BONN-TH-2006-03
hep-th/0604097

Virasoro representations and fusion for general augmented minimal models

Holger Eberle* and Michael Flohr†

Physikalisches Institut der Universität Bonn
Nußallee 12, 53115 Bonn, Germany

Abstract

In this paper we present explicit results for the fusion of irreducible and higher rank representations in two logarithmically conformal models, the augmented $c_{2,3} = 0$ model as well as the augmented Yang–Lee model at $c_{2,5} = -22/5$. We analyse their spectrum of representations which is consistent with the symmetry and associativity of the fusion algebra. We also describe the first few higher rank representations in detail. In particular, we present the first examples of consistent rank 3 indecomposable representations and describe their embedding structure.

Knowing these two generic models we also conjecture the general representation content and fusion rules for general augmented $c_{p,q}$ models.

*email: `eberle@th.physik.uni-bonn.de`
†email: `flohr@th.physik.uni-bonn.de`

Contents

1 Introduction

Logarithmic conformal field theories (logarithmic CFTs) have attracted quite a lot of attention in recent years. These theories are a generalisation of standard CFT which also allows for indecomposable action of the Virasoro modes. There are already quite a number of applications in such different fields as statistical physics (e.g. [1, 2, 3, 4]), string theory (e.g. [5, 6, 7]) and Seiberg Witten theory (e.g. [8]) which necessarily incorporate this generalisation of CFT. See [9, 10, 11, 12, 13] for an introduction to the field and a more complete list of applications. Nevertheless, studying logarithmic CFTs has only just begun because we still know only few logarithmic models explicitly and the efforts to disentangle the general structure prove to be much more complicated and tedious as in ordinary CFT (see e.g. [14, 15, 16, 17, 18, 19]). This paper hopes to give a considerable contribution to this ongoing effort as it presents and explicitly discusses the first examples of models with a much more complicated indecomposable structure up to rank 3.

The up to now most prominent examples of logarithmic CFTs have emerged from studying a specific series of the so-called minimal models in CFT, the $c_{p,1}$ models [20, 21, 22, 23, 24, 25, 13]. As standard minimal models these actually emerge to be trivial as they provide zero representation content. On the other hand, if one takes into account representations corresponding to an enlarged Kac table one encounters non-trivial models which include representations with indecomposable structure. This is the reason why we like to call these "augmented $c_{p,1}$ models". In the point of view of the Virasoro representation theory this augmentation actually comprises representations with weights in the whole infinite Kac table. On the other hand it has been found that these theories exhibit an enlarged triplet $\mathcal{W}$-algebra [26]. Wrt to this enlarged symmetry algebra the representation content of this theory can be composed into a finite set of representations associated to a finite standard cell of the Kac table. In comparison to the original minimal model of central charge $c_{p,1}$ this standard cell is enlarged by a factor of 3 on either side as if it described $c_{3p,3}$.

But there is no reason not to consider a similar augmentation in the Kac table of the general $c_{p,q}$ minimal models. This paper is devoted to the study of these general augmented $c_{p,q}$ models. We will concentrate on the Virasoro representation theory here and, thus, regard an augmentation to the whole infinite Kac table. The question whether these models also exhibit an enlarged $\mathcal{W}$-algebra has not been answered yet, although we hope to use the findings of this paper to settle this question soon [27] (see also [28, 29, 30]). The particular example of the augmented $c_{2,3} = 0$ model has already been studied in [31, 32], but up to this paper the precise structure of the Virasoro representations has been out of reach.

There have also been studies of Jordan cells with rank higher than 2 on the level of correlation functions and Ward identities in the CFT literature [33, 34, 35, 14, 16, 17, 36, 37]. However, the models and their representations discussed in this paper are the first ones where we can see explicitly how a higher rank structure appears while generating a representation as a Virasoro module from some groundstates.

We want to explore the full spectrum of these augmented $c_{p,q}$ models by successively fusing representations which we already know we have to take into the spectrum. The concept of the fusion product lies at the heart of conformal representation theory and has been subject to many thorough mathematical studies (see e.g. [38, 39, 40, 41, 42]). It governs the representation theoretic aspects of the operator product expansion and, hence, puts severe constraints on all n-point-functions in CFT. It follows that the fusion product actually dictates which set of representations of the Virasoro algebra at a certain conformal weight can be combined into a consistent CFT model. Hence, the successive application of the fusion product will actually lead us to the whole consistent representation space of the augmented $c_{p,q}$ models.

The algorithm which we use to compute these fusion products relies on the work of [43, 20]. In [43] W. Nahm showed that the main information characterising a representation can be found in a small quotient space of this representation, called the special subspace, which is finite for the large class of so-called quasirational CFTs. He actually proved that the fusion of quasirational representations leads again to a finite number of quasirational representations. In [20] M. Gaberdiel and H. Kausch used the Nahm algorithm of the proof in [43] to propose a procedure how to efficiently calculate a fusion product of two quasirational representations. This procedure was successfully applied to the augmented $c_{p,1}$ models in [20]. In this paper we present results of fusion products which have been calculated based on the implementation of the Nahm algorithm as well as this procedure for the general $c_{p,q}$ models.

In section 2 we give a short review of this Nahm algorithm and the resulting procedure how to calculate a fusion product. We also comment on our specific computer implementation. In section 3 we shortly introduce minimal models as well as their augmented generalisation. In section 4 we then discuss the easiest general augmented $c_{p,q}$ model which is not contained in the $c_{p,1}$ model series, the $c_{2,3} = 0$ model. This model exhibits a much more complicated indecomposable structure up to rank 3 in comparison to the $c_{p,1}$ models. We describe a number of rank 2 and rank 3 representations explicitly and also discuss the representation content which is consistent with the fusion product. In section 5 we discuss a second example of general augmented $c_{p,q}$ models, the augmented Yang-Lee model at $c_{2,5} = -22/5$. We actually rediscover all crucial features observed in the $c_{2,3} = 0$ model. These two examples substantiate our conjecture of the general representation content and fusion rules for general augmented $c_{p,q}$ models given in section 6. In section 7 we conclude and also give a short outlook on the expected implications of these results. In appendix A we give the basis of states which brings L_0 into Jordan diagonal form for the rank 3 representation $\mathcal{R}^{(3)}(0,1,2,5)$. Finally, the appendices B and C contain the explicit results for the fusion product calculation for both examples $c_{2,3} = 0$ and $c_{2,5} = -22/5$.

2 How to calculate fusion products

Our calculations of fusion products are based on an algorithm first developed by W. Nahm [43] to prove that the fusion of quasirational representations contains only finitely many quasirational subrepresentations and, hence, that the category of quasirational representations is stable. In [20] this procedure was generalised and it was shown how one can use this algorithm to get (computationally usually sufficient) constraints to fix the field content of the fusion product at a given level. Although the presentation of the algorithm is already very precise and thorough in [20] we nevertheless want to give a short summary here in order to make this paper self-contained. We will then present the specific properties of our particular implementation.

2.1 The Nahm algorithm

The nice and short presentation of the Nahm algorithm in [20] relies on the coproduct formula. For a holomorphic field of conformal weight h and mode expansion

$$S(w) \;=\; \sum_{l\in\mathbb{Z}+h} w^{l-h} S_{-l}$$

it is given by [40, 44]

$$\begin{aligned}
\Delta_{z,\zeta}(S_n) \;&=\; \sum_{m=1-h}^{n} \binom{n+h-1}{m+h-1}\, \zeta^{n-m}\,(S_m\otimes \mathbb{1}) \\
&\qquad + \epsilon \sum_{l=1-h}^{n} \binom{n+h-1}{l+h-1}\, z^{n-l}\,(\mathbb{1}\otimes S_l) \\
&=\; \tilde{\Delta}_{z,\zeta}(S_n) \quad \forall\, n\le 1-h \\
\Delta_{z,\zeta}(S_{-n}) \;&=\; \sum_{m=1-h}^{\infty} \binom{n+m-1}{m+h-1}\,(-1)^{m+h-1}\,\zeta^{-(n+m)}\,(S_m\otimes \mathbb{1}) \\
&\qquad + \epsilon \sum_{l=n}^{\infty} \binom{l-h}{n-h}\,(-z)^{l-n}\,(\mathbb{1}\otimes S_{-l}) \quad \forall\, n\le h \\
\tilde{\Delta}_{z,\zeta}(S_{-n}) \;&=\; \sum_{m=n}^{\infty} \binom{m-h}{n-h}\,(-\zeta)^{m-n}\,(S_{-m}\otimes \mathbb{1}) \\
&\qquad + \epsilon \sum_{l=1-h}^{\infty} \binom{n+l-1}{l+h-1}\,(-1)^{l+h-1}\, z^{-(n+l)}\,(\mathbb{1}\otimes S_l) \quad \forall\, n\le h\,,
\end{aligned}$$

where $\epsilon=-1$ if both S_m and the first field in the tensor product it is applied to are fermionic and $\epsilon=+1$ otherwise. Furthermore, z and ζ are the positions of the two fields of the tensor product which this fused operator is applied to. Due to the symmetry of the fusion product there are two alternative ways of writing the comultiplication, denoted $\Delta_{z,\zeta}$ and $\tilde{\Delta}_{z,\zeta}$. Demanding that both $\Delta_{z,\zeta}$ and $\tilde{\Delta}_{z,\zeta}$ actually yield the same result we get the fusion space of two representations $\mathcal{H}_i$ at positions z_i, $i=1,2$,

$$\mathcal{H}_1\otimes_{\rm f}\mathcal{H}_2 := (\mathcal{H}_1\otimes\mathcal{H}_2)/(\Delta_{z_1,z_2}-\tilde{\Delta}_{z_1,z_2})\,.$$

Throughout the paper we will use $\otimes_{\rm f}$ to denote the fusion product of two representations.

In this paper we will only look at representations wrt the Virasoro algebra $\mathcal{A}(L)$ which is generated by the modes L_n of the $h(L)=2$ Virasoro field L, the holomorphic energy momentum tensor of conformal field theory. We need the following subalgebras

$$\begin{aligned}
\mathcal{A}^0_-(L) \;&:=\; \langle L_{-n}|\, 0<n<h(L)\rangle \\
\mathcal{A}_{--}(L) \;&:=\; \langle L_{-n}|\, n\ge h(L)\rangle \\
\mathcal{A}_{\pm}(L) \;&:=\; \langle L_n|\, \pm n>0\rangle
\end{aligned}$$

as well as the subalgebra of words with length of at least n

$$\mathcal{A}_n(L) \;=\; \Big\langle \prod_{j=1}^{m} L_{-l_j}^{k_j} \Big| \sum_{j=1}^{m} l_j \geq n \Big\rangle \,.$$

The essential information about a representation $\mathcal{H}$ is already encoded in its "special subspace", the quotient space

$$\mathcal{H}^s \;:=\; \mathcal{H} / (\mathcal{A}_{--}(L)\, \mathcal{H}) \,.$$

We also need the family of filtrations of $\mathcal{H}$ given as quotient spaces

$$\mathcal{H}^n \;:=\; \mathcal{H} / (\mathcal{A}_{n+1}(L)\, \mathcal{H}) \,.$$

Especially for irreducible $\mathcal{H}$ this space is equal to the set of descendants up to level n.

We want to restrict to a certain type of representations, the "quasirational representations". We use the definition of a quasirational representation that it is a representation with finite special subspace. For quasirational representations of the Virasoro algebra it has been shown that [43, 20]

$$(\mathcal{H}_1 \otimes_{\mathrm{f}} \mathcal{H}_2)^n \subset \mathcal{H}_1^s \otimes \mathcal{H}_2^n \quad \wedge \quad (\mathcal{H}_1 \otimes_{\mathrm{f}} \mathcal{H}_2)^n \subset \mathcal{H}_1^n \otimes \mathcal{H}_2^s \,. \tag{1}$$

The proof uses the following Nahm algorithm which can be shown to map every state of the tensor product $\mathcal{H}_1^{\cdot\cdot} \otimes \mathcal{H}_2^{\cdot\cdot}$ to a state in the respective right hand side in (1) in a finite number of steps.

We present the algorithm only for the first equation in (1) as the other version works the same way by symmetry of the fusion product. In the following, we regard the states of the tensor product to be at positions $(z_1, z_2) = (1, 0)$. The two steps of the Nahm algorithm are then given by:

(A1) A vector $\psi_1 \otimes \psi_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is rewritten in the form

$$\psi_1 \otimes \psi_2 \;=\; \sum_i \psi_1^i \otimes \psi_2^i + \Delta_{1,0}(\mathcal{A}_{n+1}(L))\,(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

with $\psi_1^i \in \mathcal{H}_1^s$. This can be achieved by the following recursive procedure.

The crucial step is to use the nullvector conditions on ψ_1 to re-express it in the form

$$\psi_1 \;=\; \sum_j \psi_j^s + \sum_k \mathcal{A}_{--}\, \chi_k^s \,,$$

with $\psi_j^s, \chi_k^s \in \mathcal{H}_1^s$. We still need to get rid of the $\mathcal{A}_{--}$ action on the χ_k^s. We use the following formula derived from the comultiplication formula and its translation properties for $m \leq n$ [20]

$$\begin{aligned}(L_{-m} \otimes \mathbb{1}) \;=\;& \sum_{l=m}^{n} \binom{l-h}{m-h} \Delta_{1,0}(L_{-l}) \\ &- \epsilon \sum_{l=1-h}^{\infty} \binom{m+l-1}{m-h} (-1)^{h-m-1}\,(\mathbb{1} \otimes L_l) + \Delta_{1,0}(\mathcal{A}_{n+1}(L)) \,.\end{aligned}$$

This formula actually enables us to replace the $(L_{-m} \otimes \mathbb{1})$ action in $\mathcal{A}_{--}\,\chi_k^s \otimes \psi_2$ by terms where $\mathcal{A}_-^0$ or even the identity acts on the left hand vector of the tensor product. This is true as in the range $m \leq l \leq n$ the comultiplication $\Delta_{1,0}(L_{-l})$ is actually of the simple form $\mathcal{A}_-^0 \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{A}_{--}$. Now we have to take the result and repeat this procedure starting again with the re-expression of the first fields ψ_1 in the tensor product. A simple count of the strictly decreasing level of modes during the iteration shows that this algorithm has to terminate [43].

(A2) This step has to be applied to each term of the resulting sum from step (A1) separately. The input, a resulting tensor product from (A1) $\psi_1 \otimes \psi_2 \in \mathcal{H}_1^s \otimes \mathcal{H}_2$, is rewritten as

$$\psi_1 \otimes \psi_2 \quad = \quad \sum_t \psi_1^t \otimes \psi_2^t + \Delta_{1,0}(\mathcal{A}_{n+1}(L))\,(\mathcal{H}_1 \otimes \mathcal{H}_2)\;,$$

where now $\psi_1^t \in \mathcal{A}_-^0\,\mathcal{H}_1^s$ and $\psi_2^t \in \mathcal{H}_2^n$.

This is achieved by repeatedly using

$$\Delta_{1,0}(L_{-I}) \quad = \quad (\mathbb{1} \otimes L_{-l}) + \sum_k c_k\,(\mathcal{A}_-^0 \otimes L_{-I_k})$$

for a word $L_{-I} = L_{-i_1}\,L_{-i_2}\ldots$ of negative Virasoro modes with level $|I|$ and constant c_k. This recursion has to finish as the Virasoro monomials L_{-I_k} are of strictly lower level $|I_k| < |I|$. This formula is just the result of repeated use of the comultiplication formula for a monomial of modes of the same field and for the special coordinates $(z, \zeta) = (1, 0)$.

As we want to have the states in the fusion product which are projected to the subspace $(\mathcal{H}_1 \otimes_{\mathrm{f}} \mathcal{H}_2)^n$ we do not have to care about contributions $\Delta_{1,0}(\mathcal{A}_{n+1}(L))\,(\mathcal{H}_1 \otimes \mathcal{H}_2)$. It is then easy to see that iterated application of steps (A1) and (A2) will finally yield the required result (1). This algorithm terminates in a finite number of steps as the number of modes on both fields strictly decreases when re-expressing ψ_1 in step (A1) using its nullvector condition and does not increase in step (A2).

2.2 Constraints for the fusion algebra

By (1) we know that $(\mathcal{H}_1 \otimes_{\mathrm{f}} \mathcal{H}_2)^n$ is actually embedded in the easily constructed space $\mathcal{F} := \mathcal{H}_1^s \otimes \mathcal{H}_2^n$. Hence, we want to find the full set of constraints which describes $(\mathcal{H}_1 \otimes_{\mathrm{f}} \mathcal{H}_2)^n$ in $\mathcal{F}$.

The important idea of [20] was that one can find nontrivial constraints by applying $\mathcal{A}_{n+1}$ to states in $\mathcal{F}$. We then have to use the Nahm algorithm in order to map the resulting descendant states into our "standard" space $\mathcal{F}$. By definition these descendant states are divided out of $(\mathcal{H}_1 \otimes_{\mathrm{f}} \mathcal{H}_2)^n$ and, hence, are supposed to vanish. Thus, their mapping to $\mathcal{F}$ should evaluate to zero—if we acquire non-trivial expressions this simply yields the desired constraints by imposing their vanishing.

This procedure is even improved if we use the nontrivial nullvector conditions on the second field to replace the action of certain Virasoro monomials before performing the Nahm algorithm. This introduces the information about the nullvector structure on the second representation of the tensor product into the game; the information about the nullvector structure on the first representation of the tensor product has already been used in the Nahm algorithm itself.

As we noticed during our calculation it even improves the situation to include the nullvector conditions on the tensor factors in the space $\mathcal{F}$ itself.

In the following we will denote the level n at which we perform the computation with L. Certainly, one cannot perform this calculation for all of $\mathcal{A}_{L+1}$. We, hence, restricted our computation to the application of Virasoro monomials

$$\Big\langle \prod_{j=1}^{m} L_{-l_j}^{k_j} | \sum_{j=1}^{m} l_j = \tilde{L} \Big\rangle \,.$$

of equal level $\tilde{L}$. Usually we performed the calculation from $\tilde{L} = L+1$ up to a maximal $\tilde{L}_{\text{max}}$. Both L and $\tilde{L}_{\text{max}}$ are given for the respective calculations in the appendix.

As we are limited to the calculation of a finite number of constraints this procedure is only able to give a lower bound on the number of constraints and, hence, an upper bound on the number of states in the fusion product at that level. On the other hand, these constraints seem to be highly non-trivial such that already a very low $\tilde{L}_{\text{max}} > L$, often even $\tilde{L}_{\text{max}} = L+1$, is sufficient to gain all constraints which yield representations in a consistent fusion algebra. This already worked very well in [20] for the $c_{p,1}$ model case and as we will see it also works very well in the general augmented $c_{p,q}$ model case.

Now, it is especially interesting to observe the action of positive Virasoro modes on $(\mathcal{H}_1 \otimes_{\rm f} \mathcal{H}_2)^L$. The positive Virasoro modes, however, induce an action

$$L_m : (\mathcal{H}_1 \otimes_{\rm f} \mathcal{H}_2)^L \to (\mathcal{H}_1 \otimes_{\rm f} \mathcal{H}_2)^{L-m} \qquad \forall\, m \leq L \,.$$

It is important to note that the L_m map to a space of respective lower maximal level. Hence, we need to construct all spaces of lower maximal level $0 \leq n < L$. To achieve this we start with $(\mathcal{H}_1 \otimes_{\rm f} \mathcal{H}_2)^L$ and successively impose the constraints which arise in the above described way from the vanishing of the action of Virasoro monomials of level m with $n < m \leq L$ on $\mathcal{F}$.

2.3 Implementation for the $\mathbf{c_{p,q}}$ models

We have implemented the main calculational tasks for this paper, especially the Nahm algorithm and the calculation of the constraints, in C++ using the computer algebra package GiNaC [45]. We constructed new classes for the algebraic objects fields, fieldmodes, products of fieldmodes, descendant fields as well as tensor products of fields which are the basic ingredients in this algorithm. (Some of the classes have already been used in [32].)

As GiNaC does not support factorisation we used the JordanForm package of the computer algebra system Maple in order to get the Jordan diagonal form of the L_0 matrix on the resulting space as well as the matrices of base change. This Maple calculation is performed via command-line during the run of the C++ programme.

As we will see in section 3 some irreducible representations in the general augmented $c_{p,q}$ models have more than one nullvector (two, to be precise) which are completely independent, i.e. such that none of these nullvectors can be written as a descendant of the others. Hence, it is important to include both independent nullvectors into the nullvector lists which are used for replacements in the calculation as explained above. This is needed to provide the full information about the nullvector structure of the original representations which are to be fused. Sometimes one even needs to choose an $\tilde{L}_{\max}$ large enough such that the second nullvector can also become effective. The special subspace, however, is nevertheless determined by the level l of the lowest nullvector

$$\langle \psi, L_{-1}\psi, L_{-1}^2\psi, \ldots, L_{-1}^{l-1}\psi \rangle \ .$$

Besides the fusion of two irreducible representations we also implemented the possibility of fusing an irreducible representation with a rank 2 representation. Actually, this generalisation is quite straight forward. Instead of one state which generates an irreducible representation we now need two generating states. However, we have to be careful because the second generating state, the logarithmic partner, is not primary. Hence, we implemented the indecomposable action on this second generating field as additional conditions proprietary to that field (as already done in [32]). We also have to be careful to calculate the correct nullvector structure which includes besides an ordinary nullvector on the primary field the first logarithmic nullvector of the whole indecomposable representation. We have calculated these logarithmic nullvectors using the algorithm described in [32].

In order to speed up the algorithm we widely used hashing tables. This measure actually resulted in a quite equal use of computing time and memory (on a standard PC with up to 4GB memory); calculations which are on the edge of using up the memory have run times between half a day and a few days.

The performance of the implemented algorithm is, however, quite hard to benchmark as it varies very much with different input and output. Concerning the input the computing time rises with the level of the nullvectors—especially, the nullvector level of the first tensor factor is crucial. We also need much more time to compute fusion products with representations on weights that are strictly rational than corresponding ones with integer weights. And then, the performance of course depends exponentially on L as well as $\tilde{L}$, although the dependence on L is much stronger. Concerning the output the computational power of Maple is frequently the limiting factor if we have a large resulting L_0 matrix.

We have also checked the correct implementation of the algorithm by reproducing quite some fusion rules of the $c_{p,1}$ models given in [20]. In particular, we reproduced the Virasoro matrices for the example given in the appendix of [20]. We also noted that the algorithm is very sensitive and fragile such that an only small change in the parameters or the programme yields completely unreasonable results.

In contrast to the lowest $c_{p,1}$ models we have to cover a much larger parameter space with states of higher nullvectors already for the easiest general augmented $c_{p,q}$ model, the augmented $c_{2,3} = 0$ model. We, hence, decided to calculate the fusion of the lowest representations at $L = 6$ in order to be able to get results at the same L for a large parameter space. For the higher fusion as well as the fusion with rank 2 representations we had to reduce L. Details as well as the results are given in the appendices B and C.

3 Virasoro representation theory for minimal models and their extensions

In this section we want to give a short overview about the representation theory of minimal models [46, 47] as well as the augmented $c_{p,1}$ models [24, 20].

In the following we will exclusively regard representations of the Virasoro algebra which is spanned by the modes of the (holomorphic) $h = 2$ stress energy tensor L_n obeying

$$[L_m, L_n] \quad = \quad (m-n)\, L_{m+n} + \frac{c}{12}\,(m-1)\, m\,(m+1)\, \delta_{m+n,0} \qquad m,n \in \mathbb{Z}\ .$$

To any highest weight h the application of the negative Virasoro modes $L_n,\ n < 0$, freely generates the Verma module

$$\mathcal{M}(h) \quad = \quad \{L_{-n_1} \ldots L_{-n_k} v_h | n_1 \geq \ldots \geq n_k > 0,\ k \in \mathbb{Z}^+\}$$

where v_h is the highest weight state to h. In order to find the irreducible or at least indecomposable representations we need to identify the largest true subrepresentations of $\mathcal{M}(h)$ which decouple from the rest of the representation and need to construct the respective factor module.

A subrepresentation can be generated from any singular vector v in $\mathcal{M}(h)$, i.e. a vector which obeys $L_p v = 0\ \forall p > 0$ and which is hence a highest weight state of its own; and certainly we can also have unions of such representations. On the other hand a subrepresentation only decouples from the rest of the representation and can hence be factored out if it consists of nullvectors, i.e. vectors which are orthogonal to all other vectors in the Fock space of states wrt the natural sesquilinear Shapovalov form on this space. As long as we do not encounter any indecomposable structure in our representation singular vectors are at the same time nullvectors and generate subrepresentations which are null in Fock space. However, the interrelation between singular vectors and nullvectors becomes much more intricate as soon as we deal with indecomposable representations.

The Kac determinant actually parametrises the relation between conformal charges c and spectra of conformal weights $h_{r,s}$ at which we encounter nullvectors. It is hence

Table 1: Kac table for $c_{2,3} = 0$

$r \backslash s$	1	2	3	4	5
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$
4	1	$\frac{1}{8}$	0	$\frac{5}{8}$	2
5	2	$\frac{5}{8}$	0	$\frac{1}{8}$	1
6	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$
7	5	$\frac{21}{8}$	1	$\frac{1}{8}$	0
8	7	$\frac{33}{8}$	2	$\frac{5}{8}$	0
9	$\frac{28}{3}$	$\frac{143}{24}$	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$
10	12	$\frac{65}{8}$	5	$\frac{21}{8}$	1
11	15	$\frac{85}{8}$	7	$\frac{33}{8}$	2

an ingenious tool to explore interesting conformal field theories with relatively few and small representations. We are especially interested in this series of conformal field theories which emerge from the study of the Kac determinant and which, hence, exhibit a rich nullvector structure to be described below. These theories are parametrised by the conformal charges (see e.g. [47])

$$c = c_{p,q} = 1 - 6\,\frac{p-q}{pq} \qquad 1 \leq p, q \in \mathbb{Z}\ ,$$

where p and q do not have a common divisor; their highest weight spectrum is given by the weights in the Kac table

$$h_{r,s} \;=\; \frac{(pr-qs)^2 - (p-q)^2}{4pq} \quad 1 \leq r \in \mathbb{Z},\ 1 \leq s \in \mathbb{Z}\ .$$

An extract of the (infinite) Kac table for $c_{2,3} = 0$ is given in table 1.

