

# Notes on Generalised Nullvectors in logarithmic CFT

Holger Eberle\* and Michael Flohr†

Physikalisches Institut der Universität Bonn  
Nußallee 12, 53115 Bonn, Germany

## Abstract

In these notes we discuss the procedure how to calculate nullvectors in general indecomposable representations which are encountered in logarithmic conformal field theories. In particular, we do not make use of any of the restrictions which have been imposed in logarithmic nullvector calculations up to now, especially the quasi-primarity of all Jordan cell fields.

For the quite well-studied  $c_{p,1}$  models we calculate examples of logarithmic nullvectors which have not been accessible to the older methods and recover the known representation structure. Furthermore, we calculate logarithmic nullvectors in the up to now almost unexplored general augmented  $c_{p,q}$  models and use these to find bounds on their possible representation structures.

## 1 Introduction

In recent years the study of conformal field theories (CFTs) which also exhibit indecomposable structures in part of their representations has become an interesting and promising topic of research. A variety of applications of this sensible generalisation of ordinary CFTs, which are also commonly known as logarithmic CFTs, have already surfaced in statistical physics (e.g. [1, 2, 3, 4]), in Seiberg Witten theory (e.g. [5]) and even in string theory (e.g. [6, 7]). For a more complete survey of applications pursued so far as well as an introduction to the field see [8, 9, 10, 11].

Up to now the main focus of research has been put on a special class of logarithmic CFTs, the  $c_{p,1}$  models. The representation theory of their rank 2 indecomposable representations has been analysed completely in [12, 13, 14, 15] and a thorough understanding of the representations of the modular group corresponding to the enlarged triplet  $\mathcal{W}$ -algebra [16] of these models has been reached in [17, 18, 11]. Especially the  $c_{2,1} = -2$  model has been understood very well as it is isomorphic to a free field construction of the symplectic fermions [19, 20].

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\*email: eberle@th.physik.uni-bonn.de

†email: flohr@th.physik.uni-bonn.de

But going beyond representation theory we find that the calculation of explicit correlation functions proves to be much more intricate and tedious than in the ordinary CFT case [21, 22]. The construction of nullvectors, the key tool in CFT for the calculation of correlation functions, has already been addressed in [23, 24, 25, 26] for the case of indecomposable representations. However, the type of these logarithmic nullvectors calculated so far only describes a very special case. Already in the  $c_{p,1}$  models the generic logarithmic nullvectors are beyond the scope of this procedure.

On the other hand, the  $c_{p,1}$  models are only quite special representatives of the general class of augmented  $c_{p,q}$  models [18]. Although these models have already been addressed in some papers (see e.g. [27]), not much is known yet, neither about higher rank representations nor about nullvectors nor about correlation functions. There are, however, good indications that exactly these models might describe important statistical systems, such as percolation. Especially the augmented  $c_{2,3} = 0$  model seems to be of high interest in this respect [4].

The main goal of this paper is to show how to calculate logarithmic nullvectors in general, to give explicit examples and to use the information about the existence of nontrivial logarithmic nullvectors to explore the unknown structure of a more general class of logarithmic CFTs, namely the augmented  $c_{p,q}$  models.

In section 2 we give a short review of the special version of nullvector calculations which has been performed up to now in logarithmic models. In section 3 we discuss the limitations of this ansatz and propose a more general procedure which is capable of calculating all lowest logarithmic nullvectors in the  $c_{p,1}$  models. In particular, our new method does not rely on the quasi-primarity of all Jordan cell fields. In section 4 we then briefly introduce what we mean by generic augmented  $c_{p,q}$  models. We use the methods of the preceding sections with slight modifications in order to obtain constraints on what embedding structures possibly yield rank 2 indecomposable representations in these models. For this we will concentrate on the “smallest” model exhibiting this more generic behaviour, the augmented  $c_{2,3} = 0$  model. But any emerging structures should immediately generalise to any  $c_{p,q}$  model.

## 2 Jordan cells on lowest weight level

Let us briefly recall the construction of nullvectors in a logarithmic representation in which the states of the Jordan cell are all lowest weight states [23, 24, 8]. We will especially clarify the respective procedure in [8] and give a proof of the proposed logarithmic nullvector conditions. In the following, we will concentrate on Virasoro representations and try to keep close to the notations of [8].

A Jordan cell of lowest weight states with weight  $h$  of rank  $r$  is spanned by a basis of states

$$|h; n\rangle = \frac{1}{n!} \theta^n |h\rangle \quad \forall n = 0, \dots, r-1$$

on which the action of the Virasoro modes is given by

$$\begin{aligned} L_0|h;n\rangle &= h|h;n\rangle + (1 - \delta_{n,0})|h;n-1\rangle \\ &\equiv (h + \partial_\theta)|h;n\rangle, \\ L_p|h;n\rangle &= 0 \quad \forall p > 0. \end{aligned}$$

As already defined in [8],  $\theta$  is a nilpotent variable with  $\theta^r = 0$  and a handy tool to organize the Jordan cell states with the same weight. Due to their almost primary behaviour with the only defect of an additional term in the indecomposable  $L_0$  action we will call the  $|h;n\rangle$  logarithmic primary. We also note that  $|h;0\rangle$  is indeed a true primary state.

Using this information about the action of the Virasoro modes one can easily deduce that the action of any function of the Virasoro zero-mode and the central charge operator  $f(L_0, C)$  on such a Jordan cell state is given by [8]

$$f(L_0, C)|h;n\rangle = \sum_{k=0}^n \frac{1}{k!} \left( \frac{\partial^k}{\partial h^k} f(h, c) \right) |h;n-k\rangle. \quad (1)$$

On the other hand, calculation of logarithmic two-point-functions yields the following Shapovalov form of these Jordan cell states [21]

$$\langle h;k|h;l\rangle = \begin{cases} 0 & \forall l+k < r-1 \\ 1 & \forall l+k = r-1 \\ D_{l+k-r+1} & \forall l+k > r-1 \end{cases} \quad (2)$$

for constant  $D_j$ ,  $j = 1, \dots, r-1$ .

We now want to construct vectors which are null on the whole logarithmic representation. Following [8] we choose the general ansatz

$$|\chi_{h,c}^{(n)}\rangle = \sum_{j=0}^{r-1} \sum_{|\mathbf{n}|=n} b_j^{\mathbf{n}}(h, c) L_{-\mathbf{n}} |h; \partial_\theta^j a(\theta)\rangle \quad (3)$$

with  $a(\theta) = \sum_{i=0}^{r-1} a_i \frac{1}{i!} \theta^i$  and the usual multi-index notation for the modes  $L_{\mathbf{n}}$ . Choosing the states  $|h;j\rangle$  instead of  $|h; \partial_\theta^j a(\theta)\rangle$  on the right hand side would have yielded the same generality of the ansatz and in the end the same set of solutions. We prefer this special ansatz, however, for two reasons. First of all it already incorporates our knowledge that in this form of logarithmic representations there will only be a non-trivial action of the  $L_0$  modes in the end of the nullvector calculation and that this is given by derivatives wrt  $\theta$  onto lower ranks. This justifies the  $\partial_\theta^j$  part of the ansatz. On the other hand, by including lower orders of  $\theta^i$  into  $a(\theta)$  we solve for nullvectors of lower rank subrepresentations at the same time. For this we will always treat the  $a_m$  as arbitrary parameters.

Now, we can calculate the level  $n$  nullvector conditions for arbitrary  $k$  and  $\mathbf{n}_l$ ,  $|\mathbf{n}_l| = n$ , as follows (the index  $l$  indicates a suitable enumeration of the multi-indices  $\mathbf{n}$ ):

$$\begin{aligned}
& \langle h; r-1-k | L_{\mathbf{n}_l} | \chi_{h,c}^{(n)} \rangle \\
&= \sum_{j=0}^{r-1} \sum_{|\mathbf{n}|=n} b_j^{\mathbf{n}}(h,c) \sum_{m=j}^{r-1} a_m \langle h; r-1-k | L_{\mathbf{n}_l} L_{-\mathbf{n}} | h; m-j \rangle \\
&= \sum_{j=0}^{r-1} \sum_{|\mathbf{n}|=n} \sum_{m=j}^{r-1} a_m b_j^{\mathbf{n}}(h,c) \langle h; r-1-k | \sum_{t=0}^{m-j} \frac{1}{t!} \left( \frac{\partial^t}{\partial h^t} f_{\mathbf{n}_l, \mathbf{n}}(h,c) \right) | h; m-j-t \rangle \\
&= \sum_{m=0}^{r-1} a_m \sum_{j=0}^m \sum_{|\mathbf{n}|=n} \sum_{t=0}^{m-j} b_j^{\mathbf{n}}(h,c) \frac{1}{t!} \left( \frac{\partial^t}{\partial h^t} f_{\mathbf{n}_l, \mathbf{n}}(h,c) \right) \\
&\quad \times \left( \delta_{t, m-j-k} + \sum_{s=0}^{m-j-k-1} \delta_{s,t} D_{m-j-k-s} \right) \\
&=: \sum_{m=0}^{r-1} a_m \mathcal{A}'_{\mathbf{n}_l}(m, k), \tag{4}
\end{aligned}$$

where in the second step we used (1) due to the logarithmic primarity of the Jordan cell ground states  $|h; m\rangle$  as well as in the third step the Shapovalov form (2). As we want to keep the  $a_m$  as arbitrary parameters the nullvector conditions on  $|\chi_{h,c}^{(n)}\rangle$  are equivalent to the identical vanishing of all  $\mathcal{A}'_{\mathbf{n}_l}(m, k)$ .

Let us calculate the terms proportional to  $\delta_{t, m-j-k}$  in  $\mathcal{A}'_{\mathbf{n}_l}(m, k)$  first:

$$\begin{aligned}
& \sum_{j=0}^m \sum_{|\mathbf{n}|=n} \sum_{t=0}^{m-j} b_j^{\mathbf{n}}(h,c) \frac{1}{t!} \left( \frac{\partial^t}{\partial h^t} f_{\mathbf{n}_l, \mathbf{n}}(h,c) \right) \delta_{t, m-j-k} \\
&= \sum_{j=0}^{m-k} \sum_{|\mathbf{n}|=n} b_j^{\mathbf{n}}(h,c) \frac{1}{(m-k-j)!} \left( \frac{\partial^{m-k-j}}{\partial h^{m-k-j}} f_{\mathbf{n}_l, \mathbf{n}}(h,c) \right) \\
&=: \mathcal{A}_{\mathbf{n}_l}(m-k).
\end{aligned}$$

It is important to notice that the  $\mathcal{A}_{\mathbf{n}_l}(m-k)$  indeed only depend on the difference  $m-k$ .

We now show that the vanishing of the  $\mathcal{A}_{\mathbf{n}_l}(m-k)$  is necessary and already sufficient for the vanishing of the nullvector conditions in (4) by using complete induction over  $m-k$ . For  $m-k=0$  we trivially find  $\mathcal{A}'_{\mathbf{n}_l}(m, k) = \mathcal{A}_{\mathbf{n}_l}(m-k)$ . This can easily be inferred from (4) noting that in the fourth line the summation over  $t$  actually only runs from 0 to  $m-j-k$ , consequently the one over  $j$  only from 0 to  $m-k$ . On the other hand we find for general  $m-k$

$$\mathcal{A}'_{\mathbf{n}_l}(m, k) - \mathcal{A}_{\mathbf{n}_l}(m-k)$$

$$\begin{aligned}
&= \sum_{j=0}^m \sum_{|\mathbf{n}|=n} \sum_{t=0}^{m-j-k-1} b_j^{\mathbf{n}}(h, c) \frac{1}{t!} \left( \frac{\partial^t}{\partial h^t} f_{\mathbf{n}, \mathbf{n}}(h, c) \right) D_{m-j-k-t} \\
&= \sum_{j=0}^{m-k-1} \sum_{|\mathbf{n}|=n} \sum_{t=j}^{m-k-1} b_j^{\mathbf{n}}(h, c) \frac{1}{(t-j)!} \left( \frac{\partial^{t-j}}{\partial h^{t-j}} f_{\mathbf{n}, \mathbf{n}}(h, c) \right) D_{m-k-t} \\
&= \sum_{t=0}^{m-k-1} D_{m-k-t} \sum_{|\mathbf{n}|=n} \sum_{j=0}^t b_j^{\mathbf{n}}(h, c) \frac{1}{(t-j)!} \left( \frac{\partial^{t-j}}{\partial h^{t-j}} f_{\mathbf{n}, \mathbf{n}}(h, c) \right) \\
&= \sum_{t=0}^{m-k-1} D_{m-k-t} \mathcal{A}_{\mathbf{n}_l}(t) .
\end{aligned}$$

But this vanishes due to the induction assumption  $\mathcal{A}_{\mathbf{n}_l}(t) = 0$  for all  $t < m - k$ . Hence, the vanishing of  $\mathcal{A}'_{\mathbf{n}_l}(m, k)$  is equivalent to the vanishing of  $\mathcal{A}_{\mathbf{n}_l}(m - k)$  and therefore the statement is proven.

Now, if we regard these calculations with  $a_{r-1}$  as the only non-vanishing parameter we see that we still retain the whole set of conditions,  $\mathcal{A}_{\mathbf{n}_l}(r - 1 - k) = 0$  for all  $k = 0, \dots, r - 1$ . On the other hand, taking another  $a_i$  with  $i < r - 1$  as the only non-vanishing parameter automatically yields the respective set of conditions for a rank  $i + 1$  nullvector, a nullvector of a rank  $i + 1$  logarithmic subrepresentation. This indeed justifies our chosen ansatz as well as keeping the parameters  $a_m$  arbitrary.

Furthermore, this calculation shows that any nullvector wrt a (logarithmic or irreducible) representation is automatically a nullvector to any larger logarithmic representation containing the former one as a subrepresentation.

The third fact we would like to stress is that, generically, the nullvector of some rank  $r$  logarithmic representation is not a pure descendant of the Jordan cell state with rank index  $r$ , but always contains descendants of the other Jordan cell states with lower rank index, including the groundstate of the irreducible representation.

### 3 Logarithmic nullvectors for $c_{p,1}$

Already for the well-studied  $c_{p,1}$  models, however, the representations with Jordan cells on the lowest level analysed in the preceding section are not the end of the story but rather only very special cases [13]. For the generic rank 2 logarithmic representations in these models one needs a generalised way of calculating logarithmic nullvectors, which we will develop in the following. We will introduce these representations using the notational conventions of [13] and will then transfer to notations which are more adapted to our procedure.

