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Fusion & Tensoring of Conformal Field Theory and Composite Fermion Picture of Fractional Quantum Hall Effect

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Abstract

We propose a new way for describing the transition between two quantum Hall effect states with different filling factors within the framework of rational conformal field theory. Using a particular class of non-unitary theories, we explicitly recover Jain's picture of attaching flux quanta by the fusion rules of primary fields. Filling higher Landau levels of composite fermions can be described by taking tensor products of conformal theories. The usual projection to the lowest Landau level corresponds then to a simple coset of these tensor products with several U(1)-theories divided out. These two operations – the fusion map and the tensor map – can explain the Jain series and all other observed fractions as exceptional cases. Within our scheme of transitions we naturally find a field with the experimentally observed universal critical exponent 7/3.

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1 Introduction

It is well known that the wave functions of quantum Hall effect (QHE) states can be recovered from correlation functions of chiral conformal field theories (CFTs) [6, 16, 27]. We first review this principle with the slight generalization of an arbitrary background charge. Let $\phi(z)$ be a chiral scalar free field. Then $j = \partial\phi$ obeys an $\mathfrak{u}(1)$-Kac-Moody algebra

$$[j_m, j_n] = n\delta_{m+n,0}\,, \tag{1.1}$$

which is known to describe the chiral edge waves, i.e. the energy gapless excitations of the QHE states. Introduce a Virasoro field by the Sugawara construction

$$L(z) \ = :jj:(z) + \sqrt{2}\alpha_0\partial_z j(z)\,, \tag{1.2}$$

where $:\ldots:$ means normal ordering. The Virasoro algebra has then central charge $c = 1 - 24\alpha_0^2$ with $2\alpha_0$ the background charge. The local primary fields are constructed from vertex operators ψ_α of conformal weight $h(\alpha) = \alpha^2 - 2\alpha\alpha_0$, explicitly

$$\psi_\alpha = \exp\left(-\sum_{n>0}\sqrt{2}\alpha j_n\frac{z^n}{n}\right)\exp\left(-\sum_{n<0}\sqrt{2}\alpha j_n\frac{z^n}{n}\right)c(\alpha)z^{-\sqrt{2}\alpha\alpha_0}\,, \tag{1.3}$$

where $c(\alpha)$ commutes with all $j_n, n \neq 0$ and maps highest-weight states into highest-weight states. The vertex operators are well defined if the charges α are restricted to the set $\alpha_{r,s} = \frac{1}{2}(1-r)\alpha_- + \frac{1}{2}(1-s)\alpha_+$ with $r, s \in \mathbb{Z}$ and $\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1}$. A more careful study shows that under particular circumstances CFTs can be consistent and well defined local primary fields can exist even with certain rational values of r, s. This precisely happens [7] if $\frac{1}{4}\alpha_0^2 \in \mathbb{Z}_+$. In the following we denote the local primary fields and their chiral parts as $\Phi_{(n,n'|\bar{n},\bar{n}')}(z,\bar{z}) \equiv \Phi_{h_{n,n'},\bar{h}_{\bar{n},\bar{n}'}}(z,\bar{z}) = \Phi_{n,n'}(z) \otimes \Phi_{\bar{n},\bar{n}'}(\bar{z})$, where the indices may be rational.

Consider a generic correlation function of chiral vertex operators (1.3) on the plane. One has the well known result

$$\langle\Omega^*_{2\alpha_0}, \psi_{\alpha_1}(z_1)\ldots\psi_{\alpha_N}(z_N)\Omega_0\rangle = \prod_{i<j}(z_i - z_j)^{2\alpha_i\alpha_j}\,, \tag{1.4}$$

if $|z_1| > \ldots > |z_N|$ and $\sum_{i=1}^N \alpha_i = 0$, where Ω_α denotes the ground state to the superselection sector of charge α. To reproduce the non-holomorphic parts of e.g. the Laughlin wave functions [24]

$$\Psi_{\nu=\frac{1}{2p+1}} = \prod_{i<j}(z_i - z_j)^{2p+1}\exp\left(-\frac{1}{2}\sum_i |z_i|^2\right)\,, \tag{1.5}$$

we insert a term $\exp(-i\alpha\int d^2z'\bar{\rho}\phi(z'))$ into the correlator (1.4), where ϕ is again the free field and $\bar{\rho}$ is an averaged density $(\pi\alpha^2)^{-1}$. If one integrates this term over a disk of area $2\pi\alpha^2 N$, then the real part correctly yields the desired exponential term for N electrons, while the imaginary part contributes a singular phase. The latter can be eliminated by an also singular gauge transformation corresponding to the uniform external magnetic field [27]. In the following we will often neglect the exponential term and absorb the external magnetic field in $\Omega^*_{2\alpha_0}(N)$, since the integral also modifies the background charge.