The so-called minimal models are a series of such conformal field theories which manage to extract the smallest possible representation theory from the Kac table of some central charge $c_{p,q}$ by relating all weights to some standard cell $\{(r,s)|1 \leq r < q,\ 1 \leq s < p\}$ subject to the relation [46, 47]

$$h_{r,s} \;=\; h_{q-r,p-s}\ . \tag{2}$$

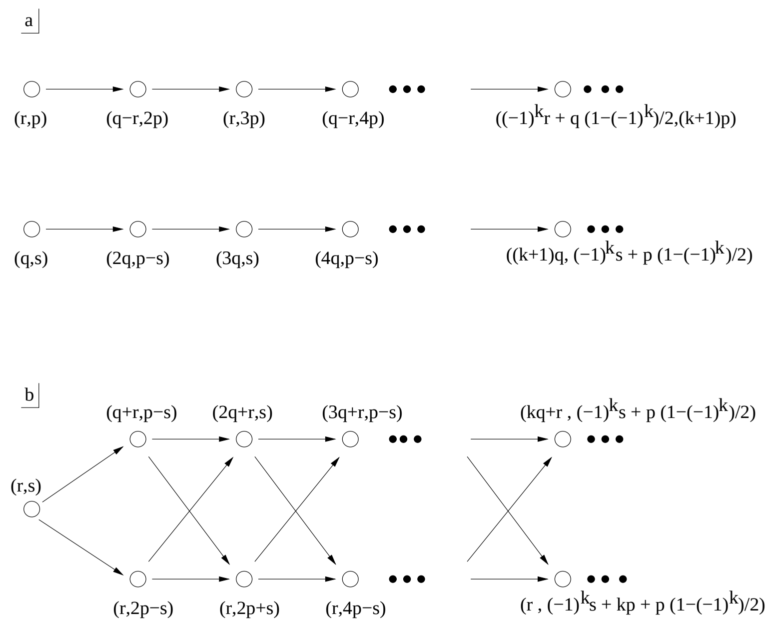


Figure 1: Nullvector embedding structure [47]

All larger higher weights are related to this standard cell by the addition of integers according to the relations [46, 47]

$$\begin{aligned} h_{r,s} &= h_{r+q,s+p} \\ h_{r,s} + rs &= h_{q+r,p-s} = h_{q-r,p+s} \\ h_{r,s} + (q-r)(p-s) &= h_{r,2p-s} = h_{2q-r,s} \end{aligned} \tag{3}$$

as long as they are in the bulk and not on the border or corners of this standard cell Kac table, i.e. as long as their indices do not obey $r = i\,q$ or $s = j\,p$ for some $i, j \in \mathbb{Z}$.

These larger weights in the Kac table bulk are exactly the weights of the nullvector descendants of the highest weights in the above standard cell. To be precise we actually find that the maximal subrepresentation of $\mathcal{M}(h)$ for h in the bulk of the Kac table is generated by two singular vectors v_1, v_2. The highest weight representations generated on v_1 and v_2, however, each contain two subrepresentations which are again both generated from two singular vectors; but actually both subrepresentations of $\mathcal{M}(v_1)$ and $\mathcal{M}(v_2)$ coincide. We therefore arrive at an embedding structure or "embedding cascade" of nullvectors as depicted in figure 1b [48, 47] whose weights are exactly the integer shifted weights appearing in the Kac table. The corresponding characters have been calculated in [49].

The irreducible representations $\mathcal{V}_{(r,s)}$ with weights $h_{r,s}$ in this standard cell and the described nullvector embedding structure have been shown to close under the following so-called BPZ fusion rules [46]

$$\mathcal{V}_{(r_1,s_1)} \otimes_{\rm f} \mathcal{V}_{(r_2,s_2)} \quad = \sum_{r_3=|r_1-r_2|+1,\text{ step }2}^{\min(r_1+r_2-1,2q-r_1-r_2-1)} \quad \sum_{s_3=|s_1-s_2|+1,\text{ step }2}^{\min(s_1+s_2-1,2p-s_1-s_2-1)} \mathcal{V}_{(r_3,s_3)} \ ,$$

where $\otimes_{\rm f}$ denotes the fusion product. We notice that the above excluded weights for $r = i\,q$ or $s = j\,p$ with $i, j \in \mathbb{Z}$, do not pop up in these fusion rules; they are hence simply ignored in these minimal models.

On the other hand, augmenting the theory with representations beyond this standard cell, especially with irreducible representations of the above excluded weights, has also led to the construction of consistent CFTs. These contain representations with non-trivial Jordan blocks and are thus examples of logarithmic CFTs. For the Virasoro representation theory of these models one actually needs the full Kac table to describe its different representations, subject only to the relation (2).

To make the terminology more precise we will call, following [32], weights whose indices obey $r = i\,q$ and $s = j\,p$ ($i, j \in \mathbb{Z}$) "on the corners of the Kac table" and weights whose indices obey (exclusively) either $r = i\,q$ or $s = j\,p$ ($i, j \in \mathbb{Z}$) "on the borders of the Kac table". All other weights which already appear in the minimal models we call "in the bulk of the Kac table". In table 1, the Kac table for $c_{2,3} = 0$, we have indicated the borders as areas with lighter shade and the corners as areas with darker shade; the bulk consists of the unshaded areas. Actually it is just this inclusion of irreducible representations with weights on the corners and the border which forces us to include states of weights of the whole Kac table into the theory.

The only well studied models up to now are contained in the series $c_{p,1}$, $p = 2, 3, \ldots$ (see e.g. [9, 20, 21, 22, 23, 24, 25]). But as these models do not contain any bulk in their Kac table, we do not expect them to be generic; indeed as we will see in this paper the existence of representations with weights in the Kac table bulk actually induces an even richer structure with indecomposable representations up to rank 3.

The nullvector embedding structure stays of course the same for representations corresponding to weights in the bulk. However, as already explained in [20] the nullvector embedding structure actually collapses to a string for representations with weights on the corners or on the border—as depicted in figure 1a. It is very important to keep in mind that the nullvectors corresponding to these higher weights in figure 1 are only true nullvectors within representations that are generated as a Virasoro module from one (!) singular vector, i.e. irreducible representations. This picture changes as soon as there appears indecomposable structure within the representation [20, 32]. Nevertheless these vectors keep their prominent role even within higher rank representations.

For later use we need to define the notion of a "weight chain" for conformal weights on the border or in the bulk. These weight chains are supposed to be a handy storage of information about the weights on successive embedding levels in the above discussed embedding structures. A weight chain for weights on the border is a list of all weights which differ just by integers, ordered by size without multiplicity (see figure 1a):

$$W^{\rm border}_{(r,p)} \quad := \quad \{h_{r,p}, h_{q-r,2p}, h_{r,3p}, \ldots\} \qquad \forall r < q \ ,$$

$$W^{\text{border}}_{(q,s)} \quad := \quad \{h_{q,s}, h_{2q,p-s}, h_{3q,s}, ...\} \qquad \forall s < p \; .$$

To form a weight chain for weights in the bulk we take a likewise list of weights differing just by integers, ordered by size without multiplicity. Then we map this list into a list of sets, the first set just consisting of the lowest weight, then every next set consisting of the next two weights. Regarding figure 1b we get

$$W^{\text{bulk}}_{(r,s)} := \{h_{r,s}, \{h_{r,2p-s}, h_{q+r,p-s}\}, \{h_{r,2p+s}, h_{2q+r,s}\}, ...\} \qquad \forall r < q, s < p \; .$$

In order to understand the generic features of augmented $c_{p,q}$ models we will study the two easiest candidates for augmented models with non-empty bulk of the Kac table in this paper, the augmented $c_{2,3} = 0$ model with Kac table given in table 1 as well as the augmented Yang-Lee model with $c_{2,5} = -22/5$. Possible candidates for higher rank representations in the augmented $c_{2,3} = 0$ model have already been explored in [32] by calculation of logarithmic nullvectors. We will show that these calculations are consistent with the findings in this article and will furthermore use these calculational tools for the explicit computation of fusion with rank 2 representations.

For the $c_{p,1}$ models it was shown that they actually have a larger $\mathcal{W}$ algebra as symmetry algebra, the triplet algebra $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$. We have strong hints that such a larger $\mathcal{W}$ algebra is also the underlying symmetry algebra of the generic augmented $c_{p,q}$ models [27]. The effective Kac table of these theories can then again be reduced to a standard cell, which is though larger as their minimal model counterpart. The standard cell is then given by $\{(r, s)|1 \leq r < n\,q,\ 1 \leq s < n\,p\}$ with n usually an odd integer larger than 1, e.g. 3 for the above mentioned triplet algebras of the $c_{p,1}$ models. In this article, however, we want to concentrate on the pure Virasoro representation theory and, hence, do not restrict our Kac table in any way.

4 Explicit discussion of the augmented $\mathbf{c_{2,3} = 0}$ model

In this section we explicitly discuss our calculations of the fusion product of representations in the $c_{2,3} = 0$ augmented model which lead us to the conjectured general fusion rules of section 6. We present examples for the newly appearing higher rank representations and also elaborate the consistency conditions for the fusion product in this case.

4.1 Higher rank representations

4.1.1 Representations of rank 2

Table 2 gives an overview over the specific properties of all rank 2 representations we have calculated for this model. The two parameters of the rank 2 representation $\mathcal{R}^{(2)}$ give the lowest weight and the weight of the logarithmic partner in this representation, i.e. the weights of the two states which generate this representation. The additional index will be explained when we discuss these particular representations (see p. 18).

Table 2: Specific properties of rank 2 representations in $c_{2,3} = 0$

	β_1	β_2	level of log. partner	level of first nullvector	level of first log. nullvector	type
$\mathcal{R}^{(2)}(5/8, 5/8)$	–	–	0	2	10	A
$\mathcal{R}^{(2)}(1/3, 1/3)$	–	–	0	3	9	A
$\mathcal{R}^{(2)}(1/8, 1/8)$	–	–	0	4	8	A
$\mathcal{R}^{(2)}(5/8, 21/8)$	-5	–	2	10	16	B
$\mathcal{R}^{(2)}(1/3, 10/3)$	$140/27$	–	3	9	18	B
$\mathcal{R}^{(2)}(1/8, 33/8)$	$-700/81$	–	4	8	20	B
$\mathcal{R}^{(2)}(0, 1)_5$	$1/3$	–	1	2	7	C
$\mathcal{R}^{(2)}(0, 1)_7$	$-1/2$	–	1	2	5	D
$\mathcal{R}^{(2)}(0, 2)_5$	–	$-5/8$	2	1	7	E
$\mathcal{R}^{(2)}(0, 2)_7$	–	$5/6$	2	1	5	F
$\mathcal{R}^{(2)}(1, 5)$	$2800/9$	–	4	6	14	C
$\mathcal{R}^{(2)}(1, 7)$	$30800/27$	$1100/9$	6	4	14	E
$\mathcal{R}^{(2)}(2, 5)$	-420	–	3	5	10	D
$\mathcal{R}^{(2)}(2, 7)$	-880	$-440/3$	5	3	10	F

Analogously, we denote an irreducible representation generated by a highest weight state of weight h by $\mathcal{V}(h)$.

The first block contains the rank 2 representations to the three different weight chains lying on the border of the Kac table, i.e. $W^{\text{border}}_{(1,2)} := \{5/8, 21/8, 85/8, \ldots\}$, $W^{\text{border}}_{(3,1)} := \{1/3, 10/3, 28/3, \ldots\}$ and $W^{\text{border}}_{(2,2)} := \{1/8, 33/8, 65/8, \ldots\}$. Both different types of rank 2 representations are depicted in figure 2. They actually exhibit precisely the same structure as the rank 2 representations of the augmented $c_{p,1}$ models described in [20]. This has already been conjectured in [32] by calculation of their first logarithmic nullvectors. Throughout this paper we stick to the graphical conventions of previous publications, see e.g. [20, 32]. As e.g. in figure 2 we take black dots to de-

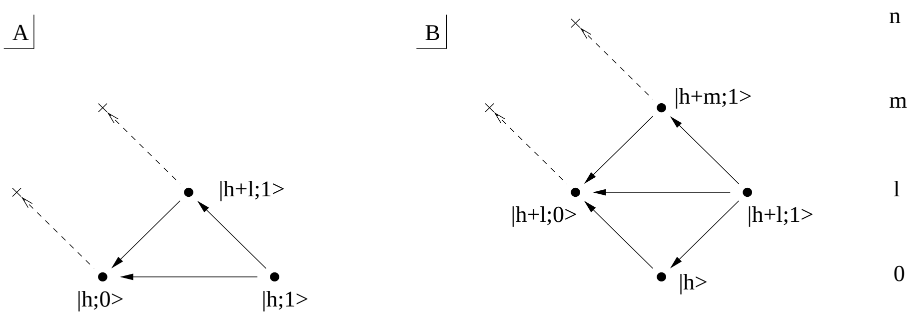


Figure 2: Rank 2 representations for weights on the border

note (sub-)singular vectors which are not null, crosses to denote nullvectors, horizontal arrows to denote indecomposable action (of L_0), arrows pointing upwards to denote a descendant relation and arrows pointing downwards to denote a non-trivial action of positive Virasoro modes. Furthermore, we indicate the levels on the right hand side of each picture.

The first three representations in table 2 are the groundstate rank 2 representation visualised in figure 2 A. They exhibit an indecomposable Jordan cell already on the zeroth level. In this case the logarithmic partner state $|h;1\rangle$ of the irreducible ground state $|h;0\rangle$ is logarithmic primary, i.e. it is primary with the exception of an indecomposable action of L_0

$$L_0|h;1\rangle \quad = \quad h\,|h;1\rangle + |h;0\rangle\ .$$

$|h;0\rangle$ actually spans an irreducible subrepresentation; its first nullvector (depicted as a cross in the figure) is hence found on the level of the next weight $h+l$ in the corresponding weight chain. The descendants of $|h;1\rangle$, however, do not form to give a nullvector at $h+l$. In order to find the first nullvector involving descendants of $|h;1\rangle$ we have to go one weight further in the weight chain. These representations are uniquely generated by the two groundstates $|h;i\rangle,\ i=0,1$; there is no need of an additional parameter to describe them.

The next three representations in table 2 are the first excited representations of these three weight chains depicted in figure 2 B. The logarithmic partner state $|h+l;1\rangle$ lies on that level l at which we would expect the first nullvector of the groundstate $|h\rangle$ if the groundstate were to span an irreducible representation. Hence, the subrepresentation generated by $|h\rangle$ is not irreducible, but only indecomposable. Actually the indecomposable action of L_0 just maps $|h+l;1\rangle$ to the singular level l descendant of $|h\rangle$ which we call $|h+l;0\rangle$ and which normally would be the first nullvector of an irreducible h representation

$$L_0|h+l;1\rangle \quad = \quad (h+l)\,|h+l;1\rangle + |h+l;0\rangle\ .$$

$|h+l;0\rangle$, on the other hand, spans an irreducible subrepresentation and yields its first nullvector on the level of the second weight after h in the weight chain, called $h+m$. In order to find the first logarithmic nullvector we have to go even one weight further in the weight chain to $h+n$. Now, the logarithmic partner field can certainly not be logarithmic primary; but as discussed in [20] it can still be made to at least vanish under all $L_p,\ p>2$. This induces one characteristic parameter $\beta=\beta_1$ in this representation due to the action of L_1; we take β to parameterise the following equation

$$(L_1)^l\ |h+l;1\rangle \quad = \quad \beta\,|h\rangle\ .$$

The structure visualised in figure 2 B is actually thought to be the generic type of rank 2 representation for weights on the border. It should be found for every two adjacent weights in a border weight chain. As in the $c_{p,1}$ model case the representations

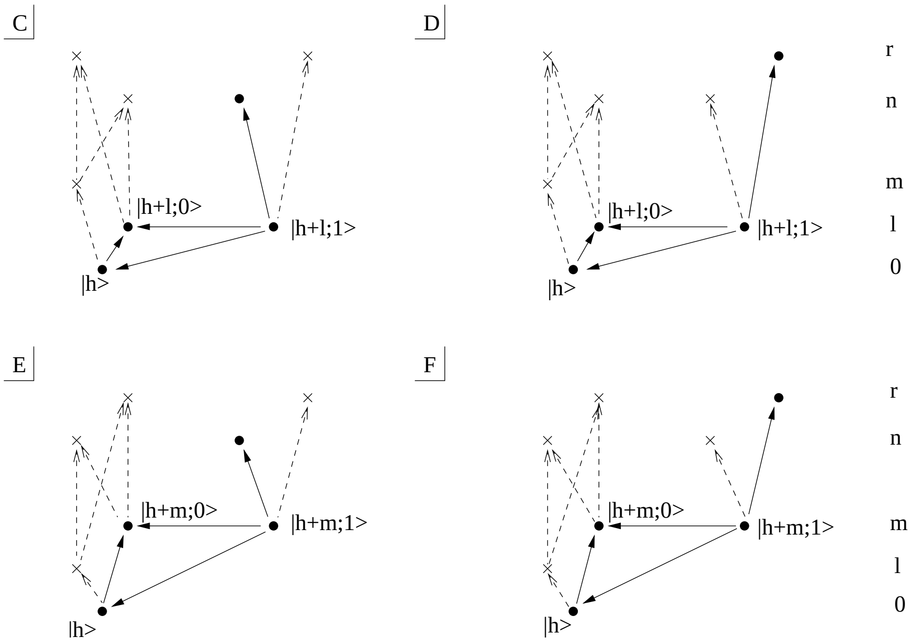


Figure 3: Rank 2 representations for weights in the bulk

of the type depicted in figure 2 A can only be found for the first and hence lowest weight of a weight chain. Furthermore, we want to stress that we can think of these rank 2 representations as being constructed by indecomposably connecting several irreducible representations. This is already suggested by figures 2 A and 2 B where the black dots represent these "towers of states" which have been irreducible representations before their indecomposable connection to a rank 2 representation. Generically, these towers of states are not irreducible subrepresentations any more, but they resemble their irreducible counterparts in terms of number of states and singular vectors. This point of view has also been worked out in [20] for rank 2 representations in the augmented $c_{p,1}$ models.

We have also successfully checked for the existence of the first logarithmic nullvectors at the levels given in table 2 using the algorithm of [32]. Even the level 20 logarithmic nullvector was now accessible to our computational power due to our explicit knowledge of β.

The second block of table 2 contains the specific properties of the lowest rank 2 representations which we found for weights in the weight chain of the Kac table bulk, i.e. $W^{\text{bulk}}_{(1,1)} := \{\{0\},\{1,2\},\{5,7\},\ldots\}$. There are four different types of rank 2 representations depicted in figure 3 which appear to be a generalisation of the situation on the border for the case of figure 2 B. This generalisation has to take into account the more complicated embedding structure of representations with weights in the bulk—the linear picture of figure 1a has to be replaced by the two string twisted picture of figure 1b. As we now have two possible nullvectors on every step of the weight chain, of which only one is rendered non-null by the existence of a logarithmic partner state of the same level, we actually also encounter cases E and F where there is a true nullvector on a level lower than the logarithmic partner state. This is new and makes the description of these particular cases more complicated.

Let us first describe the cases C and D. Starting with the lowest weight state $|h\rangle$ the first possible nullvectors are given by the next set in the weight chain at levels l and m. In the present cases the corresponding singular descendant $|h+l;0\rangle$ on $|h\rangle$ at the lower of these two levels l is rendered to be non-null by the existence of a logarithmic partner state $|h+l;1\rangle$. The indecomposable action of L_0 on $|h+l;1\rangle$ is again given by

$$L_0|h+l;1\rangle \quad = \quad (h+l)\,|h+l;1\rangle + |h+l;0\rangle\ .$$

Furthermore, the argument of [20] still applies that due to the absence of a nullvector on $|h\rangle$ on a level lower than the Jordan cell we can transform $|h+l;1\rangle$ by the addition of level l descendants of $|h\rangle$ such that it is annihilated by $L_p\ \forall p \geq 2$. Then L_1 maps $|h+l;1\rangle$ to the unique level $l-1$ descendant of $|h\rangle$ which is annihilated by $L_p\ \forall p \geq 2$. As for representations with weights on the border we take the resulting one parameter $\beta = \beta_1$ to parameterise the equation

$$(L_1)^l\ |h+l;1\rangle \quad = \quad \beta\,|h\rangle\ .$$

The second state corresponding to this set in the weight chain, the one at level m, actually stays null in the rank 2 representation. This fixes the nullvector structure on $|h\rangle$ as the embedding structure of figure 1 b tells us that the nullvectors of the next set in the weight chain, at levels n and r, are joint descendants of the states corresponding to the previous set of weights. But as the singular state at level m is already a nullvector the singular states at level n and r have to be null as well. The situation is somewhat different for the descendants of the logarithmic partner as $|h+l;1\rangle$ is the starting point of its embedding structure. The cases C and D correspond to the two possibilities of having the first logarithmic nullvector on level r respectively n. There is, however, no additional nullvector at the respective other weight. Examples for the case C are $\mathcal{R}^{(2)}(0,1)_5$ and $\mathcal{R}^{(2)}(1,5)$, for the case D $\mathcal{R}^{(2)}(0,1)_7$ and $\mathcal{R}^{(2)}(2,5)$. Again, the lowest representations play a special role as both cases are realised for lowest weight 0. Hence, we indicate the level of the logarithmic descendant which is promoted to be non-null as an index. It is important to note, however, that both cases are already distinguished by their different β values.

Let us turn to the cases E and F. These exhibit very much the same structure as the cases C and D. The crucial difference is the existence of a nullvector already on a level lower than the level of the logarithmic partner. This fact prevents us from applying the above argument how to describe the representation by only one parameter. The special cases of $\mathcal{R}^{(2)}(0,2)_5$ and $\mathcal{R}^{(2)}(0,2)_7$ can nevertheless be reduced to one parameter quite easily. As there is no non-null descendant of the lowest weight state $|0\rangle$ at level 1 the only positive Virasoro mode which can map the logarithmic partner $|2;1\rangle$ to a non-zero state is L_2. Hence, we take the one parameter $\beta = \beta_2$ to parameterise the equation

$$L_2\ |2;1\rangle \quad = \quad \beta\,|0\rangle\ .$$

This behaviour is, however, not generic. We find that we need at least two parameters to describe these two kinds of rank 2 representations in general. To see this let us regard all normal ordered monomials in Virasoro modes of length m. We are able to transform $|h+m;1\rangle$ by addition of level m descendants of $|h\rangle$ in such a way that only the application of two such Virasoro monomials does not annihilate $|h+m;1\rangle$. In particular we have:

- For $\mathcal{R}^{(2)}(1,7)$ we find the two parameters $\beta_1 = 30800/27$ and $\beta_2 = 1100/9$ parameterising

$$\begin{aligned} (L_1)^6\,|7;1\rangle &= \beta_1\,|1\rangle \\ (L_1)^3\,L_3\,|7;1\rangle &= \beta_2\,|1\rangle\ . \end{aligned}$$

  $\mathcal{R}^{(2)}(1,7)$ has been parameterised in such a way that the monomials $(L_1)^6$ and $(L_1)^3\,L_3$ are the only ones with a non-trivial action on $|7;1\rangle$. This yields the following mappings of single Virasoro modes

$$\begin{aligned} L_1\,|7;1\rangle &= \frac{11}{729}\left(\frac{857}{2}L_{-5} - 473\,L_{-4}\,L_{-1} - 721\,L_{-3}\,L_{-2} + \frac{4279}{12}\,L_{-3}\,L_{-1}^2 \right.\\ &\qquad\left. + \frac{809}{2}\,L_{-2}^2\,L_{-1} - \frac{1295}{12}\,L_{-2}\,L_{-1}^3\right)|1\rangle \\ L_2\,|7;1\rangle &= 0 \\ L_3\,|7;1\rangle &= \frac{275}{324}\left(L_{-1}^3 + 12\,L_{-2}\,L_{-1} - 24\,L_{-3}\right)|1\rangle \\ L_p\,|7;1\rangle &= 0 \qquad \forall p \geq 4\ . \end{aligned}$$

- For $\mathcal{R}^{(2)}(2,7)$ the two parameters are $\beta_1 = -880$ and $\beta_2 = -440/3$ parameterising

$$\begin{aligned} (L_1)^5\,|7;1\rangle &= \beta_1\,|2\rangle \\ (L_1)^2\,L_3\,|7;1\rangle &= \beta_2\,|2\rangle\ . \end{aligned}$$

  This yields the following mappings of single Virasoro modes

$$\begin{aligned} L_1\,|7;1\rangle &= \frac{5}{17}\left(-\frac{143}{3}L_{-4} + \frac{44}{3}\,L_{-3}\,L_{-1} + \frac{49}{3}\,L_{-2}^2 - 6\,L_{-2}\,L_{-1}^2\right)|2\rangle \\ L_2\,|7;1\rangle &= 0 \\ L_3\,|7;1\rangle &= -\frac{20}{3}\left(L_{-1}^2 - \frac{3}{2}\,L_{-2}\right)|2\rangle \\ L_p\,|7;1\rangle &= 0 \qquad \forall p \geq 4\ . \end{aligned}$$

In both cases L_3 maps $|h+m;1\rangle$ to a multiple of the unique descendant of $|h\rangle$ on level $l-1$ which is annihilated by $L_p\ \forall p \geq 2$.

