-1$. Furthermore, since the cluster decomposition property should hold, the initial conditions must also hold for arbitrarily factorized correlators, e. g., $\langle k_1 k_2 k_3 k_4 \rangle \sim \langle k_1 k_2 | 0 \rangle \langle 0 | k_3 k_4 \rangle$ in case that z_1, z_2 are well separated from z_3, z_4. However, some care has to be taken about the correct insertion of the "identity" channel, which formally can be thought of to be of the form $|0\rangle\langle 0| = \sum_{k=0}^{r-1} |h=0;k\rangle\langle h=0; r-1-k|$. It is easy to see that the cluster decomposition with the above identity channel implies (12) and that precisely one term of this identity channel survives yielding (11), where we made use of the results for two-point functions in [5].

In the beginning we mentioned that $k_i > 0$ represents a logarithmic partner field, while $k_i = 0$ is a primary field. We can subdivide the class of primary fields into two subclasses, the so called proper primary-fields and the pre-logarithmic fields. This difference between the subclasses becomes apparent if one considers the operator product expansion (OPE). In contrast to the OPE of two proper primary-fields the OPE of two pre-logarithmic shows an additional term of logarithmic behavior, cf. [12].

In the following we consider proper primary-fields only and use the term synonymous with primary field. Restricting to proper primary-fields is for simplicity only. It is possible to include pre-logarithmic fields into the theory, by making changes to the initial condition (11), (12). For instance in the well-known $c = -2$ example the initial-conditions for Jordan-rank $r = 2$ would be

$$\langle \phi\phi\phi\phi \rangle = 0 \,, \tag{13}$$

$$\langle \mu\phi\phi\phi \rangle = 0 \,, \tag{14}$$

$$\langle \mu\mu\phi\phi \rangle = F_0(x) \,, \tag{15}$$

where ϕ stands for a proper primary and μ denotes a twist field. Note that the same could be formally achieved by assigning rational values k_i to pre-logarithmic values, e. g., in this example assigning a value of $k_i = \frac{1}{2}$ to the twist fields and using (11), (12) would lead to the same initial conditions. A more precise analysis of this and how to assign correct values for the k_i can be found in [5]. Apart from the initial conditions we also need slight adaption of the "connection rules" we are going to explain in subsection 2.4.3. More comments can be found in the conclusions.

2.2 Naming conventions and assumptions

The dependence of F on the anharmonic ratio, is suppressed in the following. Further note that we do not write out the dependence on the Jordan-rank r, e. g., $\langle 1000\rangle = F_0$ (for $r = 2$) as well as $\langle 1100\rangle = \ldots = \langle 2000\rangle = F_0$, namely for $r = 3$.

Also, as already introduced above, we will denote correlators only with the shorthand

$$\langle k_1k_2k_3k_4\rangle := \left\langle \Psi_{(h_1,k_1)}(z_1)\Psi_{(h_2,k_2)}(z_2)\Psi_{(h_3,k_3)}(z_3)\Psi_{(h_4,k_4)}(z_4)\right\rangle$$

throughout the paper. Let us stress one important point here. Later on, we will talk about certain symmetries such correlators will obey. For example, we already stated that in the $r = 2$ case, $F_0 = \langle 1000\rangle = \langle 0100\rangle = \langle 0010\rangle = \langle 0001\rangle$. Such permutations are ment to solely act on the Jordan-levels k_i, not on the order of the fields, which definitely would involve non-trivial monodromies. All we mean by such symmetries is that we may distribute the individual Jordan-levels within the (indecomposable) representations appearing in a correlator in different ways. Thus, to write it out once in full detail, a permutation $\sigma \in S_4$ acts as

$$\begin{aligned}\sigma(\langle k_1k_2k_3k_4\rangle) &= \left\langle k_{\sigma(1)}k_{\sigma(2)}k_{\sigma(3)}k_{\sigma(4)}\right\rangle \\ &= \left\langle \Psi_{(h_1,k_{\sigma(1)})}(z_1)\Psi_{(h_2,k_{\sigma(2)})}(z_2)\Psi_{(h_3,k_{\sigma(3)})}(z_3)\Psi_{(h_4,k_{\sigma(4)})}(z_4)\right\rangle .\end{aligned}$$

There exist further other symmetries, such as for rank $r = 3$, where we have $F_0 = \langle 2000\rangle = \ldots = \langle 0002\rangle = \langle 1100\rangle = \langle 1010\rangle = \ldots = \langle 0011\rangle$, which again only refer to different distributions among the Jordan-levels k_i. The same conventions hold when we speak of symmetries on the structure functions $F_{k_1k_2k_3k_4}$. Thus, the order of the representations involved is always assumed to be fixed once and for all.

Of course, due to the inhomogeneous nature of the global conformal Ward identities for $K = k_1 + k_2 + k_3 + k_4 > r - 1$, the correlation functions will possess logarithms. As a consequence, the correlation functions will depend on the individual Jordan-levels k_i, as for example $\langle k, (r-k), 0, 0\rangle \propto \log(z_{12})$ and $\langle 0, k, 0, (r-k)\rangle \propto \log(z_{24})$. However, the remaining structure functions will have a higher symmetry and thus will not depend so much on the individual Jordan-levels k_i, in a similar way as the structure constants of the 2- and 3-point functions, which indeed only depend on the total Jordan-levels. In essence, this means that global conformal invariance suffices to fix the form of the 4-point functions up to structure functions, which might be regarded as some kind of reduced matrix elements, largely independent of the details of the individual k_i. The dependency on how a total Jordan-level K ist distributed on to the individual k_i is encoded to a high degree in the coefficients of the structure functions which are polynomials in the $\log(z_{ij})$.

Motivated by the general structure of 2- and 3-point functions and by the results found in the case of a LCFT of Jordan-rank $r = 2$ we assume that the general form of the solution for (6) is of the form

$$\langle k_1k_2k_3k_4\rangle = \sum_{L=0}^{K_r} \sum_{\{k'_1k'_2k'_3k'_4 : \sum_i k'_i = L\}} p^{(L)}_{k'_1k'_2k'_3k'_4}(\{l_{ij}\}) F_{k_1-k'_1,k_2-k'_2,k_3-k'_3,k_4-k'_4}(x)\,, \tag{16}$$

where $l_{ij} := \log(z_{ij})$, $K := \sum_i k_i$, $K_r := K - r + 1$. The $p^{(L)}_{k'_1k'_2k'_3k'_4}$ denote polynomials in the l_{ij} of total degree L. The remaining "reduced" structure functions $F_{k_1-k'_1,k_2-k'_2,k_3-k'_3,k_4-k'_4}$ are assumed to depend only on the crossing ratio x in order to be invariant under global conformal transformations. Of course, we put $F_{j_1j_2j_3j_4} \equiv 0$ whenever one $j_i < 0$, to make the above ansatz always well defined. In the following, certain symmetries of the F with respect to the k_i will appear, although a complete analysis of these symmetries is beyond the scope of this paper. However, our results will still yield a valid solution if the maximal possible symmetry, i.e. the structure functions depend only on the total Jordan-level, is assumed. Then, the above ansatz would simplify to

$$\langle k_1k_2k_3k_4\rangle = \sum_{L=0}^{K_r} \left(\sum_{\{k'_1k'_2k'_3k'_4 : \sum_i k'_i = L\}} p^{(L)}_{k'_1k'_2k'_3k'_4}(\{l_{ij}\}) \right) F_{(K_r-L)}(x)\,,\,, \tag{17}$$

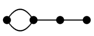
but we will use the more general ansatz (16) throughout the paper. The only symmetry we will always use is the one for F_0 which sets the initial condition for our recursions (11) and (12) and which can easily be proven by taking into account the two facts that the 3-point structure constants depend only on the total Jordan-level and that the leading OPE coefficients (i.e. the coefficient for a field of Jordan-level $k_1 + k_2$ in the OPE of two fields at Jordan-levels k_1 and k_2, respectively, whenever $k_1+k_2 \le r-1$) do not depend on the Jordan-levels at all [5]. Furthermore, it is also clear that any 4-point function with only two logarithmic fields will have the symmetry $\langle kk'00\rangle = \langle k'k00\rangle$ or $\langle 0k0k'\rangle = \langle 0k'0k\rangle$ etc., simply because such a 4-point function can only have a logarithmic dependency in z_{12} or z_{24} etc., and because the OPE of two logarithmic fields $\Psi_{(h;k)}$ and $\Psi_{(h';k')}$ is symmetric in the conformal weights h, h', and seperately in the Jordan-levels k, k', see [5]. However, clearly $\langle kk'00\rangle \ne \langle 0k0k'\rangle$. The interesting question is, whether there is a symmetry for the structure functions, $F_{kk'00}(x) = F_{k'k00}(x) \stackrel{?}{=} F_{0k0k'}(x) = F_{0k'0k}(x)$.

The highest logarithmic powers that appear in the solution are always the factors associated with the function F_0. The degree in l_{ij}, also called *logarithmic* *degree* for short, is given by

$$\deg(l_{12}^{a_1} l_{13}^{a_2} \dots l_{34}^{a_6}) = \sum_i a_i \leq K - r + 1 =: l^{\max} \,. \tag{18}$$

As discussed above, there are cases where we will find that some of the functions $F_{j_1 j_2 j_3 j_4}$ can or have to be identified with each other, e. g., we will find that $F_{2100} \equiv F_{1200}$ for $r = 3$ in the sense explained above. After identification we will always use the F-term whose index represents the lowest "number". For example we write $\langle 2100 \rangle = F_{1200}(x) + \dots$ instead of using $F_{2100}(x)$.

In many places we decided to use a graphical representation instead of writing long expressions of logarithms. The idea for this stems from [6] where it was chosen in order to give a better understanding of the contractions that can appear. Reading the diagrams is straightforward, the points stand for the four vertices and each l_{ij} is represented by a line between the vertices i and j. Permutation operators P are used to further reduce the length of the expressions, for instance

$$l_{12}^2 l_{23} l_{34} - l_{23}^2 l_{12} l_{14} = (1 - P_{(13)})$$  $. \quad (19)$

When reading such expression, one should carefully check, on what the premutation operators act: whether they act only on the polynomial in the l_{ij}, or also on the Jordan-levels of the structure functions F. The former case typically means that a symmetry for the F has been used, while the latter case does not assume such a symmetry. From section 3 on we will always use the graphical representation to present the results.

2.3 Properties of O_0, O_1

Both operators O_q are linear, nilpotent, act as derivatives on the function space and are invariant under any permutations $p \in S_4$. The function space we consider is the space of polynomials in the logarithmic functions $l_{ij} := \log|z_i - z_j|$, called $\mathcal{F}_{\log} := \mathbb{C}[l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34}]$.

For $q = 0, 1$ the operators O_q have a simple behavior, when acting on $\mathcal{F}_{\log}$:

$$O_0 : \begin{cases} \mathcal{F}_{\log} \to \mathcal{F}_{\log} \\ l_{i_1 j_1} \dots l_{i_n j_n} \mapsto \sum_{k=1}^{n} l_{i_1 j_1} \dots l_{i_{k-1} j_{k-1}} \, l_{i_{k+1} j_{k+1}} \dots l_{i_n j_n} \end{cases} , \tag{20}$$

$$O_1 : \begin{cases} \mathcal{F}_{\log} \to \mathcal{F}_{\log}[\{z_{ij}\}] \\ l_{i_1 j_1} \dots l_{i_n j_n} \mapsto \sum_{k=1}^{n} l_{i_1 j_1} \dots l_{i_{k-1} j_{k-1}} (z_{i_k} + z_{j_k}) l_{i_{k+1} j_{k+1}} \dots l_{i_n j_n} \end{cases} , \tag{21}$$

meaning that we can replace the term by a sum, where each l_{ij} is replaced by either 1 (for $q = 0$) or by $z_i + z_j$ (for $q = 1$). Thus acting with O_q on any term obviously reduces the logarithmic degree by one and by that proves (18).

An obvious question is whether the map $O_q : f \to f'$ is injective: are there any non-trivial $f \in \mathcal{F}_{\log}$ with $O_0 f = 0$ and $O_1 f = 0$? If we restrict ourselves to the function space $\mathcal{F}_{\log}$ then we find that we can exactly determine the kernel of the operator $O := (O_0, O_1)$.

As will be shown in subsection 2.6 below, the kernel is given as follows.

$$\ker_{\mathcal{F}_{\log,g}} O = \left\{ \sum_{i=0}^{g} a_i K_1^i K_2^{g-i} : a_k \in \mathbb{R} \right\} , \tag{22}$$

$$K_1 := l_{12} + l_{34} - l_{13} - l_{24} , \tag{23}$$

$$K_2 := l_{12} + l_{34} - l_{14} - l_{23} , \tag{24}$$

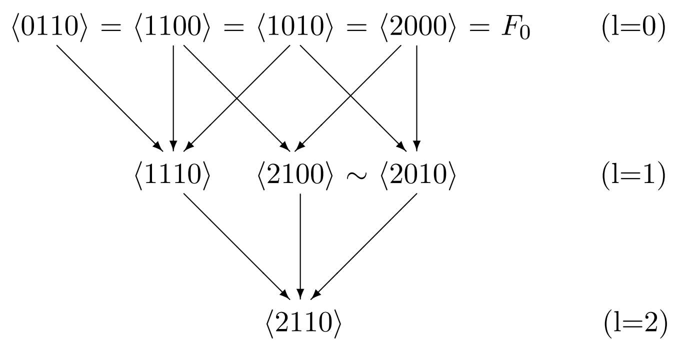
where $\mathcal{F}_{\log,g} := \{f \in \mathcal{F}_{\log} | \deg f = g\}$ denotes the space of functions with logarithmic degree g, such that $\mathcal{F}_{\log} = \bigcup_g \mathcal{F}_{\log,g}$.

