hep-th/0204154
ITP-UH-08/02

Boundary States in $c = -2$ Logarithmic Conformal Field Theory

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April 18, 2002

Abstract

Starting from first principles, a constructive method is presented to obtain boundary states in conformal field theory. It is demonstrated that this method is well suited to compute the boundary states of logarithmic conformal field theories. By studying the logarithmic conformal field theory with central charge $c = -2$ in detail, we show that our method leads to consistent results. In particular, it allows to define boundary states corresponding to both, indecomposable representations as well as their irreducible subrepresentations.

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1 Introduction

Conformal field theory in two dimensions [1] is undoubtedly one of the most important tools of modern theoretical physics with numerous applications ranging from string theory to experimentally observed and confirmed phenomena in condensed matter physics. The last few years saw a renewed interest in conformal field theories, mainly arising from two different directions. Much has been learned about conformal field theories on surfaces with a boundary [2, 3]. These are important since most real physical systems are finite in size and bounded. Surprisingly, boundary conformal field theory has also made its way into modern string theory where it presents a powerful tool for the computation of possible spectra of D-branes [4]. On the other hand, conformal field theories with logarithmic operators [5] enjoy increasing attention, see e. g. [6, 7] and references therein. Again, phenomenology yields a good reason for them, since many condensed matter systems, in particular systems involving disorder, possess density fields with scaling dimension zero (this was first observed in the treatment of dense polymers [8]). Such fields are one cause for the emergence of indecomposable representations and their associated logarithmic operators. Other causes for logarithmically diverging correlation functions which perhaps are more interesting for string theory are, e. g. twist fields in ghost systems (the problem was noted by V. Knizhnik [9] as far back as 1987) or puncture operators in Liouville theory [10].

Both directions in this field of research have reached a level of understanding close to the one of rational conformal field theory. In ordinary conformal field theory, boundary states can be obtained via a formalism presented by N. Ishibashi [11], which yields a constructive method to compute a basis of boundary states. In the case where the theory is rational J. L. Cardy [2] gave a prescription how to obtain the physical set of boundary states relating the coefficients of the linear combinations to the fusion coefficients of the bulk theory. Logarithmic conformal field theories share many properties with ordinary theories. Many notions of rationality can be generalized to such theories to a certain degree, such as characters, fusion rules, partition functions, see e. g. [12, 13, 14, 15, 16, 17, 18, 19, 20].

A question which naturally arises is whether and how a boundary theory for logarithmic conformal field theories can be obtained. I. I. Kogan and J. F. Wheater [21] were one of the first to discuss the question of boundary states and their effects in a $c = -2$ logarithmic conformal field theory. In a more recent work, S. Kawai and J. F. Wheater [22] studied boundary states of the same model by use of symplectic fermions. Further studies on $c = -2$ were conducted by Y. Ishimoto [23]. Boundaries in the framework of logarithmic conformal field theories were also discussed by A. Lewis [24] as well as by S. Moghimi-Araghi and S. Rouhani [25]. The results of the former three works on boundaries in the $c = -2$ case are all different, and partially contradictory. This demonstrates that the much more complicated representation theory of logarithmic conformal field theories poses a major obstacle to a rigorous and consistent description of boundary states in these cases. For example, they naturally contain zero-norm states which cannot be neglected from the spectrum. This and their non-trivial inner structure make it impossible to get a normalized orthogonal basis of states which usually is assumed in Ishibashi like constructions of boundary states.

In this paper, we modify the standard formalism *à la* Ishibashi in a suitable way to circumvent problems with non-normalizable states and their interpretation. Our modified approach allows to construct boundary states for logarithmic conformal field theories in a consistent way. The paper will proceed as follows:

In section two, we present an algorithm to directly calculate a complete basis of boundary states from first principles. It does not make use of any assumptions on normalizability or orthogonality of a basis of states. Each finite level of a boundary state is computed in a finite number of steps. As we will demonstrate, this algorithm is naturally adepted to work on zero-norm states and to keep the inner structure of indecomposable representations visible. Thus, our algorithm should be appropriate for applications towards generic logarithmic conformal field theories. Finally, we show that our method is completely compatible to Ishibashi's approach, yielding equivalent results for ordinary conformal field theories.

In the third section, we apply our method to the $c = -2$ logarithmic conformal field theory, the best understood example of this species. We concentrate on the $c = -2$ realization closest to the common notion of a rational theory, where the maximally extended chiral symmetry algebra is $\mathcal{W}(2, 3, 3, 3)$. We obtain a complete set of boundary states in one-to-one correspondence to the representations in this setting with respect to the extended symmetry algebra.

Section four is devoted to a detailed discussion of various properties of our boundary states. For instance, operators can be defined which relate boundary states corresponding to (irreducible) subrepresentations of indecomposable Jordan block representations to the boundary states associated to the full indecomposable representations. We comment on a possible relation between our set of boundary states and the structure of the unique local logarithmic $c = -2$ conformal field theory constructed by M. R. Gaberdiel and H. G. Kausch [17]. This is remarkable, since our set of boundary states shows a very similar structure with respect to the above-mentioned operators, although we did not start their construction from the local theory. We attempt to solve the problem of a degenerate metric of the natural pairing in the space of boundary states by introducing additional so-called weak boundary states which serve as duals of some of the proper boundary states.

In section five, we concentrate on a subset of four boundary states that corresponds to the three-dimensional space spanned by the characters of the rational $c = -2$ theory. This set is well-defined and the induced metric is non-degenerate. We show that Cardy's formalism can be applied in this situation precisely as for ordinary rational conformal field theories, and that it yields consistent results, despite the fact that the $\mathcal{S}$ matrix does not diagonalize the fusion rules in the

logarithmic conformal field theory case.
Section six repeats the above treatment for the case of the full space of boundary states. Surprisingly, Cardy's formalism still works to some extent. The partition functions are now related not to the physical characters, but to functions forming a five-dimensional representation of the modular group, which presumably can be interpreted as torus amplitudes [26]. Interestingly, Cardy's formalism fails at precisely the same point where a Verlinde formula like computation of fusion coefficients within the five-dimensional representation of the modular group breaks down. A way out seems to be a limiting procedure, which eliminates the weak boundary states introduced in section four. Unfortunately, the result of this limit is a bit ambiguous and its physical interpretation is not yet completely clear.
The paper concludes with a brief discussion, where we compare our results with earlier works emphasizing the inner consistency of our solution as well as the fact that it was derived without some of the commonly made assumptions. These assumptions, although true for ordinary rational conformal field theory, typically do not hold in the logarithmic case. Open questions and directions for future research are also discussed.

2 The method

In conformal field theories with boundaries, boundary conditions naturally can be applied. These are implemented by boundary states. Every such boundary state $\big|B\big\rangle$ to be compatible with conformal invariance has to fulfill the condition that was analyzed and given by N. Ishibashi, namely

$$\big(L_n - \overline{L}_{-n}\big)\,\big|B\big\rangle = 0. \tag{1}$$

This equation does not determine the boundary states completely because it can be read as an equation of motion for open string background in a closed string theory. Therefore, one usually analyzes boundary operators with respect to the maximally extended chiral symmetry algebra $\mathcal{W}$ with additional N fields $W^r,\ r = 1,\ldots,N$. The boundary state for this algebra has to obey in addition:

$$\big(W^r_n - (-1)^{s_r}\overline{W}^r_{-n}\big)\,\big|B\big\rangle = 0, \tag{2}$$

where s_r labels the spin of the field W^r. Let us denote a basis over a given bulk representation module $\mathcal{M}_h$ by

$$\big\{\big|l,n\big\rangle \,\big|\, l = h,\, h+1,\ldots;\, n = 1,\ldots,n_l\big\}, \tag{3}$$

where l counts the levels beginning from the highest weight h of the module and n labels a suitable basis on each level. The metric g_{mn} for this basis is given by the always well-defined and symmetric Shapovalov forms (see [15] for a prescription for logarithmic conformal field theories):

$$\delta_{ll'}g_{mn} \equiv \big\langle l,m\big|l',n\big\rangle \equiv \lim_{z\to\infty}\lim_{w\to 0} z^{2l}\big\langle\phi_{l,m}(z)\phi_{l',n}(w)\big\rangle. \tag{4}$$

Here, $\phi_{l,m}$ is the field corresponding to the state $\big|l,m\big\rangle$. In ordinary conformal field theories, the basis states are chosen orthonormalized, i. e. $g_{mn} \equiv \delta_{mn}$. However, to be more general we only demand a complete basis because the theory we especially consider contains zero-norm states. Thus, orthonormalization is not applicable. A boundary state can be written as a sum of product states $\big|l,m;\bar{l},n\big\rangle \equiv \big|l,m\big\rangle \otimes \overline{\big|\bar{l},n\big\rangle}$ of a holomorphic representation module $\mathcal{M}_h$ and a formal anti-holomorphic module $\overline{\mathcal{M}}_{\bar{h}}$:

$$\big|B\big\rangle = \sum_{l,m,\bar{l},n} c^{l\bar{l}}_{mn} \big|l,m\big\rangle \otimes \overline{\big|\bar{l},n\big\rangle} = \sum_{l,m,\bar{l},n} c^{l\bar{l}}_{mn} \big|l,m;\bar{l},n\big\rangle. \tag{5}$$