We conjecture that it actually suffices to have two parameters in order to characterise rank 2 representations of type E and F. As we have a nullvector already on the

Table 3: Number of states for $\mathcal{R}^{(3)}(0,0,1,1)$

level	number of states			total number	new null
	Jordan	Jordan	Jordan	of state	subrepresentations
	level 0	level 1	level 2		
0	1	1	–	2	–
1	2	1	1	4	–
2	3	1	1	5	1
3	5	2	2	9	–
4	8	3	3	14	–
5	11	5	4	20	2
6	17	7	6	30	–
7	23?	11?	8?	42?	2

lower level l this unique state on level $m-1$ which is annihilated by $L_p\ \forall p \geq 2$ is a descendant of this nullvector. Hence, we want to lift the restrictions by incorporating one further non-zero mapping, the mapping by $L_{m-(l-1)}$ to the unique state on level $l-1$ which is annihilated by $L_p\ \forall p \geq 2$, in order to ensure that we do not map into pure descendants of the lower nullvector. This is, indeed, equivalent to demanding that the application of all normal ordered positive Virasoro monomials of length m annihilates $|h+m;1\rangle$ except for L_1^m and $L_1^{l-1}L_{m-(l-1)}$.

Again, we want to stress that also these rank 2 representations can be thought of to be constructed by an indecomposable connection of several (to be precise four) irreducible representations. As for the rank 2 representations on the border the former irreducible representations are signified by the black dots in the corresponding figures 3 C-F.

For all bulk rank 2 representations listed in table 2 we were able to see the level of the first logarithmic nullvector already in the calculated fusion spectrum. We also confirmed this lowest logarithmic nullvector using the algorithm of [32]. It is remarkable to see that for the bulk rank 2 representations $\mathcal{R}^{(2)}(0,1)_i$ and $\mathcal{R}^{(2)}(0,2)_i$, $i=5,7$, we encounter the first nullvectors already on lower levels than the ones proposed in [32]. However, the solutions in [32] are given for general β; it is only for the special βs in table 2 that we encounter solutions even on lower levels.

4.1.2 Representations of rank 3

Representations of rank 3 only appear for weights in the bulk. In the following we will discuss the three lowest examples explicitly.

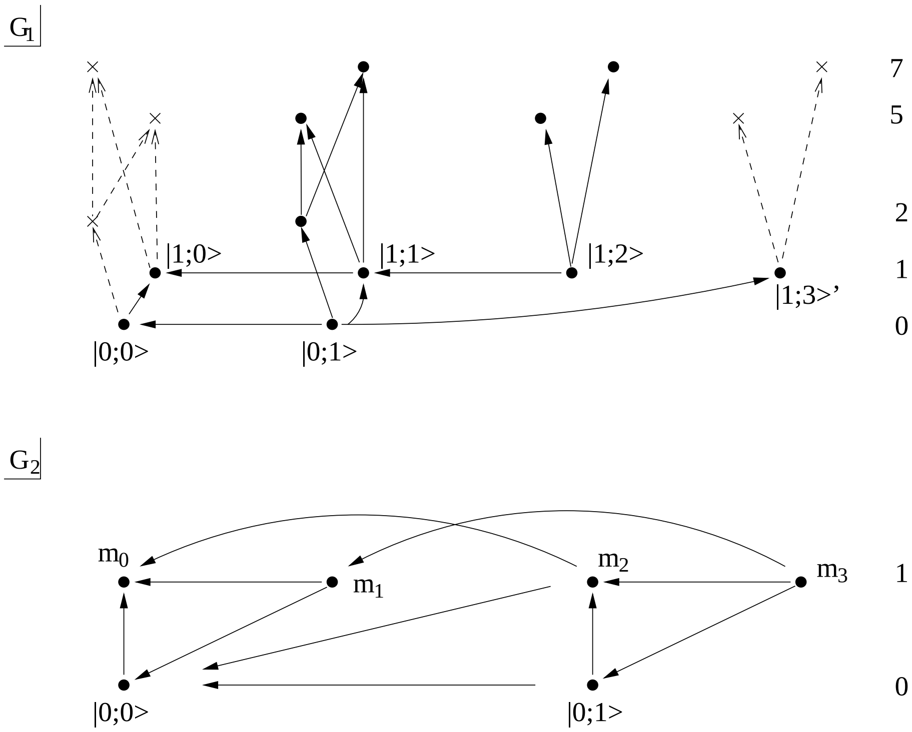


Figure 4: Two ways to visualise $\mathcal{R}^{(3)}(0,0,1,1)$

$\boldsymbol{\mathcal{R}^{(3)}(0,0,1,1)}$: Although we only encounter a rank 3 indecomposable structure in this representation, we nevertheless need four states to generate it. We will see that this is necessary and natural by two different ways of visualising the nullvector structure of $\mathcal{R}^{(3)}(0,0,1,1)$.

The first way starts out with the Jordan diagonalisation of the representation and is shown in figure 4 G_1. We have two groundstates $|0;i\rangle$, $i=0,1$, at level 0 which are interrelated by the rank 2 indecomposable action of L_0

$$L_0\,|0;1\rangle = |0;0\rangle \qquad\qquad L_0\,|0;0\rangle = 0\ .$$

On level 1, the level of the first possible nullvector on $|0;i\rangle$, the Jordan cell is enhanced to rank 3

$$\begin{aligned} L_0\,|1;2\rangle &= |1;2\rangle + |1;1\rangle \\ L_0\,|1;1\rangle &= |1;1\rangle + |1;0\rangle \\ L_0\,|1;0\rangle &= |1;0\rangle\ . \end{aligned}$$

A further fourth state of weight 1 decouples in the L_0 action

$$L_0\,|1;3\rangle' = |1;3\rangle'\ ;$$

but this seeming decoupling is deceiving as the singular descendant of $|0;1\rangle$ is actually composed of the sum of the Jordan cell state $|1;1\rangle$ and the "decoupling" state $|1;3\rangle'$; indeed the action of L_{-1} on $|0;i\rangle$, $i=0,1$, is given by

$$\begin{aligned} L_{-1}\,|0;0\rangle &= |1;0\rangle \\ L_{-1}\,|0;1\rangle &= |1;1\rangle + |1;3\rangle'\ . \end{aligned}$$

The further nullvector structure is also depicted in figure 4 G_1. We were able to calculate all states up to level 6 explicitly. The total number of states is also attainable for level 7, but in an indirect way via the second view on this representation to be discussed below. We give a list of the total number of states in table 3; we split the number according to the position of the state in the Jordan cell, which we also call the "Jordan level" of that state in the cell (following [9] the enumeration starts with 0). E.g. for level 5 there are four rank 3 Jordan cells, one additional rank 2 Jordan cell plus the four which are subcells of a rank 3 cell as well as six additional single eigenvalues.

Now we find one nullvector on level 2, the singular descendant $L_{-2}\,|0;0\rangle$. This nullvector has two singular descendants on level 5 and 7 which are also nullvectors of $|1;0\rangle = L_{-1}\,|0;0\rangle$ due to the bulk embedding structure. The "decoupling" state $|1;3\rangle'$ generates an irreducible representation with null singular vectors on level 5 and 7. These vectors as well as their descendants are the only nullvectors up to level 7. We do not yet encounter a logarithmic nullvector up to this level.

In a second way of visualising this representation we can actually view it as an indecomposable combination of rank 2 representations whose structure is given in much the same way as in figure 2 A; we only have to replace the two lowest black dots by the two rank 2 representations $\mathcal{R}^{(2)}(0,1)_5$ and the higher by $\mathcal{R}^{(2)}(2,7)$. Surprisingly, we can also put $\mathcal{R}^{(3)}(0,0,1,1)$ in a likewise form with the lower two black dots replaced by $\mathcal{R}^{(2)}(0,1)_7$ and the higher with $\mathcal{R}^{(2)}(2,5)$. Let us see how this comes about.

We choose the setup for the lowest levels as depicted in figure 4 G_2: we have two separate rank 2 representations with lowest states $|0;0\rangle$ respectively $|0;1\rangle$. $|0;0\rangle$ has a logarithmic partner at level 1, called m_1, to its singular descendant $m_0 := L_{-1}\,|0;0\rangle = |1;0\rangle$. Likewise, $|0;1\rangle$ has a logarithmic partner at level 1, called m_3, to its singular descendant $m_2 := L_{-1}\,|0;1\rangle = |1;1\rangle + |1;3\rangle'$. Furthermore, both representations are connected by the indecomposable L_0 action on $|0;1\rangle$. This directly promotes to the indecomposable action of L_0 on their L_{-1} descendants

$$L_0\,m_0 = |1;0\rangle \qquad\qquad L_0\,m_2 = |1;2\rangle + |1;0\rangle$$

as well as the consistent L_1 action

$$L_1\,m_0 = 0 \qquad\qquad L_1\,m_2 = 2\,|0;0\rangle\ .$$

Now we still have to check whether we can find m_1 and m_3 which fit this setup. We express m_1 and m_3 as linear combinations of the level 1 states $|1;j\rangle$, $j=1,2,3$ and impose the following conditions and parameters:

- $L_0\, m_1 = m_1 + m_0$,
- $L_1\, m_1 = \xi_1 |0;0\rangle$,
- $L_0\, m_3 = m_3 + m_2 + r\, m_1 + s\, m_0$,
- $L_1\, m_3 = \xi_2 |0;1\rangle + \xi_3 |0;0\rangle$.

We find that s and ξ_3 are actually irrelevant parameters which can be set to 0 using the residual freedom. r and ξ_2 are functions of ξ_1. Hence, there is one free parameter in this representation. It is most useful to just express r in terms of ξ_1 and then study the relation between ξ_1 and ξ_2 as these parameters are the defining parameters of the rank 2 representations we started with. The relation is given by

$$(12\,\xi_1 + 1)\,(12\,\xi_2 + 1) \quad = \quad 25\ .$$

There are only two solutions which fit the bulk rank 2 spectrum discussed above; they also seem to be the most natural ones

$$\begin{aligned} \xi_1 = \xi_2 \quad &= \quad -\frac{1}{2} \\ \xi_1 = \xi_2 \quad &= \quad \frac{1}{3}\ . \end{aligned}$$

These two solutions exactly give the lower level rank 2 representations which we proposed above to be the two lower dots in the figure 2 A like setup. In order to get the respective representation corresponding to the higher dot of figure 2 A we just have to count the number of states and compare these. As this higher rank 2 representation only "fills up" states which would be null in a pure rank 2 setting, but are not null in this rank 3 setting (e.g. $L_{-2}|0;1\rangle$) we do not need any further parameters to describe this representation.

Therefore there is only the one additional parameter r besides the parameters of the ingredient rank 2 representations which we need to describe $\mathcal{R}^{(3)}(0,0,1,1)$ which is given in terms of ξ_1

$$r \quad = \quad \frac{-25}{12\,\xi_1 + 1}\ .$$

Finally, we still need to explain how to determine the number of states for level 7 (as in table 3). If we want to use the total number of states to determine this higher black dot in the above setting we can decide this question knowing the total number of states of $\mathcal{R}^{(3)}(0,0,1,1)$ up to level 5 ($\mathcal{R}^{(2)}(2,5)$ and $\mathcal{R}^{(2)}(2,5)$ already differ at their third level). But then, we can in turn easily use this setting in order to determine the number of states for any higher level.

$\boldsymbol{\mathcal{R}^{(3)}(0,0,2,2)}$: The rank 3 representation $\mathcal{R}^{(3)}(0,0,2,2)$ looks much the same as the previous one. Due to the nullvector at level 1 which appears at a lower level than the final rank 3 structure we have to take some more care in fixing the freedom.

Table 4: Number of states for $\mathcal{R}^{(3)}(0,1,2,5)$

level	number of states			total number of state	new null subrepresentations
	Jordan level 0	Jordan level 1	Jordan level 2		
0	1	–	–	1	–
1	1	1	–	2	–
2	2	2	–	4	–
3	3	3	–	6	–
4	5	5	–	10	–
5	8	7	1	16	–

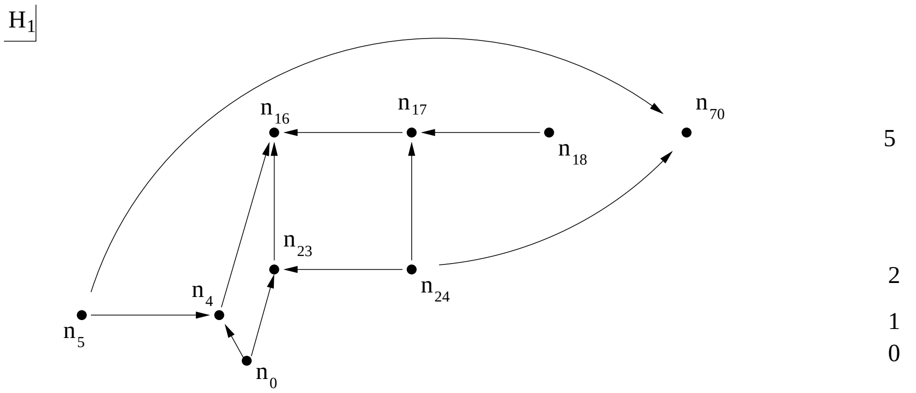
There are again two ways to decompose this rank 3 representation into rank 2 representations. Either we find twice $\mathcal{R}^{(2)}(0,2)_5$ and $\mathcal{R}^{(2)}(1,7)$ or twice $\mathcal{R}^{(2)}(0,2)_7$ and $\mathcal{R}^{(2)}(1,5)$.

$\mathcal{R}^{(\mathbf{3})}(\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{5})$: The rank 3 representation $\mathcal{R}^{(3)}(0,1,2,5)$ is the only higher rank 3 representation which was accessible to our calculations up to that level at which the rank 3 structure appears. From our knowledge of the other towers of representations it should nevertheless be fair to conjecture that most of the generic features of rank 3 representations in these $c_{p,q}$ models are already visible in this example.

In table 4 we list the number of states as calculated. We have also included the basis of states which brings L_0 into Jordan diagonal form in appendix A.

Again we find two ways to visualise the embedding structure. The Jordan diagonalisation of L_0 gives an embedding structure of the form depicted in figure 5 H_1. As the situation is even more complicated as for $\mathcal{R}^{(3)}(0,0,1,1)$ we have labeled the states according to the indexed basis which is chosen by the computer and listed in appendix A. We can see that both singular vectors on n_0, i.e. n_4 at level 1 and n_{23} at level 2, are incorporated into rank 2 Jordan cells. But nevertheless we do not encounter a first rank 3 cell until level 5. The first vector of this rank 3 Jordan cell, n_{16}, which is a true eigenvector, is given by the joint singular vector on n_4 and n_{23}. As in the $\mathcal{R}^{(3)}(0,0,1,1)$ case there is a vector at level 5 which seems to decouple from the representation. But again this decoupling in terms of the L_0 action is deceiving as this vector is a sum of descendants of n_5 and n_{24} (see appendix A for the explicit expressions). Hence, as for $\mathcal{R}^{(3)}(0,0,1,1)$ this rank 3 representation needs four generating states.

As a second way to visualise this representation we conjecture that we can construct this representation by an indecomposable combination of the four rank 2 representations $\mathcal{R}^{(2)}(0,1)_7$, twice $\mathcal{R}^{(2)}(2,5)$ and $\mathcal{R}^{(2)}(7,12)$ assembled as in figure 2 B; as before we replace the black dots in figure 2 B by the respective rank 2 representations. The picture



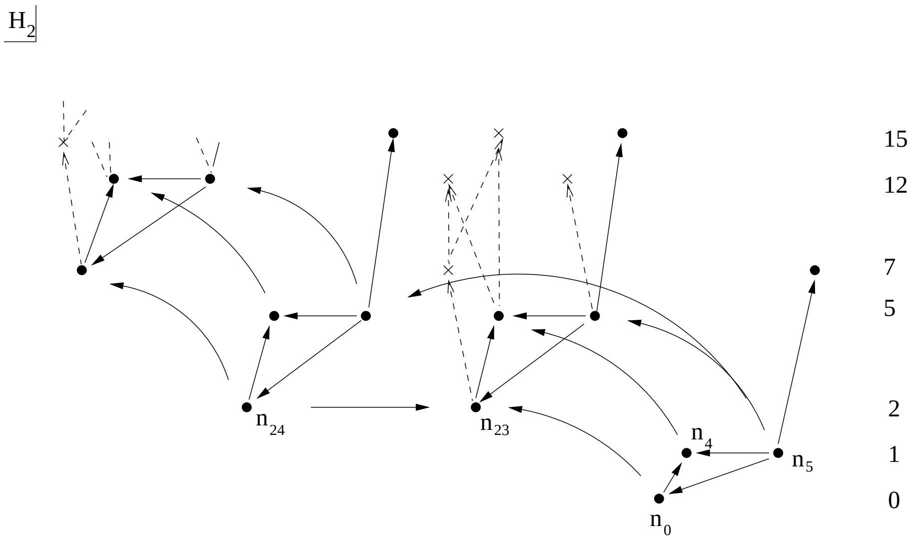


Figure 5: Two ways to visualise $\mathcal{R}^{(3)}(0, 1, 2, 5)$

that emerges is depicted in figure 5 H_2. What are the hints at such a deconstruction of $\mathcal{R}^{(3)}(0,1,2,5)$? First of all we find that

$$L_1\,n_5 \;=\; -\frac{1}{2}n_0\;;$$

this equation reproduces the defining β parameter of $\mathcal{R}^{(2)}(0,1)_7$. Hence, the two lowest generating states of $\mathcal{R}^{(3)}(0,1,2,5)$, n_0 and n_5, generate a tower of states which resembles the rank 2 representation $\mathcal{R}^{(2)}(0,1)_7$. The only difference to a true subrepresentation $\mathcal{R}^{(2)}(0,1)_7$ is that the usual first nullvector on n_0, $n_{23} = L_{-2}\,n_0$, is rendered non-null by its inclusion into an indecomposable rank 2 cell with

$$\begin{aligned} L_0\,n_{23} &= 2\,n_{23} \\ L_0\,n_{24} &= 2\,n_{24} + n_{23}\;. \end{aligned} \tag{4}$$

In the $\mathcal{R}^{(3)}(0,0,1,1)$ case we have seen an example that the indecomposable connection of two rank 2 cells of the same type of rank 2 representation produces a rank 3 cell and a seemingly decoupling further state. But this is exactly the structure we discover in this case at level 5—a rank 3 cell

$$\begin{aligned} L_0\,n_{16} &= 2\,n_{16} \\ L_0\,n_{17} &= 2\,n_{17} + n_{16} \\ L_0\,n_{18} &= 2\,n_{18} + n_{17} \end{aligned}$$

as well as a seemingly decoupling state n_{70}. Hence, we conjecture that n_{23} and n_{24} are actually both the lower generators of towers of states which both resemble $\mathcal{R}^{(2)}(2,5)$, but which are indecomposably connected according to equation (4). This structure up to level 5 is in perfect agreement with the total count of states given in table 4. Unfortunately, we cannot say anything about the embedding structure or the count of states for higher levels and, thus, the inclusion of the fourth rank 2 representation $\mathcal{R}^{(2)}(7,12)$ is highly conjectural. It is only lead by the intuition that in any rank 2 structure the first nullvectors on the true eigenstate have to have non-null corresponding states on the side of the logarithmic partner due to the non-degeneracy of the Shapovalov form.

There is, however, one further way of looking at the situation. Let us regard the fusion equation

$$\mathcal{V}(1/3)\otimes_{\rm f}\mathcal{R}^{(2)}(1,5) \;=\; \mathcal{R}^{(3)}(0,1,2,5)\;.$$

But as described before we can also view $\mathcal{R}^{(2)}(1,5)$ as a combination of four towers of states which resemble their irreducible counterparts in terms of numbers of states and singular vectors but which are indecomposably connected to form $\mathcal{R}^{(2)}(1,5)$. In this case, we can think of $\mathcal{R}^{(2)}(1,5)$ as being constructed by indecomposably connecting $\mathcal{V}(1)$, twice $\mathcal{V}(5)$ as well as $\mathcal{V}(12)$. But the fusion of its single constituents should be consistent with the fusion of $\mathcal{R}^{(2)}(1,5)$ itself. Therefore, inspecting the fusion rules (the first two calculated, see appendix B, the third inferred from the fusion rules of section 6)

$$\begin{aligned}\mathcal{V}(1/3)\otimes_{\mathrm{f}}\mathcal{V}(1) &= \mathcal{R}^{(2)}(0,1)_7\\ \mathcal{V}(1/3)\otimes_{\mathrm{f}}\mathcal{V}(5) &= \mathcal{R}^{(2)}(2,5)\\ \mathcal{V}(1/3)\otimes_{\mathrm{f}}\mathcal{V}(12) &= \mathcal{R}^{(2)}(7,12)\end{aligned}$$

we are again lead to the conjecture that we can build $\mathcal{R}^{(3)}(0,1,2,5)$ by indecomposably connecting $\mathcal{R}^{(2)}(0,1)_7$, twice $\mathcal{R}^{(2)}(2,5)$ and $\mathcal{R}^{(2)}(7,12)$.

A similar inspection of the fusion equation

$$\mathcal{V}(5/8)\otimes_{\mathrm{f}}\mathcal{R}^{(2)}(5/8,21/8) = \mathcal{R}^{(3)}(0,1,2,5)$$

leads to the conjecture that we can also construct $\mathcal{R}^{(3)}(0,1,2,5)$ by indecomposably connecting $\mathcal{R}^{(2)}(0,2)_7$, twice $\mathcal{R}^{(2)}(1,5)$ and $\mathcal{R}^{(2)}(7,15)$. This is also in perfect agreement with the total count of states up to the accessible level 5 and, furthermore, a nice generalisation of the $\mathcal{R}^{(3)}(0,0,1,1)$ case, which can also be constructed out of rank 2 representations in two ways. To see the embedding of the lowest rank 2 representation $\mathcal{R}^{(2)}(0,2)_7$ is a bit more tricky this time. For the non-trivial action of the positive Virasoro modes on the new state n_{24} at level 2 we find (after removal of some residual freedom)

$$\begin{aligned}L_1\,n_{24} &= 3\,n_5\\ L_2\,n_{24} &= -\frac{17}{12}\,n_0\,.\end{aligned} \qquad (5)$$

This state n_{24} is the logarithmic partner of the n_0 descendant $n_{23}=L_{-2}\,n_0$. From the description of $\mathcal{R}^{(2)}(0,2)_7$ we are used to these two states spanning the lowest Jordan cell. But the non-trivial mapping of L_1 in equation (5) is in clear contradiction to $\mathcal{R}^{(2)}(0,2)_7$ having a first nullvector and, hence, no state at level 1. Furthermore, we do not recover the correct β value in equation (5). But we have forgotten to take into account that due to the absence of this nullvector on level 1 the level 2 singular vector has shifted to

$$n'_{23} = n_{23}-\frac{3}{2}\,n_4\,.$$

The correct logarithmic partner to n'_{23} is given by

$$n'_{24} = n_{24}+\frac{9}{8}\,n_4-\frac{3}{2}\,n_5\,.$$

Indeed, if we calculate the action of the positive Virasoro modes for n'_{24} we find the desired properties

$$\begin{aligned}L_1\,n'_{24} &= 0\\ L_2\,n'_{24} &= \frac{5}{6}\,n_0\,.\end{aligned}$$

$\mathcal{R}^{(3)}(\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{7})$: Although the rank 3 representation $\mathcal{R}^{(3)}(0,1,2,7)$ is not accessible to our computational power we can nevertheless tackle its decomposition in the same way as the last method in the preceding case. For this argument we take the appearances of $\mathcal{R}^{(3)}(0,1,2,7)$ for granted as conjectured in appendix B. Then looking at the fusion equation

$$\mathcal{V}(5/8) \otimes_{\mathrm{f}} \mathcal{R}^{(2)}(1/8,33/8) \quad = \quad \mathcal{R}^{(3)}(0,1,2,7)$$

we conjecture that $\mathcal{R}^{(3)}(0,1,2,7)$ can be constructed by indecomposably connecting the four rank 2 representations $\mathcal{R}^{(2)}(0,1)_5$, twice $\mathcal{R}^{(2)}(2,7)$ and $\mathcal{R}^{(2)}(5,12)$. Similarly, looking at

$$\mathcal{V}(1/3) \otimes_{\mathrm{f}} \mathcal{R}^{(2)}(1/3,10/3) \quad = \quad \mathcal{R}^{(3)}(0,1,2,7) \oplus \mathcal{R}^{(2)}(1/3,10/3)$$

we can think of $\mathcal{R}^{(3)}(0,1,2,7)$ to be composed of $\mathcal{R}^{(2)}(0,2)_5$, twice $\mathcal{R}^{(2)}(1,7)$ and $\mathcal{R}^{(2)}(5,15)$.

4.2 Explicit calculation of the fusion products

We have calculated the fusion products of a large variety of representations in the augmented $c_{2,3} = 0$ model. To do this we have used the Nahm algorithm described in section 2 to determine the fusion product of irreducible and rank 1 representations with themselves as well as with the lowest lying and first excited rank 2 representations. In order to show that the fusion algebra indeed closes we have used the symmetry and associativity of the fusion product and calculated the fusion of higher rank representations. We have also used these conditions in order to perform consistency checks as described in the next subsection. The results itself are listed in appendix B.

In section 6 we want to propose a generalisation of the BPZ and $c_{p,1}$ fusion rules which is applicable to all augmented $c_{p,q}$ models and, hence, also describes in a unifying way the fusion of this model. But as these general rules look quite complex it is also possible to find simplified versions for the augmented $c_{2,3} = 0$ model, e.g.

$$\mathcal{V}(5/8) \otimes_{\mathrm{f}} \mathcal{W}(1/3|i) \quad = \quad \mathcal{W}(-1/24|i) \ ,$$

where $\mathcal{W}(h|i)$ signifies the ith element in the weight chain starting with h.

4.3 Consistency of fusion products

A consistent fusion product has to obey two main properties, symmetry and associativity. We have used both these properties for consistency checks of the chosen spectrum and the performed calculation as well as for the determination of the fusion product of higher rank representations.

