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Conformal Field Theory Properties of Two-Dimensional Percolation

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Abstract

G. M. T. Watts derived in his paper [20] that in two dimensional critical percolation the crossing probability Π_{hv} satisfies a fifth order differential equation which includes another one of third order whose independent solutions describe the physically relevant quantities $1, \Pi_h, \Pi_{hv}$.

We will show that this differential equation can be derived from a level three null vector condition of a rational $c = -24$ CFT and suggest a new interpretation of the generally known CFT properties of percolation.

1 A brief review of percolation properties

According to Langlands et al [15], critical percolation in two dimensions has interesting features in conformal field theory such as the conformal invariance of the three independant crossing probabilites $1, \Pi_h, \Pi_{hv}$. For Π_h, Cardy [5] was already able to derive an exact solution with the help of boundary

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conformal field theory which matches the numerical data to a high accuracy. Starting from this background, Watts [20] motivated how Π_{hv} can be expressed by a correlation function of boundary operators in the $Q \to 1$ limit of the Q-state Potts model and deduced a differential equation of fith order that agrees with the simulations. Additionally he observed that the three physically relevant solutions already satisfy a third order differential equation.

In the previous literature, several arguments have been given to describe the crossing probabilities in two dimensional critical percolation as conformal blocks of a four point correlation function of ($h = 0$)-operators in a $c = 0$ conformal field theory (CFT), using a second (third) level null vector to get Π_h (Π_{hv}). The most prominent are

(1.) (for $c = 0$) the Beraha numbers $Q = 4\cos^2\left(\frac{\pi}{n}\right)$ (with n usually denoted as $m + 1 = 2, 3, 4\dots$ which in most Potts models are related to the central charge by $c = 1 - \frac{6}{m(m+1)}$ [5]);

(2.) (for $c = 0$) the differential equation for Π_h can as well be derived by the Stochastic/Schramm Loewner Evolution (SLE) which strengthens the first argument;

(3.) (for $h = 0$) the ratio of the partition functions for free boundary conditions to $Z = 1$ of percolation (as suggested by Cardy [5]);

(4.) (for $c = h = 0$) the interpretation of the central charge as describing the finite size effects of the energy.

To understand the first point, we give a brief review on the Q-state Potts model. On a simply connected compact region with a piecewise differentiable boundary the horizontal crossing probability Π_h is defined through the partition function (Cardy [5], Wu [21], Kleban [12])

$$Z = \prod_{(r,r')} \left(1 + x\delta_{s(r),s(r')}\right) = \sum_G Q^{N_c} x^{N_b}, \tag{1}$$

where $x = \frac{p}{1-p}$ for $Q \to 1$ and the rightmost sum running over all possible graphs of N_b bonds in N_c clusters. By expanding it in powers of x we can extend the Q-state Potts model to $Q \in \mathbb{R}$.

Π_h describes the probability of having a connection from, e.g., one piece $X = (x_0, x_1)$ of the boundary to another disjoint part $Y = (x_2, x_3)$ where the spins are fixed to values α and β, respectively, while on the rest we have free boundary conditions (for a more detailed introduction see [3]). Hereby any region which can be mapped onto the real axis by a conformal transformation is equivalent (for corners we may get singular behavior but no discontinuities at the corresponding points). For $\alpha \neq \beta$, it is given by [12]

$$\Pi(X,Y) = \lim_{Q\to 1}\left(1 - \frac{Z_{\alpha\beta}}{Z_{\alpha\alpha}}\right). \tag{2}$$

In terms of boundary changing operators from free (f) to fixed (α,β) conditions, we get

$$Z_{\alpha,\beta} = Z_f\langle\phi_{(f|\alpha)}(x_0)\phi_{(\alpha|f)}(x_1)\phi_{(f|\beta)}(x_2)\phi_{(\beta|f)}(x_3)\rangle. \tag{3}$$

In the infinite volume limit, these quantities diverge for $Q \neq 1$, but by taking a closer look at the partition function of the Potts Model for $Q \to 1$, we find for a minimal model with central charge $c = 0$ the partition function to be $Z = 1$ in this limit.

For Π_h, the ϕ are $h_{1,2}$ boundary operators, while the results for Π_{hv} contain other boundary operators that can be identified by comparison with known Potts models (i.e. for $Q = 2, 3$) to have weight $h_{1,3}$. Another motivation for this ansatz can be found by letting the length of the segment with free boundary conditions tend to zero. Therefore we know from fusion rules, that

$$\phi_{(\alpha|f)} \times \phi_{(f|\beta)} \sim \delta_{\alpha\beta} + \phi_{(\alpha|\beta)} \tag{4}$$

which means that the fusion of two $\phi_{(1,2)}$ boundary operators yields a $\phi_{(1,3)}$ field (see Cardy [5], Kleban [12]).

