On Fusion Rules
in
Logarithmic Conformal Field Theories

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Abstract

We find the fusion rules for the $c_{p,1}$ series of logarithmic conformal field theories. This completes our attempts to generalize the concept of rationality for conformal field theories to the logarithmic case. A novelty is the appearance of negative fusion coefficients which can be understood in terms of exceptional quantum group representations. The effective fusion rules (i.e. without signs and multiplicities) resemble the BPZ fusion rules for the virtual minimal models with conformal grid given via $c = c_{3p,3}$. This leads to the conjecture that (almost) all minimal models with $c = c_{p,q}$, $(p,q) > 1$, belong to the class of rational logarithmic conformal field theories.

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1 Introduction

It is now more or less one do-decade ago since the concept of rationality of conformal field theory (CFT) made its first appearance through the minimal models of Belavin, Polyakov and Zamolodchikov [1]. Since then rational conformal field theories (RCFTs) became a main tool in modern theoretical physics.

More or less one decade later it has been shown [21] that CFTs whose correlation functions exhibit logarithmic behavior, can still be consistently defined. Recently there has been increasing interest in these logarithmic conformal field theories (LCFTs). Such LCFTs include the WZNW model on the supergroup $GL(1,1)$ [29], the $c_{p,1}$ models [14, 18, 21, 22, 23, 28], gravitationally dressed conformal field theories [2], and some critical disordered models [6, 25]. They are believed to be important for the description of certain statistical models, in particular in the theory of (multi-) critical polymers and percolation [7, 14, 30], in the quantum Hall effect [15, 25, 31], and in 2-dimensional turbulence [16, 27]. They also play a role in the so called unifying $\mathcal{W}$ algebras [3] and they might play a role in the description of normalizable zero modes for string backgrounds [11, 24].

In our paper [14], hereafter referred to as ☺, we started to generalize the concept of rationality to the case of LCFTs. There, some details have not been resolved in a completely satisfactory way. In particular, the Verlinde S matrix seems to have a block structure preventing one to use the Verlinde formula to get the fusion rules. Moreover, the quantum group structure, which underlies every RCFT [20] normally in a hidden way, becomes visible in the case of LCFTs, where in addition the values of the quantum deformation parameters $q_\pm$ are of the form $q_+ = \exp(2\pi i/p)$, $q_- = \exp(2\pi i p) = 1$. The consequences for the representation theory of LCFTs remained unclear in ☺.

The aim of this paper is to close these remaining gaps and thus complete our attempts to generalize rationality to the logarithmic case. We use the notations of ☺, and to save space the interested reader is asked to consult our earlier work for reference.

The flow of this letter is as follows: In section 2 we review the problems left in ☺. Section 3 is devoted to the consequences of exceptional representations due to the quantum group structure. We explain how the so called exceptional representations introduce some degree of freedom into the linear combinations of characters such that both, modular invariant partition functions and S matrices with "good" fusion rules, can be obtained at the same time. The constraint of integer valued fusion coefficients leads to a unique solution for the S matrix. But the linear combinations of characters do not agree with results from direct examinations of the representations and their embedding structure [18, 19]*, but are related to the latter by simple linear combinations with integer coefficients. If instead one uses the correct characters, the S matrix seems to yield bad fusion rules. This obstacle is resolved by an interesting limiting procedure. We continue in section 4 with remarks on the meaning of negative fusion coefficients and an observation regarding the effective fusion rules (forgetting the signs), which relates the $c_{p,1}$ LCFT to the would-be minimal model $c_{3p,3}$ (which does not

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*We are greatful to M.R. Gaberdiel and H.G. Kausch for pointing this out to us

exist since $3p$ and 3 are not coprime). We conclude with some speculations on a possible generalization of rational LCFTs for arbitrary $c_{np,n}$, $n > 2$, models.

We find it remarkably that we obtain simple fusion rules for these "minimal" LCFTs, despite the fact that their representation theory has some important unusual features (e.g. indecomposable highest weight representations, non-diagonalizable L_0) [18, 23, 28].

2   Characters & Partition Functions

In ☺ modular invariant partition functions have been obtained for the LCFTs with $c = c_{p,1}$. The surprising fact was that these LCFTs have a finite closing operator algebra with respect to the maximally extended chiral symmetry algebra $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$. Therefore, they also have a finite set of characters. The building blocks of the characters are the Dedekind η function $q^{1/24} \prod_{n \in \mathbb{N}} (1 - q^n)$ and the following set of ordinary and affine Θ functions:

$$\Theta_{\lambda,k} \;=\; \sum_{n \in \mathbb{Z}} q^{(2kn+\lambda^2)/4k} \,, \tag{2.1}$$

$$(\partial\Theta)_{\lambda,k} \;=\; \sum_{n \in \mathbb{Z}} (2kn+\lambda) q^{(2kn+\lambda^2)/4k} \,, \tag{2.2}$$

$$(\nabla\Theta)_{\lambda,k} \;=\; i\tau (\partial\Theta)_{\lambda,k} = \frac{1}{2\pi} \log(q) (\partial\Theta)_{\lambda,k} \,. \tag{2.3}$$

Here, $q = \exp(2\pi i\tau)$ is the modular parameter, and the last equation means that we formally rewrite $2\pi i\tau$ as $\log(q)$, if we deal with q-series expressions.

A basis for the characters of the $c_{p,1}$ model is given by the following $3p-1$ linearly independent functions

$$\chi_{\pm\lambda,p} \;=\; \frac{1}{\eta}\Theta_{\lambda,p} \pm \frac{1}{\eta}\left[(\partial\Theta)_{\lambda,p} + (\nabla\Theta)_{\lambda,p}\right] \,, \tag{2.4}$$

$$\tilde{\chi}_{\lambda,p} \;=\; \frac{\sqrt{2}}{\eta}\left[(\partial\Theta)_{\lambda,p} - (\nabla\Theta)_{\lambda,p}\right] \,, \tag{2.5}$$

where $0 \leq \lambda \leq p$. Notice, that $\chi_{-p,p} \equiv \chi_{p,p}$, and $\tilde{\chi}_{0,p} \equiv \tilde{\chi}_{p,p} \equiv 0$. The conformal dimensions of the corresponding fields are then given by

$$h_{\pm\lambda,p} = \frac{(p \pm (p-\lambda))^2 - (p-1)^2}{4p} \,. \tag{2.6}$$

In ☺, all characters $\tilde{\chi}_{\lambda,k}$ got an unphysical multiplicity of $\sqrt{2}$ in order to obtain a modular invariant partition function

$$Z_{\log}[p] = |\chi_{0,p}|^2 + |\chi_{p,p}|^2 + \sum_{\lambda=1}^{p-1} \left[\chi_{\lambda,p}\chi_{-\lambda,p}^* + \chi_{-\lambda,p}\chi_{\lambda,p}^* + |\tilde{\chi}_{\lambda,p}|^2\right] \,. \tag{2.7}$$

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Moreover, all except 2 of the characters have $\log(q)$ terms. This is insofar disturbing, as an explicit calculation of the vacuum character yields $\chi_{\text{vac},p} = (\Theta_{p-1,p} + (\partial\Theta)_{p-1,p})/(p\eta)$ without logarithmic terms, in contrast to $\chi_{p-1,p}$. Moreover, this basis of characters does not coincide with the physical one, i.e. with characters obtained from direct calculations of the number of states of the representations. The precise structure of the representations and the physical characters have been found by M.R. Gaberdiel and H.G. Kausch [18, 19].