In figure 1 we have depicted the two possible types of rank 2 Virasoro representations which appear for  $c_{p,1}$  as calculated in [13]. The dots correspond either to generating fields, i.e. fields which are not describable as descendants of some other field, or to singular descendants of these which, although they are null in the module of their

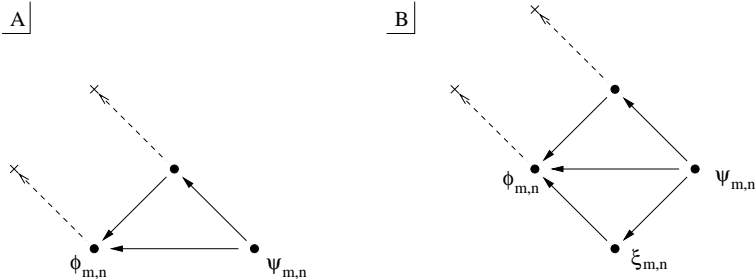


Figure 1: Types of rank 2 representations in  $c_{p,1}$  models

parent field, have a non-vanishing Shapovalov form with some other field of the whole rank 2 representation. The crosses, on the other hand, represent true nullvectors of the whole rank 2 representation. And finally, the arrows indicate which fields may be reached by the application of some polynomial in the Virasoro modes. But as remarked in the last section, nullvectors which are built in part on the second (or any higher) Jordan cell field always have to contain contributions from descendants of the primary field (and possibly lower Jordan cell fields) as well. The corresponding arrows have to be understood in this way. The naming of the fields follows the convention in [13], where the indices  $m, n$  refer to the Kac labels corresponding to the weight of the Jordan cell fields.

The case of a Jordan cell built solely on logarithmic primary fields which we discussed in section 2 and which corresponds to case **A** in figure 1 is just the exceptional case for the lowest lying representations (“lowest” in the sense of integer differences between the weights of the cyclic states). The generic rank 2 representation which is shown as case **B** in figure 1 contains an extra field  $\xi_{m,n}$  with lower weight than the Jordan cell which serves as a parent to the primary field  $\phi_{m,n}$ , the generator of the irreducible subrepresentation of the Jordan cell. Certainly,  $\phi_{m,n}$  needs to be a singular descendant of  $\xi_{m,n}$  in order to be primary. It is normalised such that the coefficient of the  $L_{-1}^l$  term is 1. Furthermore, the second field building the Jordan cell, the so-called “log-partner”  $\psi_{m,n}$ , is not logarithmic primary any more but is mapped to  $\xi_{m,n}$  by some polynomial of positive Virasoro modes. As argued in [13], if there is no additional null-vector on  $\xi_{m,n}$  on a level lower than  $\phi_{m,n}$ , this polynomial can be chosen as monomial such that

$$(L_1)^l \psi_{m,n} = \beta \xi_{m,n} \quad L_p \psi_{m,n} = 0 \quad \forall p \geq 2$$

for  $l = h(\phi_{m,n}) - h(\xi_{m,n})$  and constant  $\beta$  depending on the representation. In this setting  $L_1$  maps  $\psi_{m,n}$  to the unique  $l - 1$  descendant  $\xi^D$  of  $\xi_{m,n}$  with

$$(L_1)^{l-1} \xi^D = \beta \xi_{m,n} \quad L_p \xi^D = 0 \quad \forall p \geq 2. \quad (5)$$

This kind of representation requires a more general ansatz of logarithmic nullvectors. Loosing the prerequisite of logarithmic primarity of all Jordan cell fields we cannot

assume that only polynomials in the Virasoro null-mode and the central charge operator contribute to the matrix elements in the calculation of the nullvector conditions — we now have to take into account operators  $(L_{-1}L_1)^j$ ,  $j > 0$ , as well. Hence, the relation between the nullvector polynomials on the different Jordan cell states is not governed by the action of  $L_0$  and, thus, derivatives by  $\theta$  alone. An ansatz of the form (3) is not reasonable any more.

Instead, we propose the more general ansatz for the generic rank 2 representation

$$|\chi_{h,c}^{(n)}\rangle = \sum_{|\mathbf{n}|=n} b_1^{\mathbf{n}}(h,c) L_{-\mathbf{n}} |h;1\rangle + \sum_{|\mathbf{m}|=n+l} b_0^{\mathbf{m}}(h,c) L_{-\mathbf{m}} |h-l\rangle. \quad (6)$$

Here we choose a notation close to section 2 describing a state by its weight and enumerating Jordan cell states according to the  $L_0$  action

$$L_0 |h;n\rangle = h |h;n\rangle + (1 - \delta_{n,0}) |h;n-1\rangle.$$

The ansatz (6) certainly incorporates general level  $n$  descendants of  $\phi_{m,n} = |h;0\rangle$  as  $|h;0\rangle$  is just a level  $l$  descendant of  $\xi_{m,n} = |h-l\rangle$  itself. However, we need this more general ansatz (6) because building descendants only on the Jordan cell states we would miss out several states of the rank 2 representation which are descendants of level  $n+l$  of  $\xi_{m,n} = |h-l\rangle$ , but cannot be written as descendants of  $\phi_{m,n} = |h;0\rangle$ .

This ansatz leads to the following complete set of nullvector conditions

$$\begin{aligned} 0 &\stackrel{!}{=} \langle h;1 | L_{\mathbf{n}_i} |\chi_{h,c}^{(n)}\rangle \\ &= \sum_{|\mathbf{n}|=n} b_1^{\mathbf{n}}(h,c) \langle h;1 | L_{\mathbf{n}_i} L_{-\mathbf{n}} |h;1\rangle + \sum_{|\mathbf{m}|=n+l} b_0^{\mathbf{m}}(h,c) \langle h;1 | L_{\mathbf{n}_i} L_{-\mathbf{m}} |h-l\rangle \\ &= \sum_{|\mathbf{n}|=n} b_1^{\mathbf{n}}(h,c) \langle h;1 | F_{\mathbf{n}_i,\mathbf{n}}^{(1)}(L_{-1}^l L_1^l, L_0, C) |h;1\rangle \\ &\quad + \sum_{|\mathbf{m}|=n+l} b_0^{\mathbf{m}}(h,c) \langle h;1 | F_{\mathbf{n}_i,\mathbf{n}}^{(2)}(L_{-1}^l, L_{-1}^l L_0^s, L_{-1}^l C^t) |h-l\rangle \end{aligned} \quad (7)$$

as well as

$$\begin{aligned} 0 &\stackrel{!}{=} \langle h-l | L_{\mathbf{m}_j} |\chi_{h,c}^{(n)}\rangle \\ &= \sum_{|\mathbf{n}|=n} b_1^{\mathbf{n}}(h,c) \langle h-l | L_{\mathbf{m}_j} L_{-\mathbf{n}} |h;1\rangle + \sum_{|\mathbf{m}|=n+l} b_0^{\mathbf{m}}(h,c) \langle h-l | L_{\mathbf{m}_j} L_{-\mathbf{m}} |h-l\rangle \\ &= \sum_{|\mathbf{n}|=n} b_1^{\mathbf{n}}(h,c) \langle h-l | F_{\mathbf{m}_j,\mathbf{n}}^{(3)}(L_1^l, L_0^s L_1^l, C^t L_1^l) |h;1\rangle \\ &\quad + \sum_{|\mathbf{m}|=n+l} b_0^{\mathbf{m}}(h,c) \langle h-l | F_{\mathbf{m}_j,\mathbf{n}}^{(4)}(L_0, C) |h-l\rangle, \end{aligned} \quad (8)$$

for any  $s, t > 0$ . Several remarks are necessary. The functions  $F^{(1)}, \dots, F^{(4)}$  indicate what polynomials of Virasoro generators we can reduce the interior of the above matrix

elements to by successively using the Virasoro algebra, the level matching condition as well as properties of the states which these modes are applied to. Although we are not able to reduce these to polynomials solely of  $L_0$  and  $C$  as in section 2, these properties make possible a fair amount of reduction to functions which are polynomials only of specific combinations of the four operators  $L_{-1}$ ,  $L_0$ ,  $L_1$  and  $C$ . More specifically the function  $F^{(2)}$  actually only depends on terms proportional to  $L_{-1}^l$ ,  $L_{-1}^l L_0$ ,  $L_{-1}^l L_0^2$ ,  $\dots$  as well as  $L_{-1}^l C$ ,  $L_{-1}^l C^2$ ,  $\dots$ . This follows from the fact that to the right this function acts on a primary field, to the left however on a field which vanishes under the action of  $L_p$ ,  $p \geq 2$ , and whose weight is just at level  $l$  above  $|h-l\rangle$ .

As discussed earlier, we do not retain such nice interrelations between the nullvector polynomials as in section 2 which could be cast into the  $\theta$  calculus. But we can still find remnants of such relations as e.g. by looking at the nullvector conditions given by the application of level  $n$  descendants of  $|h;0\rangle$  onto the nullvector ansatz

$$\begin{aligned}
0 &\stackrel{!}{=} \langle h;0|L_{\mathbf{n}_j}|\chi_{h,c}^{(n)}\rangle \\
&= \sum_{|\mathbf{n}|=n} b_1^{\mathbf{n}}(h,c) \langle h;0|L_{\mathbf{n}_j} L_{-n} |h;1\rangle + \sum_{|\mathbf{m}|=n+l} b_0^{\mathbf{m}}(h,c) \langle h;0|L_{\mathbf{n}_j} L_{-m} |h-l\rangle \\
&= \sum_{|\mathbf{n}|=n} b_1^{\mathbf{n}}(h,c) \langle h;0|F_{\mathbf{n}_j,\mathbf{n}}^{(1)}(L_{-1}^l L_1^l, L_0, C) |h;1\rangle \\
&\quad + \sum_{|\mathbf{m}|=n+l} b_0^{\mathbf{m}}(h,c) \langle h;0|F_{\mathbf{n}_j,\mathbf{n}}^{(2)}(L_{-1}^l, L_{-1}^l L_0^s, L_{-1}^l C^t) |h-l\rangle \\
&= \sum_{|\mathbf{n}|=n} b_1^{\mathbf{n}}(h,c) \langle h;0|F_{\mathbf{n}_j,\mathbf{n}}^{(1)}(L_{-1}^l L_1^l, L_0, C) |h;1\rangle . \tag{9}
\end{aligned}$$

These conditions are clearly a subset of the conditions (8) as  $|h;0\rangle$  is just a descendant of  $|h-l\rangle$ . Now we make use of the Shapovalov form (2) to deduce that the only terms contributing to the matrix elements in (9) can come from contributions of  $F_{\mathbf{n}_j,\mathbf{n}}^{(1)}(L_{-1}^l L_1^l, L_0, C) |h;1\rangle$  which are proportional to  $|h;1\rangle$ . But then we can insert these vanishing equations back into (7) concluding that the terms in (7) proportional to  $D_1$  (of the Shapovalov form) already vanish on their own — a consequence of a subset of the relations (8). This is a reminiscence of the fact that in section 2 the conditions  $\mathcal{A}'_{\mathbf{n}_i}(m,k)$  can be split into the conditions  $\mathcal{A}_{\mathbf{n}_i}(m-k)$  which only depend on the difference  $m-k$ . Hence we can conclude that any logarithmic nullvector of the proposed kind does not depend on the constants of the Shapovalov form.

Now one can put the Virasoro-algebraic calculations on the computer and solve the resulting equations for possible logarithmic nullvectors. Details about the implementation can be found in appendix A. We have calculated this for the  $c_{2,1} = -2$  representation  $\mathcal{R}_{2,1}$  with lowest lying vector  $\xi_{2,1}$  at  $h=0$  and Jordan cell  $(\phi_{2,1} = L_{-1} \xi_{2,1}, \psi_{2,1})$  at  $h=1$ , a representation of type **B** (see figure 1), and found the following first nontrivial nullvector at level 6 (above the lowest lying vector)

$$\left( m_1 L_{-1}^6 + m_2 L_{-2} L_{-1}^4 + m_3 L_{-3} L_{-1}^3 + \left( \frac{16}{3} - 4m_1 + \frac{16}{3}\beta - 2m_2 \right) L_{-2}^2 L_{-1}^2 \right)$$



$$\begin{aligned}
& +(-12 - 12m_1 + 6\beta)L_{-4}L_{-1}^2 + \left(-\frac{20}{3} - 2m_3 - 16m_1 - \frac{56}{3}\beta - 2m_2\right)L_{-3}L_{-2}L_{-1} \\
& +\left(\frac{4}{3} - 16m_1 + \frac{10}{3}\beta - 2m_2\right)L_{-5}L_{-1} - 8\beta L_{-4}L_{-2} + 6\beta L_{-3}^2 - 4\beta L_{-6}\right) \xi_{2,1} \\
& +\left(L_{-1}^5 - 10L_{-2}L_{-1}^3 + 6L_{-3}L_{-1}^2 + 16L_{-2}^2L_{-1} - 12L_{-4}L_{-1} - 8L_{-3}L_{-2} + 4L_{-5}\right) \psi_{2,1}.
\end{aligned}$$

This level is indeed the expected one as the Kac table of  $c_{2,1} = -2$  gives us a third nullvector condition for  $h = 0$  exactly at level 6 as well as a corresponding second nullvector condition for  $h = 1$  at level 5. Hence, we confirm the existence of a further nontrivial nullvector at the expected level in the logarithmic rank 2 representation  $\mathcal{R}_{2,1}$  derived by different means in [13].

We notice that up to the overall normalisation of this state the nullvector polynomial applied to the second Jordan cell state  $\psi_{2,1}$  is unique. On the other hand, the nullvector polynomial on  $\xi_{2,1}$ , which serves as a correction to the effects of the indecomposable action on  $\psi_{2,1}$ , still exhibits three degrees of freedom. But we know that there is an ordinary nullvector at level 2 above  $h = 1$  in the irreducible subrepresentation whose descendants span a parameter space of dimension three at level 5 (above  $h = 1$ ). Adding such a descendant of this nullvector will certainly not alter our equations and, hence, accounts for the additional three degrees of freedom  $m_i$ ,  $i = 1, 2, 3$ .

In the same manner one can calculate logarithmic nullvectors in all rank 2 logarithmic representations in the  $c_{p,1}$  models, limited only by computer power and memory. We give a second example for a type **B** logarithmic nullvector in appendix B.

## 4 Possible logarithmic nullvectors for $\mathbf{c}_{2,3} = \mathbf{0}$

The  $c_{p,1}$  models might be the best-studied logarithmically conformal models but they still are quite special cases of the general augmented  $c_{p,q}$  models, which we still know much less about. Hence, an even more exciting question than the above construction of predicted logarithmic nullvectors surely is whether one can use these techniques to explore the shapes of the supposedly more complicated logarithmic representations in general augmented CFTs. In the following we will attack this question for the augmented model  $c_{2,3} = 0$  which seems sufficiently generic to show all the features of general augmented  $c_{p,q}$  models.