So far, it seems very reasonable to choose $c = 0$ to describe percolation, but, unfortunately, a minimal model $c_{(3,2)} = 0$ is not very interesting, since its field content only consists of two $h = 0$ fields – $\phi_{(1,1)}$ and $\phi_{(1,2)}$. Thus just taking the limit $Q \to 1$ in the Q-state Potts Model can not provide us with the necessary boundary changing operators corresponding to a $\phi_{(1,3)}$ field as suggested by Cardy [2].

In fact, if we include the $\phi_{(1,3)}$ field into the spectrum of our conformal field theory with vanishing central charge, the partition function will not be equal to one. More precisely, including this field with conformal weight $h_{1,3} = 1/3$ into the spectrum leads to a logarithmic conformal field theory, [10, 11] and references therein. The representation with this conformal weight is indecomposable, containing an irreducible sub-representation with character

$$\chi_{1,3}(q) = \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}(2n+1)q^{3(4n+1)^2/8}\,, \tag{5}$$

where $\eta(q)$ denotes the Dedekind η-function $q^{1/24}\prod_{n\geq 1}(1-q^n)$. This logarithmic conformal field theory is a so-called augmented minimal model, and it is rational in the sense that it possesses only finitely many indecomposable or irreducible representations. However, the resulting modular invariant partition function for this model is, up to terms proportional to $\log(q\bar{q})$, given by the partition function of a $c = 1$ theory[1] with radius of compactification given by $2R^2 = 1/(2 \cdot 3) = 1/6$, namely

$$Z = \frac{1}{|\eta(q)|^2} \left(|\Theta_{0,6}(q)|^2 + 2\sum_{\lambda=1}^{5} |\Theta_{\lambda,6}(q)|^2 + |\Theta_{6,6}(q)|^2 \right) , \quad (6)$$

$$\Theta_{\lambda,k}(q) = \sum_{n\in\mathbb{Z}} q^{(2kn+\lambda)^2/4k} . \quad (7)$$

The logarithmic corrections cannot be fixed in magnitude by the requirement of non-negative integer coefficients in their respective q-expansions, but we mention for completeness that

$$Z_{\rm full}[\alpha,\beta] = Z + \alpha \frac{\log(q\bar{q})}{|\eta(q)|^2} \sum_{\lambda=1}^{5} |(\partial\Theta)_{\lambda,6}(q)|^2 + \beta \log(q\bar{q})^2 |E_2(q)|^2 , \quad (8)$$

$$(\partial\Theta)_{\lambda,k}(q) = \sum_{n\in\mathbb{Z}} (2kn+\lambda) q^{(2kn+\lambda)^2/4k} , \quad (9)$$

and $E_2(q)$ is the Eisenstein series of modular weight two. Such modular invariants can be found by solving the modular differential equation, which must be satisfied by any finite-dimensional representation of the modular group in terms of modular functions (with multiplicative systems). Usually, it suffices to know one character of the conformal field theory, e.g. the vacuum character, and the spectrum, i.e. the conformal weights of all admissible irreducible or indecomposable representations [8, 9]. In any case, including the field $\phi_{(1,3)}$ from the boundary of the Kac-table of the $c_{(3,2)} = 0$ minimal model results in an enlarged theory with partition function definitely not being equal to one.

Now we will take a look at the second argument for $c = 0$ from Stochastic/Schramm Loewner Evolution (SLE). Apparently, there are no results for the horizontal-vertical-crossing since up to now SLE has only been considered for $\phi_{(1,2)}$ operators which are not feasible for Π_{hv}. Although the issues discussed above concerning the insufficient field content of the minimal model with $c = 0$ do not apply within the SLE setting, we will show that SLE does not necessarily force us to take a CFT with vanishing central charge $c = 0$.

Cardy [4] described, how SLE can be applied to calculate crossing probabilities. Simply speaking, a path evolves by a Brownian motion of speed

[1] Note that the *effective* central charge of this model is $c_{\rm eff} = c - 24h_{\rm min} = 1$.