In this section, we will use a set of characters suggested by the results of ☺, that promises good modular porperties and, perhaps, to obtain fusion rules via the Verlinde formula. At the end of section 3 we will make contact with the physical basis of characters and discuss the differences between the two sets which turn out to be related by integer valued linear combinations. The S matrix for our basis of characters is given by

$$
S_{(p)} = \left(\begin{array}{c|c} \sqrt{\frac{1}{2p}}[\cos(\frac{\pi\lambda\lambda'}{p}) + \sin(\frac{\pi\lambda\lambda'}{p})] & 0 \\ \hline 0 & -\sqrt{\frac{2}{p}}\sin(\frac{\pi\mu\mu'}{p}) \end{array} \right) . \tag{2.8}
$$

where in the upper left block we have $-p < \lambda, \lambda' \leq p$, and in the lower right block $0 < \mu, \mu' < p$. The block structure of the S matrix and the multiplicity 2 of the $\tilde{\chi}_{\lambda,p}$ characters are a hint to the quantum group structure showing up in these LCFTs.

Actually, we have 2 representations with characters $\tilde{\chi}_{\lambda,k}^{\pm}$ such that $\tilde{\chi}_{\lambda,k}^{+} + \tilde{\chi}_{\lambda,k}^{-} = \sqrt{2}\tilde{\chi}_{\lambda,k}$, the latter having vanishing quantum dimension. This precisely happens in quantum groups, if the quantum deformation parameters become roots of unity. Then, additional so called exceptional representations appear in pairs, whose quantum dimensions add up to 0 [20]. The point is that every RCFT has an underlying quantum group structure, but precisely for $c = c_{p,1}$, one of the corresponding quantum group parameters becomes $\exp(2\pi ip) = 1$ and, as already mentioned in ☺, the quantum group structure becomes visible within the RCFT itself. Notice, that the characters $\tilde{\chi}_{\lambda,p}$ have signs in their q-expansion which is a further hint to an additional quantum number. It is now tempting to check whether we can split the representations corresponding to the $\tilde{\chi}_{\lambda,k}$ characters in such a way that we obtain both, a modular invariant partition function and an S matrix from which we can calculate good fusion rules via the Verlinde formula.

In fact, this is the case. If we split each $\tilde{\chi}_{\lambda,k}$ characters into a pair and redo our analysis of ☺ with a general ansatz for all characters

$$
\chi_{\lambda,p} = \frac{1}{\eta}\left[\alpha_{\lambda,p}\Theta_{\lambda,p} + \beta_{\lambda,p}(\partial\Theta)_{\lambda,p} + \gamma_{\lambda,p}(\nabla\Theta)_{\lambda,p}\right] , \tag{2.9}
$$

$$
\tilde{\chi}_{\mu,p}^{\pm} = \frac{1}{\eta}\left[\alpha_{\mu,p}^{\pm}\Theta_{\mu,p} + \beta_{\mu,p}^{\pm}(\partial\Theta)_{\mu,p} + \gamma_{\mu,p}^{\pm}(\nabla\Theta)_{\mu,p}\right] , \tag{2.10}
$$

we have more possibilities for writing down a candidate modular invariant partition function. One remark is necessary here. We cannot just extend the S matrix to the enlarged set of characters, since the enlarged set is no longer linear independent. But since the characters are supposed to be split in such a way that adding them again yields characters to representations

with vanishing quantum dimensions, it is sufficient to include only one of the two characters, say $\tilde{\chi}^+_{\lambda,p}$ into our set for which we calculate the S matrix. The S matrix for the set of characters, where $\tilde{\chi}^-_{\lambda,p}$ replaces $\tilde{\chi}^+_{\lambda,p}$, is the same up to some signs (in particular the quantum dimensions $S^{\tilde{\lambda}}_{\text{vac}}/S^{\text{vac}}_{\text{vac}}$ change sign). This will lead to unavoidable signs in the fusion rules which we will explain in the next section. Another important consequence of this is that S is no longer unitary in the usual sense. Instead, we have that S is almost unitary with respect to the metric induced by the sesqui-linear form of the partition function, $Z_{\log}[p] = \vec{\chi}^t \cdot \mathcal{N} \cdot \vec{\chi}$, where $\mathcal{N}$ is given as

$$\mathcal{N} = (\delta_{\lambda,-\lambda'})_{-p<\lambda,\lambda'\leq p} \oplus \left(\tilde{\delta}_{\mu,\mu'}\right)_{0<\mu,\mu'<p} . \tag{2.11}$$

We then have $S\mathcal{N}S^\dagger = (\mathbb{1}_{2p}) \oplus (-\tilde{\mathbb{1}}_{p-1})$ and $S^2 = \mathbb{1}_{3p-1}$, $S\mathcal{N}S^t = \mathcal{N}$, where $\mathbb{1}_n$ denotes the $n \times n$ identity matrix, and $\tilde{\ }$ indicates parts of the matrices which correspond to $\tilde{\chi}$ characters. This reflects the fact that LCFTs do not completely factorize in left and right chiral part, since otherwise neither conformal invariance of the 4-point functions nor modular invariance of the partition function can be assured.