### 4.1 Some facts about augmented Kac tables

Minimal models manage to extract the smallest possible representation theory from the Kac table of some central charge  $c_{p,q}$  by mapping all weights to some standard cell using the relations [28]

$$\begin{aligned}
h_{r,s} &= h_{q-r,p-s} \\
h_{r,s} &= h_{r+q,s+p} \quad \forall 1 \leq r < q \quad 1 \leq s < p.
\end{aligned} \tag{10}$$

Table 1: Augmented Kac table for  $c_{2,3} = 0$

		$s$				
		1	2	3	4	5
$r$	1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7
	2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5
	3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$
	4	1	$\frac{1}{8}$	0	$\frac{5}{8}$	2
	5	2	$\frac{5}{8}$	0	$\frac{1}{8}$	1
	6	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$
	7	5	$\frac{21}{8}$	1	$\frac{1}{8}$	0
	8	7	$\frac{33}{8}$	2	$\frac{5}{8}$	0

The only weights which these relations do not relate to anything are the weights on the border of the Kac table, i.e. the weights for  $r = q$  or  $s = p$  as well as their integer multiples. But as these do not appear in any fusion rule of the other fields in the bulk of the Kac table they are simply ignored.

Augmenting the theory with fields beyond this standard cell has led to the construction of consistent CFTs containing representations with non-trivial Jordan blocks, examples of logarithmic CFTs. These theories can be associated to Kac tables which comprise the standard cell for larger parameters  $p$  and  $q$  yielding the same conformal charge; these parameters are usually odd integer multiples of  $p$  and  $q$ , i.e. we deal with the standard cell of  $c_{np,nq}$ ,  $n \in (2\mathbb{N} - 1)$ . Hence, these theories also contain fields with weights on the border of the original smaller Kac table as well as fields in the corners, which are the intersections of the borders. To give an example, we indicated the borders as areas with lighter shade, the corners as areas with darker shade in the augmented Kac table of  $c_{2,3}$  given in table 1; the bulk consists of the unshaded areas. All fields of the augmented Kac tables yield independent representations only subject to the relations (10) for the augmented cell, i.e.  $p \mapsto np$ ,  $q \mapsto nq$ . This is the result of the nontrivial fusion of the fields on the border and the corners with themselves and the fields from the bulk. Indeed, this fusion behaviour prevents the theory from just becoming a tensor product of several independent minimal model sectors.

Actually, the only well studied models up to now are contained in the series  $c_{p,1}$ ,  $p = 2, 3, \dots$  (see e.g. [13, 14, 17, 18, 8]). These models are not generic in that respect that they do not contain any nontrivial bulk in their Kac table; they only consist of fields from the border and corners.

The first and easiest example which exhibits a non-empty bulk of the Kac table is the augmented  $c_{2,3} = 0$  model with the Kac table of  $c_{6,9} = 0$  which we will explore more

closely in the following using our techniques of logarithmic nullvectors. As mentioned before, the corresponding Kac table is given in table 1.

The fact that we can restrict the representation theory of our CFT to a finite field content as with the above cells of the Kac table certainly relies on the enlargement of the symmetry algebra from a simple Virasoro algebra to a nontrivial  $\mathcal{W}$ -algebra, be it either in the minimal model or the augmented model case. In the following, however, we will restrict our focus to the Virasoro algebra and representations thereof. Still, having the possibility of such an enlarged  $\mathcal{W}$ -algebra in mind, we will mainly focus on representations connected to weights in the augmented Kac table cell keeping in mind that from the point of view of the Virasoro algebra there is an infinite tower of representations.

## 4.2 Weights on the corners and borders of the augmented Kac table of $c_{2,3} = 0$

We propose that fields associated to weights on the corner and the borders of the augmented Kac table of  $c_{2,3} = 0$  are contained in the same types of representations as the corresponding ones in the  $c_{p,1}$  models.

The weights on the corners of the Kac table,  $h = -1/24, 35/24$ , actually only appear once modulo the relations (10) and accordingly only exhibit the usual two nullvector conditions. Hence, there are no new (logarithmic) representations to be expected besides the ordinary irreducible Virasoro representation built on groundstates with these weights. Indeed, these weights give exactly the prelogarithmic fields which have been shown to be primary and to generate an irreducible representation, but not to admit an embedding into any larger indecomposable representation [29].

The weights on the borders of the Kac table actually appear in the same kind of triplets of two equal conformal weights and one which is shifted by some positive integer as we know it from the  $c_{p,1}$  models (again modulo the relations (10)). The triplets are  $T_1 := \{1/8, 1/8, 33/8\}$ ,  $T_2 := \{5/8, 5/8, 21/8\}$  and  $T_3 := \{1/3, 1/3, 10/3\}$ . We also find the same nullvector structure concerning these weights within the Kac table as we know it from the corresponding representations of the  $c_{p,1}$  models. Hence, we have checked the existence of the typical logarithmic nullvectors for all cases which were accessible to computer power and memory.

First we have checked for the first logarithmic nullvector in representation type **A** (see figure 1) and found the expected ones for all three triplets, on level 8 for  $T_1$ , on level 10 for  $T_2$  as well as on level 9 for  $T_3$ . The result for  $T_1$  can be found in appendix C.

A check for the first logarithmic nullvector in representation type **B** was only possible for the triplet  $T_2$ . In this case, we have a Jordan cell for  $h = 21/8$  with a lower lying field at  $h = 5/8$ . We find the first nontrivial logarithmic nullvector at level 16 which seems to be just at the limit of our current computing power and ability. The first logarithmic nullvectors for type **B** representations corresponding to the other two triplets are expected at even higher levels, at 18 and 20 for  $T_3$  resp.  $T_1$ .

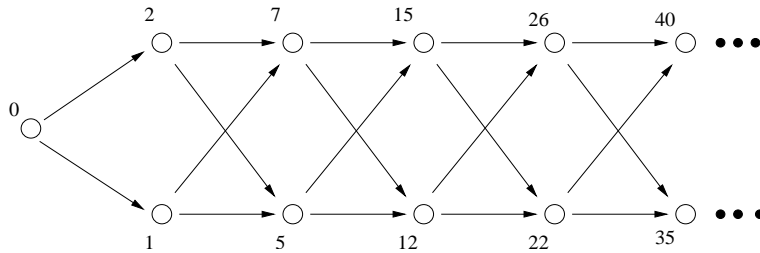


Figure 2: Embedding structure for  $h = 0$  (numbers refer to weights)

This is indeed compatible with our above proposition and a nice and nontrivial check for its validity. The above proposition is also substantiated by the analysis of possible modular functions of the corresponding  $\mathcal{W}$ -algebra [30].

### 4.3 Weights in the bulk of the augmented Kac table of $c_{2,3} = 0$

For possible logarithmic representations corresponding to weights in the bulk of the Kac table, however, we do not have any prototypes yet. Hence, we are now going to explore the main candidates for such representations and elaborate constraints on their shapes using our techniques of constructing logarithmic nullvectors. We notice two main differences to the situation on the borders.

First of all, the bulk of the augmented Kac table of  $c_{2,3} = 0$  (see table 1) exhibits an even higher abundance of equal numbers (up to integer shift) than in the  $c_{p,1}$  models, which is a clear sign of logarithmic representations there. Up to the relations (10) we actually find a nonuplet  $N = \{0, 0, 0, 1, 1, 2, 2, 5, 7\}$  of weights which are equal up to integer shift and which contain one weight with triple degeneracy,  $h = 0$ . It does not seem very likely that this nonuplet just splits into three triplets of the types analysed above. On the contrary, the analysis of the corresponding modular functions even suggests the possibility of a rank 3 logarithmic representation, and certainly predicts the existence of several more complicated rank 2 logarithmic representations constructed with weights within this set [30].

On the other hand, we have to notice that the embedding structure for nullvectors is different in the bulk in contrast to the linear embedding structure on the border (discussed in [13]). In the bulk the embedding structure is given by the more generic two string twisted picture, depicted in figure 2, which can be calculated according to general arguments in [31] or the Virasoro character formula of [32].<sup>1</sup>

Now, inspecting the nonuplet  $N$  of bulk weights we expect the usual irreducible representations to the integer weights  $h = 0, 1, 2, 5, 7$  as well as rank 2 representations corresponding to Jordan cells at weight  $h = 0, 1, 2$ . We have depicted a list of possible candidates for rank 2 representations corresponding to these bulk weights in figure 3. These pictures represent the low lying embedding structure of these candidates

<sup>1</sup>We thank A. Nichols for pointing this out to us.

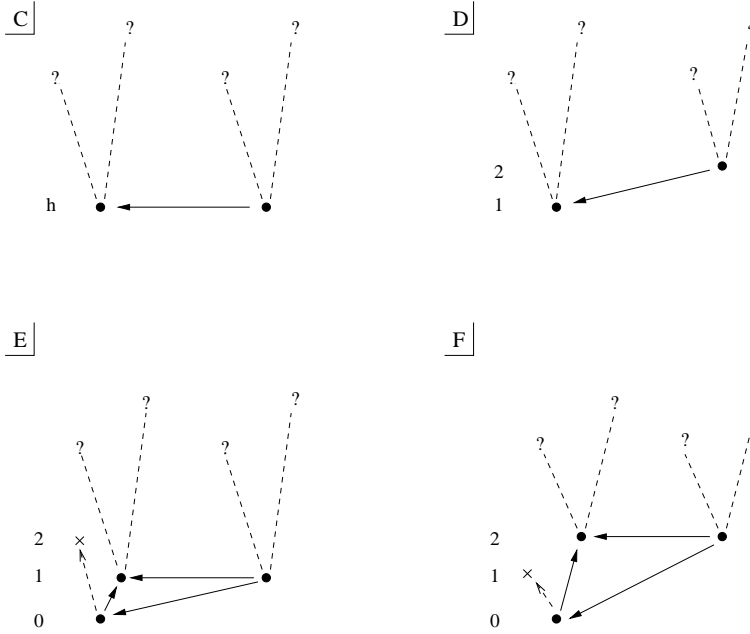


Figure 3: Candidates for rank 2 representations for weights in the bulk of  $c = 0$

using the same symbols as in section 3. Additionally, we have indicated the conformal weight on the different levels to the left of each picture as well as the unknown higher embedding structure by question marks (“?”). We have checked for the lowest nontrivial logarithmic nullvectors for all these candidates and summarise the results in table 2.

**Type C.** The calculations for the type **C** representations have been performed using the methods of section 2. For this type we even managed to calculate one rank 3 logarithmic nullvector; i.e. the first rank 3 logarithmic nullvector with lowest weight Jordan cell at  $h = 0$  appears at level 12.

**Type E.** We were able to apply the procedure of section 3 directly to the type **E** representation because we do not encounter any additional nullvector below the level of the Jordan cell and because we can take  $(L_0 - h)$  to map  $|h; 1\rangle$  to a proper singular descendant of  $|h - l\rangle$ , i.e.  $L_{-1}|h - l\rangle$ .

**Type F.** In case of the type **F** representation we actually encounter an additional nullvector below the level of the Jordan cell. This can, however, be remedied quite easily. Due to the lower lying nullvector at level 1 there is no state which  $L_1$  could map  $|h; 1\rangle$  to. But certainly  $L_2$  can take the job to map  $|h; 1\rangle$  directly down to  $|h - l\rangle$ , a mapping unique up to normalisation. This yields the new conditions

$$L_2|h; 1\rangle = \beta|h - l\rangle \quad L_p|h; 1\rangle = 0 \quad \text{for } p = 1 \text{ and } p \geq 3.$$

The singular descendant in the Jordan cell is therefore given by  $|h; 0\rangle = L_{-2}|h - l\rangle$ .

Table 2: Lowest logarithmic nullvectors of candidates for bulk representations

type	lowest weight	rank	level of lowest logarithmic nullvector
C	0	2	5
C	1	2	11
C	2	2	10
C	0	3	12
D	1	2	> 14
E	0	2	12
F	0	2	12

This feature of an additional nullvector below the level of the Jordan cell clearly shows the novelty of the bulk representations in contrast to the ones on the border described in [13]; it arises due to the more intricate embedding structure for these bulk representations compared to the embedding structure for representations on the border. A generalisation to similar cases with additional nullvectors on levels lower than the Jordan cell seems straightforward though more tedious due to the more complex embedding structure of nullvectors which are not on the “nice” level 1.

**Type D.** The case of the type **D** representation is more questionable. Here, we actually do not have a singular descendant of  $|h-l\rangle$  on the level of the Jordan cell, hence no primary state which  $(L_0 - h)$  could map  $|h; 1\rangle$  to. A priori it is not clear whether it is necessary for  $L_0$  to map  $|h; 1\rangle$  to a singular descendant of  $|h-l\rangle$ . Hence we took  $(L_0 - h)$  to map  $|h; 1\rangle$  to the only existing descendant of  $|h-l\rangle$  at level 1 which is though not singular, i.e.  $L_{-1}|h-l\rangle$ . The results of possible logarithmic nullvectors, however, do not seem to offer a particular rich structure up to the accessible levels (see table 2).

We include the explicit results for the two cases **E** and **F** in appendix D. It is quite interesting to inspect e.g. the result for type **F**. Although we actually did not impose the relation  $L_{-1}|h-l\rangle = 0$  into the computer programme, the result incorporates such a nullvector or, to be a bit more cautious, at least the independence of the result from this particular descendant; indeed, all descendants of  $L_{-1}|h-l\rangle$  just appear with free parameters. This corresponds to the additional freedom due to lower nullvectors in the irreducible subrepresentation discussed in the end of section 3. On the other hand, the second singular vector on  $|h-l\rangle$  on level 2 does not pop up in the same manner as a possible nullvector in this result; rather, the result is consistent with our ansatz where we actually impose that the level 2 singular vector on  $|h-l\rangle$  is not a nullvector for the whole rank 2 representation. Hence, we take the first observation of the independence from  $L_{-1}|h-l\rangle$  as a strong hint that a representation where both these singular descendants are not null in the whole logarithmic representation is not

favoured by our calculations.

## 5 Conclusion and outlook

In the preceding sections we have elaborated the procedure how to calculate generic logarithmic nullvectors. This generalises former computations, which assumed that all fields in Jordan cells were (logarithmic) quasi-primary. We have calculated several examples of such nullvectors in rank 2 indecomposable representations, for two  $c_{p,1}$  models as well as for the more generic augmented  $c_{2,3} = 0$  model.

Although these logarithmic nullvectors do not enable us to write down the whole indecomposable representations and hence to decide which of these representations are realised in our model, they nevertheless provide us with severe constraints about the number of states on the lower levels for all inspected candidates. As discussed for representation type  $\mathbf{F}$  one can even use the calculated logarithmic nullvector to give strong arguments for the existence of singular vectors of the irreducible subrepresentation as nullvectors of the whole indecomposable representation.