2.4 An ansatz for the equations

As mentioned before we want to recursively solve the equations (9) and (10). Since the number of terms quickly becomes huge and calculation tedious we make use of computer algebra software for performing the calculations. In the next subsection we explain in more detail what we mean by recursion. After this we show that the two equations can be reduced to a set of simpler equations and in subsection 2.4.3 we present the algorithm we used for creating an ansatz.

2.4.1 Recursion

With recursion we mean the following: we start with the initial conditions as given in (11) which corresponds to logarithmic degree $l = 0$. Then we calculate all necessary correlators which contain exactly one more logarithmic field or one field whose Jordan-level is increased exactly by one. In short this means that we determine all correlators of logarithmic degree $l = 1$. The following diagram describes which correlators need to be calculated in order to determine the correlation function for $\langle 2110 \rangle$.



The effort for calculation can be reduced, since many of the correlators are related $\sim$ to others by simple permutations, e. g., $\langle 2100 \rangle = P_{23} \langle 2010 \rangle$. Note, however, that this simple relationship via a permutation only concerns the structural form of the correlator, *not* its dependency on the conformal weights. All we mean by the relation is that the structural forms

$$\langle 2100\rangle = F_{2100}(x) - 2l_{12}F_0(x) \sim F_{2010}(x) - 2l_{13}F_0(x) = \langle 2010\rangle \qquad (25)$$

can be mapped onto each other by the permutation P_{23}, which acts on the Jordan-levels k_i and on the indices of the l_{ij}, but not on the z_{ij} and the conformal weights h_i. In this respect, it is often useful to consider the l_{ij} formally independent of the z_{ij}.

2.4.2 Breaking down into a set of equations

The operators O_0, O_1 in equations (9), (10) are linear, they act as derivatives on the correlators $\langle\ldots\rangle$ and they are invariant under any permutation $P \in S_4$ of the indices. These properties of the operators O_1, O_2 are useful to break down the two equations (9), (10) into a set of equations and this is what we will do in the following.

The correlators $\langle\ldots\rangle$ in the beforementioned equations can be replaced by formula (16). Instead of writing polynomials such as $p^{(K_r-L)}_{k'_1k'_2k'_3k'_4}$ we write $(\ldots)$ for short. In this somewhat sketchy notation the equations (9) and (10) can now be written as

$$\begin{aligned} O_q\big\{F_{k_1k_2k_3k_4} + (\ldots)^u F_{k_1-1,k_2,k_3,k_4} + (\ldots)^u F_{k_1,k_2-1,k_3,k_4} + \ldots \\ \ldots + (\ldots)^u F_{r-1,0,1,0} + (\ldots)^u F_{r-1,0,0,1} + (\ldots)^u F_0\big\} &= \\ (\ldots)F_{k_1-1,k_2,k_3,k_4} + (\ldots)F_{k_1,k_2-1,k_3,k_4} + \ldots + (\ldots)F_{r-1,0,1,0} \\ +(\ldots)F_{r-1,0,0,1} + (\ldots)F_0 \ , \end{aligned} \qquad (26)$$

where $q = 0, 1$ and where we marked the brackets $(\ldots)$ on the left hand side of the equation with a small "u" in order to point out that these are the unknown terms we want to determine. We point out again that all terms on the right hand side are known because we want to recursively solve the equations. As usual r is the Jordan-rank of the theory.

In the following we add an index to the bracket terms in order to keep in mind where the respective term stems from, e. g., we write $(\ldots)^u_{k_1-1,k_2,k_3,k_4}$ for the first $(\ldots)^u$ term in the previous equation. Using this notation and knowing that O_q operates as a derivative and that $O_qF = 0$, we find that the problem reduces to "integrating" the following set of equations

$$\begin{aligned} O_q(\ldots)^u_{k_1,k_2,k_3,k_4} &= 0 \ , \qquad (27) \\ O_q(\ldots)^u_{k_1-1,k_2,k_3,k_4} &= (\ldots)_{k_1-1,k_2,k_3,k_4} \ , \\ O_q(\ldots)^u_{k_1,k_2-1,k_3,k_4} &= (\ldots)_{k_1,k_2-1,k_3,k_4} \ , \\ &\ldots \\ O_q(\ldots)^u_{r-1,0,0,1} &= (\ldots)_{r-1,0,0,1} \ , \\ O_q(\ldots)^u_0 &= (\ldots)_0 \ . \qquad (28) \end{aligned}$$

Note that the first equation (27) and its solution is well known

$$(\ldots)^{u}_{k_1,k_2,k_3,k_4} = F_{k_1,k_2,k_3,k_4}(x) \;, \tag{29}$$

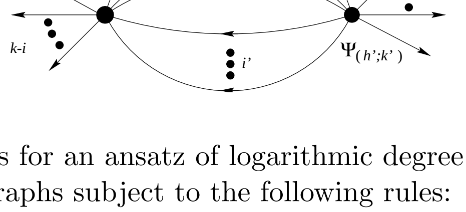
with x being the anharmonic ratio.

2.4.3 Description of the algorithm

Until now we did not specify what ansatz we fill in the left hand side of the equations (27) to (28). From OPE considerations [5] respectively from the structure of the operators O_0 and O_1 we expect the correlators to consist of terms of the type $l_{12}^{a_1} l_{13}^{a_2} \ldots l_{34}^{a_6}$, where each term comes with an coefficient which needs to be determined. More precisely, the generic structure of 2- and 3-point functions depends on the l_{ij} in a strictly polynomial form in such a way that the same is true for the operator product expansion. Thus, also the 4-point functions should depend only in a polynomial way on the l_{ij} since, asymptotically, a 4-point function decomposes into an operator product expansion times remaining 3-point functions, all of which are entirely polynomial in the l_{ij}. Unfortunately, the number of possible monomials in the l_{ij} grows heavily with the rank r of the LCFT, and thus the number of coefficients. Luckily we can reduce the number of possible terms that can show up in the following.

We do not have to take into account every logarithmic degree $a_1 + a_2 + \ldots + a_6$. The equations (20) and (21) tell us that the logarithmic degree is reduced by one if we apply O_0 or O_1. If we assume for a moment that in every equation of (27) to (28) the right hand side consists of terms of the same logarithmic degree l, then it is apparent that the terms on the left hand side have logarithmic degree $l+1$. We build all correlators recursively as explained in subsection 2.4.1 and since our initial conditions only consists of one term on the right hand side, we trivially find our assumption fulfilled. Thus by induction all terms $l_{12}^{a_1} l_{13}^{a_2} \ldots l_{34}^{a_6}$ in $(\ldots)^{u}_{n_1,n_2,n_3,n_4}$ have to have the same logarithmic degree.

As described in [6] it is helpful to use a graphical representation where each field $\Psi_{(h,k)}(z)$ in a Jordan-cell is depicted by a vertex with k outgoing lines. Contractions of logarithmic fields give rise to logarithms in the correlators, where the possible powers with which l_{ij} may occur are determined by graph combinations.



Essentially, the terms for an ansatz of logarithmic degree l are given by a sum over all admissible graphs subject to the following rules:

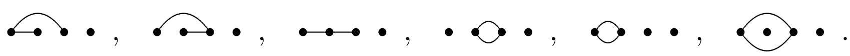
1. use at most k_i legs of a vertex for connections with other vertices,

2. the source i and the destination vertex j have to be different: $i \neq j$,
3. do connections with other logarithmic fields only (do not connect with primary fields) ,
4. create exactly l connections ,
5. write l_{ij} for every connection between two vertices i and j.

Let us have a look at a simple example. We consider a theory of Jordan-rank $r = 3$ and are interested in the structure of the correlator $\langle 2110 \rangle$ for the highest possible logarithmic degree, i. e., the F_0 term. The corresponding graph for $\langle 2110 \rangle$ is

 .

Altogether we have four legs to our disposal, but we also have to fix two of them leaving us with two free legs. If we want to know which terms can appear for logarithmic degree $l = 2$, then we have to create all 2-contractions according to the above rules. This results in the following six different graphs:



All these graphs are the result of applying the beforementioned rules to $\langle 2110 \rangle$. However, we did not draw the free legs in the above graphs as we only replace connections between vertices with l_{ij} and thus do not need these free legs for generating the ansatz.

Furthermore it should be pointed out that there are only two different truly independent graphs in the sense that they are not a mere permutation of other graphs. The first three graphs and the remaining three graphs form two equivalence classes induced by permutations of S_4.

Using the algorithm results in a maximum of $\binom{l+5}{l}$ terms that can appear. Combinatorial restrictions which we will discuss in the following can reduce this number, for instance $\langle 2211 \rangle$ for $l = 3$ does not contain a l_{34}^3 term.

2.5 Restrictions

The analysis of the results we found shows that several restrictions reduce the number of different terms that may appear in the end result.

The first restriction naturally appears during the integration process. In some cases our method for recursively constructing "higher" correlators fails. It is not possible to repair this failure in a sensible manner by adding further terms to the ansatz, but a simple identification of different functions F immediately fixes the problem.

This behavior is a general property of the theory for $r \geq 3$, as we will see in section 5. For now it is sufficient to note that $F_{k_1-1,k_2,0,0} = F_{k_1,k_2-1,0,0}$,

e. g., for $r = 3$ we get $F_{2100} = F_{1200}$ plus five more identifications by virtue of permutations.

The second restriction we encountered is the so called discrete symmetry of the correlators, which limits the dimension of the kernel. By discrete symmetry we mean that a correlator which contains at least two fields of the same Jordan-level should be invariant under any transposition that exchanges the Jordan-levels of these fields, for instance

$$P_{(12)} \langle \Psi_k \Psi_k \Psi_{k_3} \Psi_{k_4} \rangle = \langle \Psi_k \Psi_k \Psi_{k_3} \Psi_{k_4} \rangle \ . \tag{30}$$

At this point we point out again that we have formally set, without loss of generality, all conformal weights h_i to zero and wrote Ψ_k instead of $\Psi_{(h,k)}(z)$. In the next subsection we will discuss in more detail to what extent the above mentioned invariance limits the dimension of the kernel respectively show that in some cases no kernel term can show up at all.

The dimension of the kernel that finally shows up in the results is often smaller than the one we would expect for the given logarithmic degree and given discrete symmetry. The difference will show up especially if the logarithmic degree is close to the maximum degree $l^{\max} = K - r + 1$.

The reason for this difference is that the ansatz does not allow all terms of l_{ij} of a given degree to show up. For instance the correlator $\langle 2211 \rangle$ forbids the existence of terms of the type l_{34}^3 and by that limits the dimension of the kernel of degree 3. We also refer to this as combinatorial restriction, because the restriction depends on the form of the correlator, e. g., the term l_{34}^3 is not forbidden in $\langle 2221 \rangle$.

2.6 Additional constants

As we have seen in section 2.3 the kernel of the operator O is non-trivial. That means that the results may come with additional constants. In order to understand the meaning of these constants in the context of conformal field theory we rewrite the two basis terms K_1, K_2 which every element of the kernel consists of as follows:

$$K_1 := l_{12} + l_{34} - l_{13} - l_{24} = \log \left| \frac{z_{12} z_{34}}{z_{13} z_{24}} \right| \equiv \log |x| - \log |1 - x| \ , \tag{31}$$

$$K_2 := l_{12} + l_{34} - l_{14} - l_{23} = \log \left| \frac{z_{12} z_{34}}{z_{14} z_{23}} \right| \equiv \log |x| \ , \tag{32}$$

where $x = \frac{z_{12} z_{34}}{z_{14} z_{32}}$ is the anharmonic ratio of the four points. The anharmonic ratio x and its five possible involutions $\frac{1}{x}$, $1-x$, $1-\frac{1}{x}$, $\frac{1}{1-x}$, and $\frac{x}{x-1}$ result in four linearly independent functions. If we take the logarithm of the absolute value of these four functions, then we are left with only two independent solutions, namely $\log |x|$ and $\log |1 - x|$. The choice of the basis has no influence on the results and our choice of the basis K_1, K_2 is given as above.

We can turn around the argument and ask for all functions of the anharmonic ratio x, i. e., globally conformally invariant functions, which have the additional property to be strictly polynomial in the l_{ij}. These functions are all in the kernel of the operator O. On the other hand, there can be no other functions in the kernel if we restrict ourselves to polynomials of the l_{ij}, since every member of the kernel must be invariant under global conformal transformations and thus be a function of x. This proves the statement in section 2.3. However, we note that this yields only an upper bound on the size of the kernel. We will see that the size may be reduced due to further symmetries.

Equation (22) gives us the maximal dimension of the kernel for logarithmic degree l,

$$K^{(l)} := \left\{ \sum_{i=0}^{l} a_i K_1^i K_2^{d-i} : a_k \in \mathbb{R} \right\} , \tag{33}$$

$$d^{\max}(l) = \; l+1 \; . \tag{34}$$

Up to a few exceptions we will notice that the full kernel never shows up in any equation. These restrictions on the kernel are caused by the discrete symmetry and combinatorial constraints. Examples for combinatorial constraints are shown in the next two sections.