The main implication of this consistency, however, is the absence of an irreducible representation of weight $h = 0$ in the spectrum, call it $\mathcal{V}(0)$. The representation $\mathcal{V}(0)$ would be endowed with nullvectors on level 1 and 2 and would, hence, only consist of the one groundstate. Performing the Nahm algorithm of section 2 we get the following fusion products

$$\mathcal{V}(0) \otimes_{\rm f} \mathcal{V}(0) = \mathcal{V}(0) \qquad (6)$$

$$\mathcal{V}(0) \otimes_{\rm f} \mathcal{V}(h) = 0 \quad \forall\, h \in \left\{\frac{5}{8}, \frac{1}{3}, \frac{1}{8}, \frac{-1}{24}, \frac{33}{8}, \frac{10}{3}, \frac{21}{8}, 2, 1, 7, 5\right\} . \qquad (7)$$

On the other hand using just the equations (7) and the fusion rules of appendix B we arrive at

$$\begin{aligned}\Big(\mathcal{V}(0) \otimes_{\rm f} \Big(\mathcal{V}(2) \otimes_{\rm f} \mathcal{V}(2)\Big)\Big) &= \Big(\mathcal{V}(0) \otimes_{\rm f} \mathcal{V}(0)\Big) \oplus \Big(\mathcal{V}(0) \otimes_{\rm f} \mathcal{V}(2)\Big) \\ &= \mathcal{V}(0) \otimes_{\rm f} \mathcal{V}(0)\end{aligned}$$

as well as by associativity at

$$\begin{aligned}\Big(\Big(\mathcal{V}(0) \otimes_{\rm f} \mathcal{V}(2)\Big) \otimes_{\rm f} \mathcal{V}(2)\Big) &= 0 \otimes_{\rm f} \mathcal{V}(2) \\ &= 0 \,.\end{aligned}$$

(Similar equations can be obtained involving the other $\mathcal{V}(h)$ with h from (7).) This argument thus implies

$$\mathcal{V}(0) \otimes_{\rm f} \mathcal{V}(0) = 0 \,.$$

But this is in clear contradiction to (6). Fortunately, however, $\mathcal{V}(0)$ completely decouples from the rest of the fusion (as one can see in appendix B). Hence, the contradiction is easily solved by simply excluding $\mathcal{V}(0)$ from the spectrum.

On the other hand, the representations $\mathcal{R}^{(2)}(0,1)_5$ and $\mathcal{R}^{(2)}(0,1)_7$ contain a state with weight 0 which generates a subrepresentation $\mathcal{R}^{(1)}(0)_1$. This subrepresentation is indecomposable but neither is it irreducible nor does it exhibit any higher rank behaviour. It only exists as a subrepresentation as it needs the embedding into the rank 2 representation in order not to have nullvectors at both levels 1 and 2. But, nevertheless, being a subrepresentation of a representation in the spectrum it has to be included into the spectrum, too. Similarly, the representations $\mathcal{R}^{(2)}(0,2)_5$ and $\mathcal{R}^{(2)}(0,2)_7$ contain a rank 1 subrepresentation $\mathcal{R}^{(1)}(0)_2$. However, looking at the fusion rules which we calculate for these two rank 1 representations (see appendix B), especially

$$\mathcal{R}^{(1)}(0)_2 \otimes_{\rm f} \mathcal{V}(h) = \mathcal{V}(h) \quad \forall\, h \in \left\{\frac{5}{8}, \frac{1}{3}, \frac{1}{8}, \frac{-1}{24}, \frac{33}{8}, \frac{10}{3}, \frac{21}{8}, 2, 1, 7, 5\right\} ,$$

we see that the situation is after all not really too bad: $\mathcal{R}^{(1)}(0)_2$ behaves much more like the true vacuum representation as the expected $\mathcal{V}(0)$ (only regard the behaviour in (7)).

We now want to present one nice example how to use the symmetry and associativity of the fusion product in order to determine the higher rank fusion. Let us assume that we already know the fusion of irreducible representations with themselves as well as with rank 2 representations, i.e. the results we actually calculated using the Nahm algorithm (see appendix B). Then we start off with

$$\Big(\Big(\mathcal{V}(1/8)\otimes_{\mathrm{f}}\mathcal{V}(1/3)\Big)\otimes_{\mathrm{f}}\mathcal{R}^{(2)}(1/3,1/3)\Big) \quad = \quad \mathcal{R}^{(2)}(1/8,1/8)\otimes_{\mathrm{f}}\mathcal{R}^{(2)}(1/3,1/3)\ .$$

Using associativity we can also calculate

$$\begin{aligned}
&\Big(\mathcal{V}(1/8)\otimes_{\mathrm{f}}\Big((\mathcal{V}(1/3)\otimes_{\mathrm{f}}\mathcal{R}^{(2)}(1/3,1/3)\Big)\Big)\\
&\qquad = \Big(\mathcal{V}(1/8)\otimes_{\mathrm{f}}\Big(\mathcal{R}^{(3)}(0,0,2,2)\oplus\mathcal{R}^{(2)}(1/3,1/3)\Big)\Big)\\
&\qquad = 2\,\mathcal{R}^{(2)}(1/8,1/8)\oplus\mathcal{R}^{(2)}(5/8,21/8)\oplus\Big(\mathcal{V}(1/8)\otimes_{\mathrm{f}}\mathcal{R}^{(3)}(0,0,2,2)\Big)\ .
\end{aligned}$$

On the other hand we can use the symmetry as well as the associativity once more to get

$$\begin{aligned}
&\Big(\mathcal{V}(1/3)\otimes_{\mathrm{f}}\Big(\mathcal{V}(1/8)\otimes_{\mathrm{f}}\mathcal{R}^{(2)}(1/3,1/3)\Big)\Big)\\
&\quad = \Big(\mathcal{V}(1/3)\otimes_{\mathrm{f}}\Big(2\,\mathcal{R}^{(2)}(1/8,1/8)\oplus\mathcal{R}^{(2)}(5/8,21/8)\Big)\Big)\\
&\quad = 4\,\mathcal{R}^{(2)}(1/8,1/8)\oplus 2\,\mathcal{R}^{(2)}(5/8,21/8)\oplus\mathcal{V}(-1/24)\oplus 2\,\mathcal{V}(35/24)\oplus\mathcal{V}(143/24)\ .
\end{aligned}$$

By comparison we thus arrive at already two new higher rank fusion products

$$\begin{aligned}
&\mathcal{R}^{(2)}(1/8,1/8)\otimes_{\mathrm{f}}\mathcal{R}^{(2)}(1/3,1/3)\\
&\quad = 4\,\mathcal{R}^{(2)}(1/8,1/8)\oplus 2\,\mathcal{R}^{(2)}(5/8,21/8)\oplus\mathcal{V}(-1/24)\oplus 2\,\mathcal{V}(35/24)\oplus\mathcal{V}(143/24)\\
&\mathcal{V}(1/8)\otimes_{\mathrm{f}}\mathcal{R}^{(3)}(0,0,2,2)\\
&\quad = 2\,\mathcal{R}^{(2)}(1/8,1/8)\oplus\mathcal{R}^{(2)}(5/8,21/8)\oplus\mathcal{V}(-1/24)\oplus 2\,\mathcal{V}(35/24)\oplus\mathcal{V}(143/24)\ .
\end{aligned}$$

The complete list of the higher rank fusion products which we calculated is given in appendix B.

5 Explicit discussion of the augmented Yang–Lee model

Unfortunately, a complete exploration of the low lying spectrum of the next easiest general augmented model, the augmented Yang–Lee model at $c_{2,5} = -22/5$, is not yet possible due to limitations on the computational power. Nevertheless, we were able to compute most of the crucial features which we observed in the fusion of the augmented $c_{2,3} = 0$ model, including the lowest rank 2 and rank 3 representations as well as the absence of irreducible representations corresponding to the original minimal model. We also give quite some examples of fusion products which confirm the general fusion rules conjectured in section 6. The explicit results are listed in appendix C.

Table 5: Kac table for $c_{2,5} = -22/5$

r \ s	1	2	3	4	5
1	0	$\frac{11}{8}$	4	$\frac{63}{8}$	13
2	$-\frac{1}{5}$	$\frac{27}{40}$	$\frac{14}{5}$	$\frac{247}{40}$	$\frac{54}{5}$
3	$-\frac{1}{5}$	$\frac{7}{40}$	$\frac{9}{5}$	$\frac{187}{40}$	$\frac{44}{5}$
4	0	$-\frac{1}{8}$	1	$\frac{27}{8}$	7
5	$\frac{2}{5}$	$-\frac{9}{40}$	$\frac{2}{5}$	$\frac{91}{40}$	$\frac{27}{5}$
6	1	$-\frac{1}{8}$	0	$\frac{11}{8}$	4
7	$\frac{9}{5}$	$\frac{7}{40}$	$-\frac{1}{5}$	$\frac{27}{40}$	$\frac{14}{5}$
8	$\frac{14}{5}$	$\frac{27}{40}$	$-\frac{1}{5}$	$\frac{7}{40}$	$\frac{9}{5}$
9	4	$\frac{11}{8}$	0	$-\frac{1}{8}$	1
10	$\frac{27}{5}$	$\frac{91}{40}$	$\frac{2}{5}$	$-\frac{9}{40}$	$\frac{2}{5}$
11	7	$\frac{27}{8}$	1	$-\frac{1}{8}$	0
12	$\frac{44}{5}$	$\frac{187}{40}$	$\frac{9}{5}$	$\frac{7}{40}$	$-\frac{1}{5}$
13	$\frac{54}{5}$	$\frac{247}{40}$	$\frac{14}{5}$	$\frac{27}{40}$	$-\frac{1}{5}$
14	13	$\frac{63}{8}$	4	$\frac{11}{8}$	0

The Kac table of $c_{2,5} = -22/5$ is depicted in table 5. We encounter two bulk weight chains

$$\begin{aligned} W^{\text{bulk, YL}}_{(1,1)} &:= \{\{0\},\{1,4\},\{7,13\},\ldots\} \\ W^{\text{bulk, YL}}_{(2,1)} &:= \left\{\left\{-\frac{1}{5}\right\},\left\{\frac{9}{5},\frac{14}{5}\right\},\left\{\frac{44}{5},\frac{54}{5}\right\},\ldots\right\} \end{aligned}$$

as well as five border weight chains

$$\begin{aligned} W^{\text{border, YL}}_{(1,2)} &:= \left\{\frac{11}{8},\frac{27}{8},\frac{155}{8},\ldots\right\} \\ W^{\text{border, YL}}_{(2,2)} &:= \left\{\frac{27}{40},\frac{187}{40},\frac{667}{40},\ldots\right\} \\ W^{\text{border, YL}}_{(3,2)} &:= \left\{\frac{7}{40},\frac{247}{40},\frac{567}{40},\ldots\right\} \\ W^{\text{border, YL}}_{(4,2)} &:= \left\{-\frac{1}{8},\frac{63}{8},\frac{95}{8},\ldots\right\} \\ W^{\text{border, YL}}_{(5,1)} &:= \left\{\frac{2}{5},\frac{27}{5},\frac{77}{5},\ldots\right\} \end{aligned}$$

and a chain $\{-9/40, 91/40, 391/40, \ldots\}$ of weights on the corners. In table 6 we present all rank 2 representations which we found in our sample calculations (see appendix C) as well as their defining parameters. This is indeed the complete spectrum of lowest rank 2 representations to be expected according to the general considerations of section 6.

As in the $c_{2,3} = 0$ model case the lowest border rank 2 representations, given in the first block of table 6, do not need a further parameter for characterisation. Their structure is again represented by figure 2 A.

The structure of the lowest bulk rank 2 representations, given in the second block of table 6, is depicted in figure 3; their respective special type is given in the last column. The representations of type C and D exhibit their Jordan cell on the level of the first possible nullvector of the groundstate and can thus be described by just one parameter $\beta = \beta_1$

$$(L_1)^l \, |h+l;1\rangle \;=\; \beta \, |h\rangle \; ;$$

l denotes the level of the Jordan cell. Besides this, we have $L_p \, |h+l;1\rangle = 0$ for all $p \geq 2$ and, hence, all other Virasoro monomials of length l vanish applied to $|h+l;1\rangle$.

The representations of type E and F, however, have to accommodate a first nullvector on the groundstate already below the level of the Jordan cell. The same difficulties as in the $c_{2,3} = 0$ model apply. We again need two parameters β_1 and β_2 parameterising

$$\begin{aligned} (L_1)^l \, |h+l;1\rangle &= \beta_1 \, |h\rangle \\ P(L) \, |h+l;1\rangle &= \beta_2 \, |h\rangle \, , \end{aligned}$$

Table 6: Specific properties of rank 2 representations in $c_{2,5} = -22/5$

	β_1	β_2	level of log. partner	level of first nullvector	level of first log. nullvector	type
$\mathcal{R}^{(2)}(11/8, 11/8)$	–	–	0	2	18	A
$\mathcal{R}^{(2)}(27/40, 27/40)$	–	–	0	4	16	A
$\mathcal{R}^{(2)}(2/5, 2/5)$	–	–	0	5	15	A
$\mathcal{R}^{(2)}(7/40, 7/40)$	–	–	0	6	14	A
$\mathcal{R}^{(2)}(-1/8, -1/8)$	–	–	0	8	12	A
$\mathcal{R}^{(2)}(0, 1)_7$	$3/5$	–	1	4	13	C
$\mathcal{R}^{(2)}(0, 1)_{13}$	$-3/2$	–	1	4	7	D
$\mathcal{R}^{(2)}(0, 4)_7$	0	$231/50$	4	1	13	E
$\mathcal{R}^{(2)}(0, 4)_{13}$	0	$231/25$	4	1	7	F
$\mathcal{R}^{(2)}(-1/5, 9/5)_9$	$-42/125$	–	2	3	11	C
$\mathcal{R}^{(2)}(-1/5, 9/5)_{11}$	$21/50$	–	2	3	9	D
$\mathcal{R}^{(2)}(-1/5, 14/5)_9$	$21/125$	$21/50$	3	2	11	E
$\mathcal{R}^{(2)}(-1/5, 14/5)_{11}$	$-126/625$	$-63/125$	3	2	9	F

where we have taken $P(L) = L_4$ for $\mathcal{R}^{(2)}(0,4)_7$ and $\mathcal{R}^{(2)}(0,4)_{13}$ as well as $P(L) = L_2\,L_1$ for $\mathcal{R}^{(2)}(-1/5, 14/5)_9$ and $\mathcal{R}^{(2)}(-1/5, 14/5)_{11}$. All other Virasoro monomials of length l vanish applied to $|h+l;1\rangle$. This behaviour actually confirms our conjecture of section 4.1.1 that we only need two parameters for this type of representation; furthermore, the above presented parameterisation is performed exactly in the proposed way.

Again we checked for the appearance of the lowest logarithmic nullvector using the algorithm of [32]. This could be successfully done in all cases listed in table 6. For this $c_{2,5} = -22/5$ model these nullvector calculations were actually a nice and necessary check of our proposed fusion rules as this information was usually not directly accessible in the fusion spectrum due to the computational limits on L.

As a last issue in the discussion of the augmented Yang–Lee model we want to have a look at the irreducible representations with weights in the Kac-table of the corresponding non-augmented minimal model. There are two possible representations of this kind in this model, $\mathcal{V}(h=0)$ with first nullvectors on levels 1 and 4 as well as $\mathcal{V}(h=-1/5)$ with first nullvectors on levels 2 and 3. Explicit calculations with the Nahm algorithm lead to

$$\mathcal{V}(-1/5) \otimes_{\mathrm{f}} \mathcal{V}(-1/5) \;=\; \mathcal{V}(0) \oplus \mathcal{V}(-1/5) \tag{8}$$

$$\mathcal{V}(-1/5) \otimes_{\mathrm{f}} \mathcal{V}(h) \;=\; 0 \quad \forall\, h \in \left\{\frac{9}{5}, \frac{14}{5}, 4\right\}\,. \tag{9}$$

But again using only the equations of (9) and the fusion rules of appendix C we arrive at the contradicting results

$$\Big(\mathcal{V}(-1/5) \otimes_{\mathrm{f}} \Big(\mathcal{V}(14/5) \otimes_{\mathrm{f}} \mathcal{V}(14/5)\Big)\Big)$$

$$
\begin{aligned}
&= \Big(\mathcal{V}(-1/5) \otimes_{\mathrm{f}} \mathcal{V}(0)\Big) \oplus \Big(\mathcal{V}(-1/5) \otimes_{\mathrm{f}} \mathcal{V}(4)\Big) \oplus \Big(\mathcal{V}(-1/5) \otimes_{\mathrm{f}} \mathcal{V}(-1/5)\Big) \\
&\qquad \oplus \Big(\mathcal{V}(-1/5) \otimes_{\mathrm{f}} \mathcal{V}(9/5)\Big) \\
&= \Big(\mathcal{V}(-1/5) \otimes_{\mathrm{f}} \mathcal{V}(0)\Big) \oplus \Big(\mathcal{V}(-1/5) \otimes_{\mathrm{f}} \mathcal{V}(-1/5)\Big)
\end{aligned}
$$

as well as

$$
\begin{aligned}
\Big(\Big(\mathcal{V}(-1/5) \otimes_{\mathrm{f}} \mathcal{V}(14/5)\Big) \otimes_{\mathrm{f}} \mathcal{V}(14/5)\Big) &= 0 \otimes_{\mathrm{f}} \mathcal{V}(14/5) \\
&= 0
\end{aligned}
$$

which lead to

$$
\begin{aligned}
\mathcal{V}(-1/5) \otimes_{\mathrm{f}} \mathcal{V}(0) &= 0 \\
\mathcal{V}(-1/5) \otimes_{\mathrm{f}} \mathcal{V}(-1/5) &= 0 \,.
\end{aligned}
$$

A similar calculation applies to $\mathcal{V}(0)$. Thus, we have to discard $\mathcal{V}(0)$ and $\mathcal{V}(-1/5)$ from the spectrum.

But, with the same reasoning as in the augmented $c_{2,3} = 0$ model case we encounter rank 1 subrepresentations of rank 2 representations which are generated by states with weights $h = 0$ and $h = -1/5$. We therefore have to include the four rank 1 indecomposable (but not irreducible) representations $\mathcal{R}^{(1)}(0)_1$, $\mathcal{R}^{(1)}(0)_4$, $\mathcal{R}^{(1)}(-1/5)_2$ and $\mathcal{R}^{(1)}(-1/5)_3$ into the spectrum. $\mathcal{R}^{(1)}(0)_4$ actually acquires the role of the vacuum representation.

6 Representations and fusion product for general augmented $\mathbf{c_{p,q}}$ models

In this section we conjecture fusion rules for general augmented $c_{p,q}$ models; we furthermore discuss their spectrum of representations which is to be consistent with the symmetry and associativity of the fusion product. This conjecture is mainly substantiated by the thorough exploration of the full fusion algebra for the lower lying representations of the augmented model $c_{2,3} = 0$ in section 4. In addition, we have checked the proposed rules as well as the existence of the lowest rank 2 and rank 3 indecomposable representations for a considerable number of fusion products in the augmented Yang-Lee model at $c_{2,5} = -22/5$ in section 5. The explicitly calculated fusion products are listed in the appendices B respectively C.

6.1 The spectrum of representations

For weights on the border and corners of the Kac table the situation looks much the same as in the $c_{p,1}$ augmented model case [20]. States corresponding to weights on corners only generate irreducible representations. States with weights on the Kac-table border, however, also form rank 2 indecomposable representations in addition to irreducible ones. These rank 2 indecomposable representations are of the same form as described in [20, 22]. They are either generated by two groundstates whose weights are given by the lowest weight of one of the weight chains corresponding to the Kac table border, see figure 2 A, or by two successive weights in the weight chain, as in figure 2 B. For case A we actually do not need a further parameter to describe the representation; in case B one further parameter β is sufficient—it is taken to parametrise the equation

$$(L_1)^l \, |h+l;1\rangle \quad = \quad \beta \, |h\rangle \; . \tag{10}$$

As we can see in figure 2 these representations can actually be thought of to consist of several towers of states each generated from a basic state by application of negative Virasoro modes; these towers of states are very close to the corresponding irreducible representations of the same weight as they exhibit the same number of nullvectors at the levels given by the Kac table and, hence, the same number of states. They (usually) are, however, not irreducible; but, we can think of them as former irreducible representations which have been indecomposably connected to form an indecomposable representation. This indecomposable connection expresses itself by the indecomposable action of L_0 as well as the non-trivial action of a few of the positive Virasoro modes. We need three such towers to build the indecomposable representation in case A, four in case B.

The weights in the bulk of the Kac table exhibit an even richer structure as they also form rank 3 representations of the Virasoro algebra. Let us first describe the rank 2 representations. All possibilities of choosing one weight each from two adjacent sets in the weight chain will deliver the weights of the two generating states of a bulk rank 2 representation which is generically unique. Only in case that this set of two weights $h(r_1, s_1)$, $h(r_2, s_2)$ contains the lowest weight of this weight chain, let it be $h(r_1, s_1)$, there are still two possible representations with this set of weights. The additional index is given by the level w of the weight in the second set of the weight chain which is not(!) the first possible nullvector on $h(r_2, s_2)$, i.e. which is not equal to $h(r_2, s_2)+r_2\, s_2$. The nullvector structure of these bulk rank 2 representations is a generalisation of the one on the border. There are four different types of nullvector structure, depicted in figure 2. The types C and D are very similar to the case on the border and also need only one additional parameter β taken to parameterise equation (10). The novel feature of types E and F is that they exhibit a first nullvector already below the level of the first logarithmic partner. This makes it necessary to at least present two non-zero parameters to describe these representations. They have to be chosen as described for the examples in section 4. Conjecturally, two is also a sufficient number of parameters. Figure 2 also shows that all four possible types of bulk rank 2 representations can be split up into four towers of states in the above described spirit.

But the really new feature of the possible bulk representations are rank 3 representations. These need four generating states and are found in two different types. The first of these two types is generated by two states of the lowest weight of this weight chain as well as two states which are of the same weight in the second set of this weight chain. This type is realised for the two lowest rank 3 representations for a given bulk weight chain. The example $\mathcal{R}^{(3)}(0,0,1,1)$ is described in section 4.1.2 and depicted in figure 4. As discussed there, one can actually build this representation by connecting three rank 2 bulk representations roughly according to figure 2 A. Then, there is only one additional parameter which is necessary to describe the representation.

The second type, which is supposedly the generic one, is generated by states of weights from three adjacent sets in a bulk weight chain—we have to take both weights from the middle set as well as one from each of the other two. This renders all possible rank 3 representations. They are uniquely described by this set of four generating weights. The example $\mathcal{R}^{(3)}(0,1,2,5)$, its defining parameters and its composition of rank 2 representations have been discussed in section 4.1.2. This representation could be composed by indecomposably connecting four bulk rank 2 representations, roughly following figure 2 B. Unfortunately, this is the only example of this supposedly generic type of rank 3 representation which is at our grasp up to the level where the rank 3 behaviour appears for the first time. Hence, these generalisations have to be taken with the necessary caution, but nevertheless, they are motivated by similar generalisations for the rank 2 representations. However, we expect the necessity to introduce more defining parameters in cases where there appear more nullvectors on levels lower than the first rank 3 Jordan cell.

Knowing how to compose the rank 3 representations of rank 2 representations we automatically get the split-up of the whole rank 3 representation into towers of states as discussed above for the rank 2 case.

Last but not least we still have to describe the spectrum of irreducible representations corresponding to weights in the Kac table bulk. The calculations for $c_{2,3} = 0$ and $c_{2,5} = -22/5$ (see sections 4.3 and 5) show that it is not possible to include irreducible representations corresponding to weights in the bulk which appear in the Kac table of the corresponding non-augmented minimal model $c_{p,q}$, i.e. weights in the Kac table segment $1 \leq r < q$, $1 \leq s < p$. These weights exactly correspond to the lowest weight of each bulk weight chain. As shown in the examples the inclusion of these irreducible representations would simply violate the symmetry and associativity of the fusion product. On the other hand, the fusion of the border and corner irreducible representations produces rank 2 representations $\mathcal{R}^{(2)}(h(r_1,s_1),h(r_2,s_2))_w$ which contain the lowest weights of the corresponding bulk weight chain, let it be $h(r_1,s_1)$ in this case, as a generating state. This lowest weight state even generates an indecomposable subrepresentation of $\mathcal{R}^{(2)}(h(r_1,s_1),h(r_2,s_2))_w$ which does not exhibit higher rank behaviour. As they are subrepresentations of existing representations this kind of representations have to appear in the spectrum as well and will be denoted by $\mathcal{R}^{(1)}(h(r_1,s_1))_{h(r_2,s_2)-h(r_1,s_1)}$. In the fusion rules (r_1,s_1) appears to be that entry in the minimal model Kac table which does not have its nullvector at $h(r_2,s_2)$. Hence, (r_1,s_1) seems to be sufficient to describe this rank 1 representation and we can set

$$\mathcal{R}^{(1)}_{(r_1,s_1)}(h(r_1,s_1)) \quad := \quad \mathcal{R}^{(1)}(h(r_1,s_1))_{h(r_2,s_2)-h(r_1,s_1)} \; .$$

This is the first example of a case where a rank 1 indecomposable representation on a weight h appears in the spectrum although the corresponding irreducible representation on h cannot be included consistently. The other weights in the bulk of the Kac table induce ordinary irreducible representations which are consistent with the fusion algebra.

6.2 The fusion of irreducible representations

Consider first the fusion of two irreducible representations. If we describe an irreducible representation $\mathcal{V}^i_{(r,s)}(h)$ corresponding to a conformal weight h on a corner or a border in the Kac table, we choose the index (r,s) with the smallest product rs. If there are two pairs with the same product, we choose the one with larger r (although this last point is mere convention and does not effect the result). Then, the fusion product for $i,j \in \{\text{corner, border}\}$ simply amounts to the untruncated BPZ-rules

$$\left(\mathcal{V}^i_{(r_1,s_1)}(h_1) \otimes \mathcal{V}^j_{(r_2,s_2)}(h_2)\right)_{\rm f} \;=\; \sum_{r_3=|r_1-r_2|+1,\text{ step }2}^{r_1+r_2-1} \;\; \sum_{s_3=|s_1-s_2|+1,\text{ step }2}^{s_1+s_2-1} \tilde{\mathcal{V}}_{(r_3,s_3)}\Big|_{\text{rules}}. \quad (11)$$

On the right hand side, however, we do not simply encounter a sum over irreducible representations again. Some of the resulting $\tilde{\mathcal{V}}_{(r,s)}$ are automatically combined into rank 2 or rank 3 representations. The corresponding **rules** how to do this combination (indicated as constraints in the equation) are given by:

1. For (r,s) on a corner $\tilde{\mathcal{V}}_{(r,s)}$ is simply replaced by the corresponding irreducible representation $\mathcal{V}^{\text{corner}}_{(r,s)}(h(r,s))$.