Note that we allow for $l \neq \bar{l}$ at this state. Of course, it will turn out that the solution contains only states of the type $\big|l,m;l,n\big\rangle$. Now consider equation (1). The modes L_n obey the Virasoro algebra:

$$[L_m, L_n] = (m-n)\, L_{m+n} - \frac{c}{12}\left(m^3 - m\right)\delta_{m+n,0}. \tag{6}$$

From this one finds for $n \neq 2$:

$$L_n = \frac{1}{n-2}\,[L_{n-1}, L_1] \quad \text{and} \quad L_{-n} = \frac{1}{2-n}\,[L_{1-n}, L_{-1}]\,. \tag{7}$$

Thus, it is enough to check condition (1) for $n = -2, \ldots, 2$ because the equation holds automatically for $|n| \geq 3$. For a boundary state that is built on two copies of the same representation module in its holomorphic and anti-holomorphic part this statement is equivalent to the requirement that (1) holds for $n = 0, 1, 2$ while the coefficients are chosen symmetrically in m and n, i. e. $c^{l\bar{l}}_{mn} = c^{l\bar{l}}_{nm}$.
Imagine that L_0 and $\overline{L}_0$ are given in Jordan form and let us decompose them into their diagonal part $\hat{h}$ and their off-diagonal part $\hat{\delta}$, such that $L_0 = \hat{h} + \hat{\delta}$ and $\overline{L}_0 = \hat{\bar{h}} + \hat{\bar{\delta}}$. Equation (1) then reads:

$$\begin{aligned} 0 &= \left(L_0 - \overline{L}_0\right)\big|B\big\rangle \\ &= \left(\hat{h} - \hat{\bar{h}} + \hat{\delta} - \hat{\bar{\delta}}\right) \sum_{l,m,\bar{l},n} c^{l\bar{l}}_{mn} \big|l,m;\bar{l},n\big\rangle \\ &= \sum_{l,m,\bar{l},n} c^{l\bar{l}}_{mn} \left(l - \bar{l} + \hat{\delta} - \hat{\bar{\delta}}\right)\big|l,m;\bar{l},n\big\rangle. \end{aligned} \tag{8}$$

Since the basis states $\big|l,m;\bar{l},n\big\rangle$ are linearly independent, it hereby follows that the off-diagonal part $(\hat{\delta} - \hat{\bar{\delta}})\big|B\big\rangle$ has to vanish and, on the other hand, $l = \bar{l}$:

$$\big|B\big\rangle = \sum_{l,m,n} c^{l}_{mn} \big|l,m;l,n\big\rangle. \tag{9}$$

This is not quite unexpected. It is part of the result given by N. Ishibashi for ordinary conformal field theories. In the same manner the treatment of the $n = 1$ case results in the following equations:

$$
\begin{aligned}
0 &= \left(L_1 - \overline{L}_{-1}\right) \big|B\big\rangle \\
&= \left(L_1 - \overline{L}_{-1}\right) \sum_{l,m,n} c^l_{mn} \big|l, m; l, n\big\rangle \\
&= \sum_{l,m,n} c^l_{mn} \Big(\sum_a \alpha^{l\,m}{}_a \big|l-1, a; l, n\big\rangle - \sum_b \beta^{l\,n}{}_b \big|l, m; l+1, b\big\rangle \Big) \\
&= \sum_{l,m,n} \left(\alpha^{l\,a}{}_m c^l_{an} - c^{l-1}_{mb} \beta^{l-1\,b}{}_n \right) \big|l-1, m; l, n\big\rangle .
\end{aligned}
\tag{10}
$$

Here, α and β denote the coefficients in the expansion of $L_1\big|l, m\big\rangle$ and $\overline{L_{-1}\big|l, n\big\rangle}$ with respect to the basis $\big|l-1, a\big\rangle$ and $\overline{\big|l+1, b\big\rangle}$:

$$
L_1\big|l, m\big\rangle = \alpha^{l\,m}{}_a \big|l-1, a\big\rangle \quad \text{and} \quad \overline{L_{-1}\big|l, n\big\rangle} = \beta^{l\,n}{}_b \overline{\big|l+1, b\big\rangle}. \tag{11}
$$

We used Einstein's summing convention for the indices a and b in the last line. Because zero can only be created trivially out of the basis states all coefficients have to vanish identically, level by level:

$$
\alpha^{l\,a}{}_m c^l_{an} - c^{l-1}_{mb} \beta^{l-1\,b}{}_n = 0. \tag{12}
$$

For $n = 2$ it follows analogously that

$$
\varrho^{l\,a}{}_m c^l_{an} - c^{l-2}_{mb} \sigma^{l-2\,b}{}_n = 0. \tag{13}
$$

Again, we introduced the expansion coefficients ϱ and σ for the states $L_2\big|l, m\big\rangle$ and $\overline{L_{-2}\big|l, n\big\rangle}$. The condition (2) for the extended symmetry algebra $\mathcal{W}$ has to be treated in the same way. These equations reduce to a finite set as well. The additional fields in the extended chiral symmetry algebra are primary with respect to the energy-momentum tensor. Let h be the conformal weight of the field W^r:

$$
\left[L_n, W^r_m\right] = \left((h-1)m - n\right) W^r_{m+n}. \tag{14}
$$

From this we learn:

$$
W^a_n = \begin{cases} \frac{1}{(h-1)n} \left[L_n, W^r_0\right] & (n \neq 0,\ h \neq 1) \\ \left[L_{n-1}, W^r_1\right] & (h = 1) \end{cases} . \tag{15}
$$

Therefore, it is enough to check (2) for $n = 0$ (or $n = 0, \pm 1$ if $h = 1$), since the remaining conditions are treated implicitly with the help of (1):

$$
\left(W^r_0 - (-1)^{s_r} \overline{W}^r_0\right) \big|B\big\rangle = 0,\ r = 1, \ldots, N. \tag{16}
$$

This leads to additional $N + 2k$ equations, where N is the total number of fields additional to the Virasoro field and k is the number of fields of conformal weight 1 among these. In particular, in the special case of the $\mathcal{W}(2,3,3,3)$ algebra in the $c = -2$ theory the three additional fields W^a are spin-3 fields and equation (16) reduces to the three equations

$$\left(W_0^a + \overline{W}_0^a\right) \big|B\big\rangle = 0, \tag{17}$$

where a is the spinor index of $su(2)$ and takes three different values. By solving all the conditions for the coefficients c_{mn}^l we are able to find a complete basis of boundary states. We state that after having derived the coefficients for the first three levels without any inconsistencies, there are no contradictions occurring on higher levels. One can be sure that the state really exists in the way that it is a well-defined solution of (1) and (2). Indeed, this is the case for the $c = -2$ rational logarithmic conformal field theory considered here and we will show the existence of the states we derive explicitly later on. In particular, this allows to calculate the coefficients c_{mn}^l for any arbitrary given finite level l with a finite number of steps. Given an arbitrary set of representation modules $\mathcal{M}_h^1,\ m = 1, \ldots, n$ that build a Jordan block of rank n in the L_0 mode, the number of boundary states built on $\mathcal{M}_h^1 \otimes \overline{\mathcal{M}}_h^1$ derived by our method is given by n. One state is built on the whole representation module and one each for the subrepresentations, based only on the product states of these subrepresentations, respectively. This is seen very easily, namely let $\mathcal{M}_h^1$ be a subrepresentation in $\mathcal{M}_h^1$ that behaves as a true highest weight representation if taken for itself. Acting with annihilators on the states in $\mathcal{M}_h^m$ will never lead to dependencies in the equations (1) and (2).
It is obvious that in ordinary conformal field theories this formalism reproduces the usual Ishibashi results: Let $\{\big|l,m\big\rangle\}$ be an orthonormal basis of an irreducible module $\mathcal{V}$. The corresponding Ishibashi boundary state reads:

$$\big|V\big\rangle = \sum_{l,m} \big|l,m\big\rangle \otimes \overline{\mathcal{U}\big|l,m\big\rangle}. \tag{18}$$

$\mathcal{U}$ is an anti-unitary operation with the property that it acts on the modes of the extended chiral algebra as $W_n^r \mathcal{U} = (-1)^{s_r} \mathcal{U} W_n^r$ and commutes with the Virasoro modes. Since the state given in (18) satisfies the two equations (1) and (2), it has to fulfill the coefficient equations (8), (10), and (13) as well as the corresponding ones for the modes of the extended algebra (17). This means that we can construct $\big|V\big\rangle$ by applying our formalism. The difference is that we do not make explicit use of the anti-unitary operation $\mathcal{U}$. By splitting off $\mathcal{U}$ by hand afterwards, we reproduce exactly the well-known results of ordinary conformal field theories. On the other hand we introduced a generalized procedure that allows us to take care of e. g. zero-norm states in logarithmic conformal field theories. This generalization does not make any use of the properties of the states themselves other than the expansion with respect to an arbitrary but fixed basis.