$\kappa = 6$ which repeatedly hits the real axis. In a configuration where the motion starts from a point a_0 on the real axis running all over the complex upper half plane with $x_1 < a_0 < x_2$ being the end points of the crossing intervals, one of the points will be "swallowed" first. For x_1 being the first to be hit by the graph, there obviously exists a free path along the outer line of the graph, for x_2 it is quite as obvious that this is not the case. Thus the probability that there is a crossing between (a_0, x_2) to $(-\infty, x_1)$ is given by a Bessel process, described by a differential equation

$$\left(\frac{2}{x_1 - a_0}\frac{\partial}{\partial x_1} + \frac{2}{x_2 - a_0}\frac{\partial}{\partial x_2} + \frac{\kappa}{2}\frac{\partial^2}{\partial a_0^2}\right) P(x_1, x_2; a_0)\,. \tag{10}$$

From translational invariance we get $\partial a_0 = -\partial x_1 - \partial x_2$ and from conformal invariance, we know, that P is a function of the ratio $\eta = \frac{x_2 - a_0}{x_1 - a_0}$. This is exactly the same differential equation one yields with CFT for percolation from a two level null vector [5]. There is also a general expression, relating the speed of the Brownian motion κ to the central charge and thus the highest weight states of the Virasoro algebra (i.e. [1], [4])

$$c^\kappa = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}\,, \tag{11}$$

$$h^\kappa_{r,s} = \frac{(r\kappa - 4s)^2 - (\kappa - 4)^2}{16\kappa}\,. \tag{12}$$

Hence, $c = 0$ and $h_{1,2} = 0$ for $\kappa = 6$ which has been shown to describe Π_h in two dimensional critical percolation [18]. Additionally, Bauer and Bernard [1] stated a direct correspondence between the Q-state Potts model and SLE

$$Q = 4\cos^2\left(\frac{4\pi}{\kappa}\right)\,, \qquad \kappa \geq 4, \tag{13}$$

by matching the known value of the dimension of the boundary changing operator for the Q-state Potts model with $h^\kappa_{1,2}$.

The third argument makes use of the form of the partition function of the $c = 0$ model. But as we already have shown, the partition function for the augmented $c = 0$ model is not the same as for the minimal $c = 0$ model and thus especially not equal to unity. From this argument, we will show, that we do not longer have to choose $h = 0$ operators as suggested by Cardy [5].

Regarding the problem mentioned above with only a single region with fixed boundary conditions, in the $Q \to 1$ limit, we have

$$Z_\alpha = Z_f\langle\phi_{(f|\alpha)}(x_0)\phi_{(\alpha|f)}(x_1)\rangle = Z_f \times (x_0 - x_1)^{2h}. \tag{14}$$

In the minimal model, both partition functions are equal to unity, thus $h = 0$, but in the extended model, we do not know the exact form of Z_f, hence the boundary operator is not a priori fixed in its dimension.

The last point addresses the transformation back onto the original region that is described by the formula

$$\langle \phi_0(w_0)\phi_1(w_1)\ldots\rangle = \prod_i |w'(z_i)|^{-h_i}\langle \phi_0(z_0)\phi_1(z_1)\ldots\rangle. \tag{15}$$

The expression has a physical meaning in the general non scale invariance of critical systems which picks up a factor $(L/L_0)6ac$ with L being the overall size of the region, L_0 some non universal microscopic scale (i.e. the lattice spacing), c the (effective) central charge and a being dependent on the geometry (i.e. $a = -\pi/\gamma$ if the boundary operator sits in a corner with an interior angle γ, see [5],[12]). Since percolation is assumed to be scale invariant, the effect of the conformal mapping should vanish. But the physical properties of our system only depend on the differential equation arising from null vectors, thus this condition only has to hold in this sense.

We remark here that the above argument of finite size scaling effects relies on an analysis of the asymptotic behavior of the partition function. This behavior, however, depends on the central charge only modulo 24. Moreover, invariance of the correlation functions holds in any conformal field theory, as long as the Jacobian transformation factors are properly accounted for. We will see below that within our proposal, where the crossing probabilities are obtained from a CFT with non-vanishing central charge, we have quotients of correlation functions such that the final expressions have all desired properties.