On the other hand, we would like to stress again that we can regard the augmented  $c_{2,3} = 0$  model as a prototype for the general augmented  $c_{p,q}$  models. As for the series of  $c_{p,1}$  models where one encounters the same structure of singlets on the corners and triplets on the borders throughout the series, we find the same structures which we have described in section 4 for all  $c_{p,q}$  models: singlets on the corners, triplets on the borders and nonuplets in the bulk — only with a larger variety for larger models. And, most importantly, we find the same kind of relations between the levels of possible nullvectors and the differences between the weights in the  $n$ -plets as the ones described for  $c_{2,3} = 0$  in section 4.

Concerning the classification of nullvectors in logarithmic CFT [23, 25, 26], we observe that the generic nullvectors of rank 2 exist precisely at the levels one would guess from the Kac table, where the level has to be counted from the conformal weight of the Jordan cell. Within the bulk, it seems possible that even rank 3 generic nullvectors exist. However, we have only been able to calculate one example of a rank 3 logarithmic nullvector in the  $h = 0$  sector of the  $c_{2,3}$  model so far.

Thus, we made use of the information about logarithmic nullvectors to give a rough picture about how the embedding structure of rank 2 logarithmic representations might look like for the up to now almost unexplored augmented  $c_{p,q}$  models.

Certainly one can now use these explicitly constructed rank 2 nullvectors to calculate corresponding correlation functions and hence analyse the physical dynamics of these theories. This would have to proceed along the lines indicated in [24].

On the other hand, one seems to be very close to finally pinpoint the full structure of the rank 2 representations in the  $c_{2,3}$  model and, hence, in all  $c_{p,q}$  models. In order to achieve this it seems necessary to bring together the constraints on possible rank 2 representations calculated in this paper with the knowledge about the representation of the modular group corresponding to the inherent enlarged  $\mathcal{W}$ -algebra. Indeed, this

larger symmetry algebra makes it possible to uniquely fix any representation of  $c_{2,3}$  if one knows the multiplicities of states up to level 7. But it is still not clear how to combine Virasoro representations to full  $\mathcal{W}$ -representations. These considerations are subject to ongoing research [30]. Furthermore, the construction of these representations will supposedly also settle the long standing puzzle around the structure of the vacuum character in the augmented  $c_{2,3} = 0$  model (see e.g. [27]). On the other hand, it does not seem too far out of reach to construct generic  $\mathcal{W}$ -nullvectors along the lines set out in this work. The knowledge of such nullvectors would enable us to prove rationality, or at least  $C_2$ -cofiniteness, for augmented  $c_{p,q}$  models using the approach of [33].

After completion of this work the article [34] appeared which proves the equivalence between the  $c_{2,1}$  model as a vertex operator algebra and a corresponding quantum group and conjectures a similar result for general  $c_{p,1}$  models. This manifestation of the Kazhdan–Lusztig correspondence as an equivalence for the case of the  $c_{p,1}$  models leads to a highly interesting insight in the quantum group structure of these conformal field theories and might facilitate the classification of representations in these models. We hope that results as the ones about logarithmic nullvectors in this article will finally lead to a generalisation of [34] to general  $c_{p,q}$  models, although regarding our still very small knowledge about these models this seems still a long way to go.

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## A Implementation of the logarithmic nullvector calculation on the computer

The calculations performed for this article have been implemented in C++ using the computer algebra package GiNaC [35]. We constructed new classes for the algebraic objects fields, fieldmodes, products of fieldmodes as well as descendant fields. The implemented Virasoro relations (as well as possibly further commutation relations) are used for direct simplification of descendant fields towards the normal ordered standard form as soon as these are constructed. It is important to note that within the procedure the application of modes to the field has priority to the commutation of modes in order to reduce the blow-up of the number of terms within the calculation.

The calculation of matrix elements is performed in two steps. First all fieldmodes within the matrix elements are used to construct a descendant state on the ket-state which is then automatically simplified (see above). Then the correct coefficients are picked using the properties of the bra-state as well as the Shapovalov form.

The main property which has to be implemented into the fields, besides their conformal weight (and possibly fermion number), is their behaviour under the action of



nonnegative Virasoro modes. We picked conformal primarity as the standard and implemented deviations from that in a list which is pointed to by a member of each instance. E.g. for calculations of representation type **B** (see figure 1) we had to implement the indecomposable  $L_0$  action as well as the non-vanishing  $L_1$  action which maps  $|h; 1\rangle$  to the unique level  $(l - 1)$  descendant of  $|h - l\rangle$  with properties (5).

## B An explicit nullvector for $\mathbf{c}_{3,1} = -7$

As a further example for a type **B** rank 2 logarithmic nullvector (see figure 1) we give the respective nullvector with lowest lying vector at  $h = -1/4$  and Jordan cell at  $h = 7/4$  in the augmented model of  $c_{3,1} = -7$  which appears at level 10. For the sake of reasonable brevity we have set the overall normalisation to 1 and also eliminated any freedom due to the existence of lower nullvectors in the irreducible subrepresentation by setting any further free parameter to 0; the parameter  $\beta$  certainly still remains as it is a parameter of the representation as introduced in section 3:

$$\begin{aligned}
& \left( \left( \frac{7168}{27} - 512\beta \right) L_{-3} L_{-2}^2 L_{-1}^3 + \left( -\frac{1280}{81} + \frac{256}{9}\beta \right) L_{-5} L_{-2} L_{-1}^3 + \left( -\frac{21376}{27} - \frac{128}{3}\beta \right) L_{-4} L_{-3} L_{-1}^3 \right. \\
& + \left( \frac{8552}{27} + \frac{3280}{9}\beta \right) L_{-7} L_{-1}^3 + \left( -\frac{4096}{27} + \frac{512}{3}\beta \right) L_{-2}^4 L_{-1}^2 + \left( \frac{67840}{81} + \frac{256}{9}\beta \right) L_{-4} L_{-2}^2 L_{-1}^2 \\
& + \left( -\frac{18112}{81} + \frac{8192}{9}\beta \right) L_{-3}^2 L_{-2} L_{-1}^2 + \left( \frac{30080}{81} - \frac{11008}{9}\beta \right) L_{-6} L_{-2} L_{-1}^2 + (8 + 48\beta) L_{-5} L_{-3} L_{-1}^2 \\
& + \left( -\frac{3968}{27} - \frac{736}{3}\beta \right) L_{-4}^2 L_{-1}^2 + \left( \frac{24752}{81} - \frac{2992}{9}\beta \right) L_{-8} L_{-1}^2 + \left( -\frac{7168}{81} + \frac{512}{3}\beta \right) L_{-3} L_{-2}^3 L_{-1} \\
& + \left( \frac{1280}{243} + \frac{2048}{27}\beta \right) L_{-5} L_{-2}^2 L_{-1} + \left( \frac{21376}{81} + \frac{3968}{9}\beta \right) L_{-4} L_{-3} L_{-2} L_{-1} - \frac{3008}{9}\beta L_{-3}^3 L_{-1} \\
& + \left( -\frac{8552}{81} - 496\beta \right) L_{-7} L_{-2} L_{-1} + \frac{2720}{3}\beta L_{-6} L_{-3} L_{-1} - 208\beta L_{-5} L_{-4} L_{-1} + \frac{5564}{9}\beta L_{-9} L_{-1} \\
& + \left( \frac{4096}{81} - \frac{512}{9}\beta \right) L_{-2}^5 + \left( -\frac{67840}{243} - \frac{6400}{27}\beta \right) L_{-4} L_{-2}^3 + \left( -\frac{3392}{243} + \frac{1024}{27}\beta \right) L_{-3}^2 L_{-2}^2 \\
& + \left( -\frac{30080}{243} + \frac{11776}{27}\beta \right) L_{-6} L_{-2}^2 + \left( -\frac{42376}{243} - \frac{496}{27}\beta \right) L_{-5} L_{-3} L_{-2} + \left( \frac{3968}{81} + \frac{4576}{9}\beta \right) L_{-4}^2 L_{-2} \\
& + \left( -\frac{110768}{243} + \frac{15904}{27}\beta \right) L_{-8} L_{-2} + \left( \frac{21376}{81} - \frac{832}{9}\beta \right) L_{-4} L_{-3}^2 + \left( -\frac{30056}{81} + 288\beta \right) L_{-7} L_{-3} \\
& + \frac{448}{3}\beta L_{-6} L_{-4} + \left( \frac{1280}{243} + \frac{1580}{27}\beta \right) L_{-5}^2 + \left( -\frac{28672}{27} + 2128\beta \right) L_{-10} \Big) |h - l\rangle \\
& + \left( L_{-1}^8 - 28L_{-2} L_{-1}^6 + 28L_{-3} L_{-1}^5 + \frac{658}{3} L_{-2}^2 L_{-1}^4 - 132L_{-4} L_{-1}^4 - \frac{1048}{3} L_{-3} L_{-2} L_{-1}^3 \right. \\
& + \frac{976}{9} L_{-5} L_{-1}^3 - \frac{12916}{27} L_{-2}^3 L_{-1}^2 + \frac{8344}{9} L_{-4} L_{-2} L_{-1}^2 + \frac{1604}{9} L_{-3}^2 L_{-1}^2 + 240L_{-6} L_{-1}^2 \\
& + 668L_{-3} L_{-2}^2 L_{-1} - \frac{1312}{3} L_{-5} L_{-2} L_{-1} - 576L_{-4} L_{-3} L_{-1} - 186L_{-7} L_{-1} - \frac{1708}{3} L_{-4} L_{-2}^2 \\
& \left. + \frac{1225}{9} L_{-2}^4 - \frac{728}{3} L_{-3}^2 L_{-2} - 224L_{-6} L_{-2} + \frac{602}{3} L_{-5} L_{-3} + 252L_{-4}^2 - 476L_{-8} \right) |h; 1\rangle
\end{aligned}$$

## C Explicit nullvectors on the border of $\mathbf{c}_{2,3} = 0$

In the following we give the explicit form of the nullvector of type **A** for the triplet  $T_1$ , which has a Jordan cell at lowest weight  $h = 1/8$  and appears at level 8. We have set the overall normalisation to 1 in this expression; any further free parameters, however, appear as calculated (noted as  $m_i$ ). We also note again that  $\beta$  is not a free parameter of the logarithmic nullvector calculation but a parameter of the representation as

introduced in section 3:

$$\begin{aligned}
& \left( m_{21}L_{-1}^8 + m_{20}L_{-2}L_{-1}^6 + m_{19}L_{-3}L_{-1}^5 + m_{18}L_{-2}^2L_{-1}^4 + m_{17}L_{-4}L_{-1}^4 \right. \\
& + \left( \frac{64}{9} - \frac{13}{3}m_{19} - \frac{278}{3}m_{21} - 7m_{20} \right) L_{-3}L_{-2}L_{-1}^3 + \left( \frac{616}{9} - \frac{698}{9}m_{21} - 7m_{20} \right) L_{-5}L_{-1}^3 \\
& + \left( -\frac{4069}{54}m_{21} - \frac{13}{3}m_{18} - \frac{217}{12}m_{20} \right) L_{-2}^3L_{-1}^2 + \left( \frac{2870}{9} + 15m_{21} - 5m_{20} \right) L_{-6}L_{-1}^2 \\
& + \left( \frac{752}{9} - \frac{1475}{9}m_{21} - \frac{5}{2}m_{20} - \frac{13}{3}m_{17} \right) L_{-4}L_{-2}L_{-1}^2 + \left( -\frac{640}{9} - \frac{35}{12}m_{21} - \frac{1}{2}m_{17} \right) L_{-4}^2 \\
& + \left( -\frac{2164}{9} - \frac{8}{3}m_{19} - \frac{301}{9}m_{21} \right) L_{-3}^2L_{-1}^2 + \left( \frac{5425}{432}m_{21} + \frac{25}{36}m_{18} + \frac{325}{108}m_{20} \right) L_{-2}^4 \\
& + \left( -540 + 142m_{21} + \frac{10}{3}m_{18} + \frac{461}{18}m_{20} \right) L_{-5}L_{-2}L_{-1} + \left( \frac{1888}{27} + \frac{25}{36}m_{19} + \frac{955}{18}m_{21} + \right. \\
& \left. \frac{5}{3}m_{18} + 10m_{20} \right) L_{-3}L_{-2}^2L_{-1} + \left( \frac{2768}{9} + \frac{17}{6}m_{19} + 143m_{21} + \frac{5}{3}m_{17} \right) L_{-4}L_{-3}L_{-1} \\
& + \left( -\frac{614}{3} + \frac{17}{6}m_{19} + \frac{14059}{36}m_{21} + 5m_{18} + \frac{281}{6}m_{20} \right) L_{-7}L_{-1} + \left( -\frac{104}{9} + \frac{1087}{36}m_{21} \right. \\
& \left. - \frac{1}{2}m_{18} + \frac{11}{18}m_{20} + \frac{25}{36}m_{17} \right) L_{-4}L_{-2}^2 + \left( \frac{266}{27} + \frac{25}{18}m_{19} + \frac{250}{9}m_{21} + \frac{25}{18}m_{20} \right) L_{-3}^2L_{-2} \\
& + \left( -\frac{43}{3} + \frac{131}{18}m_{21} - 2m_{18} - \frac{19}{3}m_{20} \right) L_{-6}L_{-2} + \left( \frac{2482}{27} - \frac{29}{36}m_{19} + \frac{131}{4}m_{21} \right. \\
& \left. + \frac{25}{9}m_{20} \right) L_{-5}L_{-3} + \left( \frac{1220}{27} - \frac{29}{18}m_{19} - \frac{3985}{18}m_{21} - 4m_{18} - \frac{247}{9}m_{20} \right) L_{-8} \Big) |h-l\rangle \\
& + \left( L_{-1}^8 - \frac{58}{3}L_{-2}L_{-1}^6 + \frac{70}{3}L_{-3}L_{-1}^5 + \frac{1547}{18}L_{-2}^2L_{-1}^4 - \frac{347}{3}L_{-4}L_{-1}^4 - \frac{526}{9}L_{-3}L_{-2}L_{-1}^3 \right. \\
& + \frac{520}{9}L_{-5}L_{-1}^3 - \frac{589}{6}L_{-2}^3L_{-1}^2 + \frac{1157}{3}L_{-4}L_{-2}L_{-1}^2 - \frac{287}{3}L_{-3}^2L_{-1}^2 + \frac{335}{3}L_{-6}L_{-1}^2 \\
& + \frac{115}{6}L_{-3}L_{-2}^2L_{-1} - \frac{200}{3}L_{-5}L_{-2}L_{-1} + \frac{49}{3}L_{-4}L_{-3}L_{-1} - \frac{229}{12}L_{-7}L_{-1} + \frac{225}{16}L_{-4}^2 \\
& \left. - \frac{1259}{12}L_{-4}L_{-2}^2 + \frac{100}{3}L_{-3}^2L_{-2} - \frac{253}{6}L_{-6}L_{-2} - \frac{159}{4}L_{-5}L_{-3} + \frac{659}{12}L_{-4}^2 - \frac{433}{6}L_{-8} \right) |h;1\rangle
\end{aligned}$$