Unfortunately we must bother now the reader by sketching briefly a very particular set of *non-unitary* rational conformal field theories (RCFTs) – the ones with $\frac{1}{4}\alpha_0^2 \in \mathbb{Z}_+$ – since they are at the very heart of the paper.

Thus, let $\varepsilon \in \{0,1\}$ and let $\min\lambda$ denote the smallest representative of $\lambda \in \mathbb{Z}/m\mathbb{Z}$. Then for every k there exist two RCFTs, one with

$$\begin{aligned} c &= 1-24k\,, & k \in \mathbb{Z}_+/2 \\ h_{\frac{\lambda}{2k+2\varepsilon},(-)^\varepsilon\varepsilon\frac{\lambda}{2k+2\varepsilon}} &= \left[\left(\frac{\min\lambda}{2k+2\varepsilon}\right)^2 - 1\right]k + \varepsilon\left(\frac{\min\lambda}{2k+2\varepsilon}\right)^2, & \lambda \in \mathbb{Z}/(k+\varepsilon)\mathbb{Z} \\ h_{1,1} &= 0 \end{aligned} \tag{1.6}$$

which has the extended chiral symmetry algebra $\mathcal{W}(2,3k)$, and its $\mathbb{Z}_2$ orbifold

$$\begin{aligned} c &= 1-24k\,, & k \in \mathbb{Z}_+/4 \\ h_{\frac{\lambda}{4k+4\varepsilon},(-)^\varepsilon\frac{\lambda}{4k+4\varepsilon}} &= \left[\left(\frac{\min\lambda}{4k+4\varepsilon}\right)^2 - 1\right]k + \varepsilon\left(\frac{\min\lambda}{4k+4\varepsilon}\right)^2, & \lambda \in \mathbb{Z}/(4k+4\varepsilon)\mathbb{Z} \\ h_{1,1} &= 0 \\ h_{2,2} &= 3k \end{aligned} \tag{1.7}$$

with chiral symmetry algebra $\mathcal{W}(2,8k)$. Here $h_{r,s}$ denotes the Virasoro highest weight analogous to the Virasoro highest weights of degenerate models to generic central charge $c = 1-24\alpha_0^2$ given by $h_{r,s} = \frac{1}{4}\left((r\alpha_- + s\alpha_+)^2 - (\alpha_- + \alpha_+)^2\right)$. Note that in contrast to the generic degenerate Virasoro model $s = \pm r$ and r is not restricted to integers only. For further details see [3, 7]. All these RCFTs have effective central charge $c_{\mathit{eff}} = c - 24h_{min} = 1$.

The finitely many highest weight representations are highest weight representations with respect to the extended symmetry algebra. The characters which are infinite sums of Virasoro characters can be expressed in terms of Jacobi-Riemann Θ-functions divided by the usual Dedekind η-function. For example the vacuum character of the $\mathcal{W}(2,3k)$ theories is given by

$$\chi_{vac}(\tau) = \sum_{n\in\mathbb{Z}_+} \chi^{Vir}_{h_{n,n}}(\tau) = q^{\frac{1-c}{24}} \sum_{n\in\mathbb{Z}_+} \frac{q^{h_{n,n}} - q^{h_{n,-n}}}{\eta(\tau)} \tag{1.8}$$

$$= \frac{1}{2\eta(\tau)}\left(\Theta_{0,k}(\tau) - \Theta_{0,k+1}(\tau)\right), \tag{1.9}$$

where $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, $\Theta_{\lambda,k}(\tau) = \sum_{n\in\mathbb{Z}} q^{(2kn+\lambda)^2/4k}$, and $q = e^{2\pi i\tau}$. The other characters can be obtained by the modular transformation $S : \tau \mapsto -\frac{1}{\tau}$. Details including S and T matrix and fusion rules can be found in [7].

It follows that these RCFTs consist of two sectors, each a Gaussian theory, which are twisted by the vacuum representation and the h_{min} representation, common to both sectors. The both Gaussian sectors have different boundary conditions, i.e. different compactification radii $2R^2 = p/q$ and $2R'^2 = p'/q'$. We have the following condition in order to get consistent RCFTs: $p'q' - pq = \Delta$ for $k \in \mathbb{Z}_+ + \frac{1}{\Delta}$, $\Delta \in \{1,2,4\}$. Note the similarity to the unimodular equation defining $\mathrm{SL}(2,\mathbb{Z})$. Therefore, we suggestively denote these RCFTs with $\mathfrak{C}\left[\begin{smallmatrix} p' & p \\ q & q' \end{smallmatrix}\right]$. Note that the twist of two Gaussian sectors has the property that fusion of two fields in one sector yields fields in the other sector, i.e. that boundary conditions and thus statistics phases are changed.