2. Concerning the set of all (r,s) on the right hand side of (11) which correspond to weights on the border there are two possibilities how to encounter rank 2 representations. If we find twice the same weight $h(r_1,s_1) = h(r_2,s_2)$ which is the lowest of a weight chain these two need to be replaced by the rank 2 representation $\mathcal{R}^{(2)}(h(r_1,s_1),h(r_2,s_2))$. Then, if we find two weights $h(m_1,m_1)$, $h(m_2,m_2)$ of the same weight chain adjacent to each other in the chain, these need to be replaced by $\mathcal{R}^{(2)}(h(m_1,n_1),h(m_2,n_2))$. All other weights in this set simply form irreducible representations and are replaced by $\mathcal{V}^{\text{border}}_{(r,s)}(h(r,s))$.

3. For the set of all (r,s) on the right hand side of (11) which correspond to weights in the bulk we first need to identify the rank 3 representations. Rank 3 representations need four generating states of the two possible types described in the preceding subsection: either the two lowest weights and twice the same weight of the second set of a bulk weight chain; or weights from three adjacent sets of a bulk weight chain—both weights of the middle set and one from each of the other two. The last is the generic case. If we encounter a set of four weights in this manner we need to replace them by the rank 3 representation $\mathcal{R}^{(3)}(h(r_1,s_1),h(r_2,s_2),h(r_3,s_3),h(r_4,s_4))$.

4. Rank 2 representations need two generating states. For bulk representations these consist of one weight each from two adjacent sets in the weight chain. If this set of

two weights $h(r_1, s_1)$, $h(r_2, s_2)$ contains the lowest weight of this weight chain, let it be $h(r_1, s_1)$, there are still two possible representations with this set of weights. The additional index is given by the level w of the weight in the third set of the weight chain which is not (!) the first possible nullvector on $h(r_2, s_2)$, i.e. which is not equal to $h(r_2, s_2) + r_2\, s_2$. Each two weights of this form have to be replaced by the rank 2 representation $\mathcal{R}^{(2)}(h(r_1, s_1), h(r_2, s_2))_w$.

5. After extraction of all rank 3 and rank 2 representations from the set of all (r, s) within the bulk we find to each lowest weight $h(r_1, s_1)$ of a bulk weight chain a corresponding weight $h(r_2, s_2)$ of the second tuple of this weight chain. These form the rank 1 indecomposable representation $\mathcal{R}^{(1)}(h(r_1, s_1))_{h(r_2,s_2)-h(r_1,s_1)}$.

6. All (r, s) corresponding to weights in the bulk which have not been used up in the three preceding points have to be replaced by $\mathcal{V}_{(r,s)}(h(r, s))$.

This set of rules requires the following remarks:

- Unfortunately, one cannot write down the fusion rules for general augmented theories in such a nice form as for the augmented $c_{p,1}$ models [20]. First of all, we do not have border representations just on one side such that a restriction to an infinite strip of the Kac-table nicely promotes the first border conformal weights beyond that strip to corresponding rank 2 representations. Nevertheless, this property that the second conformal weight to a border conformal weight (ordered by the level of their first nullvectors, resp. products rs) somehow represents the rank 2 representation seems to persist. Secondly, for the bulk there are simply not enough entries in the Kac table in order to uniquely label all different irreducible, rank 2 and rank 3 representations.

- The fifth rule which describes the appearance of $\mathcal{R}^{(1)}(h(r_1, s_1))_{h(r_2,s_2)-h(r_1,s_1)}$ is not needed for the fusion of border and corner representations with itself. Actually, the examples show that starting with border and corner representations the whole fusion closes without the inclusion of any irreducible or rank 1 representation corresponding to weights in the bulk. It is only the appearance of these irreducible or rank 1 representations as subrepresentations of rank 2 and rank 3 representations that makes us include these in the theory. We advocate that it is only this setting they should be thought of to exist in. This point of view is strongly stressed by the existence of the above discussed rank 1 representations on the lowest weights of the bulk weight chains. Indeed, only their embedding in a rank 2 representation makes the absence of one of the first two nullvectors possible as it prevents this singular vector (which itself spans an irreducible subrepresentation of the rank 1 representation) to be a nullvector in the whole space of states.

If we want to write down the fusion product with representations corresponding to conformal weights in the bulk we need more information than the first nullvector.

For these we choose as index the set of the two pairs $\{(r_1, s_1), (r_2, s_2)\}$ with the two lowest products $r_1 s_1$, $r_2 s_2$. As there are exactly as many entries in the Kac table for these bulk representations as there are nullvectors in the nullvector cascade and as all these nullvectors in the cascade are mutually distinct, this choice is unique. Using $A(r,m) := \{|r-m|+1, |r-m|+3, \ldots, r+m-1\}$ we define the following two sets

$$\begin{aligned} & S_{\mathrm{eb}}(r,s|m_1,n_1,m_2,n_2) \\ & \quad := \quad \{(p,q)|p \in A(r,m_1), q \in A(s,n_1)\} \cap \{(p,q)|p \in A(r,m_2), q \in A(s,n_2)\} \end{aligned}$$

as well as

$$\begin{aligned} & S_{\mathrm{bb}}(r_1,s_1,r_2,s_2|m_1,n_1,m_2,n_2) \\ & := \quad \Big(\{(p,q)|p \in A(r_1,m_1), q \in A(s_1,n_1)\} \cap \{(p,q)|p \in A(r_2,m_2), q \in A(s_2,n_2)\}\Big) \\ & \cup \quad \Big(\{(p,q)|p \in A(r_2,m_1), q \in A(s_2,n_1)\} \cap \{(p,q)|p \in A(r_1,m_2), q \in A(s_1,n_2)\}\Big) \ . \end{aligned}$$

These define the parameter ranges in the following fusion rules with bulk representations, $i \in \{\text{corner, border}\}$,

$$\begin{aligned} \Big(\mathcal{V}^{i}_{(r,s)}(h_1) \otimes \mathcal{V}^{\mathrm{bulk}}_{\{(m_1,n_1),(m_2,n_2)\}}(h_2)\Big)_{\mathrm{f}} &= \sum_{(s,t)\in S_{\mathrm{eb}}(r,s|m_1,n_1,m_2,n_2)} \tilde{\mathcal{V}}_{(s,t)}\Big|_{\mathrm{rules}} \\ \Big(\mathcal{V}^{\mathrm{bulk}}_{\{(r_1,s_1),(r_2,s_2)\}}(h_1) \otimes \mathcal{V}^{\mathrm{bulk}}_{\{(m_1,n_1),(m_2,n_2)\}}(h_2)\Big)_{\mathrm{f}} &= \sum_{(s,t)\in S_{\mathrm{bb}}(r_1,s_1,r_2,s_2|m_1,n_1,m_2,n_2)} \tilde{\mathcal{V}}_{(s,t)}\Big|_{\mathrm{rules}} \ . \end{aligned}$$

The fusion rules for the rank 1 indecomposable representations actually look much the same as for border and corner representations, $i \in \{\text{corner, border}\}$,

$$\begin{aligned} \Big(\mathcal{R}^{(1)}_{(r_1,s_1)}(h_1) \otimes \mathcal{V}^{i}_{(r_2,s_2)}(h_2)\Big)_{\mathrm{f}} &= \sum_{r_3=|r_1-r_2|+1,\ \mathrm{step}\ 2}^{r_1+r_2-1} \ \sum_{s_3=|s_1-s_2|+1,\ \mathrm{step}\ 2}^{s_1+s_2-1} \tilde{\mathcal{V}}_{(r_3,s_3)}\Big|_{\mathrm{rules'}} \\ \Big(\mathcal{R}^{(1)}_{(r_1,s_1)}(h_1) \otimes \mathcal{R}^{(1)}_{(r_2,s_2)}(h_2)\Big)_{\mathrm{f}} &= \sum_{r_3=|r_1-r_2|+1,\ \mathrm{step}\ 2}^{r_1+r_2-1} \ \sum_{s_3=|s_1-s_2|+1,\ \mathrm{step}\ 2}^{s_1+s_2-1} \tilde{\mathcal{V}}_{(r_3,s_3)}\Big|_{\mathrm{rules'}} \\ \Big(\mathcal{R}^{(1)}_{(r,s)}(h_1) \otimes \mathcal{V}^{\mathrm{bulk}}_{\{(m_1,n_1),(m_2,n_2)\}}(h_2)\Big)_{\mathrm{f}} &= \sum_{(s,t)\in S_{\mathrm{eb}}(r,s|m_1,n_1,m_2,n_2)} \tilde{\mathcal{V}}_{(s,t)}\Big|_{\mathrm{rules'}} \ . \end{aligned}$$

It is only the rule (5) we have to modify slightly in this case:

5'. After extraction of all rank 3 and rank 2 representations from the set of all (r,s) each lowest weight $h(r_1,s_1)$ of a bulk weight chain corresponds to the rank 1 indecomposable representation $\mathcal{R}^{(1)}_{(r_1,s_1)}(h(r_1,s_1))$.

The reason is that treating the rank 1 indecomposable representations as generated from one state we will also get only one generating state for each rank 1 indecomposable representation in the fusion process.

6.3 The fusion with higher rank representations

Now, we can give the fusion rules with higher rank representations based on the above calculated fusion rules. These higher rank fusion rules make use of the above elaborated fact that all higher rank representations can be constructed by indecomposably connecting a number of representations of one rank lower. Certainly, these constituents are usually no true subrepresentations. But nevertheless they exhibit in sum the same number of states as the composed higher representation and even their nullvector structure survives in a certain way as "special" vectors of the new larger representation. E.g. generically one needs four $\mathcal{V}_{(r,s)}$ representations to compose a rank 2 representation.

The following general rules for fusion with higher rank representations apply successively with rising rank:

1. First split one higher rank representation into its constituents of one rank lower.
2. Calculate the fusion rules with these constituents as known before (or iterate this process until you reach a level where the fusion rules are already known).
3. If you now find the right constituents for a higher rank representation within the result (e.g. all four $\tilde{\mathcal{V}}$ for a generic rank 2 representation) you have to combine them to this higher rank representation. All other parts of the total result remain as they are. This re-introduces the indecomposable structure which has been broken up in step one where possible.
4. The one exception to the last step applies to the indecomposable rank 1 representations $\mathcal{R}^{(1)}(h(r_1,s_1))_{h(r_2,s_2)-h(r_1,s_1)}$ on the lowest weight $h(r_1,s_1)$ of a weight chain. Whenever we find that instead of $\mathcal{R}^{(1)}(h(r_1,s_1))_{h(r_2,s_2)-h(r_1,s_1)}$ the corresponding subrepresentation $\mathcal{V}(h(r_2,s_2))$ can be used as a building block of a rank 2 representation we need to replace $\mathcal{R}^{(1)}(h(r_1,s_1))_{h(r_2,s_2)-h(r_1,s_1)}$ by $\mathcal{V}(h(r_2,s_2))$. We then proceed as described in the preceding step.

This last case is only encountered if we fuse irreducible bulk representations with higher rank bulk representations. Its artificiality stems from the fact that the irreducible bulk representations only exist as subrepresentations of the rank 2 representations—it is these rank 2 representations we actually have to look at when considering fusion rules.

Let us give two examples of the $c_{2,3}=0$ fusion to show how these rules work

$$\begin{aligned}
\mathcal{V}(5/8)\otimes_{\mathrm{f}}\mathcal{R}^{(2)}(5/8,5/8) &\stackrel{\text{split-up}}{\longrightarrow} \mathcal{V}(5/8)\otimes_{\mathrm{f}}\Big(2\,\mathcal{V}(5/8)\oplus\mathcal{V}(21/8)\Big)\\
&= 2\,\mathcal{R}^{(2)}(0,2)_7\oplus\mathcal{R}^{(2)}(1,5)\\
&\stackrel{\text{re-combination}}{\longrightarrow} \mathcal{R}^{(3)}(0,0,2,2)\\
\mathcal{V}(5/8)\otimes_{\mathrm{f}}\mathcal{R}^{(2)}(1/3,1/3) &\stackrel{\text{split-up}}{\longrightarrow} \mathcal{V}(5/8)\otimes_{\mathrm{f}}\Big(2\,\mathcal{V}(1/3)\oplus\mathcal{V}(10/3)\Big)\\
&= 2\,\mathcal{V}(-1/24)\oplus\mathcal{V}(35/24)\\
&\stackrel{\text{re-combination}}{\longrightarrow} 2\,\mathcal{V}(-1/24)\oplus\mathcal{V}(35/24)\ .
\end{aligned}$$

7 Conclusion and outlook

In this paper we have given the explicit fusion rules for two examples of general augmented minimal models, the augmented $c_{2,3} = 0$ and the augmented Yang-Lee model. We have shown that these models exhibit an even richer indecomposable structure in comparison to the $c_{p,1}$ models with representations up to rank 3. We have elaborated a number of these newly appearing representations of rank 2 and 3 in detail. Especially, we have shown for some of these examples that one can construct rank 3 representations by indecomposably connecting rank 2 representations. This is a very interesting generalisation of the rank 2 representation case which can be composed by indecomposably connecting several irreducible representations.

We have also shown that the fusion rule consistency conditions of symmetry and associativity actually restrict the representation spectrum—irreducible representations corresponding to weights of the original non-augmented minimal model cannot be included into the spectrum consistently. We actually find that the vacuum representation is given by an indecomposable but not irreducible rank 1 subrepresentation of a rank 2 representation. For the $c_{2,3} = 0$ model, in particular, this settles the long standing puzzle of how to construct a consistent vacuum representation and character, at least on the level of pure Virasoro representations. This has a direct impact on models with central charge $c = 0$ for several interesting physical phenomena as e.g. percolation [50, 51, 52, 31], quenched disorder models [53, 54], or the dilute case in polymer physics [1].

These two examples already reveal a concise general structure which we think exhibits almost all of the generic structure of general augmented $c_{p,q}$ models. Accordingly, we have extrapolated this structure to a conjecture of the general representation content and fusion rules of general augmented $c_{p,q}$ models. These conjectured fusion rules still inherit much of the BPZ fusion structure. The main generalisation lies in the fact that we do not interpret the BPZ fusion rules "minimally" for these models. This "minimal" refers to the BPZ fusion rules for the non-augmented minimal models where one takes a section over all ways of applying the untruncated BPZ rules. The augmented fusion rules allow for definitely more representation content. This is, however, in perfect accordance with and directly reduces to the already known rules for the special case of augmented $c_{p,1}$ models [20] although the general rules do not look as nicely compact as the special ones in [20].

Concerning our special example of $c_{2,3} = 0$, we actually find that the numerical results of [4], table 1, which are supposed to give the low level state content of CFT representations at $c = 0$, perfectly match our representations $\mathcal{R}^{(1)}(0)_2$, $\mathcal{R}^{(1)}(0)_1$ and $\mathcal{V}(1/3)$ respectively. Although the numerical results give only relatively few levels, we conjecture that they are correctly described by the above mentioned $c_{2,3} = 0$ representations. Indeed, the spectrum of equation (5.1) in [4] is contained in a subalgebra of the augmented $c_{2,3} = 0$ model which consists of representations up to rank 2 corresponding to the first column of the Kac table (i.e. $s = 1$). The characters given in equation (5.2) of [4] agree with the field content of the irreducible representations for weights

$h = \frac{1}{3}, \frac{10}{3}, \ldots$ and with rank 1 indecomposable subrepresentations of rank 2 representations for integer weights $h = 0, 1, 2, 5, 7, \ldots$. As the $c = 0$ theory in [4] describes a XXZ quantum spin chain, the above observations open a new and very exciting connection of logarithmic CFT to the topic of quantum spin chains which is popular not only in the field of integrable systems but also seems to have an important impact on the AdS/CFT correspondence and, hence, string theory [55, 56].

We have also shown that the findings in this paper are in perfect agreement with the logarithmic nullvector calculations performed in [32]. Using the technique of the Nahm algorithm we have actually been able to pinpoint the precise structure of higher rank Virasoro representations in $c_{2,3} = 0$; various types of possible rank 2 representations had already been conjectured in [32].

On the other hand, the question of whether these models exhibit a larger $\mathcal{W}$-algebra and how the Virasoro representations combine to representations of this larger symmetry algebra still remains open. There are, however, strong hints at the existence of such an enhanced symmetry algebra coming from inspections of the corresponding representations of the modular group [27]. And even the mere fact that the Virasoro fusion rules so nicely generalise from the $c_{p,1}$ model case where we know a triplet $\mathcal{W}$-algebra to exist is a further good hint. This may be exemplified by the existence of a representation like $\mathcal{V}(7)$ in the $c_{2,3} = 0$ model whose fusion rules behave in just the same way as the ones of the "spin representation" $\mathcal{V}_{1,p}$ in the $c_{p,1}$ models. Indeed, in the $c_{p,1}$ models this spin representation behaves roughly like a square root of the $\mathcal{W}$-representation.

The representations of the modular group are already very restrictive such that for e.g. $c_{2,3} = 0$ it suffices to know the field content up to level 7 in order to precisely know the full $\mathcal{W}$-representation. We believe that this restrictiveness together with our new knowledge of the precise structure of the Virasoro representations should lead to a consistent set of modular functions which represent the right $\mathcal{W}$-characters and produce the right $\mathcal{W}$-fusion rules. This is subject to ongoing research [27].

Acknowledgements. We are very grateful to Werner Nahm and Alexander Nichols for useful discussions. Furthermore, we would like to thank the Bonn Condensed Matter Theory Group for the kind permission to use their computer cluster. This work was supported in part by the DFG Schwerpunktprogramm no. 1096 "Stringtheorie". HE acknowledges support by the Studienstiftung des Deutschen Volkes. MF is also partially supported by the European Union network HPRN-CT-2002-00325 (EUCLID).

A Explicit Jordan diagonalisation of $\mathbf{L_0}$ for $\mathcal{R}^{(3)}(0, 1, 2, 5)$

In this appendix we want to present a basis of states for the lowest five levels of $\mathcal{R}^{(3)}(0, 1, 2, 5)$ which brings the L_0 matrix in a Jordan diagonal form. The basis is denoted by n_i with a running index as assigned by the computer programme. Vectors which are not shown to be equal to descendants of some other vectors are understood to be generating states. These are n_0, n_5, n_{24} and n_{18} at levels 0, 1, 2, 5 respectively. On the right hand side we have denoted only the Jordan block in L_0 for the respective states. Different Jordan blocks are separated by lines. All other L_0 entries are zero.

states	L_0 matrix		
n_0	0		
$n_4 = L_{-1}\, n_0$	1	1	
n_5	0	1	
$n_6 = L_{-1}^2\, n_0 = L_{-1}\, n_4$	2	1	
$n_7 = L_{-1}\, n_5$	0	2	
$n_{23} = L_{-2}\, n_0$	2	1	
n_{24}	0	2	
$n_8 = L_{-2}L_{-1}\, n_0 = L_{-2}\, n_4$	3	1	
$n_9 = L_{-2}\, n_5$	0	3	
$n_{25} = L_{-1}^3\, n_0 = L_{-1}^2\, n_4$	3	1	
$n_{26} = L_{-1}^2\, n_5$	0	3	
$n_{38} = (L_{-3} + L_{-2}L_{-1})n_0 = L_{-1}\, n_{23}$	3	1	
$n_{39} = L_{-1}\, n_{24}$	0	3	
$n_{10} = L_{-3}L_{-1}\, n_0 = L_{-3}\, n_4$	4	1	
$n_{11} = L_{-3}\, n_5$	0	4	
$n_{27} = L_{-2}L_{-1}^2\, n_0 = L_{-2}L_{-1}\, n_4$	4	1	
$n_{28} = L_{-2}L_{-1}\, n_5$	0	4	
$n_{40} = L_{-2}^2\, n_0 = L_{-2}\, n_{23}$	4	1	
$n_{41} = L_{-2}\, n_{24}$	0	4	
$n_{48} = L_{-1}^4\, n_0 = L_{-1}^3\, n_4$	4	1	
$n_{49} = L_{-1}^3\, n_5$	0	4	
$n_{56} = (2L_{-4} + 2L_{-3}L_{-1} + L_{-2}L_{-1}^2)n_0 = L_{-1}^2\, n_{23}$	4	1	
$n_{57} = L_{-1}^2\, n_{24}$	0	4	
$n_{16} = \frac{4655}{31758}(L_{-4}L_{-1} - L_{-3}L_{-1}^2 + \frac{5}{3}L_{-2}L_{-1}^3 - L_{-2}^2L_{-1} - \frac{1}{4}L_{-1}^5)n_0$	5	1	0
$n_{17} = \frac{4655}{63516}\left((L_{-4} - \frac{5}{2}L_{-3}L_{-1} - L_{-2}^2 + \frac{19}{6}L_{-2}L_{-1}^2 - \frac{1}{2}L_{-1}^4)n_5 + (L_{-3} - L_{-2}L_{-1} + \frac{1}{6}L_{-1}^3)n_{24}\right)$	0	5	1
n_{18}	0	0	5
$n_{29} = L_{-3}L_{-1}^2\, n_0 = L_{-3}L_{-1}\, n_4$	5	1	
$n_{30} = L_{-3}L_{-1}\, n_5$	0	5	
$n_{42} = L_{-3}L_{-2}\, n_0 = L_{-3}\, n_{23}$	5	1	
$n_{43} = L_{-3}\, n_{24}$	0	5	
$n_{50} = L_{-2}^2L_{-1}\, n_0 = L_{-2}^2\, n_4$	5	1	
$n_{51} = L_{-2}^2\, n_5$	0	5	
$n_{58} = L_{-2}L_{-1}^3\, n_0 = L_{-2}L_{-1}^2\, n_4$	5	1	
$n_{59} = L_{-2}L_{-1}^2\, n_5$	0	5	
$n_{63} = L_{-1}^5\, n_0 = L_{-1}^4\, n_4$	5	1	
$n_{64} = L_{-1}^4\, n_5$	0	5	
$n_{68} = (6L_{-5} + 6L_{-4}L_{-1} + 3L_{-3}L_{-1}^2 + L_{-2}L_{-1}^3)n_0 = L_{-1}^3\, n_{23}$	5	1	
$n_{69} = L_{-1}^3\, n_{24}$	0	5	
$n_{70} = (L_{-4} + \frac{1}{2}L_{-3}L_{-1} - L_{-2}^2 + \frac{1}{6}L_{-2}L_{-1}^2)n_5 - (L_{-3} - L_{-2}L_{-1} + \frac{1}{6}L_{-1}^3)n_{24}$	5		

B Explicit fusion rules for $c_{2,3} = 0$

The following table contains the results of our explicit calculations of the fusion product of irreducible and rank 1 representations with each other in the augmented $c_{2,3} = 0$ model. The first column gives the representations to be fused. Then the level L at which we calculated the new representation as well as the maximal level $\tilde{L}$ up to which we took the corresponding constraints into account are given. The last column contains the result.

The computational power restricts the level L at which the fused representations are calculated, unfortunately quite severely for the higher representations. Hence, it was sometimes not possible to reach a high enough L to actually see the indecomposable structure of all component representations of the result. We nevertheless denoted the result as we would expect it according to the proposed fusion rules—to discern this guessed higher representations from the explicit results we indicated them by a question mark. But certainly our results are always in agreement with these possible higher rank representations up to the level L given in the table.