Recapitulating, we state that the assumptions on percolation should be reconsidered, since most arguments do not seem to be as strict as stated before, i.e. the central charge arguments most times refer to an effective central charge $c_{\text{eff}} = c - 24h_{\text{min}}$ where h_{min} is the weight of the ground state. Thus $c_{\text{eff}} > c$ in the case of non-unitary theories with negative weights. Thus, the arguments for $h = 0$ are either problematic due to the $c = 0$ minimal model being nearly empty or are connected with the central charge. Hence, once we agree on the proposal that we should work with the augmented, and therefore non-unitary, $c = 0$ model, we also have to deal with the effective charge in that model – which is the same for both theories,

$$c_{(6,1)} = -24 \equiv 0 = c_{(3,2)} \bmod 24\,. \tag{16}$$

2 The Watts differential equation

As already mentioned, Watts [20] derived a fifth order differential equation for Π_{hv}, starting from a $c = 0$ theory with $h_{1,2} = 0$ boundary changing operators following Cardy's ansatz for Π_h. But obviously, it is not so easy to derive the null vectors in the $c = 0$ theory, since if we say that $L_{-1}|0\rangle$ is a null state, than by just taking the generic form of a level two null state with $c_{(3,2)} = 0$, $(L_{-1}^2 - 2/3L_{-2})|0\rangle$, we see that L_{-2} is a null vector, too, and everything vanishes. Consequently, there can be no "direct" null vector on the fifth level whatsoever. Interestingly, the ansatz of a level three null vector acting on a weight 2 field yields a correct differential equation for the horizontal-vertical-crossing probability in percolation. In a $c = 0$ theory, it seems strange, that in contrary to the results for Π_h, the Π_{hv} boundary operators cannot be identified directly [12]. Considering the asymptotic behavior, one can find the correct expressions for Π_h and Π_{hv} [13] by taking linear combinations of the three physically relevant solutions of

$$\frac{d^3}{dx^3}(x(x-1))^{\frac{4}{3}}\frac{d}{dx}(x(x-1))^{\frac{2}{3}}\frac{d}{dx}F(x)\,, \tag{17}$$

where x is the crossing ratio and F the conformally mapped crossing probability. The equation factorizes into [13]

$$\left(\frac{d^2}{dx^2}(x(x-1)) + \frac{1}{2x-1}\frac{d}{dx}(2x-1)^2\right)\frac{d}{dx}(x(x-1))^{\frac{1}{3}}\frac{d}{dx}(x(x-1))^{\frac{2}{3}}\frac{d}{dx}F(x)\,, \tag{18}$$

where the rightmost part already provides us with the three expected solutions for the crossing probabilities in percolation.

This third order differential equation has neither direct interpretation as a third level null vector in a $c = 0$ theory (more precisely: there is no such vector in this theory), nor does it arise from $h = 0$ boundary operators. In contrary, we will show, that we obtain it from the null vector of an $h_{1,3} = -\frac{2}{3}$ field acting on a correlator containing $h_1 = h_2 = h_{1,3} = -\frac{2}{3}$ and $h_3 = 1$, which, for a level three null vector condition, is a unique solution.

The level 3 null vector has for $t = \frac{1}{2}(h+1) = p/q$ the generic form [17]

$$|\chi_h^{(3)}\rangle = \left(L_{-1}^3 - 2(h+1)L_{-1}L_{-2} + (h+1)(h+2)L_{-3}\right)|h\rangle\,.$$

We will be a little bit more elaborate on this subject, since there are many errors in the equations found in the canonical literature (i.e. see [6] on pages 288).

Transforming this expression into a differential operator made out of the $\mathcal{L}_{-n}$ defined by

$$\mathcal{L}_{-n}(z) = \sum_i \left(\frac{(n-1)h_i}{(z_i - z)^n} - \frac{1}{(z_i - z)^{n-1}} \partial_{z_i} \right) , \tag{19}$$

acting on a 4-point function

$$F(z, z_1, z_2, z_3) \equiv \langle \phi_h(z) \phi_{h_1}(z_1) \phi_{h_2}(z_2) \phi_{h_3}(z_3) \rangle , \tag{20}$$

yields a quite lengthy expression. Replacing again all derivatives ∂_{z_i} by expressions only containing the derivative ∂_z with respect to z and finally putting $\{z_1, z_2, z_3\} \mapsto \{0, 1, \infty\}$, results in the following ordinary third order differential equation for $F(z) \equiv F(z, 0, 1, \infty)$:

$$\begin{aligned} 0 \;=\; & \frac{\mathrm{d}^3}{\mathrm{d}z^3} F(z) + 2(h+1) \frac{2z-1}{z(z-1)} \frac{\mathrm{d}^2}{\mathrm{d}z^2} F(z) \\ + \; & (h+1) \left(\frac{h - 2h_1}{z^2} + \frac{h - 2h_2}{(z-1)^2} - 2 \frac{h_3 - h - h_1 - h_2}{z(z-1)} + \frac{h}{z(z-1)} \right) \frac{\mathrm{d}}{\mathrm{d}z} F(z) \\ + \; & h(h+1) \left(-\frac{2h_1}{z^3} - \frac{2h_2}{(z-1)^3} + \frac{(2z-1)(h + h_1 + h_2 - h_3)}{z^2 (z-1)^2} \right) F(z) \, . \end{aligned} \tag{21}$$