2 Laughlin states $\nu = 1/(2p+1)$

Let us first concentrate on QHE states with filling factor $\nu = 1/(2p+1)$, i.e. the Laughlin states. We now make the following specific choice of the fermionic non-unitary RCFTs introduced above with background charges $\alpha_0 = \sqrt{(2p+1)/2}$, i.e. $c = 1 - 24\frac{2p+1}{2}$ and $\Delta = 2$. We work with the full rational conformal field theory (RCFT), but with $\mathbb{Z}_2$ frustrated boundary conditions in one direction, and introduce the chirality constraint by hand. The frustrated partition function

$$
\begin{aligned}
Z_{1,0} &= \chi_0\chi^*_{h_{min}} + \chi_{h_{min}}\chi^*_0 \\
&+ \sum_{\lambda\in\mathbb{Z}_{2p+1}-\{0\}} \tfrac{1}{2}(\chi_{\lambda,even}\chi^*_{\lambda,odd} + c.c.) + \sum_{\lambda\in\mathbb{Z}_{2p+3}-\{0\}} \tfrac{1}{2}(\chi_{\lambda,even}\chi^*_{\lambda,odd} + c.c.)
\end{aligned}
\tag{2.1}
$$

is modular invariant under the subgroup $\Gamma(2) \subset \Gamma = \mathrm{PSL}(2,\mathbb{Z})$ generated by $T^2 : \tau \mapsto \tau + 2$ and $ST^2S : \tau \mapsto \frac{\tau}{\tau+2}$. The characters have been split into even and odd contributions modulo the fermion number $(-)^F$. In particular, the groundstate of this frustrated RCFT is twofold degenerated (one for each direction of the external magnetic field) and is created by the local primary fields $\Phi_{(0,0|1,1)} = \psi_{\alpha_0}(z) \otimes \mathbb{1}$ and $\Phi_{(1,1|0,0)}$ respective, thus is *chiral*. The conformal weight is $(h,\bar{h}) = (h_{min}, 0) = (-\frac{2p+1}{2}, 0)$ or vice versa. The N-point correlator evaluates to

$$
\langle \Omega^*_{\sqrt{2(2p+1)}}(N), \prod_{i=1}^{N} \Phi_{(0,0|1,1)}(z_i,\bar{z}_i)\Omega_0\rangle = \prod_{i<j}(z_i - z_j)^{2p+1} \exp\left(-\frac{1}{2}\sum_i |z_i|^2\right), \tag{2.2}
$$

hence nicely reproduces the Laughlin wave functions (1.5). It is remarkable that the Laughlin QHE state now appears as *groundstate* of a certain RCFT. But this is not the only remarkable fact. As we will show now, attaching of flux quanta has a beautiful realization within these RCFTs given by fusion of primary fields.

First we identify the vertex operator which describes a single magnetic flux quantum by its anyonic statistics as quasi-particle excitation. While the composite fermions are described by $\Phi_{(0,0|1,1)}(z,\bar{z})$, the flux quantum is realized by the field $\Phi_{(\frac{2p}{2p+1},\frac{2p}{2p+1}|\frac{2p}{2p+1},\frac{2p}{2p+1})}(w,\bar{w})$, whose conformal dimension is $(h,\bar{h}) = (-\frac{4p+1}{4p+2}, -\frac{4p+1}{4p+2})$. Inserting M such flux quanta into the correlator (2.2), we obtain

$$
\begin{aligned}
&\langle \Omega^*_{\sqrt{2(2p+1)}}(N)\Omega^*_{\sqrt{2/(2p+1)}}(M), \prod_{j=1}^{M} \Phi_{(\frac{2p}{2p+1},\frac{2p}{2p+1}|\frac{2p}{2p+1},\frac{2p}{2p+1})}(w_j,\bar{w}_j) \prod_{i=1}^{N} \Phi_{(0,0|1,1)}(z_i,\bar{z}_i)\Omega_0\rangle = \\
&\prod_{j<j'} |w_i - w_{j'}|^{1/(2p+1)} \prod_{i,j}(z_i - w_j) \prod_{i<i'}(z_i - z_{i'})^{2p+1} e^{-\frac{1}{2}\sum_i |z_i|^2 - \frac{1}{2(2p+1)}\sum_j |w_i|^2}.
\end{aligned}
\tag{2.3}
$$