	L	$\tilde{L}_{\max}$	Fusion product
$\mathcal{V}(5/8) \otimes_f \mathcal{V}(5/8)$	6	11	$\mathcal{R}^{(2)}(0,2)_7$
$\otimes_f \mathcal{V}(1/3)$	6	11	$\mathcal{V}(-1/24)$
$\otimes_f \mathcal{V}(1/8)$	7	8	$\mathcal{R}^{(2)}(0,1)_5$
$\otimes_f \mathcal{V}(-1/24)$	6	9	$\mathcal{R}^{(2)}(1/3,1/3)$
$\otimes_f \mathcal{V}(33/8)$	6	11	$\mathcal{R}^{(2)}(2,7)$
$\otimes_f \mathcal{V}(10/3)$	6	7	$\mathcal{V}(35/24)$
$\otimes_f \mathcal{V}(21/8)$	6	11	$\mathcal{R}^{(2)}(1,5)$
$\otimes_f \mathcal{V}(35/24)$	5	7	$\mathcal{R}^{(2)}(1/3,10/3)$
$\otimes_f \mathcal{V}(2)$	6	9	$\mathcal{V}(5/8)$
$\otimes_f \mathcal{V}(1)$	6	9	$\mathcal{V}(1/8)$
$\otimes_f \mathcal{V}(7)$	6	9	$\mathcal{V}(33/8)$
$\otimes_f \mathcal{V}(5)$	6	9	$\mathcal{V}(21/8)$
$\mathcal{V}(1/3) \otimes_f \mathcal{V}(1/3)$	6	11	$\mathcal{R}^{(2)}(0,2)_5 \oplus \mathcal{V}(1/3)$
$\otimes_f \mathcal{V}(1/8)$	6	9	$\mathcal{R}^{(2)}(1/8,1/8)$
$\otimes_f \mathcal{V}(-1/24)$	6	9	$\mathcal{R}^{(2)}(5/8,5/8) \oplus \mathcal{V}(-1/24)$
$\otimes_f \mathcal{V}(33/8)$	6	9	$\mathcal{V}(35/24)$
$\otimes_f \mathcal{V}(10/3)$	6	9	$\mathcal{R}^{(2)}(1,7) \oplus \mathcal{V}(10/3)$
$\otimes_f \mathcal{V}(21/8)$	6	9	$\mathcal{R}^{(2)}(5/8,21/8)$
$\otimes_f \mathcal{V}(35/24)$	5	7	$\mathcal{R}^{(2)}(1/8,33/8) \oplus \mathcal{V}(35/24)$
$\otimes_f \mathcal{V}(2)$	6	9	$\mathcal{V}(1/3)$
$\otimes_f \mathcal{V}(1)$	6	9	$\mathcal{R}^{(2)}(0,1)_7$
$\otimes_f \mathcal{V}(7)$	6	9	$\mathcal{V}(10/3)$
$\otimes_f \mathcal{V}(5)$	6	9	$\mathcal{R}^{(2)}(2,5)$

		L	$\tilde{L}_{\max}$	Fusion product
$\mathcal{V}(1/8)$	$\otimes_{\mathrm{f}}$ $\mathcal{V}(1/8)$	6	8	$\mathcal{R}^{(2)}(1/3,1/3)\oplus\mathcal{R}^{(2)}(0,2)_7$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(-1/24)$	6	7	$\mathcal{R}^{(3)}(0,0,1,1)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(33/8)$	6	9	$\mathcal{R}^{(2)}(1,5)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(10/3)$	6	8	$\mathcal{R}^{(2)}(5/8,21/8)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(21/8)$	5	7	$\mathcal{R}^{(2)}(1/3,10/3)\oplus\mathcal{R}^{(2)}(2,7)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(35/24)$	5	7	$\mathcal{R}^{(3)}(0,1,2,5)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(2)$	6	9	$\mathcal{V}(1/8)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(1)$	5	7	$\mathcal{V}(5/8)\oplus\mathcal{V}(-1/24)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(7)$	6	8	$\mathcal{V}(21/8)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(5)$	6	7	$\mathcal{V}(33/8)\oplus\mathcal{V}(35/24)$
$\mathcal{V}(-1/24)$	$\otimes_{\mathrm{f}}$ $\mathcal{V}(-1/24)$	5	7	$\mathcal{R}^{(3)}(0,0,2,2)\oplus\mathcal{R}^{(2)}(1/3,1/3)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(21/8)$	5	7	$\mathcal{R}^{(3)}(0,1,2,5)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(35/24)$	4	6	$\mathcal{R}^{(3)}(0,1,2,7)?\oplus\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(5)$	5	7	$\mathcal{R}^{(2)}(5/8,21/8)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(143/24)$	0	6	$\mathcal{R}^{(3)}(1,5,7,15)?\oplus\mathcal{R}^{(2)}(10/3,28/3)?$
$\mathcal{V}(33/8)$	$\otimes_{\mathrm{f}}$ $\mathcal{V}(33/8)$	6	8	$\mathcal{R}^{(2)}(0,2)_7\oplus\mathcal{R}^{(2)}(7,15)?$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(10/3)$	5	7	$\mathcal{V}(-1/24)\oplus\mathcal{V}(143/24)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(-1/24)$	5	7	$\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(21/8)$	5	7	$\mathcal{R}^{(2)}(0,1)_5\oplus\mathcal{R}^{(2)}(5,12)?$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(35/24)$	6	7	$\mathcal{R}^{(2)}(1/3,1/3)\oplus\mathcal{R}^{(2)}(10/3,28/3)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(1)$	6	9	$\mathcal{V}(21/8)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(7)$	5	7	$\mathcal{V}(5/8)\oplus\mathcal{V}(85/8)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(5)$	6	8	$\mathcal{V}(1/8)\oplus\mathcal{V}(65/8)$
$\mathcal{V}(10/3)$	$\otimes_{\mathrm{f}}$ $\mathcal{V}(10/3)$	5	7	$\mathcal{R}^{(2)}(0,2)_5\oplus\mathcal{R}^{(2)}(5,15)?\oplus\mathcal{V}(1/3)\oplus\mathcal{V}(28/3)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(-1/24)$	5	7	$\mathcal{R}^{(2)}(1/8,33/8)\oplus\mathcal{V}(35/24)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(21/8)$	5	7	$\mathcal{R}^{(2)}(1/8,1/8)\oplus\mathcal{R}^{(2)}(33/8,65/8)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(35/24)$	4	6	$\mathcal{R}^{(2)}(5/8,5/8)\oplus\mathcal{R}^{(2)}(21/8,85/8)?\oplus\mathcal{V}(-1/24)\oplus\mathcal{V}(143/24)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(5)$	3	9	$\mathcal{R}^{(2)}(0,1)_7\oplus\mathcal{R}^{(2)}(7,12)?$
$\mathcal{V}(21/8)$	$\otimes_{\mathrm{f}}$ $\mathcal{V}(21/8)$	4	6	$\mathcal{R}^{(2)}(1/3,1/3)\oplus\mathcal{R}^{(2)}(10/3,28/3)?\oplus\mathcal{R}^{(2)}(0,2)_7\oplus\mathcal{R}^{(2)}(7,15)?$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(35/24)$	4	6	$\mathcal{R}^{(3)}(0,0,1,1)\oplus\mathcal{R}^{(3)}(2,5,7,12)?$
$\mathcal{V}(35/24)$	$\otimes_{\mathrm{f}}$ $\mathcal{V}(35/24)$	3	5	$\mathcal{R}^{(3)}(0,0,2,2)\oplus\mathcal{R}^{(3)}(1,5,7,15)?\oplus\mathcal{R}^{(2)}(1/3,1/3)$ $\oplus\mathcal{R}^{(2)}(10/3,28/3)?$
$\mathcal{V}(2)$	$\otimes_{\mathrm{f}}$ $\mathcal{V}(-1/24)$	6	9	$\mathcal{V}(-1/24)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(33/8)$	6	8	$\mathcal{V}(33/8)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(10/3)$	6	7	$\mathcal{V}(10/3)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(21/8)$	6	9	$\mathcal{V}(21/8)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(35/24)$	5	7	$\mathcal{V}(35/24)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(2)$	6	9	$\mathcal{R}^{(1)}(0)_2$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(1)$	6	9	$\mathcal{R}^{(1)}(0)_1$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(7)$	6	9	$\mathcal{V}(7)$
	$\otimes_{\mathrm{f}}$ $\mathcal{V}(5)$	6	9	$\mathcal{V}(5)$
	$\otimes_{\mathrm{f}}$ $\mathcal{R}^{(1)}(0)_2$	6	9	$\mathcal{V}(2)$
	$\otimes_{\mathrm{f}}$ $\mathcal{R}^{(1)}(0)_1$	6	9	$\mathcal{V}(1)$

		L	$\tilde{L}_{\max}$	Fusion product
$\mathcal{V}(1)$	$\otimes_{\mathrm{f}}\ \mathcal{V}(-1/24)$	5	7	$\mathcal{R}^{(2)}(1/8,1/8)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(10/3)$	5	7	$\mathcal{R}^{(2)}(2,5)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(21/8)$	5	7	$\mathcal{V}(33/8)\oplus\mathcal{V}(35/24)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(35/24)$	5	7	$\mathcal{R}^{(2)}(5/8,21/8)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(1)$	6	8	$\mathcal{R}^{(1)}(0)_2\oplus\mathcal{V}(1/3)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(7)$	5	7	$\mathcal{V}(5)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(5)$	5	7	$\mathcal{V}(7)\oplus\mathcal{V}(10/3)$
$\mathcal{V}(7)$	$\otimes_{\mathrm{f}}\ \mathcal{V}(-1/24)$	5	7	$\mathcal{V}(35/24)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(10/3)$	5	7	$\mathcal{V}(1/3)\oplus\mathcal{V}(28/3)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(21/8)$	5	7	$\mathcal{V}(1/8)\oplus\mathcal{V}(65/8)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(35/24)$	4	6	$\mathcal{V}(-1/24)\oplus\mathcal{V}(143/24)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(7)$	5	7	$\mathcal{R}^{(1)}(0)_2\oplus\mathcal{V}(15)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(5)$	5	7	$\mathcal{R}^{(1)}(0)_1\oplus\mathcal{V}(12)$
$\mathcal{V}(5)$	$\otimes_{\mathrm{f}}\ \mathcal{V}(21/8)$	3	6	$\mathcal{V}(5/8)\oplus\mathcal{V}(-1/24)\oplus\mathcal{V}(85/8)\oplus\mathcal{V}(143/24)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(35/24)$	4	6	$\mathcal{R}^{(2)}(1/8,1/8)\oplus\mathcal{R}^{(2)}(33/8,65/8)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(5)$	5	7	$\mathcal{R}^{(1)}(0)_2\oplus\mathcal{V}(15)\oplus\mathcal{V}(1/3)\oplus\mathcal{V}(28/3)$
$\mathcal{R}^{(1)}(0)_2$	$\otimes_{\mathrm{f}}\ \mathcal{V}(5/8)$	6	9	$\mathcal{V}(5/8)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(1/3)$	6	9	$\mathcal{V}(1/3)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(1/8)$	6	9	$\mathcal{V}(1/8)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(-1/24)$	6	9	$\mathcal{V}(-1/24)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(33/8)$	6	9	$\mathcal{V}(33/8)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(10/3)$	6	9	$\mathcal{V}(10/3)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(21/8)$	6	9	$\mathcal{V}(21/8)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(35/24)$	6	9	$\mathcal{V}(35/24)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(2)$	6	9	$\mathcal{V}(2)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(1)$	6	9	$\mathcal{V}(1)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(7)$	6	9	$\mathcal{V}(7)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(5)$	6	9	$\mathcal{V}(5)$
	$\otimes_{\mathrm{f}}\ \mathcal{R}^{(1)}(0)_2$	6	9	$\mathcal{R}^{(1)}(0)_2$
	$\otimes_{\mathrm{f}}\ \mathcal{R}^{(1)}(0)_1$	6	9	$\mathcal{R}^{(1)}(0)_1$
$\mathcal{R}^{(1)}(0)_1$	$\otimes_{\mathrm{f}}\ \mathcal{V}(5/8)$	6	9	$\mathcal{V}(1/8)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(1/3)$	6	9	$\mathcal{R}^{(2)}(0,1)_7$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(1/8)$	6	9	$\mathcal{V}(5/8)\oplus\mathcal{V}(-1/24)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(-1/24)$	6	9	$\mathcal{R}^{(2)}(1/8,1/8)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(33/8)$	6	9	$\mathcal{V}(21/8)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(10/3)$	6	9	$\mathcal{R}^{(2)}(2,5)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(21/8)$	6	9	$\mathcal{V}(33/8)\oplus\mathcal{V}(35/24)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(35/24)$	6	9	$\mathcal{R}^{(2)}(5/8,21/8)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(2)$	6	9	$\mathcal{V}(1)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(1)$	6	9	$\mathcal{V}(2)\oplus\mathcal{V}(1/3)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(7)$	6	9	$\mathcal{V}(5)$
	$\otimes_{\mathrm{f}}\ \mathcal{V}(5)$	6	9	$\mathcal{V}(7)\oplus\mathcal{V}(10/3)$
	$\otimes_{\mathrm{f}}\ \mathcal{R}^{(1)}(0)_2$	6	9	$\mathcal{R}^{(1)}(0)_1$
	$\otimes_{\mathrm{f}}\ \mathcal{R}^{(1)}(0)_1$	6	9	$\mathcal{R}^{(1)}(0)_2\oplus\mathcal{V}(1/3)$

The next table lists our results for the fusion of irreducible and rank 1 representations with rank 2 representations. The notation stays as above.

	L	$\tilde{L}_{\max}$	Fusion product
$\mathcal{V}(5/8) \otimes_f \mathcal{R}^{(2)}(5/8,5/8)$	5	7	$\mathcal{R}^{(3)}(0,0,2,2)$
$\otimes_f \mathcal{R}^{(2)}(1/3,1/3)$	5	7	$2\,\mathcal{V}(-1/24) \oplus \mathcal{V}(35/24)$
$\otimes_f \mathcal{R}^{(2)}(1/8,1/8)$	5	7	$\mathcal{R}^{(3)}(0,0,1,1)$
$\otimes_f \mathcal{R}^{(2)}(5/8,21/8)$	5	8	$\mathcal{R}^{(3)}(0,1,2,5)$
$\otimes_f \mathcal{R}^{(2)}(1/3,10/3)$	4	6	$\mathcal{V}(-1/24) \oplus 2\,\mathcal{V}(35/24) \oplus \mathcal{V}(143/24)$
$\otimes_f \mathcal{R}^{(2)}(1/8,33/8)$	4	6	$\mathcal{R}(0,1,2,7)$?
$\otimes_f \mathcal{R}^{(2)}(0,1)_5$	4	6	$2\,\mathcal{V}(1/8) \oplus \mathcal{V}(21/8)$
$\otimes_f \mathcal{R}^{(2)}(0,1)_7$	4	6	$\mathcal{R}^{(2)}(1/8,1/8)$
$\otimes_f \mathcal{R}^{(2)}(0,2)_5$	4	6	$\mathcal{R}^{(2)}(5/8,5/8)$
$\otimes_f \mathcal{R}^{(2)}(0,2)_7$	4	7	$2\,\mathcal{V}(5/8) \oplus \mathcal{V}(33/8)$
$\otimes_f \mathcal{R}^{(2)}(1,5)$	4	6	$\mathcal{V}(1/8) \oplus 2\,\mathcal{V}(21/8) \oplus \mathcal{V}(65/8)$
$\otimes_f \mathcal{R}^{(2)}(2,5)$	4	6	$\mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{V}(65/8)$
$\otimes_f \mathcal{R}^{(2)}(1,7)$	4	8	$\mathcal{R}^{(2)}(1/8,33/8)$
$\otimes_f \mathcal{R}^{(2)}(2,7)$	4	7	$\mathcal{V}(5/8) \oplus 2\,\mathcal{V}(33/8) \oplus \mathcal{V}(85/8)$
$\mathcal{V}(1/3) \otimes_f \mathcal{R}^{(2)}(5/8,5/8)$	5	7	$\mathcal{R}^{(2)}(5/8,21/8) \oplus 2\,\mathcal{V}(-1/24)$
$\otimes_f \mathcal{R}^{(2)}(1/3,1/3)$	5	7	$\mathcal{R}^{(3)}(0,0,2,2) \oplus \mathcal{R}^{(2)}(1/3,1/3)$
$\otimes_f \mathcal{R}^{(2)}(1/8,1/8)$	4	7	$2\,\mathcal{R}^{(2)}(1/8,1/8) \oplus \mathcal{V}(35/24)$
$\otimes_f \mathcal{R}^{(2)}(5/8,21/8)$	3	10	$2\,\mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{V}(-1/24) \oplus \mathcal{V}(143/24)$
$\otimes_f \mathcal{R}^{(2)}(1/3,10/3)$	3	6	$\mathcal{R}^{(3)}(0,1,2,7)? \oplus \mathcal{R}^{(2)}(1/3,10/3)$
$\otimes_f \mathcal{R}^{(2)}(1/8,33/8)$	4	7	$\mathcal{R}^{(2)}(1/8,1/8) \oplus \mathcal{R}^{(2)}(33/8,65/8) \oplus 2\,\mathcal{V}(35/24)$
$\otimes_f \mathcal{R}^{(2)}(0,1)_5$	4	6	$\mathcal{R}^{(3)}(0,0,1,1)$
$\otimes_f \mathcal{R}^{(2)}(0,1)_7$	4	6	$2\,\mathcal{R}^{(2)}(0,1)_7 \oplus \mathcal{V}(10/3)$
$\otimes_f \mathcal{R}^{(2)}(0,2)_5$	4	7	$\mathcal{R}^{(2)}(2,5) \oplus 2\,\mathcal{V}(1/3)$
$\otimes_f \mathcal{R}^{(2)}(0,2)_7$	4	6	$\mathcal{R}^{(2)}(1/3,1/3)$
$\otimes_f \mathcal{R}^{(2)}(1,5)$	5	9	$\mathcal{R}^{(3)}(0,1,2,5)$
$\otimes_f \mathcal{R}^{(2)}(2,5)$	4	6	$2\,\mathcal{R}^{(2)}(2,5) \oplus \mathcal{V}(1/3) \oplus \mathcal{V}(28/3)$
$\otimes_f \mathcal{R}^{(2)}(1,7)$	4	8	$\mathcal{R}^{(2)}(0,1)_7 \oplus \mathcal{R}^{(2)}(7,12)? \oplus 2\,\mathcal{V}(10/3)$
$\otimes_f \mathcal{R}^{(2)}(2,7)$	4	7	$\mathcal{R}^{(2)}(1/3,10/3)$
$\mathcal{V}(1/8) \otimes_f \mathcal{R}^{(2)}(5/8,5/8)$	5	7	$\mathcal{R}^{(3)}(0,0,1,1) \oplus \mathcal{R}^{(2)}(1/3,10/3)$
$\otimes_f \mathcal{R}^{(2)}(1/3,1/3)$	5	7	$2\,\mathcal{R}^{(2)}(1/8,1/8) \oplus \mathcal{R}^{(2)}(5/8,21/8)$
$\otimes_f \mathcal{R}^{(2)}(1/8,1/8)$	5	7	$\mathcal{R}^{(3)}(0,0,2,2) \oplus 2\,\mathcal{R}^{(2)}(1/3,1/3)$
$\otimes_f \mathcal{R}^{(2)}(5/8,21/8)$	3	6	$\mathcal{R}^{(3)}(0,1,2,7)? \oplus 2\,\mathcal{R}^{(2)}(1/3,10/3)$
$\otimes_f \mathcal{R}^{(2)}(1/3,10/3)$	3	6	$\mathcal{R}^{(2)}(1/8,1/8) \oplus 2\,\mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{R}^{(2)}(33/8,65/8)$?
$\otimes_f \mathcal{R}^{(2)}(1/8,33/8)$	3	6	$\mathcal{R}^{(3)}(0,1,2,5)? \oplus \mathcal{R}^{(2)}(1/3,1/3) \oplus \mathcal{R}^{(2)}(10/3,28/3)$?
$\otimes_f \mathcal{R}^{(2)}(0,1)_5$	4	6	$2\,\mathcal{V}(5/8) \oplus \mathcal{V}(33/8) \oplus 2\,\mathcal{V}(-1/24) \oplus \mathcal{V}(35/24)$
$\otimes_f \mathcal{R}^{(2)}(0,1)_7$	4	6	$\mathcal{R}^{(2)}(5/8,5/8) \oplus 2\,\mathcal{V}(-1/24)$
$\otimes_f \mathcal{R}^{(2)}(0,2)_5$	4	6	$\mathcal{R}^{(2)}(1/8,1/8) \oplus \mathcal{V}(35/24)$
$\otimes_f \mathcal{R}^{(2)}(0,2)_7$	4	7	$2\,\mathcal{V}(1/8) \oplus \mathcal{V}(21/8)$

			L	$\tilde{L}_{\max}$	Fusion product
$\mathcal{V}(1/8)$	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1,5)$	3	6	$\mathcal{V}(5/8)\oplus 2\,\mathcal{V}(33/8)\oplus\mathcal{V}(85/8)\oplus\mathcal{V}(-1/24)$ $\oplus 2\,\mathcal{V}(-35/24)\oplus\mathcal{V}(143/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(2,5)$	4	6	$\mathcal{R}^{(2)}(1/8,33/8)\oplus 2\,\mathcal{V}(35/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1,7)$	3	8	$\mathcal{R}^{(2)}(5/8,21/8)\oplus\mathcal{V}(-1/24)\oplus\mathcal{V}(143/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(2,7)$	4	7	$\mathcal{V}(1/8)\oplus 2\,\mathcal{V}(21/8)\oplus\mathcal{V}(65/8)$
$\mathcal{V}(-1/24)$	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(5/8,5/8)$	4	6	$\mathcal{R}^{(3)}(0,1,2,5)?\oplus 2\,\mathcal{R}^{(2)}(1/3,1/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1/3,1/3)$	4	6	$2\,\mathcal{R}^{(2)}(5/8,5/8)\oplus\mathcal{R}^{(2)}(1/8,33/8)\oplus 2\,\mathcal{V}(-1/24)\oplus\mathcal{V}(35/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1/8,1/8)$	4	6	$2\,\mathcal{R}^{(3)}(0,0,1,1)\oplus\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(5/8,21/8)$	2	5	$2\,\mathcal{R}^{(3)}(0,1,2,5)?\oplus\mathcal{R}^{(2)}(1/3,1/3)\oplus\mathcal{R}^{(2)}(10/3,28/3)?$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1/3,10/3)$	2	5	$\mathcal{R}^{(2)}(5/8,5/8)\oplus 2\,\mathcal{R}^{(2)}(1/8,33/8)?\oplus\mathcal{R}^{(2)}(21/8,85/8)?$ $\oplus\mathcal{V}(-1/24)\oplus 2\,\mathcal{V}(35/24)\oplus\mathcal{V}(143/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1/8,33/8)$	3	6	$\mathcal{R}(0,0,1,1)\oplus\mathcal{R}(2,5,7,12)?\oplus 2\,\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,1)_5$	4	6	$2\,\mathcal{R}^{(2)}(1/8,1/8)\oplus\mathcal{R}^{(2)}(5/8,21/8)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,1)_7$	4	6	$2\,\mathcal{R}^{(2)}(1/8,1/8)\oplus\mathcal{V}(35/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,2)_5$	4	7	$\mathcal{R}^{(2)}(5/8,21/8)\oplus 2\,\mathcal{V}(-1/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,2)_7$	4	7	$2\,\mathcal{V}(-1/24)\oplus\mathcal{V}(35/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1,5)$	2	8	$\mathcal{R}^{(2)}(1/8,1/8)\oplus 2\,\mathcal{R}^{(2)}(5/8,21/8)\oplus\mathcal{R}^{(2)}(33/8,65/8)?$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(2,5)$	3	6	$2\,\mathcal{R}^{(2)}(5/8,21/8)\oplus\mathcal{V}(-1/24)\oplus\mathcal{V}(143/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1,7)$	2	8	$\mathcal{R}^{(2)}(1/8,1/8)\oplus\mathcal{R}^{(2)}(33/8,65/8)?\oplus 2\,\mathcal{V}(35/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(2,7)$	3	6	$\mathcal{V}(-1/24)\oplus 2\,\mathcal{V}(35/24)\oplus\mathcal{V}(143/24)$
$\mathcal{V}(33/8)$	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(5/8,5/8)$	3	9	$\mathcal{R}^{(3)}(0,1,2,7)?$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1/3,1/3)$	3	9	$\mathcal{V}(-1/24)\oplus 2\,\mathcal{V}(35/24)\oplus\mathcal{V}(143/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1/8,1/8)$	3	7	$\mathcal{R}^{(3)}(0,1,2,5)?$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(5/8,21/8)$	2	8	$\mathcal{R}^{(3)}(0,0,1,1)\oplus\mathcal{R}^{(3)}(2,5,7,12)?$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1/3,10/3)$	1	9	$2\,\mathcal{V}(-1/24)\oplus 2\,\mathcal{V}(35/24)\oplus 2\,\mathcal{V}(143/24)\oplus\mathcal{V}(323/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1/8,33/8)$	2	10	$\mathcal{R}(0,0,2,2)\oplus\mathcal{R}^{(3)}(1,5,7,15)?$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,1)_5$	3	9	$\mathcal{V}(1/8)\oplus 2\,\mathcal{V}(21/8)\oplus\mathcal{V}(65/8)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,1)_7$	3	7	$\mathcal{R}^{(2)}(5/8,21/8)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,2)_5$	3	9	$\mathcal{R}^{(2)}(1/8,33/8)?$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,2)_7$	3	9	$\mathcal{V}(5/8)\oplus 2\,\mathcal{V}(33/8)\oplus\mathcal{V}(85/8)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1,5)$	2	9	$2\,\mathcal{V}(1/8)\oplus 2\,\mathcal{V}(21/8)\oplus 2\,\mathcal{V}(65/8)\oplus\mathcal{V}(133/8)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(2,5)$	2	9	$\mathcal{R}^{(2)}(1/8,1/8)\oplus\mathcal{R}^{(2)}(33/8,65/8)?$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1,7)$	2	8	$\mathcal{R}^{(2)}(5/8,5/8)\oplus\mathcal{R}^{(2)}(21/8,85/8)?$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(2,7)$	2	9	$2\,\mathcal{V}(5/8)\oplus 2\,\mathcal{V}(33/8)\oplus 2\,\mathcal{V}(85/8)\oplus\mathcal{V}(161/8)$
$\mathcal{V}(10/3)$	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(5/8,5/8)$	2	8	$\mathcal{R}^{(2)}(1/8,1/8)\oplus\mathcal{R}^{(2)}(33/8,65/8)?\oplus 2\,\mathcal{V}(35/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1/3,1/3)$	2	8	$\mathcal{R}^{(3)}(0,1,2,7)?\oplus\mathcal{R}^{(2)}(1/3,10/3)?$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1/8,1/8)$	2	8	$2\,\mathcal{R}^{(2)}(5/8,21/8)\oplus\mathcal{V}(-1/24)\oplus\mathcal{V}(143/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,1)_5$	2	8	$\mathcal{R}^{(3)}(0,1,2,5)?$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,1)_7$	2	8	$2\,\mathcal{R}^{(2)}(2,5)?\oplus\mathcal{V}(1/3)\oplus\mathcal{V}(28/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,2)_5$	2	8	$\mathcal{R}^{(2)}(0,1)_7\oplus\mathcal{R}^{(2)}(7,12)?\oplus 2\,\mathcal{V}(10/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,2)_7$	2	8	$\mathcal{R}^{(2)}(1/3,10/3)?$