It is worth noting that a non-trivial kernel can show up in a correlator only if there is no primary field in the correlator present. This is obvious, since both kernel elements K_1, K_2 refer to all four vertices z_1, z_2, z_3, z_4.

2.6.1 Discrete symmetry for invariant F

In this subsection we are interested in the impact on the kernel by a given symmetry. Since we consider four point correlation functions exclusively there are four interesting symmetry groups only, namely S_2, $S_2 \times S_2$, S_3 and S_4. Let us study an expression first, where the function F is invariant under any permutation, e. g., $(\ldots)F_{1111}$.

We start with the smallest symmetry group $S_2 = \{1, P_{(12)}\}$, P being, as usual, a permutation of the indices. One immediately remarks that $P_{(12)}K_1 = K_2$, $P_{(12)}K_2 = K_1$ and thus a S_2 invariant kernel of logarithmic degree l has the form

$$K_{S_2}^{(l)} := \left\{ \sum_{i=0}^{l} a_i K_1^i K_2^{d-i} : a_k \in \mathbb{R}, a_k = a_{l-k} \right\} . \tag{35}$$

Therefore the maximum number of constants $d_{S_2}^{\max}$ that could appear for logarithmic degree l is

$$d_{S_2}^{\max}(l) = \lfloor \tfrac{l}{2} \rfloor + 1 \; , \tag{36}$$

where $\lfloor . \rfloor$ denotes the floor function.

If we replace the transposition $P_{(12)}$ by $P_{(34)}$ all statements stay true. Thus when restricting to kernel space $K := \bigcup_l K^{(l)}$ we have $P_{(12)} \equiv P_{(34)}$. This in turn means that the kernel is not only S_2 invariant, but automatically has full $S_2 \times S_2 = \{1, P_{(12)}, P_{(34)}, P_{(12)(34)}\}$ invariance:

$$K^{(l)}_{S_2\times S_2} \equiv K^{(l)}_{S_2} \,, \tag{37}$$

$$d^{\max}_{S_2\times S_2}(l) = \lfloor \tfrac{l}{2} \rfloor + 1 \,. \tag{38}$$

For the S_3 symmetry we note, that a S_3 invariance extends to S_4 invariance. This is because S_3 invariance in particular means $P_{(12)}$ invariance which, as explained above, also means $P_{(34)}$ invariance. By this we immediately obtain full S_4 invariance:

$$K^{(l)}_{S_3} \equiv K^{(l)}_{S_4} \,. \tag{39}$$

No linear combination of K_1, K_2 is S_4 invariant, but higher terms in K_1, K_2 have this property. The first two dimensional kernel $d = 2$ can be found for logarithmic degree $l = 6$:

$$\begin{aligned}
K^{(2)}_{S_4} &:= K_1^2 - K_1K_2 + K_2^2 \,,\\
K^{(3)}_{S_4} &:= (2K_1 - K_2)(2K_2 - K_1)(K_1 + K_2) \,,\\
K^{(4)}_{S_4} &:= (K^{(2)}_{S_4})^2 \,,\\
K^{(5)}_{S_4} &:= K^{(3)}_{S_4}K^{(2)}_{S_4} \,,\\
K^{(6,(2,2,2))}_{S_4} &:= (K^{(2)}_{S_4})^3 \,,\\
K^{(6,(3,3))}_{S_4} &:= (K^{(3)}_{S_4})^2 \,,\\
K^{(7)}_{S_4} &:= (K^{(2)}_{S_4})^2K^{(3)}_{S_4} \,.
\end{aligned} \tag{40}$$

These results are unique, up to constants and linear combinations. Of course any combination of the form $(K^{(2)}_{S_4})^i(K^{(3)}_{S_4})^j$ leads to a kernel of logarithmic degree $2i + 3j$ and we believe that the kernel space is not larger than this, though it is not important since we consider kernels up to logarithmic degree $l = 6$ only in the further course of this paper.

We expect the dimension of the S_4 invariant kernel to be the number of possible partitions of the degree in the numbers 2 and 3, e. g., $6 = 2 + 2 + 2$ as well $6 = 3 + 3$. This in turn means that every integer 6 can be represented in two different ways, leading to the following number of degrees of freedom that could appear at most for logarithmic degree l:

$$d^{\max}_{S_4}(l) = \begin{cases} \lfloor \frac{l}{6} \rfloor & ; l = 6k + 1,\ k \in \mathbb{N}_0 \\ \lfloor \frac{l}{6} \rfloor + 1 & ; \text{else} \,. \end{cases} \tag{41}$$

2.6.2 Discrete symmetry and non-invariant *F*

In the previous subsection we analyzed the structure of the kernel under symmetry groups and found that we have to consider S_2 and S_4 symmetry groups only. This holds if the function $F_{\dots}$ itself is invariant under permutations.

Things get more complicated if F_a is mapped to F_b by the permutation. For example we know that the S_2 invariant kernel for one F (F_1 for short) is one-dimensional, namely $K^{(1)}_{S_2} = \{a(K_1 + K_2) : a \in \mathbb{R}\}$. But if we have two F, which are related by the permutation, e. g., F_{1022} and F_{0122}, then the dimension of the kernel for F_{1022} becomes larger. The kernel would be

$$(cK_1 + c'K_2)F_{1022} + (c'K_1 + cK_2)F_{0122} \; ,$$

or $\mathcal{P}_{S_2}(K^{(2)}F_{1022})$ with $\mathcal{P}_{S_2} = 1 + P_{(12)}$ for short.

The kernel dimension therefore not only depends on the symmetry group and the logarithmic degree, but also on the size of the equivalence class of functions F which are involved. The size of the equivalence class is noted by F_n and the results for the logarithmic degrees $l = 1, 2, \ldots 5$ are listed in appendix A.

The simple rule that S_2 corresponds to $S_2 \times S_2$ respectively that S_3 corresponds to S_4 does not hold for $n > 1$, therefore we have to discuss all four symmetry groups in the appendix. The dimension of the kernel decreases with increasing size of the symmetry group and increases with increasing size of the equivalence class. It is interesting though not surprising, that the full kernel $K^{(l)}$ is recovered, if the size of the equivalence class $|F|$ equals the size of the symmetry group $|S|$.

3 Results for Jordan-rank $r = 2$

In this section we present and discuss the results for a logarithmic conformal field theory with Jordan-level $r = 2$. We have used the algorithm described in subsection 2.4.3 to obtain these results and though known, e. g. [4], we can write them in a more appealing form. Also we will discuss the appearance of an additional degree of freedom, which shows up for $\langle 1111 \rangle$.

We start with simply writing down the first three expressions that our algorithm provides:

$$\langle 1000 \rangle = F_0 \; , \tag{42}$$

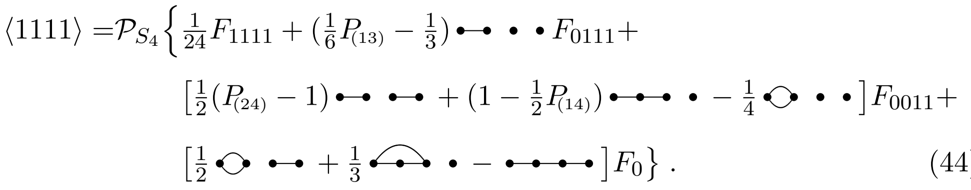
$$\langle 1100 \rangle = \mathcal{P}_{S_2}\Big\{ \tfrac{1}{2} F_{1100} - \bullet\!\!-\!\!\bullet \;\; \bullet \;\; \bullet \; F_0 \Big\} \; ,$$

$$\langle 1110 \rangle = \mathcal{P}_{S_3}\Big\{ \tfrac{1}{6} F_{1110} + (\tfrac{1}{2} P_{(13)} - 1) \bullet\!\!-\!\!\bullet \;\; \bullet \;\; \bullet \; F_{0110} + \big[\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet \;\; \bullet - \tfrac{1}{2} \bullet\!\!\bigcirc\!\!\bullet \;\; \bullet \;\; \bullet \big] F_0 \Big\} \; . \tag{43}$$

with $\mathcal{P}_X = \sum_X P_{(x)}$. Writing the results this way makes the discrete symmetry manifest, that is S_2 for $\langle 1100 \rangle$ and S_3 invariance for $\langle 1110 \rangle$.

For the correlator $\langle 1111 \rangle$ we have a logarithmic partner field at every vertex which means that we can expect getting a non-trivial kernel for the first time.

The result without kernel is given by



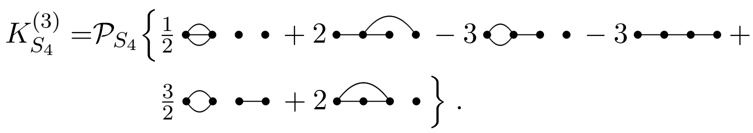
(44)

the contribution to the kernel is

$$\mathrm{Ker}_{\langle 1111\rangle} = \mathcal{P}_{S_4}\Big\{K^{(2)}_{S_2} F_{0011}\Big\} \ . \tag{45}$$

That we get a two-dimensional kernel for F_{0011} is not surprising, since there are 6 functions F belonging to the equivalence class of F_{0011} and the resulting $K^{(2)}_{S_2}$ can be read of the table for logarithmic degree 2 from the appendix.

The inverse question is more interesting, namely we are interested in understanding, why no other kernel term shows up at all. For logarithmic degree $l = 1$ the equivalence class of F_{0111} is four and thus there is no kernel term showing up. According to (41) the S_4 invariant kernel of logarithmic degree $l = 3$ should be one-dimensional. We can immediately understand why this kernel term does not show up, by looking at the graphical representation:



(46)

This shows us that terms of the form l_{12}^3 appear, which is impossible for a Jordan-rank $r = 2$ theory. Though three free legs are available the three-fold connection between vertices i and j is forbidden for $r = 2$. Of course higher Jordan-rank LCFT $r > 2$ are allowed to include such terms, but similar combinatorial restrictions will show up for $r = 3$ as well.

4 Results for Jordan-rank $r = 3$

While the general structure of the correlators for Jordan-rank $r = 2$ has been known before, nobody so far has studied the form of correlators for LCFTs beyond the case of $r = 2$. With what we have learned we can apply our methods to the case $r = 3$ in order to determine the form of all correlators for a theory of Jordan-rank $r = 3$.

Analogously to $r = 2$ the starting point for the recursion is given by

$$\langle 1100\rangle = F_0 \quad , \quad \langle 2000\rangle = F_0 \ . \tag{47}$$

The missing correlators result from applying a permutation to the correlators.

$$\langle 0012\rangle = \mathcal{P}_{S_2}\Big\{ \tfrac{1}{2}F_{0012} - \bullet\ \bullet\ \bullet\!-\!\bullet\ F_0 \Big\}\ , \tag{48}$$

$$\langle 1110\rangle = \mathcal{P}_{S_3}\Big\{ \tfrac{1}{6}F_{1110} - \tfrac{1}{2}\bullet\!-\!\bullet\ \bullet\ \bullet\ F_0 \Big\}\ , \tag{49}$$

$$\langle 2200\rangle = \mathcal{P}_{S_2\times S_2}\Big\{ \tfrac{1}{4}F_{2200} - \tfrac{1}{2}\bullet\!-\!\bullet\ \bullet\ \bullet\ F_{1200} + \tfrac{1}{2}\bullet\!=\!\bullet\ \bullet\ \bullet\ F_0 \Big\}\ , \tag{50}$$

$$\begin{aligned}\langle 1120\rangle = \mathcal{P}_{S_2}\Big\{ & \tfrac{1}{2}F_{1120} - (1+P_{(23)}+P_{(13)})\bullet\!-\!\bullet\ \bullet\ \bullet\ F_{0120} + (\tfrac{1}{2}-P_{(23)})\bullet\!-\!\bullet\ \bullet\ \bullet\ F_{1110} + \\ & \big[(1+\tfrac{3}{2}P_{(23)})\bullet\!-\!\bullet\!-\!\bullet\ \bullet - (\tfrac{1}{4}+\tfrac{1}{2}P_{(23)})\bullet\!=\!\bullet\ \bullet\ \bullet\big]F_0\Big\}\end{aligned} \tag{51}$$

where as before $\mathcal{P}_X = \sum_{x\in X} P_x$.

The above correlators do not have an additional degree of freedom because they contain at least one primary field. The simplest correlator with no primaries is

$$\langle 1111\rangle = \mathcal{P}_{S_4}\Big\{ \tfrac{1}{24}F_{1111} + (\tfrac{1}{6}P_{(13)} - \tfrac{1}{3})\bullet\!-\!\bullet\ \bullet\ \bullet\ F_{0111} + (\tfrac{1}{4}\bullet\!-\!\bullet\ \ \bullet\!-\!\bullet)F_0 \Big\}\ . \tag{52}$$

This is the first correlator for $r = 3$ which has a non-trivial kernel, namely

$$\mathrm{Ker}_{\langle 1111\rangle} = c_1 K^{(2)}_{S_4} F_0\ . \tag{53}$$

The restriction that the expression needs to be invariant under S_4 permutations is very strong and forbids any kernel terms of degree one to show up.