Indeed, the flux quanta have fractional statistics parameter $\theta/\pi = 1/(2p+1)$, and fractional charge $-e/(2p+1)$. Thus, they behave as anyons [30, 27]. In this way, we reproduce the basic excitations of the Laughlin wave functions. Of course, the anti-holomorphic part $\prod_{j<j'}(\bar{w}_j - \bar{w}_{j'})^{1/2(2p+1)}$ drops out by chiral projection but cannot be avoided due to mathematical consistency: The Berry connexion, which actually yields the anyonic statistics, is entirely determined by the on $w_i, \bar{w}_i$ dependent normalization of the wave function, see [1] or the introduction to chapter 2 in [28]. This is one reason why we have to work with the full RCFT. Note that the leading terms of correlators of primary fields are identical to the correlators of basic vertex operators, if no screening charges are needed. This especially is the case in (2.2).

We will now read Jain's idea [19, 20] (but see also [18]) of "attaching of flux quanta" literally. Thus, we let approach the coordinates $w_i \leftarrow z_i$, where for simplicity we first set $M = N$ and insert the operator product expansion (OPE) of $\Phi_{(0,0|1,1)}(z,\bar{z})\Phi_{(\frac{2p}{2p+1},\frac{2p}{2p+1}|\frac{2p}{2p+1},\frac{2p}{2p+1})}(w,\bar{w})$.

We would like to remark that OPE can mathematically rigorously be defined only for *local* fields. This further forces us to consider the full RCFT with left and right chiral parts. The OPE takes the general form

$$\begin{aligned}\Phi_{(\alpha|\beta)}(z,\bar{z})\Phi_{(\gamma|\delta)}(w,\bar{w}) &= \\ &\sum_{\zeta,\eta}(z-w)^{h(\zeta)-h(\alpha)-h(\gamma)}(\bar{z}-\bar{w})^{\bar{h}(\eta)-\bar{h}(\beta)-\bar{h}(\delta)}\mathcal{C}^{\zeta}_{\alpha\gamma}\bar{\mathcal{C}}^{\eta}_{\beta\delta}\tilde{\Phi}_{(\zeta|\eta)}(w,\bar{w})\,, \end{aligned} \tag{2.4}$$

where $\tilde{\Phi}_{(\zeta|\eta)}$ denotes a generic descendant field $f(\partial\phi,\partial^2\phi\ldots)\Phi_{(\zeta|\eta)}$. The fusion rules of our RCFT tell us which $\mathcal{W}$-conformal families will contribute to the right hand side of the OPE. It is sufficient to restrict ourselves to the term of leading order, since we are interested in the mesoscopic effects of attaching flux quanta. This is sometimes called the fusion product. Taking into account the $\mathbb{Z}_2$ structure due to the character splitting according to $(-)^F$ and the twisted boundary conditions we have

$$\begin{aligned}&\Phi_{(0,0|1,1)}\star\Phi_{(\frac{2p}{2p+1},\frac{2p}{2p+1}|\frac{1}{2p+1},\frac{1}{2p+1})}\star\Phi_{(\frac{2p}{2p+1},\frac{2p}{2p+1}|\frac{1}{2p+1},\frac{1}{2p+1})}\\ &= \Phi_{(\frac{2p+2}{2p+3},-\frac{2p+2}{2p+3}|\frac{1}{2p+1},\frac{1}{2p+1})}\star\Phi_{(\frac{2p}{2p+1},\frac{2p}{2p+1}|\frac{1}{2p+1},\frac{1}{2p+1})}+\ldots\\ &= \Phi_{(0,0|1,1)}\star\Phi_{(\frac{p+1}{2p+3},-\frac{p+1}{2p+3}|\frac{2p+2}{2p+3},-\frac{2p+2}{2p+3})}+\ldots\\ &= \Phi_{(\frac{1}{2p+3},-\frac{1}{2p+3}|\frac{2p+2}{2p+3},-\frac{2p+2}{2p+3})}+\ldots\,.\end{aligned} \tag{2.5}$$