Comparing this result to a simplified version of the differential equation given by Watts [20]

$$\left(\frac{\mathrm{d}^3}{\mathrm{d}z^3} + \frac{5(2z-1)}{z(z-1)} \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{4}{3z(z-1)} \frac{\mathrm{d}}{\mathrm{d}z} \right) F(z) = 0 \, . \tag{22}$$

we know that this equation should be reproduced by (21) for an appropriate choice of h, h_1, h_2, h_3. However, (22) does not possess a term proportional to F itself (not to one of its derivatives). Clearly, in this form, this could only be the case for $h = 0$ or $h = -1$. One can easily see, that there are no triples $\{h_1, h_2, h_3\}$ for these values of h such that (21) becomes equivalent to (22). But there is a simple and natural way out, since we know something about the generic form of a 4-point function of four primary fields. For example, any function $F(z, 0, 1, \infty)$, which is invariant under global conformal transformations, must be of the form $F(z) = z^{\mu_{01}} (z-1)^{\mu_{02}} f(z)$. Using such an ansatz in (21) and pulling the differential operators through the pre-factor yields a modified differential equation for $f(z)$. Nicely, $f(z)$ satisfies exactly (22), if we put $h = h_1 = h_2 = -2/3$ and $h_3 = -1$. This implies $c = -24$, since then the representation with highest weight $h = -2/3$ indeed possesses a null vector at level 3. Furthermore, the exponents $\mu_{01} = \mu_{02} = 1/3$ are exactly what one expects from the generic solution $\mu_{ij} = \frac{1}{3}\sum_k h_k - h_i - h_j$ of $\sum_{j\neq i}\mu_{ij} = -2h_i$, i.e. $(-2/3-2/3-2/3-1)/3+2/3+2/3 = -1+4/3 = 1/3$. To summarize, the conformal blocks of the 4-point function

$$\langle\Phi_{h=-2/3}(z)\Phi_{h_1=-2/3}(0)\Phi_{h_2=-2/3}(1)\Phi_{h_3=-1}(\infty)\rangle = z^{\mu_{01}}(1-z)^{\mu_{02}}f(z) \quad (23)$$

of the $c = -24$ theory are in one-to-one correspondence with the solutions of Watts' differential equation.

As a concluding remark we note that $h = -2/3$ corresponds to a reducible but indecomposable representation of the $c_{(6,1)} = -24$ theory. Hence, it is natural and inevitable, that correlation functions involving more than one field of this type will contain conformal blocks with logarithmic divergencies. Indeed, the Watts fifth-order differential equation has three solutions plus two with logarithmic divergencies. However, the third-order equation has only three regular solutions, which is in agreement with the fusion rules of this logarithmic CFT, where the irreducible sub-representation with highest weight $h = -2/3$ satisfies $[-2/3]*[-2/3] = [0]+[-2/3]+[1]$. Further details will be worked out later.

Additionally, $c = -24$ has a very interesting field content. Taking a look at the relevant entries of the Kac-Table (only the first row is needed, since the conformal grid of this model is obtained by formally considering the conformal grid for $c_{(18,3)} = c_{(3\cdot 6,3\cdot 1)}$)

0	$-\frac{3}{8}$	$-\frac{2}{3}$	$-\frac{7}{8}$	-1	$-\frac{25}{24}$	-1	$-\frac{7}{8}$	$-\frac{2}{3}$	$-\frac{3}{8}$	0	$\frac{11}{24}$	1	$\frac{13}{8}$	$\frac{7}{3}$	$\frac{25}{8}$	4

we encounter that the critical exponents that are assumed to come up in percolation appear shifted by 1, i.e. $h_{1,2} = -\frac{3}{8}$ and $h_{1,4} = -\frac{7}{8}$. Thus descendants of those fields could describe the physical properties of percolation. It should be mentioned that the $h_{0,0} = h_{1,6} = -\frac{25}{24}$ field appears in the table as well which is important for it not to vanish as it would in ordinary $c_{(t,1)}$ minimal models [8, 9, 14]. This field is the so-called pre-logarithmic field whose operator product expansion with itself gives rise to the indecomposable representations. All conformal weights which appear twice in the above table belong to such indecomposable representations.