To understand this behavior, let us introduce the split of characters into the partition function. It turns out that one has now two possibilities,

$$Z^+_{\log}[p] = \sum_{\lambda=-p+1}^{p} \chi_{\lambda,p}\chi^*_{-\lambda,p} + \sum_{\mu=1}^{p-1} \left[|\tilde{\chi}^+_{\mu,p}|^2 + |\tilde{\chi}^-_{\mu,p}|^2\right] , \tag{2.12}$$

$$Z^-_{\log}[p] = \sum_{\lambda=-p+1}^{p} |\chi_{\lambda,p}|^2 + \sum_{\mu=1}^{p-1} \left[\tilde{\chi}^+_{\mu,p}\tilde{\chi}^{-*}_{\mu,p} + \tilde{\chi}^-_{\mu,p}\tilde{\chi}^{+*}_{\mu,p}\right] . \tag{2.13}$$

Both cases are modular invariant for appropriate chosen linear combinations (2.9), (2.10) of the characters. Extending the S matrix by hand (the prescription for this can be found in [26]) to incorporate the split characters, we find the surprising result $S_{\text{ext}}\mathcal{N}^+S^\dagger_{\text{ext}} = \mathcal{N}^-$ and vice versa, where still $S^2_{\text{ext}} = \mathbb{1}_{4p-2}$. Thus, the action of S_{ext} intertwines $Z^+_{\log}[p]$ and $Z^-_{\log}[p]$. A closer look shows that $Z^+_{\log}[p]$ and $Z^-_{\log}[p]$ just differ by the sign of the $\log(q\bar{q})$ part in their q-series expansion. Since $\log(q\bar{q}) = 2\pi i(\tau - \bar{\tau})$, we see that successive application of $S : \tau \mapsto \frac{-1}{\tau}$ and $S^\dagger : \tau \mapsto \frac{-1}{\bar{\tau}}$ precisely yields this sign.

From the above we conclude that it is not possible to find a good extended S matrix. In fact, one can show that the method of [26] does not work in the case, where representations are split into *different* characters, instead of having just a multiplicity > 1 in the partition function. We will see later that it is possible to find a unitary (not extended) S matrix, if we incorporate the non trivial sesqui-linear form $\mathcal{N}$ in the proper way. The problem is that our characters yield an S matrix with $S^2 = \mathbb{1}$, which does not depend on the choice of a basis. What we would like is a modified matrix $\mathfrak{S}$ such that $\mathfrak{S}\mathfrak{S}^\dagger = \mathbb{1}$, $\mathfrak{S}^2 = \mathcal{N}$, i.e. $\mathcal{N}$ gives rise to non trivial charge conjugation.

So far, the correct coefficients of the linear combinations (2.9), (2.10) are not yet uniquely determined. Every quadruple $(\chi_{\lambda,p}, \chi_{-\lambda,p}, \tilde{\chi}^+_{\lambda,p}, \tilde{\chi}^-_{\lambda,p})$ is determined up to three of its coefficients (namely $\gamma_{\lambda,p} = -\gamma_{-\lambda,p}, \gamma^\pm_{\lambda,p}$), if we impose the condition of minimal integer coefficients

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for the $q, \bar{q}$-series expansion of the partition function. All other constants, i.e. all constants of terms without $\log(q)$, assume the values as given by (2.4), (2.5). The remaining free constants are determined by requiring integer valued fusion rules.

3 S Matrix & Fusion Rules

We can now either determine the remaining coefficients by directly considering the fusion rules **(case I)**, by just requiring that the S matrix should be symmetric **(case II)**, or by matching the characters with explicit calculations of the spectra of the representations [18, 19, 23, 28] **(case III)**. It was pointed out by M.R. Gaberdiel and H.G. Kausch, see [19], that the characters of the solutions to case I and case II disagree with the direct calculations. In ordinary RCFTs one would expect that all these approaches yield the same solution. This seems to be no longer true in the logarithmic case. But nonetheless, the different solutions are consistent with each other.

Getting "good" fusion rules (case I): In case I, we obtain one pair of solutions which is valid for all $p > 1$, namely $\gamma_{\lambda,p} = 0$, $\gamma_{\lambda,p}^+ = 1$ or $\gamma_{\lambda,p} = 2$, $\gamma_{\lambda,p}^+ = 3$ for all $0 < \lambda < p$, where the solution with $\gamma_{\lambda,p} = 0$ is more appealing, as we have mentioned in the introduction: all characters $\chi_{\lambda,p}$ are then free from $\log(q)$ terms. In case II, we also obtain a solution, namely $\gamma_{\lambda,p} = 0$, $\gamma_{\lambda,p}^+ = \pm i\sqrt{3}$, for all $0 < \lambda < p$. Unfortunately, this solution does not work if p is divisible by 3 due to zeros in the row/column of the S matrix corresponding to the vacuum character, such that the Verlinde formula becomes singular. Nevertheless, the second solution has some significance, as will be pointed out in the last section. Case I and case II can be viewed as specializations of the more general requirement that the S matrix fulfills $S_j^k = \pm S_k^{\dagger j}$. If one makes the (reasonable) assumption that $\gamma_{\lambda,p} = \pm\gamma_{\mu,p}$, $\gamma_{\lambda,p}^+ = \pm\gamma_{\mu,p}^+$ for all $0 < \lambda, \mu < p$ one obtains a general expression $\gamma_{\lambda,p}^+ = \pm f_\pm(\gamma_{\lambda,p})$, where $f_\pm$ are rational functions involving fractional powers. The sign label corresponds to the choice of (anti-) symmetry for the S matrix entries.

As an example we consider the $c = c_{2,1} = -2$ model. With the requirements from the last section we obtain the S matrix

$$
S_{(2)} = \begin{pmatrix}
\frac{i(xy-1)}{2(x+y)} & \frac{1}{2} & \frac{i(1-xy)}{2(x+y)} & -\frac{1}{2} & -\frac{i(x^2+1)}{x+y} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{i(1-xy)}{2(x+y)} & \frac{1}{2} & \frac{i(xy-1)}{2(x+y)} & -\frac{1}{2} & \frac{i(x^2+1)}{x+y} \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{i(y^2+1)}{x+y} & 0 & -\frac{i(y^2+1)}{x+y} & 0 & \frac{i(1-xy)}{x+y}
\end{pmatrix} , \tag{3.1}
$$

where we abbreviated $x = \gamma_{1,2}$ and $y = -\gamma_{1,2}^+$. The general solutions $y = f(x)$ are $y = \pm\sqrt{-2x^2 - 3}$ and $y = \pm\sqrt{2x^2 + 1}$. One can easily check that the solutions for x, y mentioned above yield integer valued fusion rules via the Verlinde formula

$$
N_{ij}^k = \sum_r \frac{S_i^r S_j^r (S^{-1})_r^k}{S_{\text{vac},r}} , \tag{3.2}
$$

where $S_{\text{vac},r}$ denotes the row corresponding to the vacuum representation. Notice, that the fusion coefficients may be negative. Let us denote the $\mathcal{W}$ conformal families corresponding to the characters $\chi_{\lambda,p}$, $\tilde{\chi}_{\lambda,p}$, and $\tilde{\chi}^{\pm}_{\lambda,p}$ by $[h_{\lambda,p}]$, $[\tilde{h}_{\lambda,p}]$, and $[\tilde{h}^{\pm}_{\lambda,p}]$ respectively. Choosing $y = -1, x = 0$, we get for our example the following fusion rules