We see that the leading term of the OPE is associative, but locality is only preserved, if fusion is done twice. This is expected, since we work with frustrated boundary conditions and the partition function $Z_{1,0}$ is not invariant under the S-transformation. Thus, attaching *two* flux quanta is a well defined operation within our RCFT, but attaching only one is not, since it destroys locality. It is surprising that attaching two flux quanta changes the statistics of the system from $\theta/\pi = 1/(2p+1)$ to $\theta'/\pi = 1/(2p+3)$. Moreover, since the total magnetic flux density must remain constant [19, 20, 18], the size of the system must change. This dissipation of the system (since we must decrease the electron density) will cost energy to compensate the pressure of the external magnetic field, and therefore, cool down the new QHE state. But the underlying statistics has changed such that the state cannot go to the old groundstate. On the other hand, changing the size of the system also redefines the periodicity conditions of the underlying free field (due to a shift in the magnetic length), and consequently the energy scale provided by the zero mode of the Virasoro field. The central term $\frac{c}{24}$ changes by a shift of -1, but the energy of the system renormalizes by a shift of $\frac{1}{4p+6}$ in each chiral sector, the smallest possible amount admitted by the spectrum of the RCFT. Taking both effects into account we exactly end up with the groundstate $\Phi_{(0,0|1,1)}$ of the RCFT with central charge $c = 1-12(2p+3)$. Introducing a further label for the square of the background charge α_0, we have a map

$$F:\ \Phi^{(\frac{2p+1}{2})}_{(0,0|1,1)}\stackrel{2\times\text{fusing}}{\longmapsto}\Phi^{(\frac{2p+1}{2})}_{(\frac{1}{2p+3},-\frac{1}{2p+3}|\frac{2p+2}{2p+3},-\frac{2p+2}{2p+3})}\stackrel{\text{resizing}}{\longmapsto}\Phi^{(\frac{2p+3}{2})}_{(0,0|1,1)}\,. \tag{2.6}$$

The theory with $c = 1-12(2p+1)$ has two statistics sectors with $\theta = \pi/(2p+1)$ and $\theta' = \pi/(2p+3)$. The latter is – up to an overall shift in the conformal weights – identical to the first sector of the theory with $c = 1-12(2p+3)$.

But how can we really change from one RCFT to another? Experimentally [23], one observes sharp phase transitions between the Hall plateaus with universal critical exponent 7/3. The problem is that the transition between Hall plateaus also changes the scale in the system, the magnetic length. In our present picture we further need to bridge over a gap $\Delta c = -24$ in the central charge. We propose now that this can be done with the help of still another CFT, which contains scale dependent logarithmic operators. Such theories can be consistently defined [17] and can even be rational [9]. Actually, all theories with $c = c_{p,1} = 13 - 6(p + p^{-1})$ share this property. And there is one extremely interesting candidate: The logarithmic RCFT with $c = c_{6,1} = -24$ contains a primary field $\Phi_{2,3}$ with conformal weight $h_{2,3} = 7/3$. Therefore, we conjecture that resizing the system by changing the external magnetic field can be described by tensoring our first RCFT with the $c_{6,1}$ model. The second arrow in our "fusion map" (2.6) would then provided by

$$F:\ \mathfrak{C}\begin{bmatrix} 1 & 1 \\ 2p+1 & 2p+3 \end{bmatrix} \otimes \mathfrak{C}[c_{6,1}] \mapsto \mathfrak{C}\begin{bmatrix} 1 & 1 \\ 2p+3 & 2p+5 \end{bmatrix} \otimes \mathfrak{C}[c_{3,2}]\,, \tag{2.7}$$

where we also need the non-unitary $c_{3,2} = 0$ theory to get the effective central charges matched. As is known, such non-unitary CFTs with $c = 0$ can describe (de-)localization effects due to disorder, which presumably are necessary for (un-)bounding flux quanta, see [31] and references therein. To be more specific, tensoring with the field $\Phi_{0,1}^{(c_1,6)}$ of conformal weight $h_{0,1} = -1$ maps the second sector of one RCFT to the first of the "next" theory,

$$h^{(c=1-12(2p+1))}_{\frac{\lambda}{2p+3},\frac{-\lambda}{2p+3}} + h^{(c_{6,1})}_{r,s} = h^{(c=1-12(2p+3))}_{\frac{\lambda}{2p+3},\frac{\lambda}{2p+3}} + h^{(c_{3,2})}_{r',s'} \tag{2.8}$$

with $3r' - 2s' = 6r - s$. The filling factor changes by this operation as $\nu^* = ST^2S(\nu) = \frac{\nu}{2\nu+1}$. With this we can match the central charge as well as its effective value, but not the entire spectrum of the RCFTs. This can be done by explicitly introducing disorder to the system. Assume that the disorder can be described by a random vector potential $(A_z, A_{\bar z})$ with Gaussian measure

$$P[A] = \exp\left(-\frac{1}{\sigma}\int \frac{d^2x}{\pi} A_z A_{\bar z}\right)\,,$$

which enters the action by an additional term

$$S_{disorder} = i\int \frac{d^2x}{\pi}\left(A_{\bar z}\partial\phi + A_z\bar\partial\phi\right)\,.$$