3 Boundary states in $c = -2$ rational logarithmic conformal field theory

Here, we apply the method introduced in the previous section to the rational $c = -2$ logarithmic conformal field theory. M. R. Gaberdiel and H. G. Kausch spent a lot of work in the analysis of this (bulk) theory in [13, 14, 17, 27]. F. Rohsiepe examined the physical characters of the representation modules that form the three-dimensional representation of the modular group [15]. The torus amplitudes on the other hand seem to be related to a five-dimensional representation [26] that was analyzed in [16]. The latter representation contains the smaller one as a subrepresentation. The theory contains a $\mathcal{W}(2,3,3,3)$ triplet algebra which is generated by the Virasoro modes L_n and the modes W^a_n of a triplet of spin-3 fields. With the help of two quasi-primary normal ordered fields $\Lambda = :L^2: -3/10\,\partial^2 L$ and $V^a = :LW^a: -3/14\,\partial^2 W^a$ the commutation relations read:

$$\begin{aligned}
\left[L_m, L_n\right] =& (m-n)L_{m+n} - \frac{1}{6}\left(m^3 - m\right)\delta_{m+n,0},\\
\left[L_m, W^a_n\right] =& (2m-n)W^a_{m+n},\\
\left[W^a_m, W^b_n\right] =& g^{ab}\Bigg(2\left(m-n\right)\Lambda_{m+n} + \frac{1}{20}\left(m-n\right)\left(2m^2+2n^2-mn-8\right)L_{m+n}\\
&\qquad -\frac{1}{120}m\left(m^2-1\right)\left(m^2-4\right)\delta_{m+n,0}\Bigg)\\
&+ f^{ab}_c\left(\frac{5}{14}\left(2m^2+2n^2-3mn-4\right)W^c_{m+n} + \frac{12}{5}V^c_{m+n}\right).
\end{aligned} \tag{19}$$

Here, g^{ab} is the metric and f^{ab}_c are the structure constants of $su(2)$. For our further discussion it is suitable to choose a Cartan-Weyl basis for $su(2)$ which reads W^0, $W^\pm$, such that the metric is given by $g^{00} = 1$, $g^{+-} = g^{-+} = 2$ and the non-vanishing structure constants read $f^{0\pm}_\pm = -f^{\pm 0}_\pm = \pm 1$ and $f^{+-}_0 = -f^{-+}_0 = 2$. Commutators involving the operators $\hat{\delta}$ and $\hat{h}$ read

$$\left[\hat{h}, \mathcal{O}_n\right] = \left[L_0, \mathcal{O}_n\right] \quad \text{and} \quad \left[\hat{\delta}, \mathcal{O}_n\right] = 0, \tag{20}$$

where $\mathcal{O}_n$ is any mode of the algebra. The theory contains four irreducible representations, two singlet representations, namely the vacuum representation $\mathcal{V}_0$ and $\mathcal{V}_{-1/8}$ with highest weight states Ω at $h = 0$ and μ at $h = -1/8$, respectively, and two doublet representations $\mathcal{V}_1$ and $\mathcal{V}_{3/8}$ with highest weight states at $h = 1$ and $h = 3/8$. Furthermore there exist two reducible but indecomposable representations: $\mathcal{R}_0$ is generated by a cyclic state ω at level 0 that builds a Jordan block in L_0 together with the vacuum highest weight state Ω of $\mathcal{V}_0$ and $\mathcal{R}_1$ is generated by a doublet of level 1 cyclic states $\psi^\pm$ that form Jordan blocks together with the highest weight states $\phi^\pm$ of the representation $\mathcal{V}_1$. These cyclic states have the property that they themselves are no highest weight states, i. e. $\xi^\pm \equiv -\frac{1}{2}L_1\psi^\pm$

is not zero. The representations $\mathcal{V}_0$ and $\mathcal{V}_1$ are subrepresentations of the modules $\mathcal{R}_0$ and $\mathcal{R}_1$, respectively. Due to the fact that the highest occurring weight in both of these indecomposable representations is $h = 0$, i. e. their spectra are bounded from below as in irreducible representations, these representations are also called generalized highest weight representations. It follows that the states in the two (sub-)representations $\mathcal{V}_0$ and $\mathcal{V}_1$ are zero-norm states [15]. Furthermore, $\mathcal{R}_0$ contains two subrepresentations of type $\mathcal{V}_1$ built on the two doublet states $\Psi_1^\pm$ and $\Psi_2^\pm$ (in this we follow the conventions of [14]):

$$\begin{aligned} \Psi_1^+ &= W_{-1}^+\omega, & \Psi_2^+ &= \left(W_{-1}^0 + \tfrac{1}{2}L_{-1}\right)\omega, \\ \Psi_1^- &= \left(-W_{-1}^0 + \tfrac{1}{2}L_{-1}\right)\omega, & \Psi_2^- &= W_{-1}^-\omega. \end{aligned} \tag{21}$$

The structure of the indecomposable modules $\mathcal{R}_0$ and $\mathcal{R}_1$ can be drawn schematically:

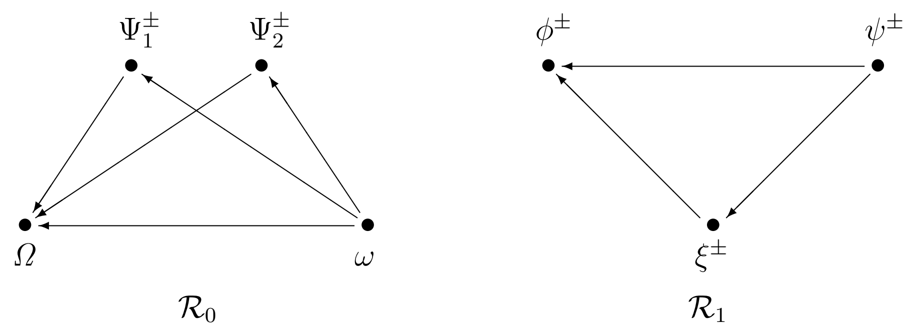

figure 1: Generalised highest weight modules $\mathcal{R}_0$ and $\mathcal{R}_1$

The points in figure 1 refer to the states on which the different (sub-)representations are built and the lines denote the action of the $\mathcal{W}$ algebra. Y. Ishimoto showed that we can choose the metric in these two representations to be given by [23]:

$$\begin{aligned} \langle \Omega|\Omega\rangle &= 0, & \langle \Omega|\omega\rangle &= 1, & \langle \omega|\omega\rangle &= d, \\ \langle \phi^+|\phi^-\rangle &= 0, & \langle \phi^+|\psi^-\rangle &= -1, & \langle \psi^+|\psi^-\rangle &= -t. \end{aligned} \tag{22}$$

Here, d and t are in principal arbitrary real numbers. This fixes the metric completely. Applying the previously introduced method we find ten boundary states:

- A single state $|V_{-1/8}\rangle$ for the pairing $\mathcal{V}_{-1/8} \otimes \overline{\mathcal{V}_{-1/8}}$
- Another state $|V_{3/8}\rangle$ for $\mathcal{V}_{3/8} \otimes \overline{\mathcal{V}_{3/8}}$

These states are the usual Ishibashi states for $\mathcal{V}_{-1/8}$ and $\mathcal{V}_{3/8}$. The situation is different for states built on $\mathcal{R}_0 \otimes \overline{\mathcal{R}_0}$. We find two independent coefficients $c_{\Omega\Omega} \equiv c^0_{\Omega\Omega}$ and $c_{\Omega\omega} \equiv c^0_{\Omega\omega}$ and hence two different solutions. This is exactly what we stated above. There exists one state for the complete module $\mathcal{R}_0$ and one for the submodule $\mathcal{V}_0$:

$$
\begin{aligned}
\big|R_0\big\rangle &= \big|c_{\Omega\Omega} = -d,\ c_{\Omega\omega} = 1\big\rangle, \\
\big|V_0\big\rangle &= \big|c_{\Omega\Omega} = 1,\ c_{\Omega\omega} = 0\big\rangle.
\end{aligned}
\tag{23}
$$

On the right hand side, the boundary states are defined via the given choice of the two free parameters, fixing all other coefficients in (9). In $\big|R_0\big\rangle$ we choose the parameter $c_{\Omega\Omega} = -d$ to be convenient for our further discussion. Remember that d is a structure constant fixed in the metric (22). Analogously, for $\mathcal{R}_1 \otimes \overline{\mathcal{R}_1}$ one derives:

$$
\begin{aligned}
\big|R_1\big\rangle &= \big|c_{\xi^+\xi^-} = -t,\ c_{\phi^+\psi^-} = 1\big\rangle, \\
\big|V_1\big\rangle &= \big|c_{\xi^+\xi^-} = 1,\ c_{\phi^+\psi^-} = 0\big\rangle,
\end{aligned}
\tag{24}
$$

where again $c_{\xi^+\xi^-} = -t$ is chosen for convenience. Note that here, the coefficients are antisymmetric with respect to interchanging the $su(2)$-spin indices.
Let us expand the states $\big|R_0\big\rangle$ and $\big|R_1\big\rangle$ into the sums (9) and re-introduce the usual anti-unitary operation $\mathcal{U}$ which acts as $W^{\pm}_n\mathcal{U} = -\mathcal{U}W^{\mp}_n$ and $W^0_n\mathcal{U} = -\mathcal{U}W^0_n$. Finally, we introduce a coefficient matrix γ that is implicitly defined by $\gamma \cdot (1 \otimes \overline{\mathcal{U}}) \equiv c$. The two states take the following explicitly written out form:

$$
\big|R_\lambda\big\rangle = \sum_{l,m,n} \gamma^{\lambda\,l}_{\ \ mn} \big|l,m\big\rangle \otimes \overline{\mathcal{U}\big|l,n\big\rangle}, \quad \lambda = 0, 1. \tag{25}
$$