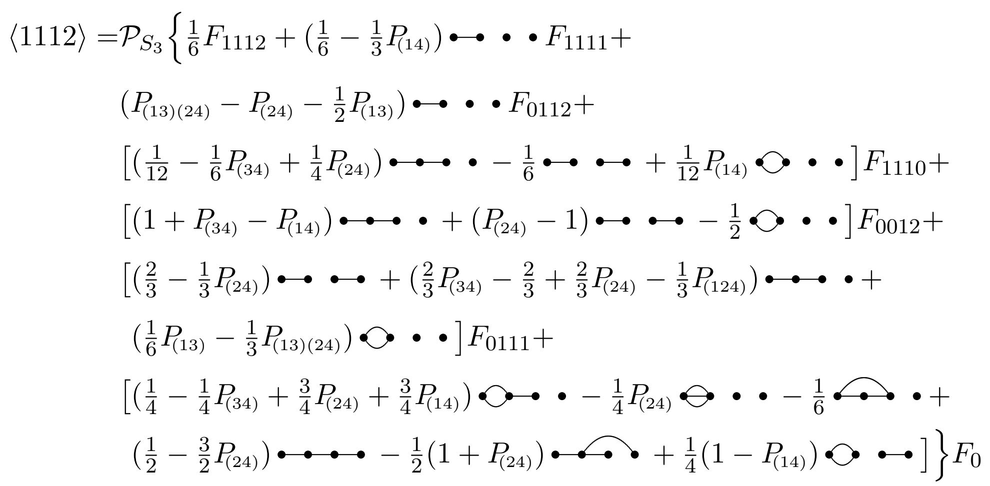
The remaining correlators containing at least a primary field are

$$\begin{aligned}\langle 2210\rangle = \mathcal{P}_{S_2}\Big\{ & \tfrac{1}{2}F_{2210} + (\tfrac{1}{2}-P_{(13)})\bullet\!-\!\bullet\ \bullet\ \bullet\ F_{2200} - (1+P_{(23)}-P_{(13)})\bullet\!-\!\bullet\ \bullet\ \bullet\ F_{1210} + \\ & \big[2\,\bullet\!-\!\bullet\!-\!\bullet\ \bullet - \bullet\!=\!\bullet\ \bullet\ \bullet\big]F_{1200} + \big[P_{(23)}\bullet\!-\!\bullet\!-\!\bullet\ \bullet + (\tfrac{1}{2}-P_{(13)})\bullet\!=\!\bullet\ \bullet\ \bullet\big]F_{1110} + \\ & \big[(-1-P_{(23)}+P_{(12)})\bullet\!-\!\bullet\!-\!\bullet\ \bullet + \tfrac{1}{2}(1+P_{(23)}+P_{(13)})\bullet\!=\!\bullet\ \bullet\ \bullet\big]F_{0120} + \\ & \big[(P_{(13)}-2)\bullet\!=\!\bullet\!-\!\bullet\ \bullet + \tfrac{1}{2}\bullet\!\equiv\!\bullet\ \bullet\ \bullet - \overset{\frown}{\bullet\!-\!\bullet\!-\!\bullet}\ \bullet\big]F_0\Big\}\end{aligned} \tag{54}$$

and

$$\begin{aligned}\langle 2220\rangle = \mathcal{P}_{S_3}\Big\{ & \tfrac{1}{6}F_{2220} + (\tfrac{1}{2}P_{(13)}-1)\bullet\!-\!\bullet\ \bullet\ \bullet\ F_{1220} + \\ & \big[P_{(23)}\bullet\!-\!\bullet\!-\!\bullet\ \bullet + (\tfrac{1}{2}-P_{(23)})\bullet\!=\!\bullet\ \bullet\ \bullet\big]F_{1120} + \\ & \big[(\tfrac{1}{2}P_{(12)}-1)\bullet\!-\!\bullet\!-\!\bullet\ \bullet + (\tfrac{1}{2}+\tfrac{1}{4}P_{(13)})\bullet\!=\!\bullet\ \bullet\ \bullet\big]F_{0220} + \\ & \big[\tfrac{1}{2}\bullet\!\equiv\!\bullet\ \bullet\ \bullet - \bullet\!=\!\bullet\!-\!\bullet\ \bullet + \tfrac{1}{3}\overset{\frown}{\bullet\!-\!\bullet\!-\!\bullet}\ \bullet\big]F_{1110} + \\ & \big[(2P_{(13)}-1)\bullet\!=\!\bullet\!-\!\bullet\ \bullet - \tfrac{1}{2}P_{(13)}\bullet\!\equiv\!\bullet\ \bullet\ \bullet - \overset{\frown}{\bullet\!-\!\bullet\!-\!\bullet}\ \bullet\big]F_{0120} + \\ & \big[\tfrac{1}{2}\overset{\frown}{\underset{\smile}{\bullet\!-\!\bullet\!-\!\bullet}}\ \bullet + \tfrac{1}{8}\bullet\!\overset{=}{=}\!\bullet\ \bullet\ \bullet + \tfrac{3}{4}\bullet\!=\!\bullet\!=\!\bullet\ \bullet - \bullet\!\equiv\!\bullet\!-\!\bullet\ \bullet\big]F_0\Big\}\ .\end{aligned} \tag{55}$$

Finally there are, up to permutations, four correlators without primary field and at least one field being of Jordan-level 2.



(56)

This correlator has 6 additional degrees of freedom:

$$\mathrm{Ker}_{\langle 1112\rangle} = \mathcal{P}_{S_3}\Big\{c_1(2K_2 - K_1)F_{0112} + K_{S_4}^{(2)}F_{1110}+ \\ \big[c_3K_1^2 + c_4(K_2^2 - K_1K_2)\big]F_{0111}+ \\ \big[c_5K_1^2 + c_6(K_2^2 - K_1K_2)\big]F_{1002}\Big\}\,. \qquad (57)$$

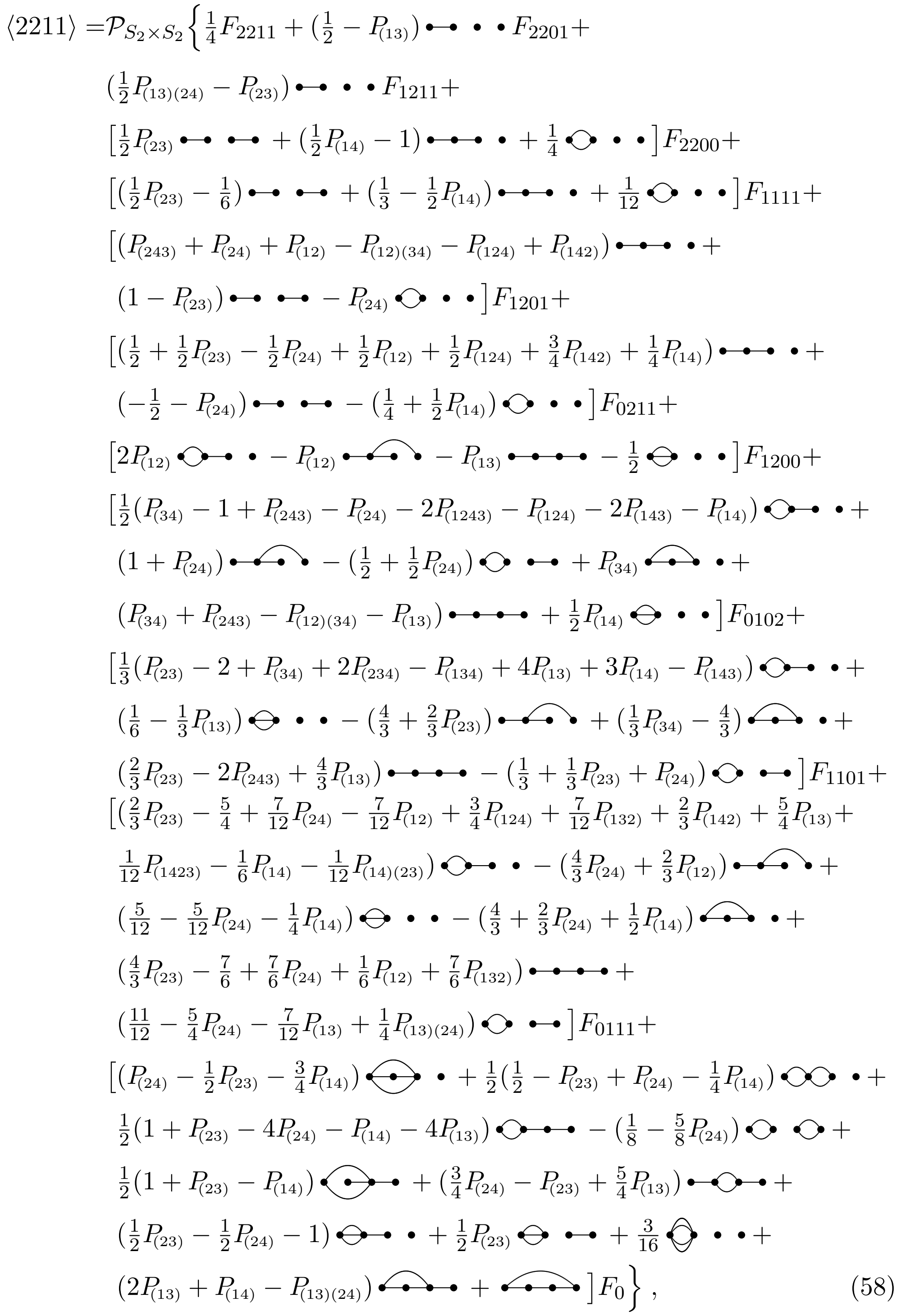
The set $\{F_{0112}, F_{1012}, F_{1102}\}$ allows a one dimensional kernel, namely $2K_2 - K_1$. Note that this kernel is not S_3 invariant.

For the self-invariant terms F_{1110} and F_0 we remarked in subsection 2.6.1 that S_3 invariance implies S_4 variance. The dimension of the S_4-invariant kernel is $n_{\max}^{S_4} = 1$ for $d = 2, 3$. The corresponding kernel term for F_{1110} shows up, combinatorial restrictions forbid the same for F_0. The only kernel of degree 3 that would have been possible is $K_{S_4}^{(3)}$, but this one includes l_{ij}^3 terms for all $1 \le i < j \le 4$, which is not compatible with the contraction rules as described in subsection 2.4.3–$\langle 2111\rangle$ cannot contain l_{34}^3 terms, cf. (46).

For the 3-element sets $\{F_{0111}, F_{1011}, F_{1101}\}$ respectively $\{F_{0012}, F_{0102}, F_{1002}\}$ we know from the kernel analysis in appendix A that there is a two-dimensional kernel.

It should be noted that $\langle 1211\rangle$ is generated by applying $P_{(12)}$ to $\langle 2111\rangle$. The same holds for the additional terms of the kernel. This means that the degrees of freedom we have for $\langle 2111\rangle$ are not available for the permutations of this correlator, e. g., $\langle 1211\rangle$.

The correlator $\langle 2211\rangle$ comes with a high number of additional degrees of freedom, some of these are restricted by combinatorial constraints. The correlator without kernel terms has the form