Since our model is Gaussian, the disorder factorizes (see e.g. [2]). Moreover, each of the two sectors of our RCFT can be affected separately by the disorder, i.e. we may introduce two different disorder couplings σ, σ'. The conformal dimensions are affected as

$$h^{disorder}_{\frac{\lambda}{2k+\varepsilon},(-)^{\varepsilon}\frac{\lambda}{2k+\varepsilon}} = h^{pure}_{\frac{\lambda}{2k+\varepsilon},(-)^{\varepsilon}\frac{\lambda}{2k+\varepsilon}} - \sigma\frac{\lambda^2}{4k+2\varepsilon}\,. \tag{2.9}$$

We denote the disordered theory by $\mathfrak{C}_{\sigma,\sigma'}\left[\begin{smallmatrix} p' & p \\ q & q' \end{smallmatrix}\right]$. It is easy to check that the spectra of both sides of (2.7) can be matched for various choices of the disorder couplings. One particular simple choice is

$$\mathfrak{C}_{\sigma,0}\begin{bmatrix} 1 & 1 \\ 2p+1 & 2p+3 \end{bmatrix} \otimes \mathfrak{C}[c_{6,1}] = \mathfrak{C}_{0,\sigma'=\frac{(2p+5)\sigma-4}{2p+1}}\begin{bmatrix} 1 & 1 \\ 2p+3 & 2p+5 \end{bmatrix} \otimes \mathfrak{C}[c_{3,2}]\,, \tag{2.10}$$

with fixpoint $\sigma' = \sigma$ for $\sigma = 1$. With this choice the disorder only affects part of the states. Moreover, there is a minimal disorder such that the spectra can match, since the disorder coupling $\sigma \geq 0$. It is $\sigma = \frac{4}{2p+5}$. This may provide an argument for the stability of the plateaus. The details of this construction will be presented elsewhere [10]. In the following we will omit the explicit notion of disorder and will understand equations of tensor products of RCFTs as disorder driven transitions from the left hand side to the right hand side.

3 Landau Levels of Composite Fermions

Jain obtains other filling factors of the form $\nu = \frac{n}{2pn\pm 1}$ by filling n Landau levels of composite fermions with $2p$ attached fluxes. The usual description of QHE states with filling $\nu = n_e/n_f$ within the CFT picture involves theories with $c_{\mathit{eff}} = n_e$. Within this scheme it is difficult to project down to the lowest Landau level which usually is done for calculation of the wave functions, and for their comparison with numerical studies. Moreover, recently it has been shown [5] that QHE states can be given as lowest-weight representations of minimal models of the $\mathcal{W}_{1+\infty}$ symmetry algebra of incompressible quantum droplets found earlier in [4, 11, 21]. This explicitly reduces the symmetry down to only one (charged) $\hat{\mathfrak{u}}(1)$ current and a classical $\mathfrak{su}(n)$ symmetry of neutral excitations.

Therefore, one should divide out additional $\hat{\mathfrak{u}}(1)$ currents from the RCFT. Consider the case $n = 2$, i.e. we have to tensorize two theories in our scheme. For example, the theory $\mathfrak{C}\left[\begin{smallmatrix}\frac{1}{2p+1} & \frac{1}{2p+3}\end{smallmatrix}\right]\otimes\mathfrak{C}\left[\begin{smallmatrix}\frac{1}{2p+1} & \frac{1}{2p+3}\end{smallmatrix}\right]/\hat{\mathfrak{u}}(1)$ has central charge $c = (1-12(2p+1))+(1-12(2p+1))-1 = 1-24(2p+1)$ and effective central charge $c_{\mathit{eff}} = 1+1-1 = 1$, hence must be one of the possible bosonic non-unitary RCFTs with $c_{\mathit{eff}} = 1$, say $\mathfrak{C}\left[\begin{smallmatrix}\tilde{p} & p\\ q & \tilde{q}\end{smallmatrix}\right]$. But counting the number of flux quanta, we find a missing defect of $2p$, precisely because the tensor product alone does not incorporate the interaction of fluxes of composite fermions of one Landau level with the charges of the particles in another. Thus, in order to provide the missing flux quanta, we have to tensorize again with $\mathfrak{C}[c_{6,1}]^{\otimes 2p}$. Then we finally end up with a bosonic RCFT of central charge $c = 1-24(2p+1)-24(2p) = 1-12\cdot 2(4p+1)$, modulo $c = 0$ contributions, in which we hide all disorder effects.