It turns out that for $\lambda = 0$, 1 the coefficient matrices γ^λ are given by the inverse metrics on the corresponding representations $\mathcal{R}_\lambda$. In this sense, the states $\big|R_\lambda\big\rangle$ can be called *generalized* Ishibashi states. They are defined free of contradictions, since:

$$
\begin{aligned}
0 &= \big(\mathcal{O}_n \pm \overline{\mathcal{O}}_{-n}\big)\,\big|R_\lambda\big\rangle \\
&= \big\langle l_1, a\big| \otimes \overline{\mathcal{U}\big\langle l_1, b\big|}\,\big(\mathcal{O}_n \pm \overline{\mathcal{O}}_{-n}\big) \sum_{l,m,n} \gamma^{\lambda\,l}_{\ \ mn} \big|l,m\big\rangle \otimes \overline{\mathcal{U}\big|l,n\big\rangle} \\
&= \sum_{l,m,n} \Big(\big\langle l_1, a\big|\mathcal{O}_n \gamma^{\lambda\,l}_{\ \ mn}\big|l,m\big\rangle\big\langle l,m\big|l_2,b\big\rangle - \big\langle l_1,a\big|\gamma^{\lambda\,l}_{\ \ mn}\big|l,m\big\rangle\big\langle l,m\big|\mathcal{O}_n\big|l_2,b\big\rangle\Big) \\
&= \big\langle l_1, a\big|\,\big[\mathcal{O}_n, \mathbb{1}^\lambda\big]\,\big|l_2, b\big\rangle
\end{aligned}
\tag{26}
$$

for the modes $\mathcal{O}_n$ of the chiral algebra. The operator $\mathbb{1}^\lambda$ defined by

$$
\mathbb{1}^\lambda \equiv \sum_{l,m,n} \gamma^{\lambda\,l}_{\ \ mn} \big|l,m\big\rangle\!\big\langle l,n\big| \tag{27}
$$

is the projector onto the representation module $\mathcal{R}_\lambda$. Indeed,

$$\begin{aligned}
\left(\mathbb{1}^\lambda\right)^2 &= \sum_{l,m,n}\sum_{k,a,b} \big|l,m\big\rangle \gamma^{\lambda\,l}_{\ mn} \underbrace{\big\langle l,n\big|k,a\big\rangle}_{\delta_{lk}g_{na}} \gamma^{\lambda\,k}_{\ ab}\big\langle k,b\big| \\
&= \sum_{l,m,n}\sum_{k,a,b} \big|l,m\big\rangle \delta_{lk}\delta_{ma}\gamma^{\lambda\,k}_{\ ab}\big\langle k,b\big| \\
&= \sum_{l,m,b} \gamma^{\lambda\,l}_{\ mb}\big|l,m\big\rangle\!\big\langle l,b\big| \;=\; \mathbb{1}^\lambda.
\end{aligned} \tag{28}$$

Hence, it commutes with the action of the algebra.
In ordinary conformal field theories there are no boundary states based on product states of different representations in their holomorphic and anti-holomorphic part because the weights of two different representations are usually disjunct sets. We have the representations $\mathcal{R}_0$ and $\mathcal{R}_1$ which contain the same weights and even their characters are equal [14]. Indeed, we find another two doublets of boundary states for the combinations $\mathcal{R}_0 \otimes \overline{\mathcal{R}_1}$ and $\mathcal{R}_1 \otimes \overline{\mathcal{R}_0}$:

$$\begin{aligned}
\big|R^{\pm}_{01}\big\rangle &= \big|c_{\Omega\xi^\pm} = 1\big\rangle, \\
\big|R^{\pm}_{10}\big\rangle &= \big|c_{\xi^\pm\Omega} = 1\big\rangle.
\end{aligned} \tag{29}$$

To summarize the results, we could identify six solutions for the boundary states that have a one-to-one correspondence to the representation modules and two doublet solutions that relate the two generalized highest weight representations to each other. These span the space of all possible boundary states in this theory.

4 Properties of the solution

This section deals with the analysis of the properties of the states derived in the last paragraph. A common problem in the treatment of boundary states is that scalar products naturally diverge unless they are equal to zero because these states are infinite sums over tensor products of bulk states possessing finite scalar products. Let us introduce the operator

$$\hat{\mathcal{N}} \equiv \hat{\delta} + \hat{\bar{\delta}}. \tag{30}$$

Recall that we have $L_0 = \hat{h} + \hat{\delta}$ and that in the $c = -2$ theory L_0 has Jordan blocks of dimension 2 at maximum for the $\mathcal{R}_0$ and $\mathcal{R}_1$ representations. Thus, $\hat{\delta}^2 = 0$, and it follows that the operator $\hat{\mathcal{N}}$ is nilpotent of degree three: $\hat{\mathcal{N}}^3 = 0$. Given this operator it is clear by use of (20) that if $\big|B\big\rangle$ is a boundary state then $\hat{\mathcal{N}}\big|B\big\rangle$ either vanishes or is a boundary state itself. We find that

$$\big|V_\lambda\big\rangle = \frac{1}{2}\hat{\mathcal{N}}\big|R_\lambda\big\rangle = -\partial\big|R_\lambda\big\rangle, \quad \lambda{=}0,1, \tag{31}$$

where $\partial \equiv \partial_d$ if acting on $\big|R_0\big\rangle$ and $\partial \equiv -\partial_t$ if acting on $\big|R_1\big\rangle$. This shows that the two states $\big|V_\lambda\big\rangle$ are well-defined. The structure is very similar to the bulk theory:

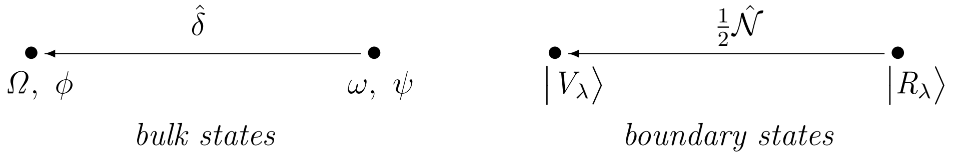

figure 2: Equivalence of $\hat{\delta}$ and $\hat{\mathcal{N}}$

Every boundary state $\big|B\big\rangle$ satisfies

$$\hat{\mathcal{N}}^2\big|B\big\rangle = \partial^2\big|B\big\rangle = 0. \tag{32}$$

This can be seen if one remembers equation (8) where we showed that a boundary state $\big|B\big\rangle$ satisfies $(\hat{\delta} - \hat{\bar{\delta}})\big|B\big\rangle = 0$. Using the nilpotency of $\hat{\delta}$ and $\hat{\bar{\delta}}$ one gains

$$0 = (\hat{\delta} - \hat{\bar{\delta}})^2\big|B\big\rangle = -2\hat{\delta}\hat{\bar{\delta}}\big|B\big\rangle = -\hat{\mathcal{N}}^2\big|B\big\rangle. \tag{33}$$

Every partition function in a theory with boundaries can be written as a linear sum of the Virasoro characters $\chi_i(q) = q^{-c/24}\,\mathrm{tr}_i\, q^{L_0}$ of its representations i. For example, the partition function of a theory with two boundaries and boundary conditions α and β, respectively, decomposes as

$$Z_{\alpha\beta}(q) = \sum_i n^i_{\alpha\beta}\chi_i(q). \tag{34}$$

Since we consider the conformal theory living on a torus, introducing boundary conditions means breaking up the torus into a cylinder and thus, there are two boundaries. With respect to the so called duality condition a physical boundary partition function $Z_{\alpha\beta}$ for given boundary conditions α and β can be written equivalently as:

$$Z_{\alpha\beta}(q) = \big\langle\alpha\big|\tilde{q}^{\frac{1}{2}\left(L_0+\overline{L}_0-\frac{c}{12}\right)}\big|\beta\big\rangle, \tag{35}$$

where $\tilde{q} \equiv \mathrm{e}^{-2\pi\mathrm{i}/\tau}$, τ being the torus parameter. Let us introduce another operator $\hat{q}$, remembering $c = -2$:

$$\hat{q} \equiv q^{\frac{1}{2}\left(L_0+\overline{L}_0+\frac{1}{6}\right)} = q^{\frac{1}{2}\left(\hat{h}+\hat{\bar{h}}+\frac{1}{6}\right)}\left[1 + \log(q)\cdot\frac{1}{2}\hat{\mathcal{N}} + \log(q)^2\cdot\frac{1}{4}\hat{\mathcal{N}}^2\right]. \tag{36}$$