$$
\begin{array}{rclcrcl}
[0] & \times & [0] & = & [0]\,, \\
[0] & \times & [-\tfrac{1}{8}] & = & [-\tfrac{1}{8}]\,, & [-\tfrac{1}{8}] & \times & [\tilde{0}] & = & [-\tfrac{1}{8}] - [\tfrac{3}{8}]\,, \\
[0] & \times & [\tfrac{3}{8}] & = & [\tfrac{3}{8}]\,, & [\tfrac{3}{8}] & \times & [\tfrac{3}{8}] & = & [0] - [\tilde{0}]\,, \\
[0] & \times & [1] & = & [1]\,, & [\tfrac{3}{8}] & \times & [1] & = & [-\tfrac{1}{8}]\,, \\
[0] & \times & [\tilde{0}] & = & [\tilde{0}]\,, & [\tfrac{3}{8}] & \times & [\tilde{0}] & = & [\tfrac{3}{8}] - [-\tfrac{1}{8}]\,, \\
[-\tfrac{1}{8}] & \times & [-\tfrac{1}{8}] & = & [0] - [\tilde{0}]\,, & [1] & \times & [1] & = & [0]\,, \\
[-\tfrac{1}{8}] & \times & [\tfrac{3}{8}] & = & [1] + [\tilde{0}]\,, & [1] & \times & [\tilde{0}] & = & -[\tilde{0}]\,, \\
[-\tfrac{1}{8}] & \times & [1] & = & [\tfrac{3}{8}]\,, & [\tilde{0}] & \times & [\tilde{0}] & = & [1] - [0] + 3[\tilde{0}]\,.
\end{array}
\tag{3.3}
$$

The choices $y = 1, x = 0$, or $y = \pm 2, x = \pm 3$, $y = \pm 2, x = \mp 3$ yield the same fusion rules up to some possible signs which correspond to exchanging $\tilde{\chi}^{+}$ with $\tilde{\chi}^{-}$. From this we can now read off the fusion rules where the $[\tilde{h}_{\lambda,p}]$ representations (in our example there is just $[\tilde{h}_{1,2}] = [\tilde{0}]$) are split. Giving only the non trivial cases we have

$$
\begin{array}{rclcrcl}
[0] & \times & [\tilde{0}^{\pm}] & = & [\tilde{0}^{\pm}]\,, \\
[-\tfrac{1}{8}] & \times & [-\tfrac{1}{8}] & = & [0] + [\tilde{0}^{-}]\,, & [\tfrac{3}{8}] & \times & [\tilde{0}^{+}] & = & [\tfrac{3}{8}]\,, \\
[-\tfrac{1}{8}] & \times & [\tfrac{3}{8}] & = & [1] + [\tilde{0}^{+}]\,, & [\tfrac{3}{8}] & \times & [\tilde{0}^{-}] & = & [-\tfrac{1}{8}]\,, \\
[-\tfrac{1}{8}] & \times & [\tilde{0}^{+}] & = & [-\tfrac{1}{8}]\,, & [1] & \times & [\tilde{0}^{\pm}] & = & [\tilde{0}^{\mp}]\,, \\
[-\tfrac{1}{8}] & \times & [\tilde{0}^{-}] & = & [\tfrac{3}{8}]\,, & [\tilde{0}^{\pm}] & \times & [\tilde{0}^{\pm}] & = & [0] + 2[1] + 2[\tilde{0}^{+}]\,, \\
[\tfrac{3}{8}] & \times & [\tfrac{3}{8}] & = & [0] + [\tilde{0}^{-}]\,, & [\tilde{0}^{\pm}] & \times & [\tilde{0}^{\mp}] & = & 2[0] + [1] + 2[\tilde{0}^{-}]\,.
\end{array}
\tag{3.4}
$$

For the last two equations one has to take into account that $[1] - [0] = ([0] + 2[1]) - (2[0] + [1])$ and $-[\tilde{0}^{\mp}] = [\tilde{0}^{\pm}]$. We note that the solution $y = \pm i\sqrt{3}, x = 0$ yields a unitary S matrix and (up to signs) again the same fusion rules, but without multiplicities, i.e. all $N^{k}_{ij} \in \{-1, 0, 1\}$. For completeness we also give the T matrix which is *non-diagonal*, a general feature of LCFTs. Again putting $x = 0, y = -1$ we obtain

$$
T_{(2)} = \begin{pmatrix}
e^{\pi i/6} & & & & \\
& e^{-\pi i/12} & & & \\
& & e^{\pi i/6} & & \\
& & & e^{11\pi i/12} & \\
-\tfrac{1}{2}e^{\pi i/6} & & \tfrac{1}{2}e^{\pi i/6} & & e^{\pi i/6}
\end{pmatrix}\,,
\tag{3.5}
$$

which fulfills $(ST)^{3} = \mathbb{1}$. The generalization of the T matrix for arbitrary p is obvious, the rows for the $[\tilde{h}_{\lambda,p}]$ representations get off-diagonal entries with $\pm\tfrac{1}{2}$ the value of the diagonal in the columns to the $[h_{\pm\lambda,p}]$ representations, i.e.

$$
T_{(p)} = \left(e^{2\pi i(h_{\lambda,p} - c/24)}\delta_{\lambda,\lambda'} + e^{2\pi i(h_{\mu-p,p} - c/24)}\left(\delta_{\mu,\mu'} + \frac{1}{2}\delta_{\mu-p,\lambda'} - \frac{1}{2}\delta_{\mu-p,-\lambda'}\right)\right)_{-p<\lambda,\lambda'\leq p,\, p<\mu,\mu'<2p}.
\tag{3.6}
$$

Here, the μ, μ' labels indicate the rows and columns to the $[\tilde{h}_{\mu,p}]$ representations, the λ, λ' labels refer to the $[h_{\lambda,p}]$ representations.