We also give the type **B** logarithmic nullvector for the triplet  $T_2$ , which has a Jordan cell at  $h = 21/8$ , a lowest weight at  $h = 5/8$  and appears at level 16. As this expression is very lengthy we have again set the overall normalisation to 1 and also eliminated any further freedom by setting any further free parameter to 0:

$$\begin{aligned}
& \left( \left( -\frac{149120}{27} - \frac{11648}{9}\beta \right) L_{-3}L_{-2}^2L_{-1}^9 + \left( -\frac{6100784}{81} - \frac{4996016}{405}\beta \right) L_{-5}L_{-2}L_{-1}^9 + \left( -\frac{564736}{27} - \right. \right. \\
& \left. \frac{2531584}{135}\beta \right) L_{-4}L_{-3}L_{-1}^9 + \left( \frac{1492754}{27} - \frac{761422}{135}\beta \right) L_{-7}L_{-1}^9 + \left( \frac{58240}{27} + \frac{11648}{27}\beta \right) L_{-2}^4L_{-1}^8 \\
& + \left( \frac{10401664}{81} + \frac{9886336}{405}\beta \right) L_{-4}L_{-2}^2L_{-1}^8 + \left( \frac{6547352}{81} + \frac{13290968}{405}\beta \right) L_{-3}^2L_{-2}L_{-1}^8 + \left( \frac{11199188}{81} \right. \\
& \left. + \frac{11666036}{405}\beta \right) L_{-6}L_{-2}L_{-1}^8 + \left( \frac{11824162}{81} + \frac{16560418}{405}\beta \right) L_{-5}L_{-3}L_{-1}^8 + \left( -\frac{2163322}{27} \right. \\
& \left. + \frac{18004358}{135}\beta \right) L_{-4}^2L_{-1}^8 + \left( \frac{25195051}{81} - \frac{19264057}{81}\beta \right) L_{-8}L_{-1}^8 + \left( -\frac{298240}{27} - \frac{55552}{27}\beta \right) L_{-3}L_{-2}^3L_{-1}^7 \\
& + \left( \frac{389529760}{243} + \frac{72031904}{243}\beta \right) L_{-5}L_{-2}^2L_{-1}^7 + \left( -\frac{43345088}{81} - \frac{4798528}{81}\beta \right) L_{-4}L_{-3}L_{-2}L_{-1}^7 \\
& + \left( -\frac{161243044}{81} + \frac{264057116}{405}\beta \right) L_{-7}L_{-2}L_{-1}^7 + \left( -\frac{3836560}{27} - \frac{1439984}{15}\beta \right) L_{-3}^3L_{-1}^7 + \left( -\frac{39224696}{81} \right. \\
& \left. + \frac{79328632}{405}\beta \right) L_{-6}L_{-3}L_{-1}^7 + \left( -\frac{30255232}{81} - \frac{420183616}{405}\beta \right) L_{-5}L_{-4}L_{-1}^7 \\
& + \left( -\frac{115606213}{81} + \frac{317139341}{405}\beta \right) L_{-9}L_{-1}^7 + \left( -\frac{546560}{81} - \frac{109312}{81}\beta \right) L_{-2}^5L_{-1}^6 \\
& + \left( -\frac{504068096}{243} - \frac{414460928}{1215}\beta \right) L_{-4}L_{-2}^3L_{-1}^6 + \left( -\frac{32395888}{243} - \frac{221287024}{1215}\beta \right) L_{-3}^2L_{-2}^2L_{-1}^6 \\
& + \left( -\frac{688257640}{243} - \frac{1894942504}{1215}\beta \right) L_{-6}L_{-2}^2L_{-1}^6 + \left( -\frac{171246124}{81} - \frac{23791292}{81}\beta \right) L_{-5}L_{-3}L_{-2}L_{-1}^6 \\
& + \left( \frac{436742516}{81} - \frac{285590284}{405}\beta \right) L_{-4}^2L_{-2}L_{-1}^6 + \left( -\frac{1311363362}{243} + \frac{618446318}{243}\beta \right) L_{-8}L_{-2}L_{-1}^6 + \\
& \left( -\frac{14349776}{81} + \frac{87406928}{81}\beta \right) L_{-4}L_{-3}L_{-1}^6 + \left( \frac{675988264}{81} - \frac{1346752232}{405}\beta \right) L_{-7}L_{-3}L_{-1}^6
\end{aligned}$$

$$\begin{aligned}
& + \left( \frac{130901740}{27} + \frac{530789812}{135} \beta \right) L_{-6} L_{-4} L_{-1}^6 + \left( \frac{57193417}{243} + \frac{323247827}{243} \beta \right) L_{-5}^2 L_{-1}^6 \\
& + \left( \frac{377046397}{27} - \frac{82026941}{135} \beta \right) L_{-10} L_{-1}^6 + \left( \frac{24149440}{81} + \frac{478912}{9} \beta \right) L_{-3} L_{-2}^4 L_{-1}^5 \\
& + \left( -\frac{6472052776}{729} - \frac{1153026056}{729} \beta \right) L_{-5} L_{-2}^3 L_{-1}^5 + \left( \frac{491223904}{81} + \frac{724362016}{405} \beta \right) L_{-4} L_{-3} L_{-2}^2 L_{-1}^5 \\
& + \left( \frac{1289170697}{81} - \frac{2296000231}{405} \beta \right) L_{-7} L_{-2}^2 L_{-1}^5 + \left( -\frac{247531144}{243} - \frac{33723704}{243} \beta \right) L_{-3}^3 L_{-2} L_{-1}^5 \\
& + \left( \frac{74820020}{9} + \frac{37930060}{9} \beta \right) L_{-6} L_{-3} L_{-2} L_{-1}^5 + \left( -\frac{349441376}{243} + \frac{8040650368}{1215} \beta \right) L_{-5} L_{-4} L_{-2} L_{-1}^5 \\
& + \left( \frac{289876117}{18} - \frac{908160193}{162} \beta \right) L_{-9} L_{-2} L_{-1}^5 + \left( -\frac{899217364}{243} - \frac{1223108068}{1215} \beta \right) L_{-5} L_{-3} L_{-1}^5 \\
& + \left( \frac{54747340}{9} - \frac{21043364}{9} \beta \right) L_{-4}^2 L_{-3} L_{-1}^5 + \left( -\frac{3343815314}{243} + \frac{1020421910}{243} \beta \right) L_{-8} L_{-3} L_{-1}^5 \\
& + \left( -\frac{4106356768}{81} + \frac{4238202096}{81} \beta \right) L_{-7} L_{-4} L_{-1}^5 + \left( \frac{6087567380}{243} - \frac{1708569956}{243} \beta \right) L_{-6} L_{-5} L_{-1}^5 \\
& + \left( \frac{1975182488}{243} - \frac{5191641784}{1215} \beta \right) L_{-4} L_{-3}^2 L_{-2} L_{-1}^4 + \left( -\frac{10951360}{243} - \frac{2190272}{243} \beta \right) L_{-2}^6 L_{-1}^4 \\
& + \left( \frac{6439826624}{729} + \frac{4398126656}{3645} \beta \right) L_{-4} L_{-2}^4 L_{-1}^4 + \left( -\frac{805947068}{729} + \frac{946244068}{3645} \beta \right) L_{-3}^2 L_{-2}^3 L_{-1}^4 \\
& + \left( \frac{9716768158}{729} + \frac{39829789006}{3645} \beta \right) L_{-6} L_{-2}^3 L_{-1}^4 + \left( \frac{1492444673}{243} - \frac{3094966463}{1215} \beta \right) L_{-5} L_{-3} L_{-2}^2 L_{-1}^4 \\
& + \left( -\frac{10110621079}{243} - \frac{300543275}{243} \beta \right) L_{-4}^2 L_{-2}^2 L_{-1}^4 + \left( \frac{12461262967}{486} - \frac{7049560643}{810} \beta \right) L_{-8} L_{-2}^2 L_{-1}^4 \\
& + \left( -\frac{23944200971}{486} - \frac{1603601245}{486} \beta \right) L_{-11} L_{-1}^5 + \left( -\frac{11528458402}{243} + \frac{25042759046}{1215} \beta \right) L_{-7} L_{-3} L_{-2} L_{-1}^4 \\
& + \left( -\frac{19090179110}{243} - \frac{62902001258}{1215} \beta \right) L_{-6} L_{-4} L_{-2} L_{-1}^4 + \left( \frac{9182523901}{1458} - \frac{9713083541}{7290} \beta \right) L_{-5} L_{-2} L_{-1}^4 \\
& + \left( -\frac{65932612501}{486} - \frac{3429904271}{486} \beta \right) L_{-10} L_{-2} L_{-1}^4 + \left( \frac{383243984}{243} + \frac{323521072}{243} \beta \right) L_{-3}^4 L_{-1}^4 \\
& + \left( \frac{55495736}{27} - \frac{829672184}{81} \beta \right) L_{-6} L_{-3}^2 L_{-1}^4 + \left( \frac{1588949072}{243} + \frac{24002270192}{1215} \beta \right) L_{-5} L_{-4} L_{-3} L_{-1}^4 \\
& + \left( \frac{957145103}{243} - \frac{23998651177}{1215} \beta \right) L_{-9} L_{-3} L_{-1}^4 + \left( -\frac{44315552}{27} - \frac{171541792}{27} \beta \right) L_{-4}^3 L_{-1}^4 \\
& + \left( \frac{4968093016}{243} + \frac{5395525736}{243} \beta \right) L_{-8} L_{-4} L_{-1}^4 + \left( \frac{2882901476}{243} - \frac{33207042922}{1215} \beta \right) L_{-7} L_{-5} L_{-1}^4 \\
& + \left( -\frac{6761458844}{243} + \frac{3531609476}{243} \beta \right) L_{-6}^2 L_{-1}^4 + \left( -\frac{5936082307}{243} - \frac{19072395649}{1215} \beta \right) L_{-12} L_{-1}^4 \\
& + \left( -\frac{260692160}{729} + \frac{1981504}{81} \beta \right) L_{-3} L_{-2}^5 L_{-1}^3 + \left( \frac{25860528968}{2187} + \frac{13853999912}{10935} \beta \right) L_{-5} L_{-2}^4 L_{-1}^3 \\
& + \left( -\frac{115094416}{9} - \frac{397874864}{81} \beta \right) L_{-4} L_{-3} L_{-2}^3 L_{-1}^3 + \left( -\frac{9183968675}{243} + \frac{60366559}{5} \beta \right) L_{-7} L_{-2}^3 L_{-1}^3 \\
& + \left( \frac{22446028}{9} + \frac{381549548}{243} \beta \right) L_{-3}^3 L_{-2}^2 L_{-1}^3 + \left( -\frac{1857923638}{243} - \frac{26692922234}{1215} \beta \right) L_{-6} L_{-3} L_{-2}^2 L_{-1}^3 \\
& + \left( \frac{18483518080}{729} - \frac{1635799888}{729} \beta \right) L_{-5} L_{-4} L_{-2}^2 L_{-1}^3 + \left( -\frac{106944789343}{2916} + \frac{123521710391}{14580} \beta \right) L_{-9} L_{-2}^2 L_{-1}^3 \\
& + \left( \frac{10262941796}{729} + \frac{4345875956}{3645} \beta \right) L_{-5} L_{-3}^2 L_{-2} L_{-1}^3 + \left( -\frac{2914640996}{81} + \frac{5112097436}{405} \beta \right) L_{-4}^2 L_{-3} L_{-2} L_{-1}^3 \\
& + \left( \frac{41031702454}{729} - \frac{71201971634}{3645} \beta \right) L_{-8} L_{-3} L_{-2} L_{-1}^3 + \left( \frac{81698877104}{243} - \frac{40731621952}{1215} \beta \right) L_{-7} L_{-4} L_{-2} L_{-1}^3 \\
& + \left( -\frac{114847983652}{729} + \frac{184898225636}{3645} \beta \right) L_{-6} L_{-5} L_{-2} L_{-1}^3 + \left( \frac{464402212315}{1458} + \frac{154322577601}{7290} \beta \right) L_{-11} L_{-2} L_{-1}^3 \\
& + \left( \frac{15314752}{3} - \frac{1668550912}{405} \beta \right) L_{-4} L_{-3}^3 L_{-1}^3 + \left( -\frac{5194154518}{81} + \frac{2708172854}{135} \beta \right) L_{-7} L_{-2}^2 L_{-1}^3 \\
& + \left( \frac{2137814356}{81} + \frac{221522588}{81} \beta \right) L_{-6} L_{-4} L_{-3} L_{-1}^3 + \left( \frac{1006401133}{729} - \frac{51304031429}{3645} \beta \right) L_{-5}^2 L_{-3} L_{-1}^3 \\
& + \left( -\frac{418142455}{9} + \frac{10153248079}{405} \beta \right) L_{-10} L_{-3} L_{-1}^3 + \left( -\frac{913003520}{81} + \frac{3109906496}{405} \beta \right) L_{-5} L_{-4} L_{-1}^3 \\
& + \left( \frac{3782161967}{81} + \frac{14793947}{135} \beta \right) L_{-9} L_{-4} L_{-1}^3 + \left( -\frac{10430827276}{243} + \frac{697278628}{405} \beta \right) L_{-8} L_{-5} L_{-1}^3 \\
& + \left( \frac{4093892420}{81} - \frac{5063706352}{405} \beta \right) L_{-7} L_{-6} L_{-1}^3 + \left( \frac{2816698331}{108} + \frac{14750597275}{324} \beta \right) L_{-13} L_{-1}^3 \\
& + \left( -\frac{3876174992}{243} - \frac{71245077136}{1215} \beta \right) L_{-5} L_{-4} L_{-3} L_{-2} L_{-1}^2 + \left( -\frac{15573821056}{2187} - \frac{5099410048}{10935} \beta \right) L_{-4} L_{-2}^5 L_{-1}^2 \\
& + \left( -\frac{240207548}{2187} - \frac{8494313084}{10935} \beta \right) L_{-3}^2 L_{-4}^2 L_{-1}^2 + \left( -\frac{47436984506}{2187} - \frac{39860881954}{2187} \beta \right) L_{-6} L_{-2}^4 L_{-1}^2 \\
& + \left( \frac{47109428591}{2187} + \frac{184063255247}{10935} \beta \right) L_{-5} L_{-3} L_{-2}^3 L_{-1}^2 + \left( \frac{34161020861}{729} - \frac{4963416739}{3645} \beta \right) L_{-4}^2 L_{-3}^2 L_{-1}^2 \\
& + \left( -\frac{5763567647}{1458} + \frac{2691154249}{162} \beta \right) L_{-8} L_{-2}^3 L_{-1}^2 + \left( -\frac{13695268444}{729} + \frac{7011515372}{3645} \beta \right) L_{-4} L_{-3}^2 L_{-2}^2 L_{-1}^2
\end{aligned}$$