Here, $q \equiv \mathrm{e}^{2\pi\mathrm{i}\tau}$. To verify the last equality one should remember the nilpotency property of $\hat{\mathcal{N}}$ and use $L_0 + \overline{L}_0 = \hat{h} + \hat{\bar{h}} + \hat{\mathcal{N}}$. By equation (32) this implies that pairings $\big\langle B\big|\hat{q}\big|C\big\rangle$ of boundary states can contain logarithmic terms proportional of order one at maximum, but never of higher order. This is not surprising, since usually, these pairings reproduce the torus amplitudes or equivalently, the characters. In ordinary conformal field theories the torus amplitudes and the characters span exactly the same representation of the modular group. We have two different representations instead, a three-dimensional one for the physical characters and a five-dimensional one presumably for the torus amplitudes including the smaller one. Their properties were examined in [13, 14, 16, 17]. The main result is that there exists elements of order $\log(q)^1$ in the latter representation but neither of them contains $\log(q)^2$ or higher order terms. It will turn out that we have to take into account states that are no boundary states to reproduce the five-dimensional representation. However, pairings of these additional states with the boundary states can contain logarithmic terms of order one at maximum as well. Let us calculate the pairings of the boundary states first, i. e. the cylinder amplitudes. They read as follows:

$$\begin{aligned} \left\langle V_{-1/8}\middle|\hat{q}\middle| V_{-1/8}\right\rangle &= \chi_{\mathcal{V}_{-1/8}}(q), \qquad & \left\langle V_{3/8}\middle|\hat{q}\middle| V_{3/8}\right\rangle &= \chi_{\mathcal{V}_{3/8}}(q), \\ \left\langle R_0\middle|\hat{q}\middle|R_0\right\rangle &= \chi_{\mathcal{R}}(q), & \left\langle R_1\middle|\hat{q}\middle|R_1\right\rangle &= \chi_{\mathcal{R}}(q). \end{aligned} \tag{37}$$

All other combinations vanish. Particularly, the six states $\left| V_0 \right\rangle$, $\left| V_1 \right\rangle$, $\left|R_{01}^{\pm}\right\rangle$, and $\left|R_{10}^{\pm}\right\rangle$ are null states with respect to the space spanned by the set of boundary states. The characters $\chi_i(q)$ were analyzed in [13, 14, 15]:

$$\begin{aligned} \chi_{\mathcal{V}_0}(q) &= \frac{1}{2\eta(q)}\left(\Theta_{1,2}(q) + (\partial\Theta)_{1,2}(q)\right), \\ \chi_{\mathcal{V}_1}(q) &= \frac{1}{2\eta(q)}\left(\Theta_{1,2}(q) - (\partial\Theta)_{1,2}(q)\right), \\ \chi_{\mathcal{V}_{-1/8}}(q) &= \frac{1}{\eta(q)}\Theta_{0,2}(q), \\ \chi_{\mathcal{V}_{3/8}}(q) &= \frac{1}{\eta(q)}\Theta_{2,2}(q), \\ \chi_{\mathcal{R}}(q) &\equiv \chi_{\mathcal{R}_0}(q) = \chi_{\mathcal{R}_1}(q) = \frac{2}{\eta(q)}\Theta_{1,2}(q). \end{aligned} \tag{38}$$

Here, $\eta(q) = q^{1/24}\prod_{n\in N}(1-q^n)$ is the Dedekind eta function and $\Theta_{r,2}(q)$ and $(\partial\Theta)_{1,2}(q) = \eta(q)^3$ are the ordinary and affine Riemann-Jacobi theta functions:

$$\begin{aligned} \Theta_{r,k}(q) &= \sum_{n\in Z} q^{(2kn+r)^2/4k}, \\ (\partial\Theta)_{r,k}(q) &= \sum_{n\in Z}(2kn+r)\, q^{(2kn+r)^2/4k}, \\ (\nabla\Theta)_{r,k}(q) &= \frac{1}{2\pi}\log(q)\,(\partial\Theta)_{r,k}(q) = \mathrm{i}\tau\,(\partial\Theta)_{r,k}(q). \end{aligned} \tag{39}$$

We also state the characters for $\mathcal{V}_0$ and $\mathcal{V}_1$ here as well as the logarithmic theta function. They will be needed further on. One learns that the boundary states reproduce the three-dimensional representation of the modular group, i. e. the physical characters. Due to the fact that most of the derived boundary states are null states the properties related to the inner structure of the indecomposable representations are not visible at this state. Therefore, we have to study the boundary states a bit more in detail. In the following we especially focus on the structural relations of the states to each other. Remember figure 2 where we showed the similarity of $\hat{\delta}$ in the bulk theory and $\hat{\mathcal{N}}$ for boundary states: Since $\hat{\mathcal{N}}$ has nilpotency degree three, the question arises if it is possible to construct a state which would not be a boundary state, such that the boundary state $\big|R_0\big\rangle$ is the image of this state under the action of $\hat{\mathcal{N}}$, and the same for $\big|R_1\big\rangle$. Unfortunately, it is only possible to find two states $\big|X_\lambda\big\rangle$ and $\big|Y_\lambda\big\rangle$ such that

$$\big|R_\lambda\big\rangle = \hat{\mathcal{N}}\big|X_\lambda\big\rangle + \big|Y_\lambda\big\rangle \quad \text{and} \quad \big|V_\lambda\big\rangle = \frac{1}{2}\hat{\mathcal{N}}\big|R_\lambda\big\rangle = \frac{1}{2}\hat{\mathcal{N}}^2\big|X_\lambda\big\rangle \quad (\lambda = 0,\, 1). \tag{40}$$

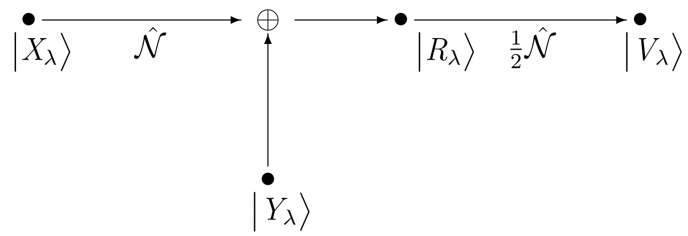

figure 3: weak boundary states

The choice of the states $\big|X_\lambda\big\rangle$ and $\big|Y_\lambda\big\rangle$ is not unique. It is possible to add states belonging to the kernel of $\hat{\mathcal{N}}$ to $\big|X_\lambda\big\rangle$ without changing anything as well as one could subtract states from $\big|X_\lambda\big\rangle$ that belong to the kernel of $\hat{\mathcal{N}}^2$ and add their images under the $\hat{\mathcal{N}}$-operation to $\big|Y_\lambda\big\rangle$. $\big|X_\lambda\big\rangle$ and $\big|Y_\lambda\big\rangle$ generate the boundary states and one can therefore call them *weak boundary states*. This is justified by looking at their scalar products with the original boundary states. $\big|X_\lambda\big\rangle$ and $\big|Y_\lambda\big\rangle$ can be chosen uniquely in such a way that:

$$\begin{aligned}
\big\langle X_\lambda\big|\hat{q}\big|V_\lambda\big\rangle &= \chi_{\mathcal{V}_\lambda}(q), \qquad & \big\langle X_\lambda\big|\hat{q}\big|R_\lambda\big\rangle &= \log(q)\cdot\chi_{\mathcal{V}_\lambda}(q),\\
\big\langle X_\lambda\big|\hat{q}\big|Y_\lambda\big\rangle &= 0, & \big\langle Y_\lambda\big|\hat{q}\big|R_\lambda\big\rangle &= \chi_{\mathcal{R}}(q) - 2\chi_{\mathcal{V}_\lambda}(q),\\
\big\langle R_\lambda\big|\hat{q}\big|R_\lambda\big\rangle &= \chi_{\mathcal{R}}(q). & &
\end{aligned} \tag{41}$$

We learn that the set of boundary states together with the two states $\big|X_\lambda\big\rangle$ reproduces the elements of the five-dimensional representation of the modular group. Unfortunately, there are terms proportional to $\log(q)\Theta_{1,2}(q)$ as well which are not physical and do not belong to the representation. Luckily, they occur in such a way that they are suppressed in certain linear combinations of the boundary states. At this state the question remains unanswered in which way the additional states $\big|X_\lambda\big\rangle$ and $\big|Y_\lambda\big\rangle$ can physically be interpreted. Before moving on to this topic we first want to concentrate on the states $\big|R_{01}^\pm\big\rangle$ and $\big|R_{10}^\pm\big\rangle$ that relate the two generalized highest weight representations to each other. Similarly to the definition of the states $\big|X_\lambda\big\rangle$ there exists states $\big|Z_{01}^\pm\big\rangle$ and $\big|Z_{10}^\pm\big\rangle$ in such a way that:

$$\big\langle Z_{01}^\pm\big|\hat{q}\big|R_{01}^\pm\big\rangle = \big\langle Z_{10}^\pm\big|\hat{q}\big|R_{10}^\pm\big\rangle = \frac{1}{2}\chi_{\mathcal{R}}(q). \tag{42}$$