If the filling factor is $\nu = n_e/n_f$, then the corresponding RCFT has central charge $c = 1-12n_e n_f$. The fact that the square of the background charge can be factorized $2\alpha_0^2 = n_e n_f$ with $(n_e, n_f) = 1$ means that the (unfrustrated) partition function of the corresponding RCFT is non-diagonal. As a consequence, there may exist a primary field $\Phi_{h,\bar{h}}(z,\bar{z})$ with weights $(h,\bar{h})$ such that $h+\bar{h}\leq h_{min}$. Then this field serves as groundstate in the frustrated theory with partition function $Z_{1,0}$ and should therefore be used for building the wave functions. Note, that dividing out the current at the same time yields the projection to the lowest Landau level, since the effective number of degrees of freedom reduces to $c_{\mathit{eff}} = 1$.

The wave function for a QHE state with $\nu = 2/(4p+1)$ is given by the following expression of correlation functions of the fermionic theory $\mathfrak{C}\left[\begin{smallmatrix}\frac{1}{2p+1} & \frac{1}{2p+3}\end{smallmatrix}\right]$, with $c = 1-12(2p+1)$ and $\mathbb{Z}_2$-twisted boundary conditions, where we denote only the left-chiral part for simplicity:

$$\begin{aligned}\Psi_{\nu=\frac{2}{4p+1}}(\{z,w\}) &= \frac{\left\langle\prod_{i=1}^N\Phi_{0,0}(z_i)\prod_{j=1}^N\Phi_{\frac{1}{2p+1},\frac{1}{2p+1}}(w_j)\right\rangle\left\langle\prod_{i=1}^N\Phi_{0,0}(w_i)\prod_{j=1}^N\Phi_{\frac{1}{2p+1},\frac{1}{2p+1}}(z_j)\right\rangle}{\left\langle\prod_{i=1}^N\Phi_{\frac{1}{2p+1},\frac{1}{2p+1}}(z_i)\right\rangle\left\langle\prod_{i=1}^N\Phi_{\frac{1}{2p+1},\frac{1}{2p+1}}(w_i)\right\rangle}\\ &= \prod_{i<j}(z_i-z_j)^{2p+1}\prod_{i<j}(w_i-w_j)^{2p+1}\prod_{i,j}(z_i-w_j)^{2p}\,. \end{aligned}\qquad(3.1)$$

This expression is just the interaction of composite fermions of one Landau level with $2p$ flux quanta of the composite fermions of the other Landau level times vice versa and self-interaction of the flux quanta divided out. In fact, the primary field $\Phi_{\frac{1}{2p+1},\frac{1}{2p+1}}(z) = :(\Phi_{\frac{2p}{2p+1},\frac{2p}{2p+1}})^{2p}:(z)$ is the leading term of the $2p$-fold normal ordered product of vertex operators describing single flux quanta. The result is identical to the wave functions proposed by Chern-Simons theory.

There are now two ways to obtain wave functions in the lowest Landau level: The first is to describe particles from Landau level n by insertion of fields $\partial^{n-1}\Phi_{0,0}(z_i)$. Thus the (unnormalized) wave function for a state with n filled Landau levels would be

$$\Psi_{\nu=\frac{n}{2pn+1}}(\{z^{(0)}, z^{(1)}, \ldots, z^{(n-1)}\}) = \left\langle \prod_{i=1}^{N/n} \Phi_{0,0}(z_i^{(0)}) \prod_{j=1}^{N/n} \partial\Phi_{0,0}(z_j^{(1)}) \ldots \prod_{k=1}^{N/n} \partial^{n-1}\Phi_{0,0}(z_k^{(n-1)}) \right\rangle . \tag{3.2}$$

For $p = 0$ this yields the correct result of the Slater determinant for n Landau levels (after left normal ordering of all $\bar{z}_i$ variables and their replacement by $2\partial_j$ to map the function back to Bargmann space). But for $p > 0$ this ansatz may be too simple. The second way admits non-analytic powers in the wave function. We use the RCFT with $c = 1 - 12n(2pn + 1)$ and there the field $\Phi_{\frac{n-1}{n},\frac{n-1}{n}}(z)$, which has minimal conformal dimension in the frustrated model with non-diagonal partition function according to the factorization $(1-c)/12 = (n)\cdot(2pn+1)$. The (unnormalized) wave function then gets the general form

$$\Psi_\nu = \langle \prod_{i=1}^{\nu N} \Phi_{\frac{n-1}{n},\frac{n-1}{n}}(z_i)\rangle = \prod_{1\leq i<j\leq \nu N} (z_i - z_j)^{1/\nu} . \tag{3.3}$$