$$
\begin{aligned}
\langle 2211\rangle =\mathcal{P}_{S_2\times S_2}\Big\{ &\tfrac{1}{4}F_{2211} + (\tfrac{1}{2} - P_{(13)})\, F_{2201}+ \\
&(\tfrac{1}{2}P_{(13)(24)} - P_{(23)})\, F_{1211}+ \\
&\big[\tfrac{1}{2}P_{(23)} + (\tfrac{1}{2}P_{(14)} - 1) + \tfrac{1}{4}\big]F_{2200}+ \\
&\big[(\tfrac{1}{2}P_{(23)} - \tfrac{1}{6}) + (\tfrac{1}{3} - \tfrac{1}{2}P_{(14)}) + \tfrac{1}{12}\big]F_{1111}+ \\
&\big[(P_{(243)} + P_{(24)} + P_{(12)} - P_{(12)(34)} - P_{(124)} + P_{(142)}) + \\
&(1 - P_{(23)}) - P_{(24)}\big]F_{1201}+ \\
&\big[(\tfrac{1}{2} + \tfrac{1}{2}P_{(23)} - \tfrac{1}{2}P_{(24)} + \tfrac{1}{2}P_{(12)} + \tfrac{1}{2}P_{(124)} + \tfrac{3}{4}P_{(142)} + \tfrac{1}{4}P_{(14)}) + \\
&(-\tfrac{1}{2} - P_{(24)}) - (\tfrac{1}{4} + \tfrac{1}{2}P_{(14)})\big]F_{0211}+ \\
&\big[2P_{(12)} - P_{(12)} - P_{(13)} - \tfrac{1}{2}\big]F_{1200}+ \\
&\big[\tfrac{1}{2}(P_{(34)} - 1 + P_{(243)} - P_{(24)} - 2P_{(1243)} - P_{(124)} - 2P_{(143)} - P_{(14)}) + \\
&(1 + P_{(24)}) - (\tfrac{1}{2} + \tfrac{1}{2}P_{(24)}) + P_{(34)} + \\
&(P_{(34)} + P_{(243)} - P_{(12)(34)} - P_{(13)}) + \tfrac{1}{2}P_{(14)}\big]F_{0102}+ \\
&\big[\tfrac{1}{3}(P_{(23)} - 2 + P_{(34)} + 2P_{(234)} - P_{(134)} + 4P_{(13)} + 3P_{(14)} - P_{(143)}) + \\
&(\tfrac{1}{6} - \tfrac{1}{3}P_{(13)}) - (\tfrac{4}{3} + \tfrac{2}{3}P_{(23)}) + (\tfrac{1}{3}P_{(34)} - \tfrac{4}{3}) + \\
&(\tfrac{2}{3}P_{(23)} - 2P_{(243)} + \tfrac{4}{3}P_{(13)}) - (\tfrac{1}{3} + \tfrac{1}{3}P_{(23)} + P_{(24)})\big]F_{1101}+ \\
&\big[(\tfrac{2}{3}P_{(23)} - \tfrac{5}{4} + \tfrac{7}{12}P_{(24)} - \tfrac{7}{12}P_{(12)} + \tfrac{3}{4}P_{(124)} + \tfrac{7}{12}P_{(132)} + \tfrac{2}{3}P_{(142)} + \tfrac{5}{4}P_{(13)} + \\
&\tfrac{1}{12}P_{(1423)} - \tfrac{1}{6}P_{(14)} - \tfrac{1}{12}P_{(14)(23)}) - (\tfrac{4}{3}P_{(24)} + \tfrac{2}{3}P_{(12)}) + \\
&(\tfrac{5}{12} - \tfrac{5}{12}P_{(24)} - \tfrac{1}{4}P_{(14)}) - (\tfrac{4}{3} + \tfrac{2}{3}P_{(24)} + \tfrac{1}{2}P_{(14)}) + \\
&(\tfrac{4}{3}P_{(23)} - \tfrac{7}{6} + \tfrac{7}{6}P_{(24)} + \tfrac{1}{6}P_{(12)} + \tfrac{7}{6}P_{(132)}) + \\
&(\tfrac{11}{12} - \tfrac{5}{4}P_{(24)} - \tfrac{7}{12}P_{(13)} + \tfrac{1}{4}P_{(13)(24)})\big]F_{0111}+ \\
&\big[(P_{(24)} - \tfrac{1}{2}P_{(23)} - \tfrac{3}{4}P_{(14)}) + \tfrac{1}{2}(\tfrac{1}{2} - P_{(23)} + P_{(24)} - \tfrac{1}{4}P_{(14)}) + \\
&\tfrac{1}{2}(1 + P_{(23)} - 4P_{(24)} - P_{(14)} - 4P_{(13)}) - (\tfrac{1}{8} - \tfrac{5}{8}P_{(24)}) + \\
&\tfrac{1}{2}(1 + P_{(23)} - P_{(14)}) + (\tfrac{3}{4}P_{(24)} - P_{(23)} + \tfrac{5}{4}P_{(13)}) + \\
&(\tfrac{1}{2}P_{(23)} - \tfrac{1}{2}P_{(24)} - 1) + \tfrac{1}{2}P_{(23)} + \tfrac{3}{16} + \\
&(2P_{(13)} + P_{(14)} - P_{(13)(24)}) + \big]F_0\Big\}\,, \qquad (58)
\end{aligned}
$$

where $\mathcal{P}_{S_2\times S_2} = 1 + P_{(12)} + P_{(34)} + P_{(12)(34)}$. The kernel of $\langle 2211\rangle$ has a dimension of 18:

$$
\begin{aligned}
\mathrm{Ker}_{\langle 2211\rangle} = \mathcal{P}_{S_2\times S_2}\Big\{ & K_{S_2}^{(1,d=1)} F_{2201} + K_{S_2}^{(1,d=1)} F_{1211} + \\
& K^{(2,d=3)} + K_{S_2}^{(2,d=2)}(F_{2200} + F_{0211} + F_{1111}) + \\
& (K_1 - K_2)^2 (K_1 + K_2)(F_{1101} + F_{0111} + F_{1200}) + \\
& \big[c(K_1 - K_2)^2 K_1 + c'(K_1 - K_2)^2 K_2 + c''(K_1 - K_2) K_1 K_2\big] F_{0102} + \\
& K_{S_4}^{(4,d=1)} F_0 \Big\}\ , \qquad (59)
\end{aligned}
$$

where we used a somewhat condensed notation and left out almost all constants. If multiple F in brackets show up you should add the necessary constants in your mind, for instance, $K_{S_2}^{(2,d=2)}(F_{2200} + F_{0211} + F_{1111})$ stands for $2 \cdot 3 = 6$ degrees of freedom. For better orientation we added a small "$d = \ldots$" index to the common kernel terms which notes their dimension.

For the sets $\{F_{2201}, F_{2210}\}$ and $\{F_{1211}, F_{2111}\}$ we get the expected one-dimensional kernel $K_{S_2}^{(1)}$. Also the results for logarithmic degree 2 are not surprising.

Things are more complicated for higher logarithmic degrees. For $d = 3$ we would have been expecting a two-dimensional kernel $K_{S_2}^{(3)}$ for F_{1101}, F_{1200}, F_{0111} and a full 4-dimensional kernel $K^{(3)}$ for F_{0102}. But here we have to take into account the combinatorial restrictions again. The basis elements of $K_{S_2}^{(3)}$, $K_{S_2}^{(3,a)} \sim K_1^3 + K_2^3$ contains $l_{12}^3, l_{13}^3, \ldots, l_{34}^3$ contributions and $K_{S_2}^{(3,b)} \sim K_1^2 K_2 + K_1 K_2^2$ comes with l_{12}^3, l_{34}^3 terms. Thus both basis elements are not allowed due to the l_{34}^3 term, but a linear combination is, namely $(K_1 - K_2)^2 (K_1 + K_2)$, which contains no l_{34}^2 term. For the set $\{F_{0102} \equiv F_{0201}, F_{1002} \equiv F_{2001}, F_{0120} \equiv F_{0210}, F_{1020} \equiv F_{2010}$ the four dimensional kernel reduces by the combinatorial constraint to a three dimensional one.

The reasoning for $d = 4$ goes along the same line. We expect from (38) a three dimensional kernel space, but also we have two restrictions. No l_{ij}^4 term may show up, not even l_{12}^4, because of the $S_2 \times S_2$ invariance. And no l_{34}^3 term is allowed. These two restrictions limit the kernel to $K = K_1 K_2 (K_1 - K_2)^2$, leaving us with a one-dimensional kernel.

$\langle 2221\rangle = \mathcal{P}_{S_3}\Big\{ \frac{1}{6} F_{2221} + (\frac{1}{6} - \frac{1}{3} P_{(14)})$  $F_{2220} + (\frac{1}{2} P_{(13)} - 1)$ $F_{1221} +$

$\big[(2 - P_{(34)} - \frac{1}{2} P_{(12)} + 2P_{(12)(34)} + P_{(14)})$ $- (1 + P_{(24)})$ $+$

$(-\frac{1}{2} - \frac{3}{4} P_{(13)})$  $\big] F_{0221} +$

$\big[(2 - P_{(24)})$ $+ (2 - 2P_{(34)} - 2P_{(12)} + 2P_{(12)(34)} - P_{(124)})$ $+$

$(P_{(13)(24)} - \frac{1}{2} P_{(13)})$  $\big] F_{1220} +$

$\big[(1 - \frac{1}{2} P_{(23)} + \frac{1}{2} P_{(243)} + P_{(34)} + P_{(234)} + P_{(24)} - P_{(14)})$ $-$ $+$

$(-P_{(24)} - \frac{1}{2} P_{(13)(24)})$  $\big] F_{1121} +$

$$\big[\tfrac{2}{3} \quad - \tfrac{2}{3} \quad - \tfrac{2}{3} \quad - \tfrac{1}{3} \quad - \tfrac{1}{6} \quad +$$

$$(\tfrac{1}{3} + \tfrac{2}{3}P_{(34)}) \quad \big] F_{1111} +$$

$$\big[\tfrac{1}{2} \quad - 2P_{(23)} \quad + P_{(23)} \quad + \quad +$$

$$(P_{(134)} - P_{(34)} - P_{(13)}) \quad \big] F_{1120} +$$

$$\big[\tfrac{1}{2}(1 - P_{(34)} - 2P_{(13)} - P_{(134)}) \quad + \quad + (1 - P_{(12)}) \quad +$$

$$\tfrac{1}{4}P_{(13)} \quad - \tfrac{1}{2} \quad + \tfrac{1}{2} \quad \big] F_{0220} +$$

$$\big[\tfrac{1}{2}(P_{(23)} - 2P_{(34)} + 2P_{(234)} - P_{(243)} + P_{(13)(24)} + P_{(12)(34)} + 3P_{(1243)} + P_{(14)(23)} +$$

$$3P_{(124)} - P_{(12)} + 2P_{(132)} + P_{(1342)} + P_{(24)} - P_{(1324)} + 4P_{(143)} - P_{(1423)}) \quad +$$

$$(1 - 2P_{(34)} + P_{(23)} + 2P_{(234)} + P_{(243)} - P_{(12)} + 2P_{(12)(34)} + P_{(1243)} - P_{(13)} +$$

$$2P_{(132)}) \quad - (1 + 2P_{(34)} + P_{(24)} + P_{(14)}) \quad +$$

$$(\tfrac{1}{2} - \tfrac{3}{2}P_{(23)} - \tfrac{1}{2}P_{(24)} - \tfrac{3}{2}P_{(14)}) \quad - (\tfrac{1}{2}P_{(23)} + P_{(14)}) \quad +$$

$$(-P_{(23)} - 2P_{(24)} - P_{(12)}) \quad \big] F_{0121} +$$

$$\big[\tfrac{2}{3} \quad - \tfrac{2}{3}P_{(14)} \quad - (\tfrac{1}{3} + P_{(24)}) \quad + (\tfrac{2}{3}P_{(34)} - \tfrac{1}{3}) \quad +$$

$$(\tfrac{4}{3} + \tfrac{2}{3}P_{(34)}) \quad - \tfrac{1}{6} \quad + \tfrac{2}{3} \quad - \quad +$$

$$(\tfrac{4}{3}P_{(14)} - \tfrac{2}{3}) \quad - \tfrac{1}{3}P_{(24)} \quad - \tfrac{2}{3} \quad \big] F_{1110} +$$

$$\big[(\tfrac{1}{4}P_{(34)} + \tfrac{1}{2}P_{(23)} - \tfrac{1}{2}P_{(234)} - \tfrac{1}{4} + \tfrac{1}{4}P_{(14)}) \quad + (P_{(23)} - P_{(34)}) \quad +$$

$$(P_{(23)} - \tfrac{1}{2} + \tfrac{1}{2}P_{(243)} + \tfrac{1}{2}P_{(24)}) \quad + (P_{(23)} - 1) \quad + \quad +$$

$$\tfrac{1}{2}(P_{(234)} - 3P_{(34)} - 3 - P_{(23)} + 2P_{(13)} - 2P_{(134)} + P_{(143)} - P_{(14)}) \quad +$$

$$(\tfrac{1}{2}P_{(234)} - 1 - P_{(34)} - \tfrac{1}{2}P_{(23)} + \tfrac{1}{2}P_{(13)} - \tfrac{1}{2}P_{(134)}) \quad + \tfrac{3}{8} \quad +$$

$$(\tfrac{3}{2}P_{(23)} - \tfrac{1}{2} - \tfrac{1}{2}P_{(143)} + \tfrac{1}{2}P_{(14)}) \quad + (\tfrac{3}{4} + \tfrac{1}{4}P_{(24)}) \quad +$$

$$(1 - P_{(13)} - P_{(14)} + P_{(1324)}) \quad \big] F_{0012} +$$

$$\big[\tfrac{1}{2}(P_{(34)} + P_{(23)} - 3P_{(243)} - P_{(12)(34)} - 6P_{(123)} - 6P_{(1234)} + P_{(132)} - 2P_{(13)(24)} +$$

$$2P_{(1324)} - 3P_{(1432)} - 2P_{(14)} + 2P_{(1423)}) \quad + (\tfrac{1}{2}P_{(24)} + \tfrac{3}{2}P_{(13)}) \quad +$$