These states have the property $\hat{\mathcal{N}}\big|Z_{mn}^\pm\big\rangle = \big|R_{mn}^\pm\big\rangle$, $m = 1-n$ and fulfill $\hat{\mathcal{N}}^2\big|Z_{mn}^\pm\big\rangle = 0$. They can be interpreted as *weak boundary states* as well.
Boundary states are strongly correlated to propagators that connect the holomorphic part (e. g. the upper half complex plane, in a very simple setting) to the formal anti-holomorphic one (the lower half plane):

$$\begin{aligned}\big|R_{01}^\pm\big\rangle &= \sum_{l,m,n} c_{mn}^l\big|l,m\big\rangle \otimes \overline{\big|l,n\big\rangle} \;\;\Leftrightarrow\;\; \hat{\mathcal{P}}_\pm\mathcal{U}^\dagger \equiv \sum_{l,m,n} c_{mn}^l\big|l,m\big\rangle\big\langle l,n\big| \quad\text{and}\\ \big|R_{10}^\pm\big\rangle &= \sum_{l,m,n} c_{mn}^l\big|l,n\big\rangle \otimes \overline{\big|l,m\big\rangle} \;\;\Leftrightarrow\;\; \hat{\mathcal{P}}_\pm^\dagger\mathcal{U}^\dagger \equiv \sum_{l,m,n} c_{mn}^l\big|l,n\big\rangle\big\langle l,m\big|.\end{aligned} \tag{43}$$

Here, $\mathcal{U}$ is the anti-unitary operator already introduced above. Because the corresponding boundary states satisfy the Ishibashi equations (1) and (2) the operators $\hat{\mathcal{P}}_\pm$ and $\hat{\mathcal{P}}_\pm^\dagger$ commute with the action of the chiral algebra:

$$\begin{aligned}0 &= \big\langle l_1,a\big| \otimes \overline{\big\langle l_2,b\big|}\,\big(L_n - \overline{L}_{-n}\big)\,\big|R_{01}\big\rangle\\ &= \sum_{l,r,s}\big\langle l_1,a\big| \otimes \overline{\big\langle l_2,b\big|}\,\big(L_n - \overline{L}_{-n}\big)\,c_{rs}^l\big|l,r\big\rangle \otimes \overline{\big|l,s\big\rangle}\\ &= \sum_{l,r,s} c_{rs}^l\Big\{\big\langle l_1,a\big|L_n\big|l,r\big\rangle\big\langle l,s\big|l_2,b\big\rangle - \big\langle l_1,a\big|l,r\big\rangle\big\langle l,s\big|L_n\big|l_2,b\big\rangle\Big\}\\ &= \big\langle l_1,a\big|\big[L_n,\hat{\mathcal{P}}\mathcal{U}^\dagger\big]\big|l_2,b\big\rangle\\ &= \big\langle l_1,a\big|\big[L_n,\hat{\mathcal{P}}\big]\mathcal{U}^\dagger\big|l_2,b\big\rangle.\end{aligned} \tag{44}$$

We learn that $\hat{\mathcal{P}}$ commutes with the Virasoro modes. Analogously one can show that $\hat{\mathcal{P}}$ commutes with the modes W_n^a and thus, the statement is proved. This means that given a boundary state $\big|B\big\rangle$ the state $\hat{\mathcal{P}}\big|B\big\rangle$ is again a boundary state or equal to zero. It turns out that the action of the operators $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}^\dagger$ on the bulk states in the representation modules $\mathcal{R}_0$ and $\mathcal{R}_1$ is given by:

$$\begin{aligned}&\hat{\mathcal{P}}_\pm^\dagger\big|\omega\big\rangle = \big|\xi^\pm\big\rangle, &&\hat{\mathcal{P}}_\pm^\dagger\big|\Omega\big\rangle = 0, &&\hat{\mathcal{P}}_+\big|\psi^\pm\big\rangle = -\big|\Psi_2^\pm\big\rangle,\\ &\hat{\mathcal{P}}_\pm\big|\xi^\mp\big\rangle = \pm\big|\Omega\big\rangle, &&\hat{\mathcal{P}}_\pm\big|\phi\big\rangle = 0, &&\hat{\mathcal{P}}_-\big|\psi^\pm\big\rangle = \big|\Psi_1^\pm\big\rangle.\end{aligned} \tag{45}$$

Especially, these operators decompose the off-diagonal part $\hat{\delta}$ of L_0:

$$\hat{\delta} = \begin{cases} \hat{\mathcal{P}}\hat{\mathcal{P}}^\dagger & \text{on} \quad \mathcal{R}_0 \\ \hat{\mathcal{P}}^\dagger\hat{\mathcal{P}} & \text{on} \quad \mathcal{R}_1 \end{cases} . \tag{46}$$

By use of this equality it is easy to agree on the existence of the *mixed states*:

$$\begin{aligned} \big|R_{01}^\pm\big\rangle &= \hat{\mathcal{P}}_\pm\big|R_1\big\rangle = \hat{\overline{\mathcal{P}}}{}^\dagger_\pm\big|R_0\big\rangle, & \big|V_0\big\rangle &= \hat{\overline{\mathcal{P}}}_\mp\big|R_{01}^\pm\big\rangle = \hat{\mathcal{P}}_\mp\big|R_{10}^\pm\big\rangle, \\ \big|R_{10}^\pm\big\rangle &= \hat{\overline{\mathcal{P}}}_\pm\big|R_1\big\rangle = \hat{\mathcal{P}}^\dagger_\pm\big|R_0\big\rangle, & \big|V_1\big\rangle &= \hat{\mathcal{P}}^\dagger_\mp\big|R_{01}^\pm\big\rangle = \hat{\overline{\mathcal{P}}}{}^\dagger_\mp\big|R_{10}^\pm\big\rangle. \end{aligned} \tag{47}$$

The action of the operators $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}^\dagger$ and their anti-holomorphic partners on the states $\big|X_\lambda\big\rangle$ and $\big|Y_\lambda\big\rangle$ shows that they are not independent but that $\big|Y_\lambda\big\rangle$ can be derived from the states $\big|X_\lambda\big\rangle$:

$$\big|Y_0\big\rangle = \hat{\mathcal{P}}\hat{\overline{\mathcal{P}}}\big|X_1\big\rangle \quad \text{and} \quad \big|Y_1\big\rangle = \hat{\mathcal{P}}^\dagger\hat{\overline{\mathcal{P}}}{}^\dagger\big|X_0\big\rangle. \tag{48}$$

Therefore, the states $\big|X_\lambda\big\rangle$ are the generating states for the boundary states involving the indecomposable representations. On the other hand, we can now justify the denomination *weak boundary states* in the sense that they produce zero under the action of certain operations $\hat{\mathcal{A}} \in \{\hat{\mathcal{N}},\, \hat{\mathcal{P}}\hat{\mathcal{P}}^\dagger\}$:

$$\hat{\mathcal{A}}\left(\mathcal{O}_n \pm \overline{\mathcal{O}}_{-n}\right)\big|X_\lambda\big\rangle = 0. \tag{49}$$

The relations between the boundary states under the action of these operators look schematically like the following. On the right hand side we state the embedding scheme of the representation $\mathcal{R}$ of the local logarithmic conformal field theory for the $c = -2$ model which looks exactly the same:

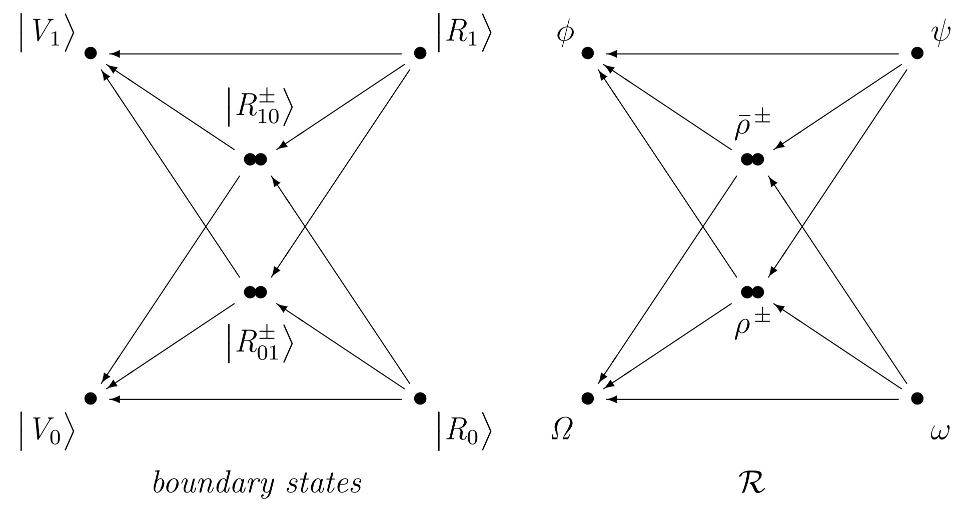

figure 4: boundary states vs. local theory

The perfect one-to-one correspondence between these two diagrams suggests that there is a deeper relation between the local theory and the one with boundaries. This is very interesting, especially because the boundary states were derived completely independent to the local theory. One the other hand, that there has to be at least a link between the two theories is already clear from the very beginning. The local theory fuses together a chiral and an anti-chiral copy of the rational $c = -2$ theory. To keep locality, certain states have to be devided out, namely the image of $L_0 - \overline{L}_0$. Equation (1) treated for $n = 0$ states that exactly the states with non-vanishing norm in the range of $L_0 - \overline{L}_0$ are not allowed to contribute to the boundary states.