In general, case I yields (up to some irrelevant signs) the following expression for the S matrix:

$$
S_{(p)} = \left(
\begin{array}{cc|cc|c}
\sqrt{\frac{1}{2p}}\exp(\frac{\pi i \lambda\lambda'}{p}) & & & & i\sqrt{\frac{2}{p}}\sin(\frac{\pi\mu\mu'}{p}) \\
& & & & \overline{0 \ldots 0} \\
& \sqrt{\frac{1}{2p}}\exp(\frac{\pi i \lambda\lambda'}{p}) & & & -i\sqrt{\frac{2}{p}}\sin(\frac{\pi\mu\mu'}{p}) \\
& & & & \overline{0 \ldots 0} \\
\hline
-i\sqrt{\frac{2}{p}}\sin(\frac{\pi\mu\mu'}{p}) & \begin{matrix}0\\\vdots\\0\end{matrix} & i\sqrt{\frac{2}{p}}\sin(\frac{\pi\mu\mu'}{p}) & \begin{matrix}0\\\vdots\\0\end{matrix} & -i\sqrt{\frac{2}{p}}\sin(\frac{\pi\mu\mu'}{p})
\end{array}
\right) . \tag{3.7}
$$

Here, $-p < \lambda, \lambda' \le p$ and in all blocks $0 < \mu, \mu' < p$. Therefore, the S matrix has an interesting structure. It consists of blocks which by itself are well known. Indeed, the blocks

$$
i\sqrt{\frac{2}{p}}\left(\sin(\frac{\pi\mu\mu'}{p})\right)_{0<\mu,\mu'<p} \tag{3.8}
$$

are nothing else than (up to an irrelevant overall factor $-i$) the S matrix of the current algebra $A_1^{(1)}$ at level $p+2$. The other block,

$$
\sqrt{\frac{1}{2p}}\left(\exp(\frac{\pi i \lambda\lambda'}{p})\right)_{-p<\lambda,\lambda'\le p} \tag{3.9}
$$

resembles (up to the same factor $-i$) the S matrix of a $c = 1$ model with compactification radius $2R^2 = p$. Consequently, these blocks yield good fusion rules by themselves denoted by $\mathcal{S}_{\lambda\mu}^{\nu}$ and $\mathcal{E}_{\lambda\mu}^{\nu}$ respectively. Thus, it is easy to see that $S_{(p)}$ yields integer valued fusion rules which actually turn out to be linear combinations of the form $N_{ij}^k = a\mathcal{E}_{ij}^k + b\mathcal{S}_{ij}^k$ with $a, b \in \{-1, 0, 1\}$.

Getting a "good" S matrix (case II): Let us next discuss the solution of case II. As a matter of fact, putting $\gamma_{\lambda,p}^+ = \pm i\sqrt{3}$, $\gamma_{\lambda,p} = 0$ always yields real, symmetric, unitary S matrices, which we denote by $S_{(p)}^{\pm}$. Unfortunately, they have vanishing entries in some rows and columns, in particular the row and column to the vacuum representation, if and only if p is divisible by 3. One can show that this problem can partially be overcome, if one considers the matrix $\mathfrak{S}_{(p)} = \frac{1}{\sqrt{2}}(\sqrt{i}S_{(p)}^+ + \sqrt{-i}S_{(p)}^-)$, which is the average of the two S matrices to the split characters $\tilde{\chi}_{\lambda,p}^+$ and $\tilde{\chi}_{\lambda,p}^-$ respectively. The precise form of this average is dictated by the conditions $\mathfrak{S}\mathfrak{S}^\dagger = \mathbb{1}$ and $\mathfrak{S}^2 = \mathcal{N}'$ with

$$
\mathcal{N}' = (\delta_{\lambda,-\lambda'})_{-p<\lambda,\lambda'\le p} \oplus \left(-\tilde{\delta}_{\mu,\mu'}\right)_{0<\mu,\mu'<p} . \tag{3.10}
$$

The charge conjugation might seem unusual, but since $\tilde{\chi}^+_{\lambda,p} = -\tilde{\chi}^-_{\lambda,p}$, we get exactly $\mathcal{N}'$ if $C(\tilde{\chi}^+_{\lambda,p}) = \tilde{\chi}^-_{\lambda,p}$ and vice versa.

If one calculates fusion rules with the help of the $\mathfrak{S}$ matrix, one finds for $p \neq 3p'$ integer valued fusion coefficients $N^k_{ij} \in \{-1, 0, 1\}$. If $p = 3p'$, one finds instead rational fusion coefficients $N^k_{ij} \in \{0, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1\}$, which is a bit disturbing. The origin of these factors of $\frac{1}{3}$ might be understood as follows:

It has been pointed out in $\smiley$ that the field content of the $c_{p,1}$ LCFTs can be read off from the conformal grid of $c_{3p,3}$. The mapping is $h_{\lambda,p} = h(3p,3)_{1,p-\lambda}$, $h_{-\lambda,p} = h(3p,3)_{1,3p-\lambda}$ and $\tilde{h}_{\lambda,p} = h(3p,3)_{1,p+\lambda}$. Naively, this "minimal model" does not exist [1], because $3p$ and 3 are certainly not coprime. On the other hand, the famous BPZ fusion rules for a minimal model with $c = c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$ and fields $\phi_{r,r'}$ of conformal dimensions $h(p,q)_{r,r'} = \frac{1}{4pq}\left[(pr'-qr)^2 - (p-q)^2\right]$ are

$$[h(p,q)_{r,r'}] \times [h(p,q)_{s,s'}] = \sum_{\substack{n=|r-s|+1 \\ n+r+s-1\equiv 0\,\mathrm{mod}\,2}}^{\min(q-1,r+s-1)} \sum_{\substack{n'=|r'-s'|+1 \\ n'+r'+s'-1\equiv 0\,\mathrm{mod}\,2}}^{\min(p-1,r'+s'-1)} [h(p,q)_{n,n'}] \qquad (3.11)$$

and are well defined for arbitrary p,q. In the case that p,q are coprime, the S matrix for the characters of the minimal model with $c = c_{p,q}$ is [5]

$$S(p,q)^{(s,s')}_{(r,r')} = (-)^{rs'+r's+1} 2\sqrt{\frac{2}{pq}} \sin\left(\pi\frac{p}{q}rs\right) \sin\left(\pi\frac{q}{p}r's'\right), \qquad (3.12)$$

where $1 \leq r,s \leq q-1$ and $1 \leq r',s' \leq p-1$. Let us now assume that $p = \alpha^{\#p}p'$, $q = \alpha^{\#q}q'$ with p',q' coprime. Of course, one of the powers is 1, but it is convenient to use this symmetric notation. Then we can define a matrix

$$S(p,q)^{(s,s')}_{(r,r')} = (-)^{rs'+r's+1} 2\sqrt{\frac{2}{pq}} \sin\left(\pi\frac{p}{\alpha^{\#p}q}rs\right) \sin\left(\pi\frac{q}{\alpha^{\#q}p}r's'\right). \qquad (3.13)$$

One can now prove that this S matrix yields BPZ fusion rules (3.11) for arbitrary p,q. The surprise is that $S(3p,3)$ induces the fusion rules of our LCFTs with $c = c_{p,1}$, if we forget about all signs and multiplicities. We call such fusion rules *effective* fusion rules. Moreover, $\mathfrak{S}_{(p)}{}^l_k$ equals $S(3p,3)^l_k$ up to phases (correct relabeling implicitly understood), if p is not divisible by 3. In this case $\#(3p) = \#3 = 1$, $\alpha = 3$. The above formula (3.13) shows that for p divisible by 3, extra powers have to be introduced into the denominator of one of the sine terms.