		L	$\tilde{L}_{\max}$	Fusion product
$\mathcal{V}(21/8)$	$\otimes_f\ \mathcal{R}^{(2)}(5/8,5/8)$	2	8	$\mathcal{R}^{(3)}(0,1,2,5)?\oplus\mathcal{R}^{(2)}(1/3,1/3)\oplus\mathcal{R}^{(2)}(10/3,28/3)?$
	$\otimes_f\ \mathcal{R}^{(2)}(1/3,1/3)$	2	8	$\mathcal{R}^{(2)}(1/8,1/8)\oplus 2\,\mathcal{R}^{(2)}(5/8,21/8)\oplus\mathcal{R}^{(2)}(33/8,65/8)?$
	$\otimes_f\ \mathcal{R}^{(2)}(1/8,1/8)$	2	8	$\mathcal{R}^{(3)}(0,1,2,7)?\oplus 2\,\mathcal{R}^{(2)}(1/3,10/3)?$
	$\otimes_f\ \mathcal{R}^{(2)}(0,1)_5$	2	8	$\mathcal{V}(5/8)\oplus 2\,\mathcal{V}(33/8)\oplus\mathcal{V}(85/8)\oplus\mathcal{V}(-1/24)\oplus 2\,\mathcal{V}(35/24)$
				$\oplus\mathcal{V}(143/24)$
	$\otimes_f\ \mathcal{R}^{(2)}(0,1)_7$	2	8	$\mathcal{R}^{(2)}(1/8,33/8)?\oplus 2\,\mathcal{V}(35/24)$
	$\otimes_f\ \mathcal{R}^{(2)}(0,2)_5$	2	8	$\mathcal{R}^{(2)}(5/8,21/8)\oplus\mathcal{V}(-1/24)\oplus\mathcal{V}(143/24)$
	$\otimes_f\ \mathcal{R}^{(2)}(0,2)_7$	2	8	$\mathcal{V}(1/8)\oplus 2\,\mathcal{V}(21/8)\oplus\mathcal{V}(65/8)$
$\mathcal{V}(2)$	$\otimes_f\ \mathcal{R}^{(2)}(5/8,5/8)$	5	7	$\mathcal{R}^{(2)}(5/8,5/8)$
	$\otimes_f\ \mathcal{R}^{(2)}(1/3,1/3)$	5	7	$\mathcal{R}^{(2)}(1/3,1/3)$
	$\otimes_f\ \mathcal{R}^{(2)}(1/8,1/8)$	5	7	$\mathcal{R}^{(2)}(1/8,1/8)$
	$\otimes_f\ \mathcal{R}^{(2)}(5/8,21/8)$	4	7	$\mathcal{R}^{(2)}(5/8,21/8)$
	$\otimes_f\ \mathcal{R}^{(2)}(1/3,10/3)$	4	7	$\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_f\ \mathcal{R}^{(2)}(1/8,33/8)$	4	7	$\mathcal{R}^{(2)}(1/8,33/8)$
	$\otimes_f\ \mathcal{R}^{(2)}(0,1)_5$	4	6	$\mathcal{R}^{(2)}(0,1)_5$
	$\otimes_f\ \mathcal{R}^{(2)}(0,1)_7$	4	6	$\mathcal{R}^{(2)}(0,1)_7$
	$\otimes_f\ \mathcal{R}^{(2)}(0,2)_5$	4	6	$\mathcal{R}^{(2)}(0,2)_5$
	$\otimes_f\ \mathcal{R}^{(2)}(0,2)_7$	4	6	$\mathcal{R}^{(2)}(0,2)_7$
	$\otimes_f\ \mathcal{R}^{(2)}(1,5)$	4	6	$\mathcal{R}^{(2)}(1,5)$
	$\otimes_f\ \mathcal{R}^{(2)}(2,5)$	4	6	$\mathcal{R}^{(2)}(2,5)$
	$\otimes_f\ \mathcal{R}^{(2)}(1,7)$	6	8	$\mathcal{R}^{(2)}(1,7)$
	$\otimes_f\ \mathcal{R}^{(2)}(2,7)$	6	8	$\mathcal{R}^{(2)}(2,7)$
$\mathcal{V}(1)$	$\otimes_f\ \mathcal{R}^{(2)}(5/8,5/8)$	5	7	$\mathcal{R}^{(2)}(1/8,1/8)\oplus\mathcal{V}(35/24)$
	$\otimes_f\ \mathcal{R}^{(2)}(1/3,1/3)$	5	7	$\mathcal{R}^{(3)}(0,0,1,1)$
	$\otimes_f\ \mathcal{R}^{(2)}(1/8,1/8)$	5	7	$\mathcal{R}^{(2)}(5/8,5/8)\oplus 2\,\mathcal{V}(-1/24)$
	$\otimes_f\ \mathcal{R}^{(2)}(5/8,21/8)$	4	7	$\mathcal{R}^{(2)}(1/8,33/8)\oplus 2\,\mathcal{V}(35/24)$
	$\otimes_f\ \mathcal{R}^{(2)}(1/3,10/3)$	3	6	$\mathcal{R}^{(3)}(0,1,2,5)?$
	$\otimes_f\ \mathcal{R}^{(2)}(1/8,33/8)$	4	5	$\mathcal{R}^{(2)}(5/8,21/8)\oplus\mathcal{V}(-1/24)\oplus\mathcal{V}(143/24)$
	$\otimes_f\ \mathcal{R}^{(2)}(0,1)_5$	4	6	$\mathcal{R}^{(2)}(0,2)_7\oplus\mathcal{R}^{(2)}(1/3,1/3)$
	$\otimes_f\ \mathcal{R}^{(2)}(0,1)_7$	4	6	$\mathcal{R}^{(2)}(0,2)_5\oplus 2\,\mathcal{V}(1/3)$
	$\otimes_f\ \mathcal{R}^{(2)}(0,2)_5$	4	7	$\mathcal{R}^{(2)}(0,1)_7\oplus\mathcal{V}(10/3)$
	$\otimes_f\ \mathcal{R}^{(2)}(0,2)_7$	4	7	$\mathcal{R}^{(2)}(0,1)_5$
	$\otimes_f\ \mathcal{R}^{(2)}(1,5)$	4	6	$\mathcal{R}^{(2)}(2,7)?\oplus\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_f\ \mathcal{R}^{(2)}(2,5)$	4	6	$\mathcal{R}^{(2)}(1,7)?\oplus 2\,\mathcal{V}(10/3)$
	$\otimes_f\ \mathcal{R}^{(2)}(1,7)$	4	8	$\mathcal{R}^{(2)}(2,5)\oplus\mathcal{V}(1/3)\oplus\mathcal{V}(28/3)$
	$\otimes_f\ \mathcal{R}^{(2)}(2,7)$	4	7	$\mathcal{R}^{(2)}(1,5)$
$\mathcal{V}(7)$	$\otimes_f\ \mathcal{R}^{(2)}(5/8,5/8)$	4	8	$\mathcal{R}^{(2)}(1/8,33/8)$
	$\otimes_f\ \mathcal{R}^{(2)}(1/3,1/3)$	4	8	$\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_f\ \mathcal{R}^{(2)}(1/8,1/8)$	4	9	$\mathcal{R}^{(2)}(5/8,21/8)$
	$\otimes_f\ \mathcal{R}^{(2)}(5/8,21/8)$	3	7	$\mathcal{R}^{(2)}(1/8,1/8)\oplus\mathcal{R}^{(2)}(33/8,65/8)?$
	$\otimes_f\ \mathcal{R}^{(2)}(1/3,10/3)$	3	6	$\mathcal{R}^{(2)}(1/3,1/3)\oplus\mathcal{R}^{(2)}(10/3,28/3)?$
	$\otimes_f\ \mathcal{R}^{(2)}(1/8,33/8)$	3	7	$\mathcal{R}^{(2)}(5/8,5/8)\oplus\mathcal{R}^{(2)}(21/8,85/8)?$

			L	$\tilde{L}_{\max}$	Fusion product
$\mathcal{V}(7)$	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(0,1)_5$	4	6	$\mathcal{R}^{(2)}(1,5)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(0,1)_7$	4	6	$\mathcal{R}^{(2)}(2,5)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(0,2)_5$	6	7	$\mathcal{R}^{(2)}(1,7)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(0,2)_7$	5	7	$\mathcal{R}^{(2)}(2,7)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(1,5)$	4	6	$\mathcal{R}^{(2)}(0,1)_5 \oplus \mathcal{R}^{(2)}(5,12)?$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(2,5)$	4	6	$\mathcal{R}^{(2)}(0,1)_7 \oplus \mathcal{R}^{(2)}(7,12)?$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(1,7)$	3	6	$\mathcal{R}^{(2)}(0,2)_5 \oplus \mathcal{R}^{(2)}(5,15)?$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(2,7)$	4	7	$\mathcal{R}^{(2)}(0,2)_7 \oplus \mathcal{R}^{(2)}(7,15)?$
$\mathcal{V}(5)$	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(5/8,5/8)$	3	7	$\mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{V}(-1/24) \oplus \mathcal{V}(143/24)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(1/3,1/3)$	3	7	$\mathcal{R}^{(3)}(0,1,2,5)?$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(1/8,1/8)$	3	7	$\mathcal{R}^{(2)}(1/8,33/8)? \oplus 2\,\mathcal{V}(35/24)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(5/8,21/8)$	2	8	$\mathcal{R}^{(2)}(5/8,5/8) \oplus \mathcal{R}^{(2)}(21/8,85/8)? \oplus 2\,\mathcal{V}(-1/24)$
					$\oplus 2\,\mathcal{V}(143/24)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(1/3,10/3)$	2	8	$\mathcal{R}^{(3)}(0,0,1,1) \oplus \mathcal{R}^{(3)}(2,5,7,12)?$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(1/8,33/8)$	2	8	$\mathcal{R}^{(2)}(1/8,1/8) \oplus \mathcal{R}^{(2)}(33/8,65/8)? \oplus 2\,\mathcal{V}(35/24)$
					$\oplus \mathcal{V}(323/24)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(0,1)_5$	3	8	$\mathcal{R}^{(2)}(2,7)? \oplus \mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(0,1)_7$	3	9	$\mathcal{R}^{(2)}(1,7)? \oplus 2\,\mathcal{V}(10/3)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(0,2)_5$	3	8	$\mathcal{R}^{(2)}(2,5) \oplus \mathcal{V}(1/3) \oplus \mathcal{V}(28/3)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(0,2)_7$	3	8	$\mathcal{R}^{(2)}(1,5)?$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(1,5)$	2	8	$\mathcal{R}^{(2)}(0,2)_7 \oplus \mathcal{R}^{(2)}(7,15)? \oplus \mathcal{R}^{(2)}(1/3,1/3)$
					$\oplus \mathcal{R}^{(2)}(10/3,28/3)?$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(2,5)$	2	8	$\mathcal{R}^{(2)}(0,2)_5 \oplus \mathcal{R}^{(2)}(5,15)? \oplus 2\,\mathcal{V}(1/3) \oplus 2\,\mathcal{V}(28/3)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(1,7)$	2	8	$\mathcal{R}^{(2)}(0,1)_7 \oplus \mathcal{R}^{(2)}(7,12)? \oplus 2\,\mathcal{V}(10/3) \oplus \mathcal{V}(55/3)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(2,7)$	2	8	$\mathcal{R}^{(2)}(0,1)_5 \oplus \mathcal{R}^{(2)}(5,12)?$
$\mathcal{R}^{(1)}(0)_2$	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(5/8,5/8)$	5	7	$\mathcal{R}^{(2)}(5/8,5/8)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(1/3,1/3)$	5	7	$\mathcal{R}^{(2)}(1/3,1/3)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(1/8,1/8)$	5	7	$\mathcal{R}^{(2)}(1/8,1/8)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(5/8,21/8)$	4	7	$\mathcal{R}^{(2)}(5/8,21/8)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(1/3,10/3)$	4	7	$\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(1/8,33/8)$	4	7	$\mathcal{R}^{(2)}(1/8,33/8)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(0,1)_5$	5	7	$\mathcal{R}^{(2)}(0,1)_5$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(0,1)_7$	5	7	$\mathcal{R}^{(2)}(0,1)_7$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(0,2)_5$	5	7	$\mathcal{R}^{(2)}(0,2)_5$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(0,2)_7$	5	7	$\mathcal{R}^{(2)}(0,2)_7$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(1,5)$	4	7	$\mathcal{R}^{(2)}(1,5)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(2,5)$	4	7	$\mathcal{R}^{(2)}(2,5)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(1,7)$	6	7	$\mathcal{R}^{(2)}(1,7)$
	$\otimes_{\rm f}$	$\mathcal{R}^{(2)}(2,7)$	6	7	$\mathcal{R}^{(2)}(2,7)$

	L	$\tilde{L}_{\max}$	Fusion product
$\mathcal{R}^{(1)}(0)_1 \otimes_{\mathrm{f}} \mathcal{R}^{(2)}(5/8,5/8)$	5	7	$\mathcal{R}^{(2)}(1/8,1/8) \oplus \mathcal{V}(35/24)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(1/3,1/3)$	5	7	$\mathcal{R}^{(3)}(0,0,1,1)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(1/8,1/8)$	5	7	$\mathcal{R}^{(2)}(5/8,5/8) \oplus 2\,\mathcal{V}(-1/24)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(5/8,21/8)$	4	7	$\mathcal{R}^{(2)}(1/8,33/8) \oplus 2\,\mathcal{V}(35/24)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(1/3,10/3)$	4	7	$\mathcal{R}^{(3)}(0,1,2,5)?$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(1/8,33/8)$	4	7	$\mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{V}(-1/24) \oplus \mathcal{V}(143/24)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(0,1)_5$	5	7	$\mathcal{R}^{(2)}(0,2)_7 \oplus \mathcal{R}^{(2)}(1/3,1/3)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(0,1)_7$	5	7	$\mathcal{R}^{(2)}(0,2)_5 \oplus 2\,\mathcal{V}(1/3)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(0,2)_5$	5	7	$\mathcal{R}^{(2)}(0,1)_7 \oplus \mathcal{V}(10/3)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(0,2)_7$	5	7	$\mathcal{R}^{(2)}(0,1)_5$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(1,5)$	4	7	$\mathcal{R}^{(2)}(2,7)? \oplus \mathcal{R}^{(2)}(1/3,10/3)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(2,5)$	4	7	$\mathcal{R}^{(2)}(1,7)? \oplus 2\,\mathcal{V}(10/3)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(1,7)$	4	7	$\mathcal{R}^{(2)}(2,5) \oplus \mathcal{V}(1/3) \oplus \mathcal{V}(28/3)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(2,7)$	4	7	$\mathcal{R}^{(2)}(1,5)$

In order to extract the fusion of higher rank representations we used the symmetry and associativity properties of the fusion product along the lines described in section 4.3. For these calculations we applied our explicit results of the Nahm algorithm in the form stated in the tables above. The results are listed below.

	Fusion product
$\mathcal{R}^{(2)}(5/8,5/8) \otimes_{\mathrm{f}} \mathcal{R}^{(2)}(5/8,5/8)$	$2\,\mathcal{R}^{(3)}(0,0,2,2) \oplus \mathcal{R}^{(3)}(0,1,2,5) \oplus \mathcal{R}^{(2)}(1/3,1/3)$ $\oplus \mathcal{R}^{(2)}(10/3,28/3)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(1/3,1/3)$	$\mathcal{R}^{(2)}(1/8,1/8) \oplus 2\,\mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{R}^{(2)}(33/8,65/8)$ $\oplus 4\,\mathcal{V}(-1/24) \oplus 2\,\mathcal{V}(35/24)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(1/8,1/8)$	$2\,\mathcal{R}^{(3)}(0,0,1,1) \oplus \mathcal{R}^{(3)}(0,1,2,7) \oplus 2\,\mathcal{R}^{(2)}(1/3,10/3)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(5/8,21/8)$	$\mathcal{R}^{(3)}(0,0,2,2) \oplus 2\,\mathcal{R}^{(3)}(0,1,2,5) \oplus \mathcal{R}^{(3)}(1,5,7,15)$ $\oplus 2\,\mathcal{R}^{(2)}(1/3,1/3) \oplus 2\,\mathcal{R}^{(2)}(10/3,28/3)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(1/3,10/3)$	$2\,\mathcal{R}^{(2)}(1/8,1/8) \oplus 2\,\mathcal{R}^{(2)}(5/8,21/8) \oplus 2\,\mathcal{R}^{(2)}(33/8,65/8)$ $\oplus \mathcal{R}^{(2)}(85/8,133/8) \oplus 2\,\mathcal{V}(-1/24) \oplus 4\,\mathcal{V}(35/24)$ $\oplus 2\,\mathcal{V}(143/24)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(1/8,33/8)$	$\mathcal{R}^{(3)}(0,0,1,1) \oplus 2\,\mathcal{R}^{(3)}(0,1,2,7) \oplus \mathcal{R}^{(3)}(2,5,7,12)$ $\oplus 2\,\mathcal{R}^{(2)}(1/3,10/3) \oplus \mathcal{R}^{(2)}(28/3,55/3)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(0,1)_5$	$2\,\mathcal{R}^{(2)}(1/8,1/8) \oplus \mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{V}(-1/24) \oplus 2\,\mathcal{V}(35/24)$ $\oplus \mathcal{V}(143/24)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(0,1)_7$	$2\,\mathcal{R}^{(2)}(1/8,1/8) \oplus \mathcal{R}^{(2)}(1/8,33/8) \oplus 2\,\mathcal{V}(35/24)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(0,2)_5$	$2\,\mathcal{R}^{(2)}(5/8,5/8) \oplus \mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{V}(-1/24) \oplus \mathcal{V}(143/24)$
$\otimes_{\mathrm{f}} \mathcal{R}^{(2)}(0,2)_7$	$2\,\mathcal{R}^{(2)}(5/8,5/8) \oplus \mathcal{R}^{(2)}(1/8,33/8)$

		Fusion product
$\mathcal{R}^{(2)}(1/3,1/3)$	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(1/3,1/3)$	$2\,\mathcal{R}^{(3)}(0,0,2,2)\oplus\mathcal{R}^{(3)}(0,1,2,7)\oplus 2\,\mathcal{R}^{(2)}(1/3,1/3)$
		$\oplus\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(1/8,1/8)$	$4\,\mathcal{R}^{(2)}(1/8,1/8)\oplus 2\,\mathcal{R}^{(2)}(5/8,21/8)\oplus\mathcal{V}(-1/24)$
		$\oplus 2\,\mathcal{V}(35/24)\oplus\mathcal{V}(143/24)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(5/8,21/8)$	$2\,\mathcal{R}^{(2)}(1/8,1/8)\oplus 4\,\mathcal{R}^{(2)}(5/8,21/8)\oplus 2\,\mathcal{R}^{(2)}(33/8,65/8)$
		$\oplus 2\,\mathcal{V}(-1/24)\oplus 2\,\mathcal{V}(35/24)\oplus 2\,\mathcal{V}(143/24)\oplus\mathcal{V}(323/24)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(1/3,10/3)$	$\mathcal{R}^{(3)}(0,0,2,2)\oplus 2\,\mathcal{R}^{(3)}(0,1,2,7)\oplus\mathcal{R}^{(3)}(1,5,7,12)$
		$\oplus\mathcal{R}^{(2)}(1/3,1/3)\oplus 2\,\mathcal{R}^{(2)}(1/3,10/3)\oplus\mathcal{R}^{(2)}(10/3,28/3)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(1/8,33/8)$	$2\,\mathcal{R}^{(2)}(1/8,1/8)\oplus 2\,\mathcal{R}^{(2)}(5/8,21/8)\oplus 2\,\mathcal{R}^{(2)}(33/8,65/8)$
		$\oplus\mathcal{R}^{(2)}(85/8,133/8)\oplus 2\,\mathcal{V}(-1/24)\oplus 4\,\mathcal{V}(35/24)$
		$\oplus 2\,\mathcal{V}(143/24)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,1)_5$	$2\,\mathcal{R}^{(3)}(0,0,1,1)\oplus\mathcal{R}^{(3)}(0,1,2,5)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,1)_7$	$2\,\mathcal{R}^{(3)}(0,0,1,1)\oplus\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,2)_5$	$\mathcal{R}^{(3)}(0,1,2,5)\oplus 2\,\mathcal{R}^{(2)}(1/3,1/3)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,2)_7$	$2\,\mathcal{R}^{(2)}(1/3,1/3)\oplus\mathcal{R}^{(2)}(1/3,10/3)$
$\mathcal{R}^{(2)}(1/8,1/8)$	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(1/8,1/8)$	$2\,\mathcal{R}^{(3)}(0,0,2,2)\oplus\mathcal{R}^{(3)}(0,1,2,5)\oplus 4\,\mathcal{R}^{(2)}(1/3,1/3)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(5/8,21/8)$	$\mathcal{R}^{(3)}(0,0,1,1)\oplus 2\,\mathcal{R}^{(3)}(0,1,2,7)\oplus\mathcal{R}^{(3)}(2,5,7,12)$
		$\oplus 4\,\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(1/3,10/3)$	$2\,\mathcal{R}^{(2)}(1/8,1/8)\oplus 4\,\mathcal{R}^{(2)}(5/8,21/8)\oplus 2\,\mathcal{R}^{(2)}(33/8,65/8)$
		$\oplus 2\,\mathcal{V}(-1/24)\oplus 2\,\mathcal{V}(35/24)\oplus 2\,\mathcal{V}(143/24)\oplus\mathcal{V}(323/24)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(1/8,33/8)$	$\mathcal{R}^{(3)}(0,0,2,2)\oplus 2\,\mathcal{R}^{(3)}(0,1,2,5)\oplus\mathcal{R}^{(3)}(1,5,7,15)$
		$\oplus 2\,\mathcal{R}^{(2)}(1/3,1/3)\oplus 2\,\mathcal{R}^{(2)}(10/3,28/3)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,1)_5$	$2\,\mathcal{R}^{(2)}(5/8,5/8)\oplus\mathcal{R}^{(2)}(1/8,33/8)\oplus 4\,\mathcal{V}(-1/24)$
		$\oplus 2\,\mathcal{V}(35/24)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,1)_7$	$2\,\mathcal{R}^{(2)}(5/8,5/8)\oplus\mathcal{R}^{(2)}(5/8,21/8)\oplus 4\,\mathcal{V}(-1/24)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,2)_5$	$2\,\mathcal{R}^{(2)}(1/8,1/8)\oplus\mathcal{R}^{(2)}(1/8,33/8)\oplus 2\,\mathcal{V}(35/24)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,2)_7$	$2\,\mathcal{R}^{(2)}(1/8,1/8)\oplus\mathcal{R}^{(2)}(5/8,21/8)$
$\mathcal{R}^{(2)}(0,1)_5$	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,1)_5$	$2\,\mathcal{R}^{(2)}(0,2)_7\oplus\mathcal{R}^{(2)}(2,7)\oplus 2\,\mathcal{R}^{(2)}(1/3,1/3)$
		$\oplus\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,1)_7$	$\mathcal{R}^{(3)}(0,0,2,2)\oplus 2\,\mathcal{R}^{(2)}(1/3,1/3)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,2)_5$	$\mathcal{R}^{(3)}(0,0,1,1)\oplus\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,2)_7$	$2\,\mathcal{R}^{(2)}(0,1)_5\oplus\mathcal{R}^{(2)}(1,5)$
$\mathcal{R}^{(2)}(0,1)_7$	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,1)_7$	$2\,\mathcal{R}^{(2)}(0,2)_5\oplus\mathcal{R}^{(2)}(2,5)\oplus 4\,\mathcal{V}(1/3)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,2)_5$	$2\,\mathcal{R}^{(2)}(0,1)_7\oplus\mathcal{R}^{(2)}(1,7)\oplus 2\,\mathcal{V}(10/3)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,2)_7$	$\mathcal{R}^{(3)}(0,0,1,1)$
$\mathcal{R}^{(2)}(0,2)_5$	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,2)_5$	$2\,\mathcal{R}^{(2)}(0,2)_5\oplus\mathcal{R}^{(2)}(2,5)\oplus\mathcal{V}(1/3)\oplus\mathcal{V}(28/3)$
	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,2)_7$	$\mathcal{R}^{(3)}(0,0,2,2)$
$\mathcal{R}^{(2)}(0,2)_7$	$\otimes_{\rm f}\ \mathcal{R}^{(2)}(0,2)_7$	$2\,\mathcal{R}^{(2)}(0,2)_7\oplus\mathcal{R}^{(2)}(2,7)$