Further support for our conjecture that the rational logarithmic conformal field theory with central charge $c = c_{(6,1)} = -24$ might describe percolation is given by the following remarkable observation. The partition function of this theory is equivalent to the partition function eq. (8) of the extended $c = 0$ theory discussed above. More precisely, we have [8, 9] that

$$Z_{c_{(6,1)}=-24}[\alpha] = Z_{\text{full}}[\alpha, \beta = 0]\,. \quad (24)$$

Therefore, the non-logarithmic parts of the two partition functions, which actually count the states, are identical.

On the other hand, this is not entirely surprising. Many arguments, which favour a conformal field theory with vanishing central charge for the description of two-dimensional percolation, rely on the modular properties of the partition function. These properties cannot fix the central charge uniquely, but only modulo 24. Surely enough, $c_{(6,1)} = -24 \equiv 0 = c_{(3,2)}$ mod 24, and the effective central charges are equal to one for both theories.

If we still want to describe percolation as a $c = 0$ theory and still do not want to reject the interpretation of Watts' differential equation as a Level three null vector, we may construct a tensorized CFT consisting of the $c = -24$ and a $c = 24$ part. Therefore, any correlation function or field factorizes into two parts, one for each of the two CFTs, i.e. $\Phi_H(z) = \Phi_{h,c=-24}(z) \otimes \Phi_{H-h,c=+24}(z)$. However, since the 4-point function

$$F_{c=-24}(z) = \langle \Phi_{-2/3}(z)\Phi_{-2/3}(0)\Phi_{-2/2}(1)\Phi_{-1}(\infty)\rangle_{c=-24}$$

already yields as solutions the desired crossing probabilities, the second factor,

$$G_{c=+24}(z) = \langle \Phi_h(z)\Phi_{h_1}(0)\Phi_{h_2}(1)\Phi_{h_3}(\infty)\rangle_{c=+24}$$

should be trivial. To make the picture perfect, we could try to achieve

$$f(z) = F_{c=-24}(z)G_{c=+24}(z) \qquad \Longrightarrow \qquad G_{c=+24}(z) = z^{-1/3}(z-1)^{-1/3}\,.$$

The easiest way to get that result is to assume that $G(z)$ is, essentially, a 3-point function $\langle \Phi_{1/3}(z)\Phi_{1/3}(0)\Phi_{1/3}(1)\mathbb{I}(\infty)\rangle_{c=-24}$. It remains to clarify whether such a correlator exists and is non-vanishing in a $c = +24$ theory.

But what about the results already derived and proven consistent with numerical simulations for Π_h if percolation was described by a $c_{(1,6)} = -24$ theory?

As already mentioned above, the horizontal crossing probability is determined by a second order differential equation interpreted as a level two null vector condition arising from $\phi_{(1,2)}$ which has the weight $h = h_{1,2} = -\frac{3}{8}$ in this case. Therefore we have to solve

$$\left(\frac{1}{t}\frac{d^2}{dz^2} + \frac{2z-1}{z(z-1)}\frac{d}{dz} - \frac{h_1}{z^2} - \frac{h_2}{(z-1)^2} + \frac{h+h_1+h_2-h_3}{z(z-1)}\right)F(z) = 0. \tag{25}$$

with $t = \frac{p}{q}$ being related to the central charge and thus determining h.

From the numerical simulation of Langlands et. al. [15] we know, that Cardy's formula [5] for Π_h derived from a level two null vector in a $c = 0$ minimal model should be the outcome. Thus we know that $F(z)$ should be of the form ${}_2F_1(1/3, 2/3, 4/3, z)$. A simple calculation yields $h_1 = h_2 = h_3 = h_{1,4} - \frac{7}{8}$ and $F_1(z) = (z(z-1))^{\frac{1}{4}} \cdot z^{\frac{1}{3}} {}_2F_1(1/3, 2/3, 4/3, z)$ as well as $F_2(z) = (z(z-1))^{\frac{1}{4}}$ as the second solution. Hence in comparison to Cardy, the crossing probability for percolation is given by their quotient F_1/F_2. Now our solution for Π_h has exactly the same properties as described in [15] and thus is zero for $z \to 1$ and one for $z \to 0$, as desired.[2] The normalization is obtained by considering the identity

$$\frac{3\Gamma\left(\frac{2}{3}\right)}{\Gamma^2\left(\frac{1}{3}\right)} {}_2F_1(1/3, 2/3, 4/3, z) = 1 - \frac{3\Gamma\left(\frac{2}{3}\right)}{\Gamma^2\left(\frac{1}{3}\right)}(1-z)^{\frac{1}{3}} {}_2F_1(1/3, 2/3, 4/3, 1-z). \tag{26}$$

Hence the correct normalization constant must be $\frac{3\Gamma\left(\frac{2}{3}\right)}{\Gamma^2\left(\frac{1}{3}\right)}$.