Getting "good" characters (case III): The solution given above does yield good fusion rules but has the disadvantage that the basis of characters is still not the one observed in physics, i.e. which one gets by explicitly considering representations and counting states. The latter task has been dealt with in [18, 23, 28]. The result is

$$\chi_{0,p} = \frac{1}{\eta}\Theta_{0,p}, \qquad (3.14)$$

$$\chi_{p,p} = \frac{1}{\eta}\Theta_{p,p}, \tag{3.15}$$

$$\chi_{\lambda,p}^{+} = \frac{1}{p\eta}\left[(p-\lambda)\Theta_{\lambda,p} + (\partial\Theta)_{\lambda,p}\right], \tag{3.16}$$

$$\chi_{\lambda,p}^{-} = \frac{1}{p\eta}\left[\lambda\Theta_{\lambda,p} - (\partial\Theta)_{\lambda,p}\right], \tag{3.17}$$

$$\tilde{\chi}_{\lambda,p}^{+} = \frac{1}{\eta}\left[\Theta_{\lambda,p} + i\alpha\lambda(\nabla\Theta)_{\lambda,p}\right], \tag{3.18}$$

$$\tilde{\chi}_{\lambda,p}^{-} = \frac{1}{\eta}\left[\Theta_{\lambda,p} - i\alpha(p-\lambda)(\nabla\Theta)_{\lambda,p}\right], \tag{3.19}$$

where $0 < \lambda < p$. The conformal weights are (in the same order) $h(p,1)_{1,p}$, $h(p,1)_{1,2p}$, $h(p,1)_{p-\lambda}$, $h(p,1)_{3p-\lambda}$, and $h(p,1)_{p+\lambda}$. One easily sees that the partition function

$$Z_{\log}[p,\alpha] = |\chi_{0,p}|^2 + |\chi_{p,p}|^2 + \sum_{\lambda=1}^{p}\left[\chi_{\lambda,p}^{+}\tilde{\chi}_{\lambda,p}^{+*} + \chi_{\lambda,p}^{+*}\tilde{\chi}_{\lambda,p}^{+} + \chi_{\lambda,p}^{-}\tilde{\chi}_{\lambda,p}^{-*} + \chi_{\lambda,p}^{-*}\tilde{\chi}_{\lambda,p}^{-}\right] \tag{3.20}$$

$$= \frac{1}{\eta\eta^*}\left\{|\Theta_{0,p}|^2 + |\Theta_{p,p}|^2 + \sum_{\lambda=1}^{p}\left[2|\Theta_{\lambda,p}|^2 + i\alpha\left((\partial\Theta)_{\lambda,p}(\nabla\Theta)_{\lambda,p}^* - (\partial\Theta)_{\lambda,p}^*(\nabla\Theta)_{\lambda,p}\right)\right]\right\}$$

is modular invariant for all $\alpha \in \mathbb{R}$. Notice the important fact that the partition function remains modular invariant even for $\alpha = 0$ and then equals the standard $c = 1$ Gaussian model partition function $Z(\sqrt{p/2})$. This in particular means that we have a modular invariant partition function even in the case where the characters do not form a closed finite dimensional representation of the modular group by themselves. One can now again calculate an S matrix for one set of linear independent characters, e.g. $\{\chi_{0,p}, \chi_{p,p}, \chi_{\lambda,p}^{\pm}, [(p+x-\lambda)\tilde{\chi}_{\lambda,p}^{+}+(\lambda-x)\tilde{\chi}_{\lambda,p}^{-}]\}$. This S matrix and the fusion rules obtained via the Verlinde formula (3.2) depend on the value of α. Clearly, the S matrix becomes singular for $\alpha \to 0$, but the fusion rules remain well defined. The limes $\alpha \to 0$ just puts several of the fusion coefficients to zero. It turns out that in general only the fusion rules in the limit $\alpha \to 0$ are consistent and integer valued.

As an example let us again consider the case $c = c_{2,1} = -2$. The S matrix reads

$$S_{(2,\alpha)} = \begin{pmatrix} \frac{1}{2\alpha} & \frac{1}{4} & \frac{1}{2\alpha} & -\frac{1}{4} & -\frac{1}{4\alpha} \\ 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ -\frac{1}{2\alpha} & \frac{1}{4} & -\frac{1}{2\alpha} & -\frac{1}{4} & \frac{1}{4\alpha} \\ -1 & \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ -2\alpha & 1 & 2\alpha & -1 & 0 \end{pmatrix}. \tag{3.21}$$

The S matrix $S_{(p,\alpha)}$ in general is neither symmetric nor unitary but again fulfills $S_{(p,\alpha)}^2 = \mathbb{1}$. The general expression for the S matrix is cumbersome, but can easily obtained by applying an appropriate base change of characters to, e.g., the matrix $S_{(p)}$ given in (3.7). Defining $N_{jk}^{i}(\alpha)$ via the Verlinde formula (3.2), the fusion rules are obtained by the limes $N_{jk}^{i} = \lim_{\alpha\to 0} N_{jk}^{i}(\alpha)$. Care must then be taken in interpreting the right hand sides of fusion

products: Since the characters are non longer linearly independent in this limit, we may have character identities which translate to relations among the representations. For $\alpha = 0$ one has at least that $2[h(p,1)_{1,p-\lambda}] + 2[h(p,1)_{1,3p-\lambda}] = [h(p,1)_{1,p+\lambda}]$, $0 < \lambda < p$, e.g. the relation $[\tilde{0}] = 2[0] + 2[1]$ in our example. The interpretation of signs remains the same as before. The fusion rules now read

$$
\begin{array}{rclcl}
[0] & \times & [\Phi] & = & [\Phi]\,, \\
[-\tfrac{1}{8}] & \times & [-\tfrac{1}{8}] & = & 2[0] + 2[1] = [\tilde{0}]\,, \\
[-\tfrac{1}{8}] & \times & [\tfrac{3}{8}] & = & 2[0] + 2[1] = [\tilde{0}]\,, \\
[-\tfrac{1}{8}] & \times & [1] & = & [\tfrac{3}{8}]\,, \\
[-\tfrac{1}{8}] & \times & [\tilde{0}] & = & 2[-\tfrac{1}{8}] + 2[\tfrac{3}{8}]\,, \\
[\tfrac{3}{8}] & \times & [\tfrac{3}{8}] & = & 2[0] + 2[1] = [\tilde{0}]\,,
\end{array}
\qquad
\begin{array}{rclcl}
[\tfrac{3}{8}] & \times & [1] & = & [-\tfrac{1}{8}]\,, \\
[\tfrac{3}{8}] & \times & [\tilde{0}] & = & 2[-\tfrac{1}{8}] + 2[\tfrac{3}{8}]\,, \\
[1] & \times & [1] & = & [0]\,, \\
[1] & \times & [\tilde{0}] & = & 4[0] + 4[1] - [\tilde{0}] = [\tilde{0}]\,, \\
[\tilde{0}] & \times & [\tilde{0}] & = & 8[0] + 8[1] = 4[\tilde{0}]\,.
\end{array}
\tag{3.22}
$$