$$\begin{aligned}
& + \left( \frac{21517454756}{729} - \frac{116959711288}{3645} \beta \right) L_{-7} L_{-3} L_{-2}^2 L_{-1}^2 + \left( \frac{196664238547}{729} + \frac{542209846357}{3645} \beta \right) L_{-6} L_{-4} L_{-2}^2 L_{-1}^2 \\
& + \left( -\frac{238400926535}{2916} - \frac{104126564893}{2916} \beta \right) L_{-5}^2 L_{-2}^2 L_{-1}^2 + \left( \frac{855093533125}{2916} + \frac{606232113211}{14580} \beta \right) L_{-10} L_{-2}^2 L_{-1}^2 \\
& + \left( \frac{1639307648}{729} - \frac{5639533168}{3645} \beta \right) L_{-3}^4 L_{-2} L_{-1}^2 + \left( -\frac{16056755168}{243} + \frac{3324232744}{243} \beta \right) L_{-6} L_{-3}^2 L_{-2} L_{-1}^2 \\
& + \left( \frac{9611840}{729} + \frac{1922368}{729} \beta \right) L_{-2}^7 L_{-1}^2 + \left( -\frac{10283821591}{1458} + \frac{377475370451}{7290} \beta \right) L_{-9} L_{-3} L_{-2} L_{-1}^2 \\
& + \left( \frac{520593056}{81} + \frac{54200032}{3} \beta \right) L_{-4}^3 L_{-2} L_{-1}^2 + \left( -\frac{119998885100}{729} - \frac{62785077796}{729} \beta \right) L_{-8} L_{-4} L_{-2} L_{-1}^2 \\
& + \left( \frac{8363078144}{729} + \frac{380846363054}{3645} \beta \right) L_{-7} L_{-5} L_{-2} L_{-1}^2 + \left( \frac{82219813000}{729} - \frac{29138179912}{729} \beta \right) L_{-6}^2 L_{-2} L_{-1}^2 \\
& + \left( \frac{31541761250}{729} + \frac{97639549658}{3645} \beta \right) L_{-12} L_{-2} L_{-1}^2 + \left( \frac{3724220342}{729} + \frac{16033283606}{3645} \beta \right) L_{-5} L_{-3}^3 L_{-1}^2 \\
& + \left( -\frac{676716122}{81} - \frac{135149054}{135} \beta \right) L_{-4}^2 L_{-3}^2 L_{-1}^2 + \left( \frac{39555209293}{729} + \frac{95016781}{3645} \beta \right) L_{-8} L_{-3}^2 L_{-1}^2 \\
& + \left( \frac{11257024304}{243} + \frac{15187070696}{1215} \beta \right) L_{-7} L_{-4} L_{-3} L_{-1}^2 + \left( -\frac{13835770486}{729} - \frac{20728972126}{3645} \beta \right) L_{-6} L_{-5} L_{-3} L_{-1}^2 \\
& + \left( \frac{161403296071}{1458} + \frac{185298212569}{7290} \beta \right) L_{-11} L_{-3} L_{-1}^2 + \left( -\frac{1353700318}{9} - \frac{5011610942}{81} \beta \right) L_{-6} L_{-4}^2 L_{-1}^2 \\
& + \left( \frac{39014405471}{729} + \frac{81704195681}{3645} \beta \right) L_{-5}^2 L_{-4} L_{-1}^2 + \left( -\frac{1251898711}{27} - \frac{3014837587}{405} \beta \right) L_{-10} L_{-4} L_{-1}^2 \\
& + \left( -\frac{54758596976}{729} - \frac{189462959732}{3645} \beta \right) L_{-9} L_{-5} L_{-1}^2 + \left( \frac{26091481631}{243} + \frac{81857918359}{1215} \beta \right) L_{-8} L_{-6} L_{-1}^2 \\
& + \left( \frac{41764853429}{243} - \frac{59364317437}{1215} \beta \right) L_{-6} L_{-5} L_{-2}^2 L_{-1} + \left( \frac{519772644341}{729} + \frac{536982902849}{3645} \beta \right) L_{-14} L_{-1}^2 \\
& + \left( \frac{16951960}{243} - \frac{9326296}{81} \beta \right) L_{-3} L_{-2}^6 L_{-1} + \left( -\frac{1718220067}{729} + \frac{565209697}{729} \beta \right) L_{-5} L_{-2}^5 L_{-1} \\
& + \left( \frac{611230072}{243} + \frac{3131873176}{1215} \beta \right) L_{-4} L_{-3} L_{-2}^4 L_{-1} + \left( \frac{16790069291}{648} - \frac{3114171113}{72} \beta \right) L_{-7} L_{-2}^4 L_{-1} \\
& + \left( \frac{508133378}{243} - \frac{522868522}{1215} \beta \right) L_{-3}^3 L_{-2}^3 L_{-1} + \left( -\frac{3350726791}{243} + \frac{16970704619}{1215} \beta \right) L_{-6} L_{-3} L_{-2}^3 L_{-1} \\
& + \left( -\frac{6183346072}{243} - \frac{12019334032}{1215} \beta \right) L_{-5} L_{-4} L_{-2}^3 L_{-1} + \left( \frac{537159191}{648} - \frac{2545333679}{3240} \beta \right) L_{-9} L_{-3}^2 L_{-1} \\
& + \left( -\frac{7490314333}{243} - \frac{5707344769}{1215} \beta \right) L_{-5} L_{-3}^2 L_{-2}^2 L_{-1} + \left( \frac{4251155539}{81} - \frac{2417418029}{405} \beta \right) L_{-4}^2 L_{-3} L_{-2}^2 L_{-1} \\
& + \left( -\frac{13774642501}{486} + \frac{47574031139}{2430} \beta \right) L_{-8} L_{-3} L_{-2}^2 L_{-1} + \left( -\frac{30597896944}{81} + \frac{7602405812}{405} \beta \right) L_{-7} L_{-4} L_{-2}^2 L_{-1} \\
& + \left( -\frac{1818452026}{81} + \frac{2487870947}{405} \beta \right) L_{-7}^2 L_{-1}^2 + \left( -\frac{598093659731}{1944} - \frac{118832263337}{9720} \beta \right) L_{-11} L_{-2}^2 L_{-1} \\
& + \left( -\frac{1127428304}{81} + \frac{2005731856}{405} \beta \right) L_{-4} L_{-3}^3 L_{-2} L_{-1} + \left( \frac{8937898306}{81} - \frac{9374211074}{405} \beta \right) L_{-7} L_{-3}^2 L_{-2} L_{-1} \\
& + \left( -\frac{2175490966}{81} - \frac{7301230378}{405} \beta \right) L_{-6} L_{-4} L_{-3} L_{-2} L_{-1} + \left( \frac{5686048555}{486} + \frac{71774155933}{2430} \beta \right) L_{-5}^2 L_{-3} L_{-2} L_{-1} \\
& + \left( \frac{5640483061}{162} - \frac{25935548597}{810} \beta \right) L_{-10} L_{-3} L_{-2} L_{-1} + \left( \frac{2068510520}{81} - \frac{175247464}{405} \beta \right) L_{-5} L_{-4}^2 L_{-2} L_{-1} \\
& + \left( \frac{30251263}{18} - \frac{780131773}{270} \beta \right) L_{-9} L_{-4} L_{-2} L_{-1} + \left( \frac{1652402846}{81} - \frac{13911205078}{405} \beta \right) L_{-8} L_{-5} L_{-2} L_{-1} \\
& + \left( -\frac{2116132367}{27} + \frac{3248810563}{135} \beta \right) L_{-7} L_{-6} L_{-2} L_{-1} + \left( -\frac{8015375435}{648} - \frac{40699798345}{648} \beta \right) L_{-13} L_{-2} L_{-1} \\
& + \left( -\frac{124276184}{81} - \frac{369101384}{405} \beta \right) L_{-3}^5 L_{-1} + \left( \frac{474109532}{27} + \frac{439356476}{45} \beta \right) L_{-6} L_{-3}^3 L_{-1} \\
& + \left( \frac{813558668}{81} - \frac{5307013348}{405} \beta \right) L_{-5} L_{-4} L_{-2}^2 L_{-1} + \left( -\frac{1380887179}{54} + \frac{6613469743}{270} \beta \right) L_{-9} L_{-3}^2 L_{-1} \\
& + \left( -\frac{185916220}{9} + \frac{20241916}{3} \beta \right) L_{-4}^3 L_{-3} L_{-1} + \left( \frac{242801510}{9} - \frac{4142537918}{135} \beta \right) L_{-8} L_{-4} L_{-3} L_{-1} \\
& + \left( -\frac{2519102860}{81} + \frac{1743712978}{81} \beta \right) L_{-7} L_{-5} L_{-3} L_{-1} + \left( -\frac{48696580}{81} - \frac{10819957036}{405} \beta \right) L_{-6}^2 L_{-3} L_{-1} \\
& + \left( \frac{3430860452}{81} + \frac{18602113922}{405} \beta \right) L_{-12} L_{-3} L_{-1} + \left( \frac{7433166527}{54} - \frac{2647655137}{270} \beta \right) L_{-7} L_{-4}^2 L_{-1} \\
& + \left( -\frac{4287087352}{81} + \frac{2955645208}{81} \beta \right) L_{-6} L_{-5} L_{-4} L_{-1} + \left( -\frac{887650963}{54} - \frac{8540950477}{270} \beta \right) L_{-11} L_{-4} L_{-1} \\
& + \left( \frac{1877772049}{243} - \frac{1317572437}{243} \beta \right) L_{-5}^3 L_{-1} + \left( \frac{10351048265}{81} + \frac{14282331911}{405} \beta \right) L_{-10} L_{-5} L_{-1} \\
& + \left( \frac{382719419}{9} - \frac{178157449}{3} \beta \right) L_{-9} L_{-6} L_{-1} + \left( -\frac{45624853453}{324} + \frac{38349308507}{1620} \beta \right) L_{-8} L_{-7} L_{-1} \\
& + \left( -\frac{45585350849}{162} - \frac{100134503837}{3240} \beta \right) L_{-15} L_{-1} + \left( \frac{22715000}{243} + \frac{4543000}{243} \beta \right) L_{-2}^8 \\
& + \left( -\frac{2052937000}{729} - \frac{548807000}{729} \beta \right) L_{-4} L_{-2}^6 + \left( \frac{1729008875}{1458} + \frac{700883575}{1458} \beta \right) L_{-3}^2 L_{-2}^5
\end{aligned}$$