This result is remarkable, since it contains the two fields for critical exponents in percolation, i.e. $h_{1,2} = -\frac{3}{8}$ and $h_{1,4} = -\frac{7}{8}$.

Another important thing to be considered are the results of SLE for percolation, showing the equivalence of Cardy's formula and the results for $\kappa = 6$. At first we have to state that the frequently cited proof of Smirnow [18] only holds for site percolation on a triangular lattice, and according to himself and Werner [19], the method used in [18] can not be applied directly to bond percolation on the square lattice as discussed in this paper. Additionally, we know that at one point in the derivation of the differential equation for the SLE_κ-process, namely the identification evolution operator $\mathcal{A}$ with a level two null vector of a CFT [1], the assumption, that $h_{1,2} = 0$ is made. It has consequences on the relation between the coefficients of the differential equation (κ, c and $h_{1,2}$) and the evolution operator,

$$\mathcal{A} = -2L_{-2} + \frac{\kappa}{2}L_{-1}^2 \quad vs. \quad L_{-2} - \frac{3}{2(2h_{1,2}+1)}L_{-1}^2. \tag{27}$$

Hence, we know that

$$\frac{\kappa}{4} = \frac{3}{2(2h_{1,2}+1)}. \tag{28}$$

Obviously, this leaves us with $\kappa = 6$ if we restrict ourselves to $h = 0$ in our ansatz for percolation (or equivalently $c_{(3,2)} = 0$ which means $\frac{3}{2(2h_{1,2}+1)} = \frac{1}{t} = \frac{q}{p} = \frac{3}{2}$). But since there are no compulsory conditions to justify this ansatz as explained before, we may question why we should not try $h = -\frac{3}{8}$ and

[2] This means that if we consider a rectangle whose corners are mapped clockwise in decreasing order to the z_i with $r := (z_3 - z_0)/(z_1 - z_0)$, $r \to 0$ and $r \to \infty$, respectively. Note that $0 < z < 1$ and therefore the correct mapping on the upper complex plane is taking $z_0 \to z$, $z_1 \to 0$, $z_2 \to \infty$ and $z_3 \to 1$.

thus $\kappa = 24$ or $h = 4$ and $\kappa = \frac{2}{3}$. Unfortunately, up to now there has been no investigation of these SLEs.

There is, however, one possibility to try to elucidate this question further. In [16], a generalization of the SLE process related to percolation is proposed, which yields generalized probabilities, depending on a parameter b and given by the formula

$$\Pi(b;z) = z^{b+\frac{1}{6}}{}_2F_1(\frac{1}{6}+b, \frac{1}{2}+b; 1+2b; z)\,. \tag{29}$$

Obviously, $b = 1/6$ reproduces the case relevant for percolation, and thus this is referred to as a generalization of Cardy's formula. It is clear that (29) cannot be given in terms of 4-point functions for all values of b for one and the same CFT with fixed central charge, But we can try to check, whether (29) can be reproduced by 4-point functions of CFTs whose central charges c depend on the choice b. We restrict ourselves to the case of positive rational $b \in \mathbb{Q}$, $b = p/q$. We then further require that all four fields in the correlator shall be degenerate primary fields, i.e. have conformal weights $h_{r,s}(c)$ from the Kac-table.

Thus, we have to match the general solution of the second-order differential equation (25) for a level two null field with the desired expression (29). This leads to the result

$$F(z) = [z(1-z)]^{-\frac{2}{3}h}\Pi(b;z)\,, \qquad h_1 = \tfrac{36b^2-(4h-1)^2}{24(2h+1)}\,, \qquad h_2 = h_3 = -\tfrac{h(2h-1)}{3(2h+1)}\,, \tag{30}$$

where we used that $t = \frac{3}{2}(2h+1)^{-1}$. Now, $h = h_{1,2}$ is a member of the Kac-table by construction, but we have to check, whether h_1 and h_2 can also be chosen from the same Kac-table, since c is already fixed by the choice of h via $c = 13 - 6(t + 1/t)$. Let us assume that $b = p/q > 0$, and that $h_2 = h_{r,s}$, $h_3 = h_{r',s'}$. Plugging $h = h_{1,2}$ into the solutions for h_1 and h_2, and then solving for s and s', respectively, leads to the diophantine equations

$$s = t\left(r \pm 2\frac{p}{q}\right)\,, \qquad s' = t\left(r' \pm \frac{1}{3}\right)\,. \tag{31}$$

There are various solutions to this, but clearly $t = \mathrm{lcm}(3, pq)$ and thus $c = c_{(t,1)}$ will it always make possible to find positive r, r', s, s' such that all conformal weights are from the Kac-table.