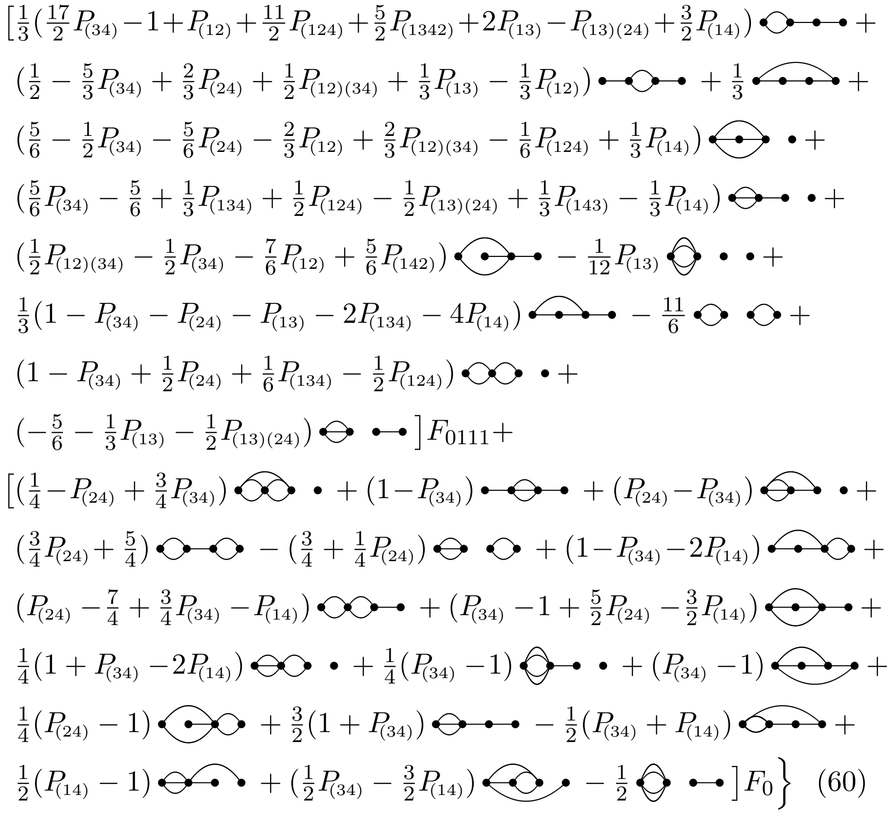
$$\tfrac{1}{2}(3P_{(24)} - P_{(23)} - 3P_{(234)} - P_{(12)} + P_{(12)(34)} + P_{(123)}) \quad + \tfrac{3}{2}P_{(24)} \quad +$$

$$\tfrac{1}{2}(2P_{(23)} - P_{(34)} + P_{(12)(34)} - P_{(132)} - P_{(14)}) \quad + \tfrac{1}{2}(P_{(243)} - P_{(23)} - P_{(234)} +$$

$$P_{(124)}) \quad + (2P_{(1234)} - P_{(23)} - P_{(234)} + P_{(243)} - P_{(124)}) \quad +$$

$$\tfrac{1}{2}(P_{(124)} + P_{(13)} - P_{(134)} - P_{(13)(24)} - P_{(14)} + P_{(1423)}) \quad +$$

$$(1 + P_{(34)} + P_{(24)} + P_{(13)}) \quad + (2 - P_{(34)}) \quad \big] F_{0120} +$$



$$
\begin{aligned}
&\big[\tfrac{1}{3}(\tfrac{17}{2}P_{(34)}-1+P_{(12)}+\tfrac{11}{2}P_{(124)}+\tfrac{5}{2}P_{(1342)}+2P_{(13)}-P_{(13)(24)}+\tfrac{3}{2}P_{(14)})\ +\\
&(\tfrac{1}{2}-\tfrac{5}{3}P_{(34)}+\tfrac{2}{3}P_{(24)}+\tfrac{1}{2}P_{(12)(34)}+\tfrac{1}{3}P_{(13)}-\tfrac{1}{3}P_{(12)})\ +\tfrac{1}{3}\ +\\
&(\tfrac{5}{6}-\tfrac{1}{2}P_{(34)}-\tfrac{5}{6}P_{(24)}-\tfrac{2}{3}P_{(12)}+\tfrac{2}{3}P_{(12)(34)}-\tfrac{1}{6}P_{(124)}+\tfrac{1}{3}P_{(14)})\ +\\
&(\tfrac{5}{6}P_{(34)}-\tfrac{5}{6}+\tfrac{1}{3}P_{(134)}+\tfrac{1}{2}P_{(124)}-\tfrac{1}{2}P_{(13)(24)}+\tfrac{1}{3}P_{(143)}-\tfrac{1}{3}P_{(14)})\ +\\
&(\tfrac{1}{2}P_{(12)(34)}-\tfrac{1}{2}P_{(34)}-\tfrac{7}{6}P_{(12)}+\tfrac{5}{6}P_{(142)})\ -\tfrac{1}{12}P_{(13)}\ +\\
&\tfrac{1}{3}(1-P_{(34)}-P_{(24)}-P_{(13)}-2P_{(134)}-4P_{(14)})\ -\tfrac{11}{6}\ +\\
&(1-P_{(34)}+\tfrac{1}{2}P_{(24)}+\tfrac{1}{6}P_{(134)}-\tfrac{1}{2}P_{(124)})\ +\\
&(-\tfrac{5}{6}-\tfrac{1}{3}P_{(13)}-\tfrac{1}{2}P_{(13)(24)})\ \big]F_{0111}+\\
&\big[(\tfrac{1}{4}-P_{(24)}+\tfrac{3}{4}P_{(34)})\ +(1-P_{(34)})\ +(P_{(24)}-P_{(34)})\ +\\
&(\tfrac{3}{4}P_{(24)}+\tfrac{5}{4})\ -(\tfrac{3}{4}+\tfrac{1}{4}P_{(24)})\ +(1-P_{(34)}-2P_{(14)})\ +\\
&(P_{(24)}-\tfrac{7}{4}+\tfrac{3}{4}P_{(34)}-P_{(14)})\ +(P_{(34)}-1+\tfrac{5}{2}P_{(24)}-\tfrac{3}{2}P_{(14)})\ +\\
&\tfrac{1}{4}(1+P_{(34)}-2P_{(14)})\ +\tfrac{1}{4}(P_{(34)}-1)\ +(P_{(34)}-1)\ +\\
&\tfrac{1}{4}(P_{(24)}-1)\ +\tfrac{3}{2}(1+P_{(34)})\ -\tfrac{1}{2}(P_{(34)}+P_{(14)})\ +\\
&\tfrac{1}{2}(P_{(14)}-1)\ +(\tfrac{1}{2}P_{(34)}-\tfrac{3}{2}P_{(14)})\ -\tfrac{1}{2}\ \big]F_0\Big\} \qquad (60)
\end{aligned}
$$

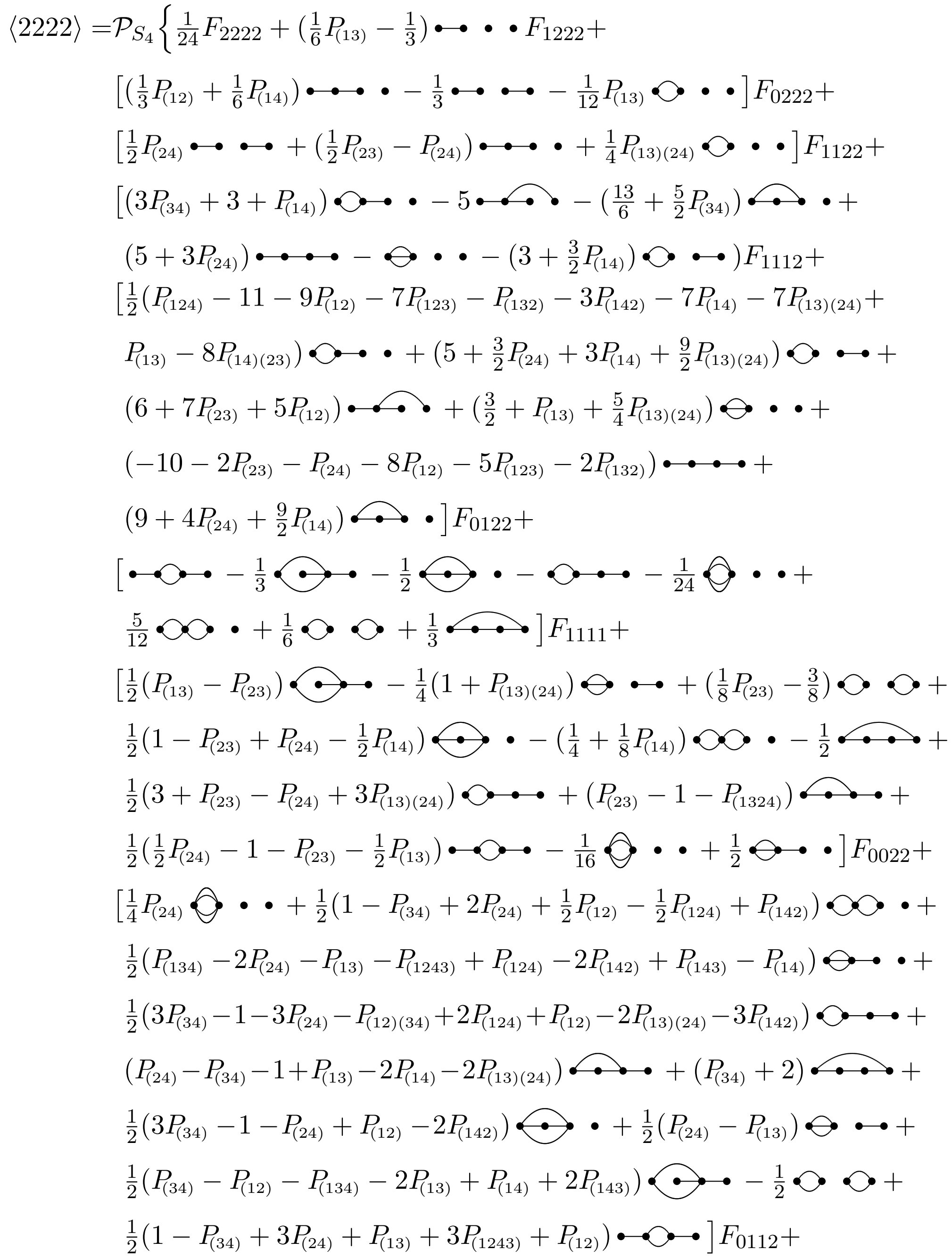
Interestingly the dimension of the kernel for $\langle 2221\rangle$ is also 18 and by that not larger than the kernel for $\langle 2211\rangle$. Naively one would expect that the kernel dimension increases with growing Jordan-level $K := \sum_i k_i$. On the other hand the larger symmetry group (S_3 instead of $S_2 \times S_2$) reduces the kernel size which can even lead to a smaller kernel, as we will see in the case of $\langle 2222\rangle$.

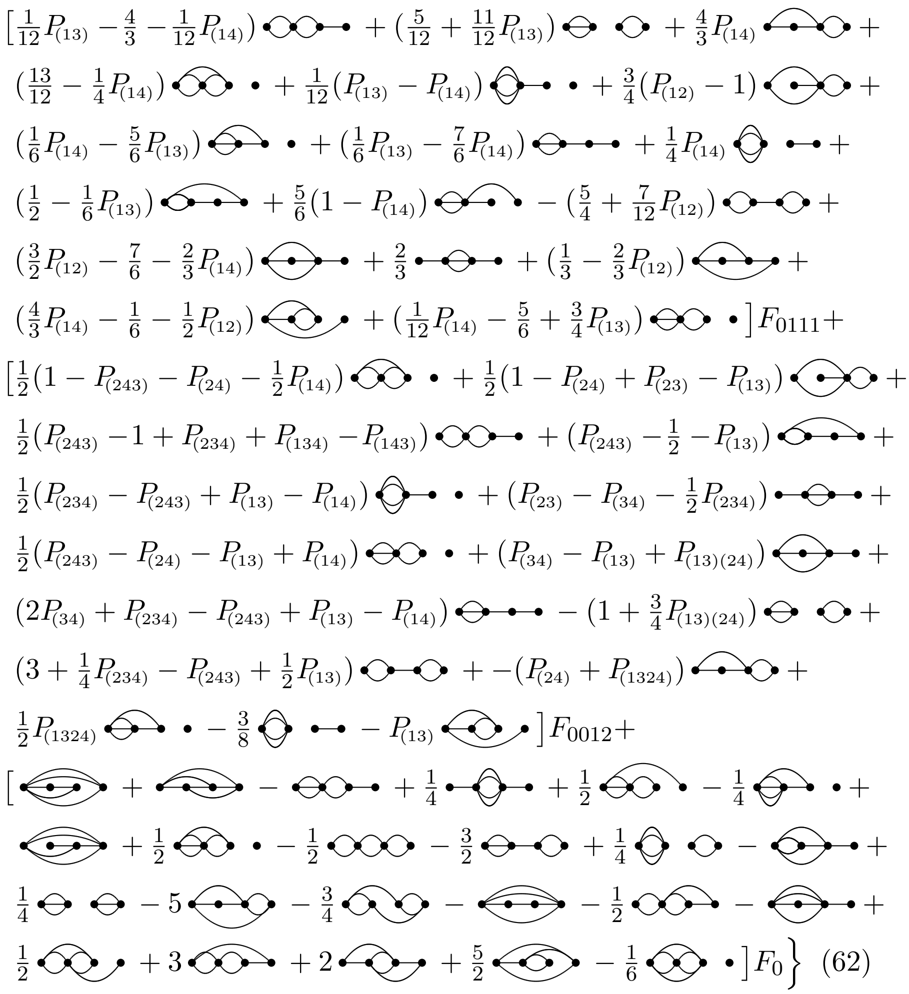
$$
\begin{aligned}
\mathrm{Ker}_{\langle 2221\rangle} =&\mathcal{P}_{S_3}\Big\{(2K_2-K_1)F_{1221}|^{d=1}+\\
&\big[cK_1^2+c'(K_2^2-K_1K_2)\big](F_{1220}|^{d=2}+F_{2111}|^{d=2}+F_{0221}|^{d=2})+\\
&K^{(3)}F_{0121}|^{d=4}+K^{(3)}_{S_4}F_{1111}|^{d=1}+\\
&\big[cK_2(K_1-K_2)(K_1-2K_2)+c'K_1^2(K_1-2K_2)\big]F_{0220}|^{d=2}+\\
&\big[cK_2(K_1-K_2)(K_1-2K_2)+c'K_1^2(K_1-2K_2)\big]F_{2110}|^{d=2}+\\
&K_1^2K_2(K_1-K_2)F_{0111}|^{d=1}+K_1^2K_2(K_1-K_2)F_{0120}|^{d=1}+\\
&K_1^2K_2(K_1-K_2)F_{1002}|^{d=1}\Big\} \qquad (61)
\end{aligned}
$$

There is not much surprise for most results. For logarithmic degree 1 and 2 we get for the sets containing three F a $d = 1$ respectively a $d = 2$ kernel. For degree 3 we have the self-invariant term F_{1111} with a S_4 symmetry and two $d = 2$ kernels for $\{F_{0220}, F_{2020}, F_{2200}\}$ and $\{F_{2110}, F_{1210}, F_{1120}\}$. There is also a set containing six F, which results in a full four dimensional $K^{(3)}$ kernel.