Again, we can reintroduce the split representations $[\tilde{h}^{\pm}]$ back into the fusion rules, though their characters coincide in the limit $\alpha \to 0$. This yields the following modifications

$$
\begin{array}{rclcl}
[-\tfrac{1}{8}] & \times & [-\tfrac{1}{8}] & = & [\tilde{0}^+]\,, \\
[-\tfrac{1}{8}] & \times & [\tfrac{3}{8}] & = & [\tilde{0}^-]\,, \\
[\tfrac{3}{8}] & \times & [\tfrac{3}{8}] & = & [\tilde{0}^+]\,,
\end{array}
\qquad
\begin{array}{rclcl}
[1] & \times & [\tilde{0}^{\pm}] & = & [\tilde{0}^{\mp}]\,, \\
[\tilde{0}^{\pm}] & \times & [\tilde{0}^{\pm}] & = & 2[\tilde{0}^+] + 2[\tilde{0}^-]\,, \\
[\tilde{0}^{\pm}] & \times & [\tilde{0}^{\mp}] & = & 2[\tilde{0}^+] + 2[\tilde{0}^-]\,.
\end{array}
\tag{3.23}
$$

It is a simple task to check that these fusion rules are "compatible" with the ones given before, (3.3), (3.4), if the correct identifications are taken into account. For instance, we have $[0]_I + [\tilde{0}^-]_I = [\tilde{0}^+]_{III}$, $[1]_I + [\tilde{0}^+]_I = [\tilde{0}^-]_{III}$ in our example, relating the base within the representations of case I to the base in case III.

Combining the results: To summarize, we found several possibilities to get the fusion rules of LCFTs from the modular behavior of the partition function and its building blocks, the characters. Since there is no unique basis of characters in the case of LCFTs (due to the integer valued differences between conformal weights of several representations), several possible choices of a basis arose. What we called cases I and II above used bases in which the fusion rules can be obtained from the full characters, including logarithmic terms, such that these characters form a closed finite dimensional representation of the modular group. But these bases are not the physical ones. In case III we used the physical basis of characters, but could obtain fusion rules via Verlinde formula only in the limit of vanishing logarithmic terms. Although the partition function remains modular invariant in this limit, the characters do not longer form a finite dimensional representation of the modular group. Fortunately, the different fusion rules turned out to be compatible with each other. Although, case III deals with the physical characters, its fusion rules are difficult to read due to relations among the representations in the limit of vanishing logarithmic terms. The best way to obtain the fusion rules is therefore to combine case I and case III. The recipe is to calculate the fusion rules via case I (where the logarithmic terms prohibit "obscuring" relations) and then make a transformation into the physical base of case III. We checked explicitly all models up to $p = 6$ and obtained compatible and consistent results. In particular, these results agree with direct calculations in [19].

A closer look reveals that all $[\tilde{h}]$ representations actually fall into triplets of the form $([h(p,1)_{1,r}], [\tilde{h}(p,1)_{1,2p-r}], [h(p,1)_{1,2p+r}])$ and, in the physical basis, $[h(p,1)_{1,r}] \subset [\tilde{h}(p,1)_{1,2p-r}]$, $[h(p,1)_{1,2p+r}] \subset [\tilde{h}(p,1)_{1,2p-r}]$. This fact connects the fusion rules of case I and case III with the BPZ fusion rules of case II. Indeed, replacing in the physical basis $[\tilde{h}(p,1)_{1,2p-r}] = [h(p,1)_{1,r}] + [h(p,1)_{1,2p+r}] + [\hat{h}(p,1)_{1,2p-r}]$ transforms the fusion rules precisely to BPZ form. We emphasize that this is purely symbolical, since the $[\tilde{h}(p,1)^{\pm}_{1,2p-r}]$ representations are indecomposable [18]. Nonetheless, this shows once more that the $c_{p,1}$ LCFTs have a structure very similar to minimal models.

4 On The Space of 2-dimensional Field Theories

We do not understand up to now, whether our view of the $c_{p,1}$ LCFTs as certain "minimal models for non coprime $3p, 3$" has any deeper meaning. But we emphasize the striking fact that all the structures known for RCFTs and in particular for the minimal models have counterparts in rational LCFTs. By finding characters, partition functions, and – finally – the fusion rules for LCFTs, we complete our attempts to generalize *rationality* to LCFTs. Since (3.13) is valid for arbitrary p, q, we conjecture that there should exist a whole class of generalized (logarithmic) CFTs, for which our $c_{p,1}$ LCFTs are just the first examples. For instance, one might think of enlarged minimal models (thereby no longer minimal in the precise meaning of this term) $\mathcal{M}(\alpha p, \alpha q)$ with p, q coprime and central charge still $c_{p,q}$. The $c_{p,1}$ LCFTs are then just the enlarged models $\mathcal{M}(3p, 3)$, where simply the "minimal" models with $\mathcal{M}(p, 1)$ are empty.

We remark that the enlarged models $\mathcal{M}(2p, 2q)$ can be interpreted as $N = 1$ supersymmetric extensions of the $N = 0$ minimal models $\mathcal{M}(p, q)$, where the free Grassman parts have been canceled (otherwise, the central charge had to be shifted by $+\frac{1}{2}$). We remind the reader that for unitary minimal $N = 1$ supersymmetric models $p - q = 2$, and that the Ramond sector contains only for p, q even a globally supersymmetric invariant "vacuum" state. Is it just a coincidence that the model $\mathcal{M}(3 \cdot 2, 3 \cdot 1)$ with $c = -2$ has a hidden $N = 2$ supersymmetry [30]? It might be worthwhile to investigate further how supersymmetric and logarithmic CFTs are related. This might shed some light on the representation theory of $N = 2$ supersymmetric conformal field theories (SCFTs) in general and more specifically the question of the relationship between rationality and unitarity for $N = 2$ SCFTs. We conjecture that non-unitary $N = 2$ SCFTs can only be rational, if they are *logarithmic* CFTs, since all attempts so far to find decompositions of their partition functions into finitely many products of characters, i.e. $Z = \sum_{i,j \in \Lambda} \chi_i^* \mathcal{N}_{i,j} \chi_j$ with $|\Lambda| < \infty$ and χ_i modular forms of weight 0, have failed [9]. The $c = -2$ model would be the first example of such a case, namely the non-free part of an $N = 2$ SCFT with central charge (shifted by the free Grassmann contribution by $+2$) $c = 0$.