Finally, we observe that $F(z)$ is only proportional to the desired quantity $\Pi(b;z)$. Again, we would like to have that the quotient of the two conformal blocks, or correlations functions, gives the probability, $\Pi(b;z) = F_1(z)/F_2(z)$. To this end, we would need that $F_1 = F$ and that F_2 is simply $F_2(z) = [z(1-$

$z)]^{-\frac{2}{3}h}$. This is possible, if the charge balance, in a free field realisation of the CFT, adds up to the background charge, such that no screening integrations, which lead to a non-trivial $F_2(z)$, are necessary. This yields us a further condition, since the charges are

$$\alpha_{r,s} = \frac{1}{2}(r-1)\sqrt{t} + \frac{1}{2}(1-s)\frac{1}{\sqrt{t}}\,, \qquad \alpha_0 = \frac{1}{2}\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\,. \tag{32}$$

We must have $\alpha_{1,2} + \alpha_{r,s} + 2\alpha_{r',s'} = 2\alpha_0$. There is no good solution to this, but one easily can check that $\alpha_{1,2} + 3\alpha_{r',s'} = 2\alpha_0$ is automatically fulfilled. We therefore arrive at the result that for all $b = p/q > 0$, a logarithmic CFT with central charge $c = c_{(t,1)}$, $t = \mathrm{lcm}(3, pq)$, reproduces the generalized version of Cardy's formula as follows: Since t is always divisible by three, we put $t = 3t'$, $t' \in \mathbb{N}$, and have

$$\Pi(b;z) = \frac{\langle \phi_{(1,2)}(z)\phi_{(1,3t'(1\pm 2b))}(0), \phi_{(1,2t')}(1)\phi_{(1,2t')}(\infty)\rangle}{\langle \phi_{(1,2)}(z)\phi_{(1,2t')}(0), \phi_{(1,2t')}(1)\phi_{(1,2t')}(\infty)\rangle}\,. \tag{33}$$

Note that $3t'(1\pm 2b)$ is always an integer. For $b < 1$, we can choose the minus sign, otherwise, we should choose the plus sign. Both cases are within the augmented Kac-table for the rational logarithmic models with central charge $c_{(3t',1)}$.

Interestingly, the known solution for $b = 1/6$ in terms of a CFT with $c_{(3,2)} = 0$ cannot be extended in a unified fashion to a series of CFTs for all rational b. Although this is no rigorous proof, this result might indicate that our proposal is more natural.

3 Conclusion and perspective

In this paper, we have shown that if we want to describe two dimensional bond percolation within a conformal field theory, using a level three null vector condition to get a differential equation for horizontal-vertical crossing probability Π_{hv} that fits the numerical data, we have to take $c = -24$. This solution is unique.

Additionally, there are no strict arguments contradicting our result, even not from the derivation of the horizontal crossing probability Π_h whose form has already been proven in the literature, since it can be explained in our $c = -24$ CFT proposal as well. Hence the question remains if we should consider percolation being rather a $c = -24$ than the commonly assumed $c = 0$ theory. Although we have presented several arguments indicating that our proposal is more natural, and that some arguments in favour of the $c = 0$

theory are problematic (particularly the serious issue of having a partition function $Z = 1$ and simultaneously a $\phi_{(1,3)}$ field in the spectrum of the $c = 0$ theory), we do not have a strict proof for our solution.

But there are still open questions that arise when considering SLE. The two most obvious are

- Can we describe the horizontal vertical crossing probability using the SLE formalism?

- Is there an SLE corresponding to bond percolation on the square lattice? If yes, what are its properties? Is the proof explicit in both directions? Does it endorse or destroy the ansatz of $c = -24$?

Besides the discussion whether one or the other ansatz is correct, another important issue is to investigate in more detail the close relationship between conformal field theories whose central charges differ by multiples of 24, especially why $c = -24$ and $c = 0$ have so many similar properties concerning percolation.

Note added: After completion of this work, we were kindly informed that our first question has been answered recently by Julien Dubedat [7]. There is a rigorous derivation of Watts' crossing formula in terms of certain SLE processes which depend on more parameters than the classical κ-SLE mentioned above. This immediately raises the new question whether such generalized SLE processes have a description in terms of conformal field theory. Perhaps, our work might hint towards a positive answer to this question.

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