			Fusion product
$\mathcal{R}^{(3)}(0,0,1,1)$	$\otimes_{\mathrm{f}}$	$\mathcal{V}(5/8)$	$2\,\mathcal{R}^{(2)}(1/8,1/8) \oplus \mathcal{R}^{(2)}(5/8,21/8)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(1/3)$	$2\,\mathcal{R}^{(3)}(0,0,1,1) \oplus \mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(1/8)$	$2\,\mathcal{R}^{(2)}(5/8,5/8) \oplus \mathcal{R}^{(2)}(1/8,33/8) \oplus 4\,\mathcal{V}(-1/24) \oplus 2\,\mathcal{V}(35/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(-1/24)$	$4\,\mathcal{R}^{(2)}(1/8,1/8) \oplus 2\,\mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{V}(-1/24)$ $\oplus 2\,\mathcal{V}(35/24) \oplus \mathcal{V}(143/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(2)$	$\mathcal{R}^{(3)}(0,0,1,1)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(1)$	$\mathcal{R}^{(3)}(0,0,2,2) \oplus 2\,\mathcal{R}^{(2)}(1/3,1/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(7)$	$\mathcal{R}^{(3)}(0,1,2,5)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(5)$	$\mathcal{R}^{(3)}(0,1,2,7) \oplus 2\,\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(1)}(0)_2$	$\mathcal{R}^{(3)}(0,0,1,1)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(1)}(0)_1$	$\mathcal{R}^{(3)}(0,0,2,2) \oplus 2\,\mathcal{R}^{(2)}(1/3,1/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(5/8,5/8)$	$\mathcal{R}^{(2)}(5/8,5/8) \oplus 2\,\mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{R}^{(2)}(21/8,85/8)$ $\oplus 4\,\mathcal{R}^{(2)}(1/8,1/8) \oplus 2\,\mathcal{R}^{(2)}(1/8,33/8)$ $\oplus 2\,\mathcal{V}(-1/24) \oplus 4\,\mathcal{V}(35/24) \oplus 2\,\mathcal{V}(143/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1/3,1/3)$	$4\,\mathcal{R}^{(3)}(0,0,1,1) \oplus 2\,\mathcal{R}^{(3)}(0,1,2,5) \oplus \mathcal{R}^{(2)}(1/3,1/3)$ $\oplus 2\,\mathcal{R}^{(2)}(1/3,10/3) \oplus \mathcal{R}^{(2)}(10/3,28/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(1/8,1/8)$	$4\,\mathcal{R}^{(2)}(5/8,5/8) \oplus 2\,\mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{R}^{(2)}(1/8,1/8)$ $\oplus 2\,\mathcal{R}^{(2)}(1/8,33/8) \oplus \mathcal{R}^{(2)}(33/8,65/8)$ $\oplus 8\,\mathcal{V}(-1/24) \oplus 4\,\mathcal{V}(35/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,1)_5$	$2\,\mathcal{R}^{(3)}(0,0,2,2) \oplus \mathcal{R}^{(3)}(0,1,2,7) \oplus 4\,\mathcal{R}^{(2)}(1/3,1/3)$ $\oplus 2\,\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,1)_7$	$2\,\mathcal{R}^{(3)}(0,0,2,2) \oplus \mathcal{R}^{(3)}(0,1,2,5) \oplus 4\,\mathcal{R}^{(2)}(1/3,1/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,2)_5$	$2\,\mathcal{R}^{(3)}(0,0,1,1) \oplus \mathcal{R}^{(3)}(0,1,2,7) \oplus 2\,\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(2)}(0,2)_7$	$2\,\mathcal{R}^{(3)}(0,0,1,1) \oplus \mathcal{R}^{(3)}(0,1,2,5)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(3)}(0,0,1,1)$	$4\,\mathcal{R}^{(3)}(0,0,2,2) \oplus 2\,\mathcal{R}^{(3)}(0,1,2,5) \oplus \mathcal{R}^{(3)}(0,0,1,1)$ $\oplus 2\,\mathcal{R}^{(3)}(0,1,2,7) \oplus \mathcal{R}^{(3)}(2,5,7,12)$ $\oplus 8\,\mathcal{R}^{(2)}(1/3,1/3) \oplus 4\,\mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(3)}(0,0,2,2)$	$\mathcal{R}^{(3)}(0,0,2,2) \oplus 2\,\mathcal{R}^{(3)}(0,1,2,5) \oplus \mathcal{R}^{(3)}(1,5,7,15)$ $\oplus 4\,\mathcal{R}^{(3)}(0,0,1,1) \oplus 2\,\mathcal{R}^{(3)}(0,1,2,7) \oplus 2\,\mathcal{R}^{(2)}(1/3,1/3)$ $\oplus 4\,\mathcal{R}^{(2)}(1/3,10/3) \oplus 2\,\mathcal{R}^{(2)}(10/3,28/3)$
$\mathcal{R}^{(3)}(0,0,2,2)$	$\otimes_{\mathrm{f}}$	$\mathcal{V}(5/8)$	$2\,\mathcal{R}^{(2)}(5/8,5/8) \oplus \mathcal{R}^{(2)}(1/8,33/8)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(1/3)$	$\mathcal{R}^{(3)}(0,1,2,5) \oplus 2\,\mathcal{R}^{(2)}(1/3,1/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(1/8)$	$2\,\mathcal{R}^{(2)}(1/8,1/8) \oplus \mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{V}(-1/24)$ $\oplus 2\,\mathcal{V}(35/24) \oplus \mathcal{V}(143/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(-1/24)$	$\mathcal{R}^{(2)}(1/8,1/8) \oplus 2\,\mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{R}^{(2)}(33/8,65/8)$ $\oplus 4\,\mathcal{V}(-1/24) \oplus 2\,\mathcal{V}(35/24)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(2)$	$\mathcal{R}^{(3)}(0,0,2,2)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(1)$	$\mathcal{R}^{(3)}(0,0,1,1) \oplus \mathcal{R}^{(2)}(1/3,10/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(7)$	$\mathcal{R}^{(3)}(0,1,2,7)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(5)$	$\mathcal{R}^{(3)}(0,1,2,5) \oplus \mathcal{R}^{(2)}(1/3,1/3) \oplus \mathcal{R}^{(2)}(10/3,28/3)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(1)}(0)_2$	$\mathcal{R}^{(3)}(0,0,2,2)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(1)}(0)_1$	$\mathcal{R}^{(3)}(0,0,1,1) \oplus \mathcal{R}^{(2)}(1/3,10/3)$

	Fusion product
$\mathcal{R}^{(3)}(0,0,2,2) \otimes_{\rm f} \mathcal{R}^{(2)}(5/8,5/8)$	$4\,\mathcal{R}^{(2)}(5/8,5/8) \oplus 2\,\mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{R}^{(2)}(1/8,1/8)$ $\oplus 2\,\mathcal{R}^{(2)}(1/8,33/8) \oplus \mathcal{R}^{(2)}(33/8,65/8) \oplus 2\,\mathcal{V}(-1/24)$ $\oplus 2\,\mathcal{V}(35/24) \oplus 2\,\mathcal{V}(143/24) \oplus \mathcal{V}(323/24)$
$\otimes_{\rm f} \mathcal{R}^{(2)}(1/3,1/3)$	$\mathcal{R}^{(3)}(0,0,1,1) \oplus 2\,\mathcal{R}^{(3)}(0,1,2,5) \oplus \mathcal{R}^{(3)}(2,5,7,12)$ $\oplus 4\,\mathcal{R}^{(2)}(1/3,1/3) \oplus 2\,\mathcal{R}^{(2)}(1/3,10/3)$
$\otimes_{\rm f} \mathcal{R}^{(2)}(1/8,1/8)$	$\mathcal{R}^{(2)}(5/8,5/8) \oplus 2\,\mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{R}^{(2)}(21/8,85/8)$ $\oplus 4\,\mathcal{R}^{(2)}(1/8,1/8) \oplus 2\,\mathcal{R}^{(2)}(1/8,33/8)$ $\oplus 2\,\mathcal{V}(-1/24) \oplus 4\,\mathcal{V}(35/24) \oplus 2\,\mathcal{V}(143/24)$
$\otimes_{\rm f} \mathcal{R}^{(2)}(0,1)_5$	$2\,\mathcal{R}^{(3)}(0,0,1,1) \oplus \mathcal{R}^{(3)}(0,1,2,5) \oplus \mathcal{R}^{(2)}(1/3,1/3)$ $\oplus 2\,\mathcal{R}^{(2)}(1/3,10/3) \oplus \mathcal{R}^{(2)}(10/3,28/3)$
$\otimes_{\rm f} \mathcal{R}^{(2)}(0,1)_7$	$2\,\mathcal{R}^{(3)}(0,0,1,1) \oplus \mathcal{R}^{(3)}(0,1,2,7) \oplus 2\,\mathcal{R}^{(2)}(1/3,10/3)$
$\otimes_{\rm f} \mathcal{R}^{(2)}(0,2)_5$	$2\,\mathcal{R}^{(3)}(0,0,2,2) \oplus \mathcal{R}^{(3)}(0,1,2,5) \oplus \mathcal{R}^{(2)}(1/3,1/3) \oplus$ $\mathcal{R}^{(2)}(10/3,28/3)$
$\otimes_{\rm f} \mathcal{R}^{(2)}(0,2)_7$	$2\,\mathcal{R}^{(3)}(0,0,2,2) \oplus \mathcal{R}^{(3)}(0,1,2,7)$
$\otimes_{\rm f} \mathcal{R}^{(3)}(0,0,2,2)$	$4\,\mathcal{R}^{(3)}(0,0,2,2) \oplus 2\,\mathcal{R}^{(3)}(0,1,2,5) \oplus \mathcal{R}^{(3)}(0,0,1,1)$ $\oplus 2\,\mathcal{R}^{(3)}(0,1,2,7) \oplus \mathcal{R}^{(3)}(2,5,7,12) \oplus 2\,\mathcal{R}^{(2)}(1/3,1/3)$ $\oplus 2\,\mathcal{R}^{(2)}(1/3,10/3) \oplus 2\,\mathcal{R}^{(2)}(10/3,28/3) \oplus \mathcal{R}^{(2)}(28/3,55/3)$
$\mathcal{R}^{(3)}(0,1,2,5) \otimes_{\rm f} \mathcal{V}(5/8)$	$2\,\mathcal{R}^{(2)}(5/8,21/8) \oplus \mathcal{R}^{(2)}(1/8,1/8) \oplus \mathcal{R}^{(2)}(33/8,65/8)$
$\otimes_{\rm f} \mathcal{V}(1/3)$	$2\,\mathcal{R}^{(3)}(0,1,2,5) \oplus \mathcal{R}^{(2)}(1/3,1/3) \oplus \mathcal{R}^{(2)}(10/3,28/3)$
$\otimes_{\rm f} \mathcal{V}(1/8)$	$\mathcal{R}^{(2)}(5/8,5/8) \oplus 2\,\mathcal{R}^{(2)}(1/8,33/8) \oplus \mathcal{R}^{(2)}(21/8,85/8)$ $\oplus 2\,\mathcal{V}(-1/24) \oplus 4\,\mathcal{V}(35/24) \oplus 2\,\mathcal{V}(143/24)$
$\otimes_{\rm f} \mathcal{V}(-1/24)$	$2\,\mathcal{R}^{(2)}(1/8,1/8) \oplus 4\,\mathcal{R}^{(2)}(5/8,21/8) \oplus 2\,\mathcal{R}^{(2)}(33/8,65/8)$ $\oplus 2\,\mathcal{V}(-1/24) \oplus 2\,\mathcal{V}(35/24) \oplus 2\,\mathcal{V}(143/24) \oplus \mathcal{V}(323/24)$
$\otimes_{\rm f} \mathcal{V}(2)$	$\mathcal{R}^{(3)}(0,1,2,5)$
$\otimes_{\rm f} \mathcal{V}(1)$	$\mathcal{R}^{(3)}(0,1,2,7) \oplus 2\,\mathcal{R}^{(2)}(1/3,10/3)$
$\otimes_{\rm f} \mathcal{R}^{(1)}(0)_2$	$\mathcal{R}^{(3)}(0,1,2,5)$
$\otimes_{\rm f} \mathcal{R}^{(1)}(0)_1$	$\mathcal{R}^{(3)}(0,1,2,7) \oplus 2\,\mathcal{R}^{(2)}(1/3,10/3)$
$\otimes_{\rm f} \mathcal{R}^{(2)}(0,2)_7$	$\mathcal{R}^{(3)}(0,0,1,1) \oplus 2\,\mathcal{R}^{(3)}(0,1,2,5) \oplus \mathcal{R}^{(3)}(2,5,7,12)$

C Explicit fusion rules for $\mathbf{c_{2,5} = -22/5}$

In the following table we have collected the results of our explicit calculations of the fusion product of irreducible representations in the augmented $c_{2,5} = -22/5$ model. These results are certainly not complete, but mainly serve to compute the lowest higher rank representations as well as to check the proposed fusion rules on a variety of examples. The notations are as before.

			L	$\tilde{L}_{\max}$	Fusion product
$\mathcal{V}(11/8)$	$\otimes_{\mathrm{f}}$	$\mathcal{V}(11/8)$	5	7	$\mathcal{R}^{(2)}(0,4)_{13}$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(27/40)$	5	7	$\mathcal{R}^{(2)}(-1/5,14/5)_{11}$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(2/5)$	5	7	$\mathcal{V}(-9/40)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(7/40)$	5	7	$\mathcal{R}^{(2)}(-1/5,9/5)_{9}$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(-1/8)$	7	8	$\mathcal{R}^{(2)}(0,1)_{7}$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(-9/40)$	4	6	$\mathcal{R}^{(2)}(2/5,2/5)$
$\mathcal{V}(27/40)$	$\otimes_{\mathrm{f}}$	$\mathcal{V}(27/40)$	5	7	$\mathcal{R}^{(2)}(0,4)_{13}\oplus\mathcal{R}^{(2)}(-1/5,9/5)_{9}$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(2/5)$	5	7	$\mathcal{R}^{(2)}(-1/8,-1/8)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(7/40)$	5	7	$\mathcal{R}^{(2)}(0,1)_{7}\oplus\mathcal{R}^{(2)}(-1/5,14/5)_{11}$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(-1/8)$	5	7	$\mathcal{R}^{(2)}(-1/5,9/5)_{9}\oplus\mathcal{R}^{(2)}(2/5,2/5)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(-9/40)$	4	6	$\mathcal{R}^{(3)}(0,0,1,1)$
$\mathcal{V}(2/5)$	$\otimes_{\mathrm{f}}$	$\mathcal{V}(11/8)$	5	7	$\mathcal{V}(-9/40)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(27/40)$	5	7	$\mathcal{R}^{(2)}(-1/8,-1/8)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(2/5)$	5	7	$\mathcal{R}^{(2)}(0,4)_{7}\oplus\mathcal{R}^{(2)}(-1/5,9/5)_{11}\oplus\mathcal{V}(2/5)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(7/40)$	4	6	$\mathcal{R}^{(2)}(7/40,7/40)\oplus\mathcal{V}(-9/40)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(-1/8)$	4	6	$\mathcal{R}^{(2)}(-1/8,-1/8)\oplus\mathcal{R}^{(2)}(27/40,27/40)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(-9/40)$	3	5	$\mathcal{R}^{(2)}(7/40,7/40)\oplus\mathcal{R}^{(2)}(11/8,11/8)\oplus\mathcal{V}(-9/40)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(1)}(0)_1$	5	7	$\mathcal{R}^{(2)}(0,1)_{13}\oplus\mathcal{R}^{(2)}(-1/5,14/5)_{9}$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(1)$	3	9	$\mathcal{R}^{(2)}(0,1)_{13}\oplus\mathcal{R}^{(2)}(-1/5,14/5)_{9}$
$\mathcal{V}(7/40)$	$\otimes_{\mathrm{f}}$	$\mathcal{V}(7/40)$	4	6	$\mathcal{R}^{(2)}(0,4)_{13}\oplus\mathcal{R}^{(2)}(-1/5,9/5)_{9}\oplus\mathcal{R}^{(2)}(2/5,2/5)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(-1/8)$	4	6	$\mathcal{R}^{(3)}(0,0,1,1)\oplus\mathcal{R}^{(2)}(-1/5,14/5)_{11}$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(-9/40)$	4	6	$\mathcal{R}^{(3)}(-1/5,-1/5,9/5,9/5)\oplus\mathcal{R}^{(2)}(2/5,2/5)$
$\mathcal{V}(-1/8)$	$\otimes_{\mathrm{f}}$	$\mathcal{V}(-1/8)$	3	5	$\mathcal{R}^{(3)}(-1/5,-1/5,9/5,9/5)\oplus\mathcal{R}^{(2)}(0,4)_{13}$ $\oplus\mathcal{R}^{(2)}(2/5,2/5)$
	$\otimes_{\mathrm{f}}$	$\mathcal{V}(-9/40)$	4	6	$\mathcal{R}^{(3)}(0,0,1,1)\oplus\mathcal{R}^{(3)}(-1/5,-1/5,14/5,14/5)$
$\mathcal{V}(-9/40)$	$\otimes_{\mathrm{f}}$	$\mathcal{V}(-9/40)$	4	5	$\mathcal{R}^{(3)}(0,0,4,4)\oplus\mathcal{R}^{(3)}(-1/5,-1/5,9/5,9/5)$ $\oplus\mathcal{R}^{(2)}(2/5,2/5)$
$\mathcal{V}(14/5)$	$\otimes_{\mathrm{f}}$	$\mathcal{V}(14/5)$	2	8	$\mathcal{R}^{(1)}(0)_4\oplus\mathcal{R}^{(1)}(-1/5)_2$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(1)}(-1/5)_2$	2	8	$\mathcal{V}(1)\oplus\mathcal{V}(14/5)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(1)}(-1/5)_3$	2	8	$\mathcal{V}(4)\oplus\mathcal{V}(9/5)$
$\mathcal{V}(4)$	$\otimes_{\mathrm{f}}$	$\mathcal{V}(4)$	2	9	$\mathcal{R}^{(1)}(0)_4$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(1)}(0)_1$	2	9	$\mathcal{V}(1)$
	$\otimes_{\mathrm{f}}$	$\mathcal{R}^{(1)}(0)_4$	2	9	$\mathcal{V}(4)$

References

[1] H. Saleur, *Polymers and percolation in two dimensions and twisted $N = 2$ supersymmetry*, *Nucl. Phys.* **B382** (1992) 486–531, [hep-th/9111007].

[2] V. Gurarie, M. Flohr, and C. Nayak, *The Haldane-Rezayi quantum Hall state and conformal field theory*, *Nucl. Phys.* **B498** (1997) 513–538, [cond-mat/9701212].

[3] P. Ruelle, *A $c = -2$ boundary changing operator for the Abelian sandpile model*, *Phys. Lett.* **B539** (2002) 172–177, [hep-th/0203105].

[4] P. A. Pearce, V. Rittenberg, and J. de Gier, *Critical $Q = 1$ Potts model and Temperley-Lieb stochastic processes*, cond-mat/0108051.

[5] E. Gravanis and N. E. Mavromatos, *Impulse action on D-particles in Robertson-Walker space times, higher-order logarithmic conformal algebras and cosmological horizons*, hep-th/0106146.

[6] I. Bakas and K. Sfetsos, *PP-waves and logarithmic conformal field theories*, *Nucl. Phys.* **B639** (2002) 223–240, [hep-th/0205006].

[7] N. Lambert, H. Liu, and J. Maldacena, *Closed strings from decaying D-branes*, hep-th/0303139.

[8] M. A. I. Flohr, *Logarithmic conformal field theory or how to compute a torus amplitude on the sphere*, in *From fields to strings* (M. Shifman, ed.), vol. 2, pp. 1201–1256, 2004. hep-th/0407003.

[9] M. Flohr, *Bits and pieces in logarithmic conformal field theory*, *Int. J. Mod. Phys.* **A18** (2003) 4497–4592, [hep-th/0111228].

[10] M. R. Gaberdiel, *An algebraic approach to logarithmic conformal field theory*, *Int. J. Mod. Phys.* **A18** (2003) 4593–4638, [hep-th/0111260].

[11] S. Kawai, *Logarithmic conformal field theory with boundary*, *Int. J. Mod. Phys.* **A18** (2003) 4655–4684, [hep-th/0204169].

[12] A. Nichols, *$SU(2)_k$ logarithmic conformal field theories.* PhD thesis, University of Oxford, 2002. hep-th/0210070.

[13] M. Flohr and M. R. Gaberdiel, *Logarithmic torus amplitudes*, *J. Phys.* **A39** (2006) 1955–1968, [hep-th/0509075].

[14] M. A. I. Flohr, *Singular vectors in logarithmic conformal field theories*, *Nucl. Phys.* **B514** (1998) 523–552, [hep-th/9707090].

[15] M. Flohr, *Null vectors in logarithmic conformal field theory*, hep-th/0009137.

[16] M. Flohr, *Operator product expansion in logarithmic conformal field theory*, *Nucl. Phys.* **B634** (2002) 511–545, [hep-th/0107242].

[17] M. Flohr and M. Krohn, *Four-point functions in logarithmic conformal field theories*, hep-th/0504211.

[18] S. Moghimi-Araghi, S. Rouhani, and M. Saadat, *Logarithmic conformal field theory through nilpotent conformal dimensions*, *Nucl. Phys.* **B599** (2001) 531–546, [hep-th/0008165].

[19] S. Moghimi-Araghi, S. Rouhani, and M. Saadat, *Use of nilpotent weights in logarithmic conformal field theories*, *Int. J. Mod. Phys.* **A18** (2003) 4747–4770, [hep-th/0201099].

[20] M. R. Gaberdiel and H. G. Kausch, *Indecomposable fusion products*, *Nucl. Phys.* **B477** (1996) 293–318, [hep-th/9604026].

[21] M. R. Gaberdiel and H. G. Kausch, *A rational logarithmic conformal field theory*, *Phys. Lett.* **B386** (1996) 131–137, [hep-th/9606050].

[22] F. Rohsiepe, *On reducible but indecomposable representations of the Virasoro algebra*, hep-th/9611160.

[23] M. R. Gaberdiel and H. G. Kausch, *A local logarithmic conformal field theory*, *Nucl. Phys.* **B538** (1999) 631–658, [hep-th/9807091].

[24] M. A. I. Flohr, *On modular invariant partition functions of conformal field theories with logarithmic operators*, *Int. J. Mod. Phys.* **A11** (1996) 4147–4172, [hep-th/9509166].

[25] M. A. I. Flohr, *On fusion rules in logarithmic conformal field theories*, *Int. J. Mod. Phys.* **A12** (1997) 1943–1958, [hep-th/9605151].

[26] H. G. Kausch, *Extended conformal algebras generated by a multiplet of primary fields*, *Phys. Lett.* **B259** (1991) 448–455.

[27] H. Eberle, M. Flohr, and A. Nichols. In preparation.

[28] N. Carqueville and M. Flohr, *Nonmeromorphic operator product expansion and C_2-cofiniteness for a family of $\mathcal{W}$-algebras*, *J. Phys.* **A39** (2006) 951–966, [math-ph/0508015].

[29] B. L. Feigin, A. M. Gainutdinov, A. M. Semikhatov, and I. Y. Tipunin, *Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center*, hep-th/0504093.

[30] B. L. Feigin, A. M. Gainutdinov, A. M. Semikhatov, and I. Y. Tipunin, *Kazhdan–Lusztig correspondence for the representation category of the triplet $\mathcal{W}$-algebra in logarithmic conformal field theory*, math.qa/0512621.

[31] M. Flohr and A. Müller-Lohmann, *Notes on non-trivial and logarithmic CFTs with $c = 0$*, *J. Stat. Mech.* **0604** (2006) P002, [hep-th/0510096].

[32] H. Eberle and M. Flohr, *Notes on generalised nullvectors in logarithmic CFT*, *Nucl. Phys.* **B741** (2006) 441–466, [hep-th/0512254].

[33] J. S. Caux, I. I. Kogan, and A. M. Tsvelik, *Logarithmic operators and hidden continuous symmetry in critical disordered models*, *Nucl. Phys.* **B466** (1996) 444–462, [hep-th/9511134].

[34] Z. Maassarani and D. Serban, *Non-unitary conformal field theory and logarithmic operators for disordered systems*, *Nucl. Phys.* **B489** (1997) 603–625, [hep-th/9605062].

[35] A. M. Ghezelbash and V. Karimipour, *Global conformal invariance in D dimensions and logarithmic correlation functions*, *Phys. Lett.* **B402** (1997) 282–289, [hep-th/9704082].

[36] J. Rasmussen, *Jordan cells in logarithmic limits of conformal field theory*, hep-th/0406110.

[37] J. Rasmussen, *On conformal Jordan cells of finite and infinite rank*, *Lett. Math. Phys.* **73** (2005) 83–90, [hep-th/0408029].

[38] G. Felder, J. Fröhlich, and G. Keller, *Braid matrices and structure constants for minimal conformal models*, *Commun. Math. Phys.* **124** (1989) 647–664.

[39] G. Felder, J. Fröhlich, and G. Keller, *On the structure of unitary conformal field theory. 2. Representation theoretic approach*, *Commun. Math. Phys.* **130** (1990) 1.

[40] M. Gaberdiel, *Fusion in conformal field theory as the tensor product of the symmetry algebra*, *Int. J. Mod. Phys.* **A9** (1994) 4619–4636, [hep-th/9307183].

[41] J. Lepowsky, *From the representation theory of vertex operator algebras to modular tensor categories in conformal field theory*, *Proc. Nat. Acad. Sci.* **102** (2005) 5304–5305, [math.qa/0504311].

[42] C. Schweigert, J. Fuchs, and I. Runkel, *Categorification and correlation functions in conformal field theory*, math.ct/0602079.

[43] W. Nahm, *Quasirational fusion products*, *Int. J. Mod. Phys.* **B8** (1994) 3693–3702, [hep-th/9402039].

[44] M. Gaberdiel, *Fusion rules of chiral algebras*, *Nucl. Phys.* **B417** (1994) 130–150, [hep-th/9309105].

[45] C. Bauer, A. Frink, and R. Kreckel, *Introduction to the* GiNaC *framework for symbolic computation within the C++ programming language*, *J. Symbolic Computation* **33** (2002) 1, [cs.sc/0004015]. The GiNaC library is available at http://www.ginac.de.

[46] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, *Nucl. Phys.* **B241** (1984) 333–380.

[47] P. D. Francesco, P. Mathieu, and D. Senechal, *Conformal field theory.* Springer, New York, 1997.

[48] B. L. Feigin and D. B. Fuks, *Verma modules over the Virasoro algebra*, *Funct. Anal. Appl.* **17** (1983) 241–242.

[49] A. Rocha-Caridi, *Vacuum vector representations of the Virasoro algebra*, in *Vertex Operators in Mathematics and Physics* (J. Lepowsky, ed.), p. 451, Springer, 1985.

[50] G. M. T. Watts, *A crossing probability for critical percolation in two dimensions*, *J. Phys.* **A29** (1996) L363, [cond-mat/9603167].

[51] J. Cardy, *Lectures on conformal invariance and percolation*, math-ph/0103018.

[52] M. A. I. Flohr and A. Müller-Lohmann, *Conformal field theory properties of two-dimensional percolation*, *J. Stat. Mech.* **0512** (2005) P004, [hep-th/0507211].

[53] V. Gurarie and A. W. W. Ludwig, *Conformal algebras of 2D disordered systems*, *J. Phys.* **A35** (2002) L377–L384, [cond-mat/9911392].

[54] V. Gurarie and A. W. W. Ludwig, *Conformal field theory at central charge* $c = 0$ *and two dimensional critical systems with quenched disorder*, hep-th/0409105.

[55] J. A. Minahan and K. Zarembo, *The Bethe-ansatz for* $N = 4$ *super Yang-Mills*, *JHEP* **03** (2003) 013, [hep-th/0212208].

[56] N. Beisert, V. Dippel, and M. Staudacher, *A novel long range spin chain and planar* $N = 4$ *super Yang-Mills*, *JHEP* **07** (2004) 075, [hep-th/0405001].