$$\begin{aligned}
& + \left( \frac{31484615917}{2916} + \frac{13690595345}{2916} \beta \right) L_{-6} L_{-2}^5 + \left( -\frac{141033886615}{5832} - \frac{54089482475}{5832} \beta \right) L_{-5} L_{-3} L_{-2}^4 \\
& + \left( \frac{16209468985}{648} + \frac{4917725989}{648} \beta \right) L_{-4}^2 L_{-2}^4 + \left( -\frac{69278656301}{1296} - \frac{5818122955}{432} \beta \right) L_{-8} L_{-2}^4 \\
& + \left( \frac{327789298}{243} - \frac{182443906}{243} \beta \right) L_{-4} L_{-2}^3 L_{-2}^3 + \left( \frac{579707603}{54} + \frac{1499507321}{162} \beta \right) L_{-7} L_{-3} L_{-2}^3 \\
& + \left( -\frac{71894257961}{486} - \frac{27674494435}{486} \beta \right) L_{-6} L_{-4} L_{-2}^3 + \left( \frac{529286724439}{5832} + \frac{171551292269}{5832} \beta \right) L_{-5}^2 L_{-2}^3 \\
& + \left( -\frac{219343338799}{1944} - \frac{48802331837}{1944} \beta \right) L_{-10} L_{-2}^3 + \left( -\frac{752201776}{243} - \frac{99365984}{243} \beta \right) L_{-3}^4 L_{-2}^2 \\
& + \left( \frac{3167581984}{81} + \frac{164797772}{81} \beta \right) L_{-6} L_{-2}^3 L_{-2}^2 + \left( \frac{7333242724}{243} + \frac{5959030124}{243} \beta \right) L_{-5} L_{-4} L_{-3} L_{-2}^2 \\
& + \left( -\frac{3744239456}{81} - \frac{4563477511}{162} \beta \right) L_{-9} L_{-3} L_{-2}^2 + \left( -\frac{4568992472}{81} - \frac{1329441208}{81} \beta \right) L_{-4}^3 L_{-2}^2 \\
& + \left( \frac{98011788952}{243} + \frac{23350859840}{243} \beta \right) L_{-8} L_{-4} L_{-2}^2 + \left( -\frac{20981999923}{243} - \frac{24546873745}{486} \beta \right) L_{-7} L_{-5} L_{-2}^2 \\
& + \left( -\frac{12537042125}{243} + \frac{2293822643}{243} \beta \right) L_{-6}^2 L_{-2}^2 + \left( \frac{148867854119}{972} + \frac{25735300993}{972} \beta \right) L_{-12} L_{-2}^2 \\
& + \left( \frac{3397474222}{243} + \frac{592140590}{243} \beta \right) L_{-5} L_{-3}^3 L_{-2} + \left( -\frac{153629552}{81} - \frac{157430272}{81} \beta \right) L_{-4}^2 L_{-3} L_{-2} \\
& + \left( -\frac{18698585200}{243} - \frac{3030629912}{243} \beta \right) L_{-8} L_{-3}^2 L_{-2} + \left( \frac{332973250}{81} + \frac{334271030}{81} \beta \right) L_{-7} L_{-4} L_{-3} L_{-2} \\
& + \left( \frac{4006295476}{243} + \frac{1534401548}{243} \beta \right) L_{-6} L_{-5} L_{-3} L_{-2} + \left( -\frac{39235245287}{243} - \frac{11785966714}{243} \beta \right) L_{-11} L_{-3} L_{-2} \\
& + \left( \frac{3943923671}{27} + 51922221 \beta \right) L_{-6} L_{-4}^2 L_{-2} + \left( -\frac{56023386371}{486} - \frac{17628497017}{486} \beta \right) L_{-5}^2 L_{-4} L_{-2} \\
& + \left( \frac{18665820475}{162} + \frac{4831631945}{162} \beta \right) L_{-10} L_{-4} L_{-2} + \left( \frac{132951996853}{486} + \frac{49107342479}{486} \beta \right) L_{-9} L_{-5} L_{-2} \\
& + \left( -\frac{2535966295}{18} - \frac{424001521}{6} \beta \right) L_{-8} L_{-6} L_{-2} + \left( -\frac{3527974355}{324} - \frac{2781374389}{324} \beta \right) L_{-7}^2 L_{-2} \\
& + \left( -\frac{256267624133}{486} - \frac{53727010357}{486} \beta \right) L_{-14} L_{-2} + \left( \frac{62662600}{81} + \frac{33837512}{81} \beta \right) L_{-4} L_{-3}^4 \\
& + \left( \frac{12081754}{81} - \frac{386061646}{81} \beta \right) L_{-7} L_{-3}^3 + \left( -\frac{1235827894}{81} + \frac{324696958}{81} \beta \right) L_{-6} L_{-4} L_{-2}^3 \\
& + \left( -\frac{734163464}{81} - \frac{27208715}{81} \beta \right) L_{-5}^2 L_{-3}^3 + \left( \frac{1700122117}{54} - \frac{141939863}{18} \beta \right) L_{-10} L_{-2}^3 \\
& + \left( -\frac{644584063}{162} - \frac{1563558899}{162} \beta \right) L_{-5} L_{-4}^2 L_{-3} + \left( -\frac{2365946831}{81} + \frac{391994255}{81} \beta \right) L_{-9} L_{-4} L_{-3} \\
& + \left( \frac{41998314095}{324} + \frac{14424238267}{324} \beta \right) L_{-8} L_{-5} L_{-3} + \left( -\frac{5696336410}{81} - \frac{1244830214}{81} \beta \right) L_{-7} L_{-6} L_{-3} \\
& + \left( -\frac{50875270591}{648} - \frac{5501974607}{162} \beta \right) L_{-13} L_{-3} + \left( \frac{141665042}{9} + \frac{45794666}{9} \beta \right) L_{-4}^4 \\
& + \left( -\frac{16720323782}{81} - \frac{4469878486}{81} \beta \right) L_{-8} L_{-4}^2 + \left( \frac{4719078932}{81} + \frac{1981388674}{81} \beta \right) L_{-7} L_{-5} L_{-4} \\
& + \left( \frac{3329574508}{27} + \frac{774237704}{27} \beta \right) L_{-6}^2 L_{-4} + \left( -\frac{24444294973}{81} - \frac{7245075107}{81} \beta \right) L_{-12} L_{-4} \\
& + \left( -\frac{17795208008}{243} - \frac{5697658762}{243} \beta \right) L_{-6} L_{-5}^2 + \left( \frac{172133662723}{648} + \frac{2558624480}{27} \beta \right) L_{-11} L_{-5} \\
& + \left( -\frac{2482312006}{81} - \frac{359398637}{81} \beta \right) L_{-10} L_{-6} + \left( -\frac{11445305005}{324} - \frac{11100560395}{648} \beta \right) L_{-9} L_{-7} \\
& + \left( \frac{88700463031}{486} + \frac{21332312435}{486} \beta \right) L_{-8}^2 + \left( -\frac{273742696867}{324} - \frac{76299006809}{324} \beta \right) L_{-16} \Big) |h-l\rangle \\
& + \left( L_{-1}^{14} - \frac{511}{6} L_{-2} L_{-1}^{12} + 273 L_{-3} L_{-1}^{11} + \frac{31031}{12} L_{-2}^2 L_{-1}^{10} - 3575 L_{-4} L_{-1}^{10} - \frac{71357}{6} L_{-3} L_{-2} L_{-1}^9 \right. \\
& + \frac{156871}{9} L_{-5} L_{-1}^9 - \frac{7702123}{216} L_{-2}^3 L_{-1}^8 + \frac{2429141}{18} L_{-4} L_{-2} L_{-1}^8 - \frac{29315}{18} L_{-3}^2 L_{-1}^8 \\
& - \frac{330200}{9} L_{-6} L_{-1}^8 + \frac{3232229}{18} L_{-3} L_{-2}^2 L_{-1}^7 - \frac{14411306}{27} L_{-5} L_{-2} L_{-1}^7 - \frac{190736}{3} L_{-4} L_{-3} L_{-1}^7 \\
& + \frac{1113553}{12} L_{-7} L_{-1}^7 + \frac{299830531}{1296} L_{-4}^2 L_{-1}^6 - \frac{87916465}{54} L_{-4} L_{-2} L_{-1}^6 - \frac{1101737}{54} L_{-3} L_{-2} L_{-1}^6 \\
& + \frac{25678282}{27} L_{-6} L_{-2} L_{-1}^6 + \frac{11453869}{108} L_{-5} L_{-3} L_{-1}^6 + \frac{1494307}{3} L_{-4}^2 L_{-1}^6 - \frac{18846499}{18} L_{-8} L_{-1}^6 \\
& - \frac{120020173}{108} L_{-3} L_{-2}^3 L_{-1}^5 + \frac{9826873}{2} L_{-5} L_{-2}^2 L_{-1}^5 + \frac{12821156}{9} L_{-4} L_{-3} L_{-2} L_{-1}^5 \\
& - \frac{52971385}{24} L_{-7} L_{-2} L_{-1}^5 + \frac{858578}{9} L_{-3}^3 L_{-1}^5 + \frac{340520}{9} L_{-6} L_{-3} L_{-1}^5 - \frac{16430486}{9} L_{-5} L_{-4} L_{-1}^5 \\
& \left. + \frac{53355355}{18} L_{-9} L_{-1}^5 - \frac{563493749}{864} L_{-2}^4 L_{-1} + \frac{779378095}{108} L_{-4} L_{-2}^3 L_{-1}^4 + \frac{1494401}{3} L_{-3}^2 L_{-2} L_{-1}^4 \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{20491817}{3}L_{-6}L_{-2}L_{-1}^4 - \frac{165034463}{72}L_{-5}L_{-3}L_{-2}L_{-1}^4 - \frac{118077817}{18}L_{-4}L_{-2}L_{-1}^4 \\
& + \frac{558714659}{36}L_{-8}L_{-2}L_{-1}^4 - \frac{5919766}{9}L_{-4}L_{-3}L_{-1}^4 + \frac{173851973}{72}L_{-7}L_{-3}L_{-1}^4 \\
& - \frac{3001604}{3}L_{-6}L_{-4}L_{-1}^4 + \frac{209551}{3}L_{-5}L_{-1}^4 + \frac{19970203}{9}L_{-10}L_{-1}^4 + \frac{370750733}{144}L_{-3}L_{-2}L_{-1}^3 \\
& - \frac{88749761}{6}L_{-5}L_{-2}L_{-1}^3 - \frac{69331496}{9}L_{-4}L_{-3}L_{-2}L_{-1}^3 + \frac{588707659}{48}L_{-7}L_{-2}L_{-1}^3 \\
& - \frac{47120023}{54}L_{-3}L_{-2}L_{-1}^3 + \frac{4427710}{3}L_{-6}L_{-3}L_{-2}L_{-1}^3 + \frac{149796614}{9}L_{-5}L_{-4}L_{-2}L_{-1}^3 \\
& - \frac{1533491879}{54}L_{-9}L_{-2}L_{-1}^3 + \frac{52642805}{36}L_{-5}L_{-3}L_{-1}^3 + 2193925L_{-4}L_{-3}L_{-1}^3 \\
& - \frac{244608971}{18}L_{-8}L_{-3}L_{-1}^3 - \frac{68361415}{12}L_{-7}L_{-4}L_{-1}^3 + \frac{104958070}{9}L_{-6}L_{-5}L_{-1}^3 \\
& - \frac{3755911505}{144}L_{-11}L_{-1}^3 + \frac{359158375}{576}L_{-2}L_{-1}^6 - \frac{1422954571}{144}L_{-4}L_{-2}L_{-1}^4 - \frac{110349109}{72}L_{-3}L_{-2}L_{-1}^3L_{-2}^2 \\
& + \frac{242776861}{18}L_{-6}L_{-3}L_{-2}L_{-1}^2 + \frac{417557591}{48}L_{-5}L_{-3}L_{-2}L_{-1}^2 + \frac{215671777}{12}L_{-4}L_{-2}L_{-1}^2 \\
& - \frac{1133758667}{24}L_{-8}L_{-2}L_{-1}^2 + \frac{9402171}{2}L_{-4}L_{-3}L_{-2}L_{-1}^2 - \frac{46173362}{3}L_{-7}L_{-3}L_{-2}L_{-1}^2 \\
& + \frac{8619874}{3}L_{-6}L_{-4}L_{-2}L_{-1}^2 - 346185L_{-5}L_{-2}L_{-1}^2 - 12764834L_{-10}L_{-2}L_{-1}^2 + \frac{781726}{9}L_{-3}L_{-2}L_{-1}^4 \\
& - \frac{6265157}{3}L_{-6}L_{-3}L_{-2}L_{-1}^2 - \frac{17503249}{4}L_{-5}L_{-4}L_{-3}L_{-2}L_{-1}^2 + \frac{851252395}{72}L_{-9}L_{-3}L_{-1}^2 \\
& - \frac{10377505}{3}L_{-4}L_{-1}^3 + \frac{88750081}{2}L_{-8}L_{-4}L_{-1}^2 - \frac{107002967}{48}L_{-7}L_{-5}L_{-1}^2 - 15108105L_{-6}L_{-2}L_{-1}^2 \\
& + \frac{421427881}{12}L_{-12}L_{-1}^2 - \frac{429866875}{288}L_{-3}L_{-2}L_{-1}^5 + \frac{1485768463}{144}L_{-5}L_{-2}L_{-1}^4 \\
& + \frac{75890573}{9}L_{-4}L_{-3}L_{-2}L_{-1}^3 - \frac{3917780737}{288}L_{-7}L_{-2}L_{-1}^3 + \frac{47264245}{36}L_{-3}L_{-2}L_{-1}^3 \\
& - \frac{14009867}{3}L_{-6}L_{-3}L_{-2}L_{-1}^2 - \frac{47369767}{2}L_{-5}L_{-4}L_{-2}L_{-1}^2 + \frac{3093452827}{72}L_{-9}L_{-2}L_{-1}^2 \\
& - \frac{114347681}{24}L_{-5}L_{-3}L_{-2}L_{-1}^2 - \frac{46986829}{6}L_{-4}L_{-3}L_{-2}L_{-1}^2 + \frac{171211201}{4}L_{-8}L_{-3}L_{-2}L_{-1} \\
& + \frac{452772625}{24}L_{-7}L_{-4}L_{-2}L_{-1} - \frac{96873449}{3}L_{-6}L_{-5}L_{-2}L_{-1} + \frac{2348340155}{32}L_{-11}L_{-2}L_{-1} \\
& - \frac{8346652}{9}L_{-4}L_{-3}L_{-1}^3 + \frac{36927503}{12}L_{-7}L_{-3}L_{-1}^2 + 1760132L_{-6}L_{-4}L_{-3}L_{-1} - \frac{41087977}{12}L_{-6}L_{-2}L_{-1}^4 \\
& + \frac{2278813}{12}L_{-5}L_{-3}L_{-1} + 12560898L_{-10}L_{-3}L_{-1} + 5520381L_{-5}L_{-4}L_{-1} + \frac{16051875}{32}L_{-3}L_{-2}L_{-1}^4 \\
& - \frac{504079789}{18}L_{-9}L_{-4}L_{-1} - \frac{114043262}{3}L_{-8}L_{-5}L_{-1} + 17921683L_{-7}L_{-6}L_{-1} + \frac{178458875}{96}L_{-4}L_{-2}L_{-1}^5 \\
& + \frac{3300324455}{144}L_{-13}L_{-1} - \frac{11116875}{128}L_{-2}^7 - \frac{111748133}{32}L_{-5}L_{-3}L_{-2}^3 - \frac{136597993}{24}L_{-4}L_{-3}^2 \\
& + \frac{262660881}{16}L_{-8}L_{-2}^3 - \frac{11084647}{4}L_{-4}L_{-3}L_{-2}^2 + \frac{273622811}{32}L_{-7}L_{-3}L_{-2}^2 - 22330L_{-6}L_{-4}L_{-2}^2 \\
& + \frac{690417}{4}L_{-5}L_{-2}^2 + \frac{69379331}{12}L_{-10}L_{-2}^2 - \frac{1549555}{12}L_{-4}L_{-3}L_{-2} + \frac{4463795}{2}L_{-6}L_{-3}L_{-2} \\
& + \frac{43957095}{8}L_{-5}L_{-4}L_{-3}L_{-2} - \frac{249418197}{16}L_{-9}L_{-3}L_{-2} + \frac{60703909}{18}L_{-4}L_{-2}^3 - \frac{17478125}{4}L_{-8}L_{-2}^3 \\
& - \frac{531624373}{12}L_{-8}L_{-4}L_{-2} + \frac{245476273}{96}L_{-7}L_{-5}L_{-2} + \frac{90955697}{6}L_{-6}L_{-2}^2 - \frac{844913671}{24}L_{-12}L_{-2} \\
& + \frac{8812909}{24}L_{-5}L_{-3}^3 + \frac{6587483}{6}L_{-4}L_{-3}^2 - \frac{103274297}{24}L_{-7}L_{-4}L_{-3} + 2522646L_{-6}L_{-5}L_{-3} \\
& - \frac{210841967}{24}L_{-11}L_{-3} + \frac{3034556}{3}L_{-6}L_{-4}^2 - 525399L_{-5}L_{-4}^2 - 17243744L_{-10}L_{-4} \\
& + \frac{64762145}{6}L_{-9}L_{-5} + 6664196L_{-8}L_{-6} - \frac{400057469}{96}L_{-7}^2 - \frac{254131871}{12}L_{-14} \Big) |h; 1\rangle.
\end{aligned}$$