The characters of the LCFTs studied so far are mixed expressions of modular forms of weight 0 and 1. We call such theories LCFTs of degree 1. There are also indications

that one can even further conjecture that in general non-unitary rational $N = 2k$ SCFTs are logarithmic of degree k. This might explain the appearance of certain modular weights in partition functions of such models, which have been obtained without considering the theories as LCFTs. Work in this direction will be reported elsewhere [17]

There is an alternative view of the $c_{p,1}$ LCFTs, which might illuminate our understanding of the general space of 2-dimensional field theories (2dFTs), and how CFTs are embedded in this larger space. At least the series of $c_{p,1}$ LCFTs can be understood as limiting points of certain series of non-unitary minimal models.

Let us consider the two sequences $\mathcal{M}((\alpha+1)p, \alpha q)$ and $\mathcal{M}(\alpha p, (\alpha+1)q,$ with $\alpha \in \mathbb{Z}_+$. If $\alpha \to \infty$, we see that the central charges tend to

$$\lim_{\alpha \to \infty} 1 - 6\frac{((\alpha+1)p - \alpha q)^2}{\alpha(\alpha+1)pq} = \lim_{\alpha \to \infty} 1 - 6\frac{(\alpha p - (\alpha+1)q)^2}{\alpha(\alpha+1)pq} = 1 - 6\frac{(p-q)^2}{pq} = c_{p,1}, \quad (4.1)$$

while the *effective* central charges, defined as $c_{\text{eff}} = c - 24h_{\min}$, $h_{\min}$ being the smallest highest weight, tend to

$$\lim_{\alpha \to \infty} 1 - 6\frac{1}{\alpha(\alpha+1)pq} = 1, \quad (4.2)$$

in agreement with the fact that the $c_{p,1}$ LCFTs have effective central charge $c_{\text{eff}} = 1$ (see ☺). Let us consider the characters of these minimal models,

$$\chi_{r,s}^{((\alpha+1)p,\alpha q)} = \frac{1}{\eta}\left[\Theta_{(\alpha+1)pr - \alpha qs, \alpha(\alpha+1)pq} - \Theta_{(\alpha+1)pr + \alpha qs, \alpha(\alpha+1)pq}\right], \quad (4.3)$$

$$\chi_{r,s}^{(\alpha p,(\alpha+1)q)} = \frac{1}{\eta}\left[\Theta_{\alpha pr - (\alpha+1)qs, \alpha(\alpha+1)pq} - \Theta_{\alpha pr + (\alpha+1)qs, \alpha(\alpha+1)pq}\right]. \quad (4.4)$$

If $\alpha \to \infty$, the level of the Θ functions approaches arbitrarily close a number containing a square factor, $\alpha^2 pq$. In such cases, we have the following identity

$$\sum_{\mu=0}^{\alpha-1} \Theta_{\alpha(\lambda+2k\mu),\alpha^2 k} = \Theta_{\lambda,k}, \quad (4.5)$$

which actually is valid for arbitrary values of k, λ, but makes sense as an identity between modular forms only for $k, \lambda \in \mathbb{Z}/2$. This identity allows us in the limit to (approximately) collect characters of the huge minimal models (i.e. having a huge field content Λ with its number $|\Lambda|$ proportional to α^2) to a very much smaller set. But we must be very careful with our limiting procedure. Considering the difference between the two series, we see that

$$\Theta_{(\alpha+1)pr-\alpha qs, \alpha(\alpha+1)pq} - \Theta_{\alpha pr-(\alpha+1)qs, \alpha(\alpha+1)pq} \xrightarrow{\alpha \gg 1} \frac{pr-qs}{\alpha} \frac{\partial}{\partial(\alpha(pr-qs))}\Theta_{\alpha(pr-qs),\alpha(\alpha+1)pq},$$
$$(4.6)$$

which certainly vanishes in the limit. But we have to renormalize this expression to assure integer coefficients in the power series, otherwise we cannot treat it as a correction term

for *finite* α. The renormalization factor is usually chosen to be $(\partial\Theta)_{\lambda,k} = \frac{2k}{2\pi i\tau}\partial_\lambda\Theta_{\lambda,k}$, such that we recover the affine Θ functions. Plugging this into (4.5), we obtain (by choosing an appropriate sequence of pairs r, s such that $\lambda + 2pq\mu = pr - qs$)

$$\frac{2\alpha^2 pq}{2\pi i\tau}\sum_{\mu=0}^{\alpha-1}\frac{1}{\alpha}\frac{\partial}{\partial(\alpha\lambda)}\Theta_{\alpha(\lambda+2pq\mu),\alpha^2pq} = \frac{2pq}{2\pi i\tau}\frac{\partial}{\partial\lambda}\sum_{\mu=0}^{\alpha-1}\Theta_{\alpha(\lambda+2pq\mu),\alpha^2pq} = (\partial\Theta)_{\lambda,pq}\,. \tag{4.7}$$

Therefore, we conclude that the characters for the theory at the limit point have additional terms proportional to $(\partial\Theta)_{\lambda,pq}$. The different non-trivial factorizations $k = pq$ yield the different non-diagonal partition functions $Z_{\log}[p/q]$ for $c_{pq,1}$ models, given in ☺. This is not to be confused with the case where our two series of minimal models approach $\mathcal{M}(p, q)$. In this latter case we expect that the limit point actually is the LCFT $\mathcal{M}(3p, 3q)$.

This draws a new picture for the space of general 2dFTs. We certainly have 2dFTs $\supset$ CFTs $\supset$ RCFTs. The border between the space of CFTs and the space of non-conformal 2dFTs appears to be made out of LCFTs, which have as a subspace the border between RCFTs and non-conformal 2dFTs, namely the hereby established class of rational LCFTs. In a formal way we might write LCFTs $= \partial\overline{\text{CFTs}}$. Since the Zamolodchikov metric is known to remain regular at this border, LCFTs might serve as a new tool for describing transitions between different CFTs and also the more general transitions from a CFT into the non-conformal region and vice versa.

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