As expected combinatorial constraints show up the first time for degree four, because of the last vertex having one leg only and thus disallowing any l_{i4}^4

($i = 1, 2, 3$) term. For degree actually a $d = 3$ kernel would have been possible, but eliminating all l_{i4}^4 terms means that the kernel has to be reduced to a one-dimensional kernel each. Also note that the only possible kernel term of degree 5 would have been $K_{S_4}^{(5)}$ which does not show up, because of the same combinatorial restriction.





$$\Big[\tfrac{1}{12}P_{(13)} - \tfrac{4}{3} - \tfrac{1}{12}P_{(14)}\big) \quad + \big(\tfrac{5}{12} + \tfrac{11}{12}P_{(13)}\big) \quad + \tfrac{4}{3}P_{(14)} \quad +$$
$$\big(\tfrac{13}{12} - \tfrac{1}{4}P_{(14)}\big) \quad + \tfrac{1}{12}\big(P_{(13)} - P_{(14)}\big) \quad + \tfrac{3}{4}\big(P_{(12)} - 1\big) \quad +$$
$$\big(\tfrac{1}{6}P_{(14)} - \tfrac{5}{6}P_{(13)}\big) \quad + \big(\tfrac{1}{6}P_{(13)} - \tfrac{7}{6}P_{(14)}\big) \quad + \tfrac{1}{4}P_{(14)} \quad +$$
$$\big(\tfrac{1}{2} - \tfrac{1}{6}P_{(13)}\big) \quad + \tfrac{5}{6}\big(1 - P_{(14)}\big) \quad - \big(\tfrac{5}{4} + \tfrac{7}{12}P_{(12)}\big) \quad +$$
$$\big(\tfrac{3}{2}P_{(12)} - \tfrac{7}{6} - \tfrac{2}{3}P_{(14)}\big) \quad + \tfrac{2}{3} \quad + \big(\tfrac{1}{3} - \tfrac{2}{3}P_{(12)}\big) \quad +$$
$$\big(\tfrac{4}{3}P_{(14)} - \tfrac{1}{6} - \tfrac{1}{2}P_{(12)}\big) \quad + \big(\tfrac{1}{12}P_{(14)} - \tfrac{5}{6} + \tfrac{3}{4}P_{(13)}\big) \quad \Big]F_{0111} +$$
$$\Big[\tfrac{1}{2}\big(1 - P_{(243)} - P_{(24)} - \tfrac{1}{2}P_{(14)}\big) \quad + \tfrac{1}{2}\big(1 - P_{(24)} + P_{(23)} - P_{(13)}\big) \quad +$$
$$\tfrac{1}{2}\big(P_{(243)} - 1 + P_{(234)} + P_{(134)} - P_{(143)}\big) \quad + \big(P_{(243)} - \tfrac{1}{2} - P_{(13)}\big) \quad +$$
$$\tfrac{1}{2}\big(P_{(234)} - P_{(243)} + P_{(13)} - P_{(14)}\big) \quad + \big(P_{(23)} - P_{(34)} - \tfrac{1}{2}P_{(234)}\big) \quad +$$
$$\tfrac{1}{2}\big(P_{(243)} - P_{(24)} - P_{(13)} + P_{(14)}\big) \quad + \big(P_{(34)} - P_{(13)} + P_{(13)(24)}\big) \quad +$$
$$\big(2P_{(34)} + P_{(234)} - P_{(243)} + P_{(13)} - P_{(14)}\big) \quad - \big(1 + \tfrac{3}{4}P_{(13)(24)}\big) \quad +$$
$$\big(3 + \tfrac{1}{4}P_{(234)} - P_{(243)} + \tfrac{1}{2}P_{(13)}\big) \quad + -\big(P_{(24)} + P_{(1324)}\big) \quad +$$
$$\tfrac{1}{2}P_{(1324)} \quad - \tfrac{3}{8} \quad - P_{(13)} \quad \Big]F_{0012} +$$
$$\Big[\quad + \quad - \quad + \tfrac{1}{4} \quad + \tfrac{1}{2} \quad - \tfrac{1}{4} \quad +$$
$$\quad + \tfrac{1}{2} \quad - \tfrac{1}{2} \quad - \tfrac{3}{2} \quad + \tfrac{1}{4} \quad - \quad +$$
$$\tfrac{1}{4} \quad - 5 \quad - \tfrac{3}{4} \quad - \quad - \tfrac{1}{2} \quad - \quad +$$
$$\tfrac{1}{2} \quad + 3 \quad + 2 \quad + \tfrac{5}{2} \quad - \tfrac{1}{6} \quad \Big]F_0 \Big\} \qquad (62)$$

We saw that the transition from $\langle 2211\rangle$ to $\langle 2221\rangle$ did not increase the dimension of the kernel mainly because of the increase of the discrete symmetry group from $S_2 \times S_2$ to S_3. This transition to $\langle 2222\rangle$ enlarges the symmetry group from S_3 to S_4 and by that even reduces the dimension of the kernel to 13.

$$\begin{aligned}
\mathrm{Ker}_{\langle 2222\rangle} =& \mathcal{P}_{S_4}\Big\{K^{(2)}_{S_2} F_{1122}|^{d=2} + \\
& \big[cK_2(K_1 - K_2)(K_1 - 2K_2) + c' K_1^2(K1 - 2K_2)\big] F_{0122}|^{d=2} + \\
& K^{(3)}_{S_4} F_{1112}|^{d=1} + K^{(4)}_{S_2} F_{0022}|^{d=3} + K^{(4)}_{S_4} F_{1111}|^{d=1} + \\
& \big[cK_1^4 + c' K_2^2 (K_1 - K_2)^2 + c''(K_1^3 K_2 - 2K_1 K_2^3 + K_2^4)\big] F_{0112}|^{d=3} + \\
& K_1 K_2 (K_1 - K_2)^2 (K_1 + K_2) F_{0012}|^{d=1}\Big\}
\end{aligned} \qquad (63)$$

There is no kernel of logarithmic degree one because at most we have four F in a set and the S_4 symmetry then is forbidden according to appendix A. Additional combinatorial constraints start with logarithmic degree 5: no l_{ij}^5 for $1 \le i < j \le 4$ is allowed to show up.

For F_{0012} plus the five permutations there would have been a $d = 3$ kernel, but the given linear combination is the only one which eliminates all l_{ij}^5 terms. For

$\{F_{0111}, F_{1011}, F_{1101}, F_{1110}\}$ only $K_{S_4}^{(5,d=1)}$ would be possible, but is ruled by the combinatorial restriction.

That there is a kernel for $d = 6$ is a bit unexpected. On the one hand we have (40) a two dimensional kernel, but on the other there are two constraints which need to be satisfied, namely all l_{ij} to the power of 5 and to the power of 6 have to be eliminated. The given combination fulfills both restrictions and thus gives us an additional degree of freedom for F_0.

5 Exact results for two logarithmic fields

The easiest non-trivial case is the one, where we have two logarithmic fields and two primaries. For this case the correlator $\langle k_1 k_2 00\rangle$ for $k_1, k_2 > 0$ can be solved exactly for arbitrary Jordan-rank r.

The correlator for Jordan-rank r has the following form

$$\langle k_1 k_2 00\rangle = F_{k_1,k_2,0,0} + c_1 l_{12} F_{k_1-1,k_2,0,0} + c_2 l_{12} F_{k_1,k_2-1,0,0} + \dots \; . \tag{64}$$

As described in subsection 2.5 it is possible to identify some of the appearing F-terms with each other. In this case it turns out that it is easy to find the identifications that stems from the integration process by inserting the above ansatz in equation (10). This leads to

$$O_1 \langle k_1 k_2 00\rangle = -2z_1 \langle k_1-1, k_2, 0, 0\rangle - 2z_2 \langle k_1, k_2-1, 0, 0\rangle \; , \tag{65}$$

and considering the terms of the lowest order in $\{l_{ij}\}$ only we get

$$\begin{aligned}(z_1 + z_2)(c_1 F_{k_1-1,k_2,0,0} + c_2 F_{k_1,k_2-1,0,0}) + \mathcal{O}(l_{12})\\ = -2z_1 F_{k_1-1,k_2,0,0} - 2z_2 F_{k_1,k_2-1,0,0} + \mathcal{O}(l_{12}) \; .\end{aligned} \tag{66}$$

We immediately see that these equations do not have a solution. As before we can circumvent the problem by reducing the complexity of the equations, which can be accomplished by identification of some of the functions F. Here we can solve equation (66) by using the following identifications

$$F_{k_1-1,k_2,0,0} \equiv F_{k_1,k_2-1,0,0} \; . \tag{67}$$

This in perfect agreement with the results presented so far for $r = 3$. Because of having the the above identifications we are left with only one function F for each logarithmic degree of $\langle k_1 k_2 00\rangle$. Using (9) yields after a short calculation the full result for a correlator of Jordan-rank r with two primary fields:

$$\langle k_1 k_2 00\rangle = \sum_{n=0}^{k_1+k_2-(r-1)} \frac{(-2)^n}{n!} l_{12}^n F_{k_1+k_2-(r-1)-n,r-1,0,0} \; . \tag{68}$$

As a consistency check we can compare the above result with the one presented in [5], respectively [19]. For the two-point correlation function the first paper gives the following result

$$\langle \Psi_{k_1}(z_1)\Psi_{k_2}(z_2)\rangle = \sum_{\ell=0}^{k_1+k_2} \frac{(-2)^\ell}{\ell!} l_{12}^\ell D_{(h_1=0,h_2=0,k_1+k_2-\ell)} \ . \tag{69}$$

where we slightly adapted the notation and have set the conformal weights h_1, h_2 to zero. The $D_{(\dots)}$ are called "structure constants" and have the property that $D_{(h,h;k)} = 0$ for $k < r-1$. In other words the index ℓ in (69) effectively runs from 0 to $k_1 + k_2 - (r-1)$ and thus (68) and (69) are of identical structure. This means, that the polynomial dependence on the logarithms l_{ij} is exactly the same and that precisely the same number of free structure constants $D_{(\dots)}$ or structure functions $F_{\dots}(x)$ are needed.

6 Summary and discussion

In the scope of this paper we analyzed the influence of the global conformal symmetries in form of the global conformal Ward identities on 4-point correlation functions in arbitrary logarithmic conformal field theory. While it is not possible to completely determine the correlators, this does not even work in the CFT case, it is possible to fix the generic structure of the correlators.

The presented algorithm can be used to calculate the generic structure of 4-point correlators. Within this paper we restricted ourselves to combinations of proper primary and logarithmic fields, but did mention how to adjust the algorithm in order to extend the algorithm to pre-logarithmic fields.

We explicitly gave the results for, up to permutations, all correlators of Jordan-rank $r = 2, 3$. In some of the results we found additional constants which were identified as elements of the kernel O. Furthermore we discussed various restrictions which limit the number of terms that can appear in an ansatz or which lead to lesser degrees of freedom in the kernel. Also we found that integration sometimes requires that some functions F need to be identified with each other.

There are almost no computations of LCFT 4-pt functions in the literature. We can only compare the structure of our results with some of the functions given in [8]. However, Gaberdiel and Kausch compute, in the specific example of the $c = -2$ LCFT, the full non-chiral 4-pt functions. They can do this specifically for 4-pt functions which involve at least one or two twist fields. These latter fields are pre-logarithmic fields, and therefore not proper primary. Such functions are not coverd by the explicit results in our work, although it is – in principle – a straight-forward task to generalize our initial conditions to be applicable to such cases as well. The only 4-pt function we can compare directly with is the one of four logarithmic partner fields of the identity. The form of this function in [8] clearly agrees with our generic form for $\langle 1111\rangle$ in the $r = 2$ case. Also, their 4-pt functions of type $\langle \mu\mu 00\rangle$ involving two twist fields and two proper primaries agree in their form with what we expect our ansatz would yield with the appropriate initial conditions: The structure of such 4-pt functions is equivalent to the one of type $\langle 1000\rangle$, again for $r = 2$.

Finally we gave explicit results for the case of exactly two logarithmic fields for arbitrary Jordan-rank r. Studying this very simple case showed us why we need to identify some of the functions F with each other. Also we did a consistency check of the result and showed that equation (68) is equivalent to the one presented in [5].