## D Explicit nullvectors in the bulk of $\mathbf{c}_{2,3} = 0$

In the following we give the explicit form of the nullvector of type **E** which has a Jordan cell at  $h = 1$ , lowest weight  $h = 0$  and appears at level 12. Again, for the sake of brevity, we have set the overall normalisation to 1 and also eliminated any further freedom by setting any further free parameter to 0; again, the parameter  $\beta$  remains as

it is a parameter of the representation as introduced in section 4:

$$\begin{aligned}
& \left( -\frac{44800}{27}L_{-3}L_{-2}^4L_{-1} + \left(\frac{358528}{27} - \frac{4000}{9}\beta\right)L_{-5}L_{-2}^3L_{-1} + \left(\frac{117920}{27} + \frac{49600}{9}\beta\right)L_{-4}L_{-3}L_{-2}^2L_{-1} \right. \\
& + \left(-\frac{1543136}{81} - \frac{814208}{81}\beta\right)L_{-7}L_{-2}^2L_{-1} + \left(-\frac{572912}{81} - \frac{100000}{27}\beta\right)L_{-3}^3L_{-2}L_{-1} \\
& + \left(\frac{620576}{9} - \frac{217408}{9}\beta\right)L_{-6}L_{-3}L_{-2}L_{-1} + \left(-\frac{1214624}{27} + \frac{243424}{9}\beta\right)L_{-5}L_{-4}L_{-2}L_{-1} \\
& + \left(\frac{3934496}{81} - \frac{341888}{27}\beta\right)L_{-9}L_{-2}L_{-1} + \left(\frac{551312}{27} + \frac{396352}{9}\beta\right)L_{-5}L_{-3}^2L_{-1} \\
& + \left(-\frac{91424}{9} - \frac{116416}{3}\beta\right)L_{-4}^2L_{-3}L_{-1} + \left(-\frac{619904}{27} - 1408\beta\right)L_{-8}L_{-3}L_{-1} \\
& + \left(\frac{548672}{9} + \frac{769984}{9}\beta\right)L_{-7}L_{-4}L_{-1} + \left(-\frac{4270240}{81} - \frac{532160}{9}\beta\right)L_{-6}L_{-5}L_{-1} \\
& + \left(\frac{3200}{9} - \frac{44800}{27}\beta\right)L_{-4}L_{-2}^4 + \left(\frac{10400}{3} + \frac{11200}{9}\beta\right)L_{-3}^2L_{-2}^3 + \left(-\frac{2984320}{243} + \frac{585856}{27}\beta\right)L_{-6}L_{-2}^3 \\
& + \left(-\frac{1611424}{81} - \frac{1122080}{27}\beta\right)L_{-5}L_{-3}L_{-2}^2 + \left(-\frac{63488}{27} + 36864\beta\right)L_{-4}L_{-3}^2L_{-2} \\
& + \left(\frac{309376}{9} + \frac{72832}{3}\beta\right)L_{-11}L_{-1} + \left(\frac{571936}{81} + \frac{304064}{9}\beta\right)L_{-8}L_{-2}^2 + \left(\frac{195488}{27} - \frac{9920}{3}\beta\right)L_{-4}^2L_{-2}^2 \\
& + \left(\frac{229504}{81} - \frac{5155328}{81}\beta\right)L_{-7}L_{-3}L_{-2} + \left(-\frac{1020352}{81} - \frac{1000192}{9}\beta\right)L_{-6}L_{-4}L_{-2} \\
& + \left(\frac{451616}{27} + \frac{709312}{9}\beta\right)L_{-5}^2L_{-2} + \left(-\frac{200192}{9} - \frac{279040}{3}\beta\right)L_{-10}L_{-2} + \left(\frac{40096}{81} - \frac{318016}{27}\beta\right)L_{-3}^4 \\
& + \left(-\frac{693152}{81} + \frac{1014976}{9}\beta\right)L_{-6}L_{-3}^2 + \left(-\frac{1646560}{81} + \frac{3016960}{27}\beta\right)L_{-9}L_{-3} + \left(\frac{1521088}{81} - \frac{762496}{9}\beta\right)L_{-6}^2 \\
& + \left(-\frac{35168}{9} + \frac{95168}{3}\beta\right)L_{-4}^3 + \left(\frac{128672}{27} - \frac{892096}{9}\beta\right)L_{-8}L_{-4} + \left(-\frac{758240}{81} + \frac{6217312}{81}\beta\right)L_{-7}L_{-5} \\
& + \left(\frac{306176}{27} - \frac{234080}{3}\beta\right)L_{-5}L_{-4}L_{-3} + \left(-\frac{947200}{27} + \frac{758080}{9}\beta\right)L_{-12} \Big) |h-l\rangle \\
& + \left(L_{-1}^{11} - 44L_{-2}L_{-1}^9 + 88L_{-3}L_{-1}^8 + \frac{1804}{3}L_{-2}^2L_{-1}^7 - 836L_{-4}L_{-1}^7 - 1320L_{-3}L_{-2}L_{-1}^6 \right. \\
& + \frac{16456}{9}L_{-5}L_{-1}^6 - \frac{85184}{27}L_{-2}^3L_{-1}^5 + \frac{107536}{9}L_{-4}L_{-2}L_{-1}^5 - \frac{12232}{9}L_{-3}^2L_{-1}^5 + \frac{4640}{9}L_{-6}L_{-1}^5 \\
& + \frac{52448}{9}L_{-3}L_{-2}^2L_{-1}^4 - \frac{168112}{9}L_{-5}L_{-2}L_{-1}^4 + 720L_{-4}L_{-3}L_{-1}^4 - 1360L_{-7}L_{-1}^4 + \frac{53504}{9}L_{-2}^4L_{-1}^3 \\
& - \frac{373888}{9}L_{-4}L_{-2}^2L_{-1}^3 + \frac{59072}{9}L_{-3}^2L_{-2}L_{-1}^3 - \frac{13760}{9}L_{-6}L_{-2}L_{-1}^3 - \frac{56896}{9}L_{-5}L_{-3}L_{-1}^3 \\
& + 15520L_{-4}^2L_{-1}^3 - \frac{214816}{9}L_{-8}L_{-1}^3 - \frac{22784}{3}L_{-3}L_{-2}^3L_{-1}^2 + \frac{120320}{3}L_{-5}L_{-2}^2L_{-1}^2 \\
& - \frac{7040}{3}L_{-4}L_{-3}L_{-2}L_{-1}^2 + \frac{3904}{3}L_{-7}L_{-2}L_{-1}^2 + \frac{22016}{9}L_{-3}^3L_{-1}^2 + 3008L_{-6}L_{-3}L_{-1}^2 \\
& - \frac{46016}{3}L_{-5}L_{-4}L_{-1}^2 + \frac{161152}{9}L_{-9}L_{-1}^2 - \frac{8192}{3}L_{-2}^5L_{-1} + 29696L_{-4}L_{-2}^3L_{-1} \\
& - 4672L_{-3}^2L_{-2}^2L_{-1} - \frac{2048}{3}L_{-6}L_{-2}^2L_{-1} + \frac{27392}{3}L_{-5}L_{-3}L_{-2}L_{-1} - \frac{94208}{3}L_{-4}^2L_{-2}L_{-1} \\
& + 53120L_{-8}L_{-2}L_{-1} + \frac{1984}{3}L_{-4}L_{-3}^2L_{-1} + \frac{24832}{3}L_{-7}L_{-3}L_{-1} - \frac{45952}{3}L_{-6}L_{-4}L_{-1} \\
& - \frac{3776}{3}L_{-5}^2L_{-1} + 22784L_{-10}L_{-1} + \frac{4096}{3}L_{-3}L_{-2}^4 - 9984L_{-5}L_{-2}^3 + 128L_{-4}L_{-3}L_{-2}^2 \\
& + \frac{3328}{3}L_{-7}L_{-2}^2 - \frac{3968}{3}L_{-3}^3L_{-2} - \frac{5632}{3}L_{-6}L_{-3}L_{-2} + \frac{33536}{3}L_{-5}L_{-4}L_{-2} \\
& - \frac{39808}{3}L_{-9}L_{-2} + \frac{4480}{3}L_{-5}L_{-2}^3 - \frac{3200}{3}L_{-4}^2L_{-3} - 4480L_{-8}L_{-3} \\
& \left. - \frac{6656}{3}L_{-7}L_{-4} + 8064L_{-6}L_{-5} - \frac{48256}{3}L_{-11} \right) |h; 1\rangle .
\end{aligned}$$

The explicit form of the nullvector of type  $\mathbf{F}$  which has a Jordan cell at  $h = 2$ , lowest weight  $h = 0$  and also appears at level 12 is given below. As we need the full beauty of this result in the argument of section 4 any free parameter appears as calculated (noted as  $m_i$ ); only the overall normalisation we have set to 1:

$$\left( m_{76}L_{-1}^{12} + m_{75}L_{-2}L_{-1}^{10} + m_{74}L_{-3}L_{-1}^9 + m_{73}L_{-2}^2L_{-1}^8 + m_{72}L_{-4}L_{-1}^8 \right.$$

$$\begin{aligned}
& +m_{71}L_{-3}L_{-2}L_{-1}^7 + m_{70}L_{-5}L_{-1}^7 + m_{69}L_{-2}^3L_{-1}^6 + m_{68}L_{-4}L_{-2}L_{-1}^6 + m_{67}L_{-3}^2L_{-1}^6 \\
& +m_{66}L_{-6}L_{-1}^6 + m_{65}L_{-3}L_{-2}^2L_{-1}^5 + m_{64}L_{-5}L_{-2}L_{-1}^5 + m_{63}L_{-4}L_{-3}L_{-1}^5 \\
& +m_{62}L_{-7}L_{-1}^5 + m_{61}L_{-2}^4L_{-1}^4 + m_{60}L_{-4}L_{-2}^2L_{-1}^4 + m_{59}L_{-3}^2L_{-2}L_{-1}^4 \\
& +m_{58}L_{-6}L_{-2}L_{-1}^4 + m_{57}L_{-5}L_{-3}L_{-1}^4 + m_{56}L_{-4}^2L_{-1}^4 + m_{55}L_{-8}L_{-1}^4 \\
& +m_{54}L_{-3}L_{-2}^3L_{-1}^3 + m_{53}L_{-5}L_{-2}^2L_{-1}^3 + m_{52}L_{-4}L_{-3}L_{-2}L_{-1}^3 + m_{51}L_{-7}L_{-2}L_{-1}^3 \\
& +m_{50}L_{-3}^3L_{-1}^3 + m_{49}L_{-6}L_{-3}L_{-1}^3 + m_{48}L_{-5}L_{-4}L_{-1}^3 + m_{47}L_{-9}L_{-1}^3 + m_{46}L_{-2}^5L_{-1}^2 \\
& +m_{45}L_{-4}L_{-2}^3L_{-1}^2 + m_{44}L_{-3}^2L_{-2}^2L_{-1}^2 + m_{43}L_{-6}L_{-2}^2L_{-1}^2 + m_{42}L_{-5}L_{-3}L_{-2}L_{-1}^2 \\
& +m_{41}L_{-4}^2L_{-2}L_{-1}^2 + m_{40}L_{-8}L_{-2}L_{-1}^2 + m_{39}L_{-4}L_{-3}^2L_{-1}^2 + m_{38}L_{-7}L_{-3}L_{-1}^2 \\
& +m_{37}L_{-6}L_{-4}L_{-1}^2 + m_{36}L_{-5}^2L_{-1}^2 + m_{35}L_{-10}L_{-1}^2 + m_{34}L_{-3}L_{-2}^4L_{-1} \\
& +m_{33}L_{-5}L_{-2}^3L_{-1} + m_{32}L_{-4}L_{-3}L_{-2}^2L_{-1} + m_{31}L_{-7}L_{-2}^2L_{-1} + m_{30}L_{-3}^3L_{-2}L_{-1} \\
& +m_{29}L_{-6}L_{-3}L_{-2}L_{-1} + m_{28}L_{-5}L_{-4}L_{-2}L_{-1} + m_{27}L_{-9}L_{-2}L_{-1} + m_{26}L_{-5}L_{-2}^3L_{-1} \\
& +m_{25}L_{-4}^2L_{-3}L_{-1} + m_{24}L_{-8}L_{-3}L_{-1} + m_{23}L_{-7}L_{-4}L_{-1} + m_{22}L_{-6}L_{-5}L_{-1} + \\
& m_{21}L_{-11}L_{-1} - 4096L_{-4}L_{-2}^4 + 3072L_{-3}^2L_{-2}^3 + (12800 - 8192\beta)L_{-6}L_{-2}^3 \\
& +(-32960 + 14336\beta)L_{-5}L_{-3}L_{-2}^2 + (30976 + 6144\beta)L_{-4}^2L_{-2}^2 + (-25088 - 17920\beta)L_{-8}L_{-2}^2 \\
& +(-4736 - 18432\beta)L_{-4}L_{-3}^2L_{-2} + (18560 + 32640\beta)L_{-7}L_{-3}L_{-2} \\
& +(-73216 + 50176\beta)L_{-6}L_{-4}L_{-2} + (54464 - 36608\beta)L_{-5}^2L_{-2} + (-496 + 5760\beta)L_{-3}^4 \\
& +(-18432 + 48128\beta)L_{-10}L_{-2} + (12096 - 54784\beta)L_{-6}L_{-2}^3 + (2432 + 40448\beta)L_{-5}L_{-4}L_{-3} \\
& +(-11648 - 59008\beta)L_{-9}L_{-3} + \left(-\frac{22144}{3} - 17408\beta\right)L_{-4}^3 + (27264 + 55040\beta)L_{-8}L_{-4} \\
& +(-5120 - 39040\beta)L_{-7}L_{-5} + (-4288 + 41472\beta)L_{-6}^2 + (31552 - 35328\beta)L_{-12} \Big) |h-l\rangle \\
& + \left( L_{-1}^{10} - \frac{130}{3}L_{-2}L_{-1}^8 + \frac{284}{3}L_{-3}L_{-1}^7 + \frac{5152}{9}L_{-2}^2L_{-1}^6 - 776L_{-4}L_{-1}^6 - 1488L_{-3}L_{-2}L_{-1}^5 \right. \\
& + \frac{6232}{3}L_{-5}L_{-1}^5 - \frac{8320}{3}L_{-2}^3L_{-1}^4 + \frac{29440}{3}L_{-4}L_{-2}L_{-1}^4 - \frac{2752}{3}L_{-3}^2L_{-1}^4 + 640L_{-6}L_{-1}^4 \\
& + \frac{21376}{3}L_{-3}L_{-2}^2L_{-1}^3 - \frac{67264}{3}L_{-5}L_{-2}L_{-1}^3 + \frac{800}{3}L_{-4}L_{-3}L_{-1}^3 - 800L_{-7}L_{-1}^3 + 4096L_{-2}^4L_{-1}^2 \\
& - 23552L_{-4}L_{-2}^2L_{-1}^2 + 992L_{-3}^2L_{-2}L_{-1}^2 - 3840L_{-6}L_{-2}L_{-1}^2 + 8896L_{-5}L_{-3}L_{-1}^2 \\
& - 9024L_{-8}L_{-1}^2 - 10240L_{-3}L_{-2}^3L_{-1} + 48768L_{-5}L_{-2}^2L_{-1} + 4160L_{-4}L_{-3}L_{-2}L_{-1} \\
& - 4992L_{-7}L_{-2}L_{-1} + 18048L_{-6}L_{-3}L_{-1} - 23552L_{-5}L_{-4}L_{-1} + 30528L_{-9}L_{-1} \\
& - 8192L_{-4}L_{-2}^3 + 10240L_{-3}^2L_{-2}^2 + 1024L_{-6}L_{-2}^2 - 60800L_{-5}L_{-3}L_{-2} \\
& + 47104L_{-4}^2L_{-2} - 30720L_{-8}L_{-2} - 6144L_{-4}L_{-2}^3 + 15232L_{-7}L_{-3} \\
& \left. - 39424L_{-6}L_{-4} + 36992L_{-5}^2 - 24576L_{-10} \right) |h;1\rangle .
\end{aligned}$$

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