5 Partition function

Looking at (37) and taking only the non-null states into account, one immediately finds the three-dimensional representation of the modular group in this case. Let us try to apply Cardy's method to obtain the physically relevant boundary conditions. For the set $\{\chi_{\mathcal{R}_0}, \chi_{\mathcal{R}_1}, \chi_{\mathcal{V}_{-1/8}}, \chi_{\mathcal{V}_{3/8}}\}$, the $\mathcal{S}$ and $\mathcal{T}$ matrices that give the transformations of the characters under the modular transformations $\tau \to 1/\tau$ and $\tau \to \tau + 1$ respectively are dealt with in [15]. There are six proper choices due to the fact that there are four independent representation modules whose characters form only a three-dimensional representation of the modular group. One of the possibilities is:

$$\mathcal{S} = \begin{pmatrix} \frac{\mathrm{i}}{2} & -\frac{\mathrm{i}}{2} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{\mathrm{i}}{2} & \frac{\mathrm{i}}{2} & \frac{1}{4} & -\frac{1}{4} \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\pi/6} & 0 & 0 \\ \mathrm{e}^{\mathrm{i}\pi/6} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-\mathrm{i}\pi/12} & 0 \\ 0 & 0 & 0 & -\mathrm{e}^{-\mathrm{i}\pi/12} \end{pmatrix}. \tag{50}$$

The associated charge conjugation matrix $\mathcal{C}$ is the permutation matrix that permutes the first two lines. It reads:

$$\mathcal{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{51}$$

Altogether, these matrices satisfy $\mathcal{S}^4 = 1$ and $(\mathcal{S}\mathcal{T})^3 = \mathcal{S}^2 = \mathcal{C}$. Taking the set of boundary basis states to be $\{|R_0\rangle, |R_1\rangle, |V_{-1/8}\rangle, |V_{3/8}\rangle\}$ we arrive at the starting point for Cardy's method. Firstly, we have to construct the vacuum boundary state $|\omega\rangle$ which can be written in terms of the basis states as

$$|\boldsymbol{\omega}\rangle = a|R_0\rangle + b|R_1\rangle + c|V_{-1/8}\rangle + d|V_{3/8}\rangle. \tag{52}$$

The remaining problem is that $\mathcal{S}_0^j$ is required to be positive valued. This is obviously not the case, the coefficients are even complex. We therefore introduce the conjugate vacuum representation boundary state $|\omega^\vee\rangle$:

$$|\boldsymbol{\omega}^\vee\rangle = a^*|R_0\rangle + b^*|R_1\rangle + c^*|V_{-1/8}\rangle + d^*|V_{3/8}\rangle, \tag{53}$$

and look at boundary conditions of the form $(\alpha^\vee, \beta)$, i. e. on the one hand side of the cylinder we apply the condition $\alpha^\vee$ instead of α and on the other hand the condition β. With this, it is an easy task to calculate the boundary states *à la* Cardy:

$$|\boldsymbol{i}\rangle = \sum_j \frac{S_i^j}{\sqrt{S_0^j}} |j\rangle. \tag{54}$$

Here, $|j\rangle$ denotes the boundary basis state belonging to the representation j and $|\boldsymbol{i}\rangle$ labels the physical relevant boundary state that corresponds to a bulk Hamiltonian that contains only the representation i in its spectrum. The physical boundary conditions finally read:

$$\begin{aligned}
|\boldsymbol{\omega}\rangle &= \tfrac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\pi/4}|R_0\rangle - \tfrac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\pi/4}|R_1\rangle + \tfrac{1}{2}|V_{-1/8}\rangle + \tfrac{i}{2}|V_{3/8}\rangle,\\
|\boldsymbol{\psi}\rangle &= -\tfrac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\pi/4}|R_0\rangle + \tfrac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\pi/4}|R_1\rangle + \tfrac{1}{2}|V_{-1/8}\rangle + \tfrac{\mathrm{i}}{2}|V_{3/8}\rangle,\\
|\boldsymbol{\mu}\rangle &= \sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}|R_0\rangle - \sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}|R_1\rangle + |V_{-1/8}\rangle - \mathrm{i}|V_{3/8}\rangle,\\
|\boldsymbol{\nu}\rangle &= -\sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4}|R_0\rangle + \sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}|R_1\rangle + |V_{-1/8}\rangle - \mathrm{i}|V_{3/8}\rangle.
\end{aligned} \tag{55}$$

Here, the states are labeled in correspondence to the cyclic states of the underlying bulk representation. The conjugate states are given by complex conjugation of the coefficients. The boundary states (55) are not uniquely defined but rather chosen up to a $\mathbb{Z}_4$ symmetry in the coefficient phases. Of course, it is as well possible to start from any of the other five proper definitions of the $\mathcal{S}$ and $\mathcal{T}$ matrices which lead to the same solutions. By construction it is clear that the partition function coefficients with respect to these states are equal to the fusion rules that are related to the elements of the $\mathcal{S}$ matrix by the Verlinde formula [28]:

$$n_{i^\vee j}^k = N_{ij}^k = \sum_r \frac{\mathcal{S}_r^i \mathcal{S}_r^j \mathcal{S}_k^r}{\mathcal{S}_0^r}. \tag{56}$$

Usually, the $\mathcal{S}$ matrix diagonalizes the fusion rules. As indicated in [15] this is not the case. Instead, the fusion matrices are transformed into block-diagonal form.

To summarize the result of this section we were able to show that the standard Cardy procedure works perfectly well in the $c = -2$ theory on the character representation of the modular group. It seems natural that this could be generalized to other such theories.

6 The five-dimensional representation

Now, we want to focus on the five-dimensional representation of the modular group. To do so, we study the complete set of boundary states plus the two generating states $\big|X_\lambda\big\rangle$. The representation was investigated in [13, 14, 17, 16]. In [16], an approach based on ideas of S. D. Mathur *et al.* [29, 30] was taken, which we will follow here: Let the linearly independent set of characters be given by the set

$$\Big\{\chi_{\mathcal{V}_0},\,\chi_{\mathcal{V}_{-1/8}},\,\chi_{\mathcal{V}_1},\chi_{\mathcal{V}_{3/8}},\,2\chi_{\widetilde{\mathcal{R}}}\equiv\frac{2}{\eta}\left[\Theta_{1,2}+\mathrm{i}\alpha\,(\nabla\Theta)_{1,2}\right]\Big\},\tag{57}$$

where $\alpha\in\mathbb{R}$ is arbitrary. The corresponding $\mathcal{S}$ matrix that transforms the characters under $\tau\to-1/\tau$ reads:

$$\mathcal{S}=\begin{pmatrix}\frac{1}{2\alpha}&\frac14&\frac{1}{2\alpha}&-\frac14&-\frac{1}{4\alpha}\\1&\frac12&1&\frac12&0\\-\frac{1}{2\alpha}&\frac14&-\frac{1}{2\alpha}&-\frac14&\frac{1}{4\alpha}\\-1&\frac12&-1&\frac12&0\\-2\alpha&1&2\alpha&-1&0\end{pmatrix}.\tag{58}$$

In order to find the elements of the five-dimensional representation we had to introduce partners to the boundary null-states that themselves are no boundary states and serve as duals for the null-states. However, we want to have the possibility that at least in a physical limit these states vanish. To do so let us not consider the states $\big|\,V_\lambda\big\rangle$ and their corresponding bra-states $\big\langle X_\lambda\big|$ but instead use the two renormalized states

$$\big|\,V_\lambda\big\rangle\longrightarrow\frac{2\pi}{\sqrt{2}\alpha}\big|\,V_\lambda\big\rangle\;\;\text{and}\;\;\big\langle X_\lambda\big|\longrightarrow\frac{\sqrt{2}\alpha}{2\pi}\big\langle X_\lambda\big|.\tag{59}$$

It is obvious that the pairings $\big\langle X_\lambda\big|\hat{q}\big|\,V_\lambda\big\rangle$ do not change for any choice of α, in particular for $\alpha=2\pi/\sqrt{2}$ for which we obtain the original states and that, on the other hand, the pairings $\big\langle X_\lambda\big|\hat{q}\big|R_\lambda\big\rangle$ get an additional pre-factor $(\sqrt{2}\alpha)/(2\pi)$ such that we obtain

$$\big\langle X_\lambda\big|\hat{q}\big|R_\lambda\big\rangle=\frac{\sqrt{2}\alpha}{2\pi}\log(q)\chi_{\mathcal{V}_\lambda}(q)\;\;\text{and}\;\;\big\langle X_\lambda\big|\hat{q}\big|\,V_\lambda\big\rangle=\chi_{\mathcal{V}_\lambda}(q).\tag{60}$$

Following the Cardy formalism the physical vacuum boundary state is up to a choice of phases in ordinary conformal field theories given by

$$\big|\boldsymbol{\Omega}\big\rangle=\sum_j\sqrt{\mathcal{S}_0^j}\,\big|j\big\rangle.\tag{61}$$