The comparison can be extended to three-point correlators. For instance we can consider the terms of logarithmic degree $l = 2$ of the correlator $\langle 2110\rangle$ in a Jordan-rank $r = 3$ theory, cf. equation (51):

$$\langle 2110\rangle\,|_{l=2} = \big[-\tfrac{1}{2}(l_{12}^2 + l_{13}^2 + l_{23}^2) + 3l_{12}l_{13} + l_{12}l_{23} + l_{13}l_{23}\big]F_0\ . \tag{70}$$

As a comparison we evaluate formula (3.11) in [5] and get for $l = 2$ the same result,

$$\langle 211\rangle\,|_{l=2} = \big[-\tfrac{1}{2}(l_{12}^2 + l_{13}^2 + l_{23}^2) + 3l_{12}l_{13} + l_{12}l_{23} + l_{13}l_{23}\big]C_{(h_1,h_2,h_3;k=0)}\ , \tag{71}$$

except that F_0 has to be replaced by the structure constant $C_{(h_1,h_2,h_3;k=0)}$. We once more note that we suppressed any direct but trivial dependence on the conformal weights, so actually, we should compare with $C_{(0,0,0;k=0)}$. However, our results are, up to the omitted prefactor $\prod_{i<j} z_{ij}^{\mu_{ij}}$, valid and independent of the values of the conformal weights h_i. For the other correlators like $\langle 2210\rangle$ et cetera we also confirmed that the results match if we restrict us to the highest logarithmic degree, which corresponds to $l^{\max} = k_1 + k_2 + k_3 - r + 1$. As we will see in the following it is interesting to study the case where $l < l^{\max}$. We use $\langle 2110\rangle$ as an example again, but this time we consider the term of $l = 1$ only:

$$\begin{aligned}\langle 2110\rangle\,|_{l=1} =& F_{1020}(l_{13} - l_{12} - l_{23}) + F_{1110}(l_{23} - l_{12} - l_{13}) + \\ & F_{1200}(l_{12} - l_{13} - l_{23})\ .\end{aligned} \tag{72}$$

We remind the reader that the above result includes the usual identifications such as $F_{2100} \equiv F_{1200}$. The structure of the formula in [5] makes it obvious that for $l = 1$ only one structure constant shows up and thus the corresponding term is

$$\langle 211\rangle\,|_{l=1} = -(l_{12} + l_{13} + l_{23})\,C_{(h_1=0,h_2=0,h_3=0;k=1)}\ , \tag{73}$$

where we again set the conformal weights to zero and slightly adjusted the notation. Though looking differently at first glance we can achieve the same form of the result if we demand that the following extended identifications hold too, namely

$$F_{1200} \equiv F_{1020} \equiv F_{1110}\ . \tag{74}$$

This means that we do not only regain the F_0 terms, but that we can reclaim all information, provided that we do all necessary identifications. With “necessary” we mean that we have to identify all $F_{\dots}$ terms of the same logarithmic degree.

We already encountered one situation where we had to identify several functions F with each other: the initial conditions (11) where we identified $F_0 \equiv F_0' \equiv \ldots$ by virtue of the cluster decomposition argument.

This evokes the question whether this form of massive identifications of functions F is necessary or useful in the context of some physical theory respectively what conditions could force us to massively reduce the number of functions F. It is clear that the special case where all conformal weights h_i are equal to each other has an additional symmetry, since we can freely exchange the fields. In this case, we definitely expect that a large number of such identifications should take place.

Furthermore, one can quickly check that the given solutions remain valid after identifying remaining free structure functions because any remaining such function can be arbitrarily chosen as long as no further constraints such as local conformal symmetry are invoked. Due to the recursive dependence of the solutions for total Jordan-level K on the ones for level $K' < K$, identifications are consistent only if restricted to functions $F_{k_1k_2k_3k_4}$, $F_{k_1'k_2'k_3'k_4'}$ with $k_1 + k_2 + k_3 + k_4 = k_1' + k_2' + k_3' + k_4'$. However, a more detailed analysis which identifications should be present in the general case, i. e., for arbitrary values of the conformal weights h_i, will be left to future work.

Of course, when all four fields in the 4-point function are logarithmic, we cannot expect that the resulting polynomials in the l_{ij} can be matched with the ones of 2- and 3-point functions. But one might attempt to make the following comparison.

The structure functions $F_{k_1k_2k_3k_4}(x)$ are ultimately composed out of (a suitable generalization of) conformal blocks which depend on the internal propagator in the 4-point function. Crossing symmetry of the 4-point function imply that the structure functions possess for each asymptotic region $|x| < 1$, $|1 - x| < 1$, or $1/|x| < 1$ expansions of the schematic form

$$F_{(h_1,k_1)(h_2,k_2)(h_3,k_3)(h_4,k_4)}(x) \sim \sum_{(h,k)} C^{(h,k)}_{(h_i,k_i)(h_j,k_j)} C_{(h,k)(h_l,k_l)(h_m,k_m)} + \ldots \quad (75)$$

for all permutations $\{i, j, l, m\}$ of $\{1, 2, 3, 4\}$, which must all be expansions of the same analytical functions. These expansions involve the 3-point structure constants as well as the OPE structure constants. In the logarithmic case, these structure "constants" are matrix valued with coefficients in $\mathbb{C}[\{l_{ij}\}]$. In the notation used in this paper, $C_{(h_1,k_1)(h_2,k_2)(h_3,k_3)} = \langle k_1k_2k_3\rangle$ where on both sides all terms of the form $z_{ij}^{\mu_{ij}}$ depending in the canonical way on the conformal weights are omitted. In the r-dimensional Jordan-cell space, this defines matrices $(C_{k_1})_{k_2k_3}$ labeled by the first Jordan-level and with indices given by the second and third Jordan-level. In the same way, the propagator defines a matrix $(D)_{k_1k_2} = \langle k_1k_2\rangle$. The OPE structure "constants" are then given by the matrix product

$$(C_{k_1})_{k_2}^{\ k_3} = (C_{k_1})_{k_2k}(D^{-1})^{kk_3} \quad (76)$$

involving the inverse propagator. Now, one can compute the leading orders of the different expansions of the 4-point structure functions which will yield different polynomials in the l_{ij} with coefficients given by rational functions of the 2- and 3-point structure constants $D_{(h,h;p)}$ and $C_{(h_i h_j,h;q)}$. Two observations can now be made:

Firstly, the three expansions for the s-, t- and u-channel, i. e., for $|x| < 1$, $|1-x| < 1$ and $1/|x| < 1$ all differ. They lead to different polynomials. It is easy to check in simple examples that certain monomials in the l_{ij} may appear only in one of the expansions. This always happens for 4-point functions of the form $\langle k_1 k_2 k_3 k_4 \rangle$ with all $k_i > 0$ but not all k_i equal.

Secondly, the polynomials in l_{ij} with coefficients given by the structure functions $F_{k_1 k_2 k_3 k_4}(x)$ cannot be matched to any of the three expansions. On the contrary, the 4-point functions will involve all the different monomials in the l_{ij} and in particular all the ones which do not appear in all the expansions, but in only one of them. It is therefore much more difficult to match the 4-point structure functions to expressions in the 3- and 2-point structure constants or to suggest further identifications as they can easily be read off in the case of 4-point functions of type $\langle k_1 k_2 00 \rangle$ or $\langle k_1 k_2 k_3 0 \rangle$. In fact, it is not straightforward how the three different expansions should be combined for a comparison of coefficients in case all four fields are logarithmic. A further complication is given by the freedom to change the polynomials in the l_{ij} by elements in the kernel of the operator O or, equivalently, by a redefinition of the structure function coefficients. But we believe that it would be very interesting to investigate the consequences of crossing symmetry for the structure functions of LCFT 4-point functions, because this might yield severe restrictions on the number of functions which have to be determined by other means, for example local conformal invariance. This is an important task for future work in order to greatly ease the full computation of 4-point correlation functions in LCFT of rank $r > 2$.

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A Overview of the kernel terms

The following tables contain the kernel terms that can show up for a logarithmic degree from one to five. In addition to the logarithmic degree the kernel depends on the number of functions F that are involved and also on the discrete symmetry. As we are considering four point functions only we are left with four different symmetry groups.

The format of the entries is the same as in subsection 2.6 with the small addition of the dimension d of the kernel. It is interesting how similar the entries for the different logarithmic degrees are, the only exception being the entry for the pair (F_3, S_3) respectively (F_{12}, S_4). Also note that each column contains the full kernel, namely if and only if $|F| = |S|$, where S denotes the symmetry group and $|S|$ its cardinality .

"$\leftrightarrow$" means that these entries have to be identical as shown in (37), (39).

Log.deg 1	S_2		$S_{2\times 2}$	S_3		S_4
F_1	$K_{S_2}^{(1),d=1}$	$\leftrightarrow$	$K_{S_2}^{(1),d=1}$	0	$\leftrightarrow$	0
F_2	$K^{(1),d=2}$		$K_{S_2}^{(1),d=1}$	—		—
F_3	—		—	$(*)\|^{d=1}$		—
F_4	—		$K^{(1),d=2}$	—		0
F_6	—		—	$K^{(1),d=2}$		$K_{S_2}^{(1),d=1}$
F_{12}	—		—	—		$(*)\|^{d=1}$
F_{24}	—		—	—		$K^{(1),d=2}$

$(*) = 2K_2 - K_1|^{d=1}$

Log.deg 2	S_2		$S_{2\times 2}$	S_3		S_4
F_1	$K_{S_2}^{(2),d=2}$	$\leftrightarrow$	$K_{S_2}^{(2),d=2}$	$K_{S_4}^{(2),d=1}$	$\leftrightarrow$	$K_{S_4}^{(2),d=1}$
F_2	$K^{(2),d=3}$		$K_{S_2}^{(2),d=2}$	—		—
F_3	—		—	$(*)\|^{d=2}$		—
F_4	—		$K^{(2),d=3}$	—		$K_{S_4}^{(2),d=1}$
F_6	—		—	$K^{(2),d=3}$		$K_{S_2}^{(2),d=2}$
F_{12}	—		—	—		$(*)\|^{d=2}$
F_{24}	—		—	—		$K^{(2),d=3}$

$(*) = cK_1^2 + c'(K_2^2 - K_1K_2)|^{d=2}$

Log.deg 3	S_2		$S_{2\times 2}$	S_3		S_4
F_1	$K_{S_2}^{(3),d=2}$	$\leftrightarrow$	$K_{S_2}^{(3),d=2}$	$K_{S_4}^{(3),d=1}$	$\leftrightarrow$	$K_{S_4}^{(3),d=1}$
F_2	$K^{(3),d=4}$		$K_{S_2}^{(3),d=2}$	—		—
F_3	—		—	$(*)\|^{d=2}$		—
F_4	—		$K^{(3),d=4}$	—		$K_{S_4}^{(3),d=1}$
F_6	—		—	$K^{(3),d=4}$		$K_{S_2}^{(3),d=2}$
F_{12}	—		—	—		$(*)\|^{d=2}$
F_{24}	—		—	—		$K^{(3),d=4}$

$(*) = cK_1^2(K_1 - 2K_2) + c'K_2(K_1 - K_2)(K_1 - 2K_2)|^{d=2}$

Log.deg 4	S_2		$S_{2\times 2}$	S_3		S_4
F_1	$K_{S_2}^{(4),d=3}$	$\leftrightarrow$	$K_{S_2}^{(4),d=3}$	$K_{S_4}^{(4),d=1}$	$\leftrightarrow$	$K_{S_4}^{(4),d=1}$
F_2	$K^{(4),d=5}$		$K_{S_2}^{(4),d=3}$	—		—
F_3	—		—	$(*)\|^{d=3}$		—
F_4	—		$K^{(4),d=5}$	—		$K_{S_4}^{(4),d=1}$
F_6	—		—	$K^{(4),d=5}$		$K_{S_2}^{(4),d=5}$
F_{12}	—		—	—		$(*)\|^{d=3}$
F_{24}	—		—	—		$K^{(4),d=5}$

$(*) = cK_1^4 + c'K_2^2(K_1 - K_2)^2 + c''(K_1^3K_2 - 2K_1K_2^3 + K_2^4)$

Log.deg 5	S_2		$S_{2\times 2}$	S_3		S_4
F_1	$K_{S_2}^{(5),d=3}$	$\leftrightarrow$	$K_{S_2}^{(5),d=3}$	$K_{S_4}^{(5),d=1}$	$\leftrightarrow$	$K_{S_4}^{(5),d=1}$
F_2	$K^{(5),d=6}$		$K_{S_2}^{(5),d=3}$	—		—
F_3	—		—	$(*)\|^{d=3}$		—
F_4	—		$K^{(5),d=6}$	—		$K_{S_4}^{(5),d=1}$
F_6	—		—	$K^{(5),d=6}$		$K_{S_2}^{(5),d=3}$
F_{12}	—		—	—		$(*)\|^{d=3}$
F_{24}	—		—	—		$K^{(5),d=6}$

$$\begin{aligned}(*) = {}& c(-2K_1^3K_2^2 + 8K_1^2K_2^3 - 11K_1K_2^4 + 5K_2^5)+\\ & c'(-K_1^4K_2 + 4K_1^3K_2^2 - 6K_1^2K_2^3 + 4K_1K_2^4)+\\ & c''(8K_1^5 + 1K_1^4K_2 - 10K_1^2K_2^3 + 20K_1K_2^4 - 20K_2^5)|^{d=3}\end{aligned}$$

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