Here, j runs as in the previous section over the representation modules and the corresponding boundary basis states. In our case, this has to be treated more carefully. First of all, since the concerned elements of the $\mathcal{S}$ matrix are not positive we again have to introduce a conjugate vacuum representation $\big|\boldsymbol{\Omega}^\vee\big\rangle$ in order to be able to follow Cardy's argumentation. If we naively compute the boundary states we find that they do not exactly reproduce the characters for which the $\mathcal{S}$ matrix is written down in the above form. Recall that we have the states $\big|Z_{\lambda(1-\lambda)}\big\rangle$ at our disposal. By putting together all these arguments and following the standard formalism one can write down the following boundary states:

$$
\begin{aligned}
\big|\boldsymbol{\Omega}\big\rangle &= \frac{1}{2\sqrt{\alpha}}\Big\{\sqrt{2}\left(\big|V_0\big\rangle + \big|V_1\big\rangle\right) + \mathrm{i}\left(\big|R_0\big\rangle - \big|R_1\big\rangle\right) \\
&\quad + \left(\big|R_{01}\big\rangle + \big|R_{10}\big\rangle\right)\Big\} + \frac{1}{2}\left(\big|V_{-1/8}\big\rangle + \mathrm{i}\,\big|V_{3/8}\big\rangle\right), \\
\big|\boldsymbol{\mu}\big\rangle &= \sqrt{2\alpha}\left\{\big|V_0\big\rangle + \big|V_1\big\rangle\right\} + \left(\big|V_{-1/8}\big\rangle - \mathrm{i}\,\big|V_{3/8}\big\rangle\right), \\
\big|\boldsymbol{\phi}\big\rangle &= \frac{1}{2\sqrt{\alpha}}\Big\{-\sqrt{2}\left(\big|V_0\big\rangle + \big|V_1\big\rangle\right) + \mathrm{i}\left(-\big|R_0\big\rangle + \big|R_1\big\rangle\right) \\
&\quad + \left(-\big|R_{01}\big\rangle + \big|R_{10}\big\rangle\right)\Big\} + \frac{1}{2}\left(\big|V_{-1/8}\big\rangle + \mathrm{i}\,\big|V_{3/8}\big\rangle\right), \\
\big|\boldsymbol{\nu}\big\rangle &= -\sqrt{2\alpha}\left\{\big|V_0\big\rangle + \big|V_1\big\rangle\right\} + \left(\big|V_{-1/8}\big\rangle - \mathrm{i}\,\big|V_{3/8}\big\rangle\right), \\
\big|\boldsymbol{\omega}\big\rangle &= 2\sqrt{2\alpha^3}\Big\{-\big|V_0\big\rangle + \big|V_1\big\rangle\Big\} + 2\left(\big|V_{-1/8}\big\rangle + \mathrm{i}\,\big|V_{3/8}\big\rangle\right),
\end{aligned}
\tag{62}
$$

First of all, the coefficients for the six boundary basis states that belong to only one representation module each fulfill the Cardy law, i. e. are up to the phases given by $\mathcal{S}_k^j/\sqrt{\mathcal{S}_0^j}$. Note that the two states corresponding to the indecomposable representations are treated by the same matrix elements up to a phase change. Secondly, one recognizes terms proportional to the mixed states that are added by hand. Their presence is explained as they happen to be counter terms in order to get the partition function coefficients to satisfy the Verlinde formula as in the ordinary cases. They appear due to the fact that the characters are not what one gets out of the pairings. Nevertheless, the representation of the modular group is the same, of course. Indeed, by calculating the partition functions for given boundary conditions $(i^\vee, j)$ it turns out that

$$
Z_{i^\vee j}(q) = \sum_k n_{i^\vee j}^k \chi_k(q) \tag{63}
$$

with $n_{i^\vee j}^k$ being *nearly* equal to the fusion coefficients derived from the given $\mathcal{S}$ matrix via the Verlinde formula (56). These fusion coefficients are equal to the physical ones up to some identifications, i. e. $2\mathcal{V}_0 + 2\mathcal{V}_1 \equiv \mathcal{R}_0 \equiv \mathcal{R}_1$ concerning the number of states on each level and in the physical limit $\alpha \longrightarrow 0$ under which the contributions of the non-boundary states and the mixed states vanish.

The problem with the partition function is that one finds terms proportional to $\log(q)\Theta_{1,2}(q)$. Fortunately, they come with a pre-factor α and thus vanish in the limit $\alpha \longrightarrow 0$. One even finds that under this limit $n^k_{i^\vee j} = N^k_{ij}$. Again, the fusion matrices are not diagonalized by the $\mathcal{S}$ matrix but instead transformed in block-diagonal form. However, it is not possible to apply this limit to the boundary states themselves since they get divergent. This is related to the observation in [16] of the same problem occurring for the $\mathcal{S}$ matrix, namely that even though it reproduces the fusion coefficients in the limit $\alpha \longrightarrow 0$, it is not possible to apply this limit to the matrix itself due to the fact that in this limit the set of characters gets linearly dependent.

7 Discussion

We presented a mathematically consistent way on how to treat boundary states in logarithmic conformal field theories and applied it to the rational $c = -2$ logarithmic conformal field theory. The advantage of the invented method is it's simplicity: We only make use of an arbitrary basis for each representation module and the expansion of a given state with respect to this basis. In particular, this basis does not have to be orthonormal. As a side-effect the algorithm derives the inverse metric on each representation module. The algorithm turns out to be finite in the sense that the components of the boundary states can be derived up to any given finite level in a finite number of steps.

We could identify ten states that obey the Ishibashi boundary state conditions and that can be arranged in a scheme very similar to the embedding scheme of the local theory proposed by M. R. Gaberdiel and H. G. Kausch. Six of these states turned out to be null states in the space of boundary states. The remaining four together with the corresponding $\mathcal{S}$ matrix can be treated by the standard Cardy formalism in order to obtain the physical relevant boundary conditions concerning the three-dimensional representation of the modular group. On the other hand, we could identify additional states in such a way that their pairings with the boundary states together with the non-vanishing boundary state-boundary state pairings reproduce the five-dimensional representation of the modular group. By referring to these additional states as the dual states corresponding to the boundary null-states we were able to apply a slightly modified version of the Cardy formalism in this case and obtained at least in a physical limit the wanted relation between the partition function coefficients and the fusion rules of the bulk theory. For the application of this limit, the same problems arise as for the $\mathcal{S}$ matrix that transforms the bulk characters. It is remarkable that the Cardy formalism works in both cases. The meaning of this, however, is still unknown but it is worth noting in this context that we could identify exactly the same elements that are presumed to form the set of torus amplitudes in the bulk theory. The investigation of the deeper meaning of the additional so-called

weak boundary states is left for future work.
In ordinary rational conformal field theories, solving Cardy's consistency condition reduces to finding non-negative integer-valued matrix representations (so-called NIM-representations) of the Verlinde algebra [3], see also [31]. Notice that our five-dimensional solution for the $c = -2$ logarithmic conformal field theory does involve negative integers. These occur in a very similar fashion as in the computation of fusion matrices along the lines of [16]. This demonstrates that logarithmic conformal field theories, although they share many properties with rational conformal field theories, cannot entirely be put on equal footing with them. However, the negative integers are not as bad as they initially may appear, since they precisely reflect the linear dependencies among the boundary states which appear in the above discussed limiting procedure. If these dependencies are taken into account in the correct way, the final solution can be written without negative integers. Unfortunately, this last step has to be done by hand, since the computation via the $\mathcal{S}$ matrix and Cardy's ansatz inevitabely will lead to some negative integer coefficients. It remains an interesting open question, in which sense more general solutions than the NIM-representations should be taken into account for settings slightly more general than ordinary rational conformal field theory.
Y. Ishimoto conjectured that for every indecomposable representation of rank 2, there exists exactly one boundary state [23]. Our analysis shows that this conjecture holds in the $c = -2$ case, even though not strictly. We derived two boundary states for the indecomposable representations each where one only refers to the contained subrepresentation, respectively. This seems in contradiction to the stated conjecture. On the other hand, one of these two states turns out to be a null state in the space of boundary states.
In one of the first works on this topic, I. I. Kogan and J. F. Wheater tried to fix the zero-norm state problem by a perturbative procedure. By doing this, they introduced a physical limit as well that looks much like our's, namely they multiplied the vacuum Ishibashi boundary state by a factor of $1/\epsilon$ and did the limiting in the calculation of the pairings. The trouble they ran into is that the characters they arrive at are not the ones that are really observed. A more severe problem is based in the perturbation process, namely if one introduces a non-vanishing scalar product of the bulk vacuum state with itself, then the L_0 mode does not behave well any longer, i. e. the Shapovalov forms would turn out to be non-symmetric. Nevertheless, the principle idea of introducing such a limit is still the same.
S. Kawai and J. F. Wheater tried to solve the boundary problem by introducing symplectic fermions. With this, they found six boundary states and were able to relate them to the bulk properties in the usual way by defining the bra and ket states completely independent of each other. Another issue was that they could either choose a set of states that corresponded to the local theory or to the chiral theory.

None of the cited works, however, discussed the mixed states that intertwine the two different indecomposable bulk representations. The existence of these states is justified by the comparison to the local theory. There is still some work to do, but we conjecture that there has to be a deeper fundamental relation between the boundary and the local theory.

Acknowledgement: The work of M.F. is supported by the DFG string network (SPP no. 1096), Fl 259/2-1.

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