We may extract the holomorphic part $\prod_{i<j}(z_i - z_j)^{[1/\nu]}$ from the wave function (3.3), which for the Jain series is given by $[1/\nu] = 2p$, i.e. a pure Jastrow factor. The remaining non-analytic part lives on an n-fold covering of the complex plane which reflects the number of Landau levels. Expanding it in such a way that the overall asymmetry of the wavefunction is assured and (after left normal ordering of all still fractional powers modulo one) replacing $z_i^{a/n} \mapsto (2\partial_i)^a$ for $0 < a < n$ yields the wave function projected down to the lowest Landau level. If one skips all negative powers in the expansion of the non-analytic part, one exactly recovers the Jain wave functions for $\nu = \frac{n}{2pn\pm1}$, i.e. a Slater determinant of n Landau levels multiplied with a Jastrow factor[i]. Keeping the negative powers (where also the rôle of z_i and $2\partial_i$ has to be exchanged) yields additional terms, which can be viewed as higher order corrections of the Slater-Jastrow wave functions.

Note that formally one has two choices for $[1/\nu]$, $1/\nu \notin \mathbb{Z}$, namely $\lfloor 1/\nu \rfloor$ and $\lceil 1/\nu \rceil$, which are not equivalent on the level of our formal expansion of the non-analytic remainder, where the overall asymmetry of the wave function has to be kept. This may be understood as a mixing of the two polarization possibilities of the particles. The partially polarized states are just given by the sum of the two possible wave functions. This is supported by the observation that the mixing is exactly half-half for the states with filling $\nu = 2/d$ which are known to be unpolarized, and one-zero for the Laughlin states which are known to be fully polarized (here $1/\nu$ is an integer). In fact, $\Psi_{\nu=\frac{2}{4p\pm1}}$ can be splitted as $\Psi_{\nu=\frac{1}{2p}}\Psi_{\nu=\pm2}$ or as $\Psi_{\nu=\frac{1}{2p\pm1}}\Psi_{\nu=\mp2}$.

The RCFTs we use in our second approach are the fermionic theories $\mathfrak{C}\begin{bmatrix} r & n \\ 2pn+1 & s\end{bmatrix}$ with $n(2pn+1)-rs = \pm2$ for n odd, and the bosonic theories $\mathfrak{C}\begin{bmatrix} r & n/2 \\ 2pn+1 & s\end{bmatrix}$ with $n(2pn+1)/2-rs = \pm1$ for n even. Filling additional Landau levels of composite fermions corresponds to

$$T:\ \mathfrak{C}\begin{bmatrix} r & n \\ 2pn+1 & s\end{bmatrix} \otimes \mathfrak{C}\begin{bmatrix} r' & m \\ 2pm+1 & s'\end{bmatrix} \otimes \mathfrak{C}[c_{6,1}]^{\otimes 2pmn}/\hat{\mathfrak{u}}(1) =$$

[i]The filling fractions $\frac{n}{2pn-1}$ are obtained by formally exchanging the rôle of z_i and $2\partial_i$ in the Slater part of the wave function, i.e. setting $\Psi_{\nu=-n}(\{z\}) = \Psi_{\nu=n}(\{\bar{z}\})$.

$$\mathfrak{C}\begin{bmatrix} \tilde{r}/2 & (n+m)/2 \\ 2p(n+m)+1 & \tilde{s} \end{bmatrix} \otimes \mathfrak{C}[c_{3,2}]^{\otimes 2pmn}\,, \tag{3.4}$$

for n, m odd and similar for n or m even where bosonic and fermionic theories have to be replaced accordingly. The filling factor transforms due to this operation as $\nu^* = \frac{n+1}{2p(n+1)+1} = (ST^{2p}S)T^m(ST^{-2p}S)(\nu)$. Note that the modular transformation on ν for filling composite particle Landau levels is quite complicated and does depend on the number $2p$ of flux quanta already attached. This is reflected in the fact that our operation in the space of RCFTs involves a $2pmn$-fold tensor product. Actually, the correct number of additional flux quanta is not entirely fixed by the condition that the number of Landau levels is strictly additive. We will show later that this can explain the appearance of some observed non-Jain series.

But first we give a naive argument on the number of additional flux quanta. Consider two QHE states with $\nu = n_e/n_f$ and $\nu' = n'_e/n'_f$, where without loss of generality $n_f = 2p_f + 1$, $n'_f = 2p'_f + 1$. Joining these two states in the manner described above to get a new one, we introduce interaction between the charges n_e and the fluxes $2p'_f$ and vice versa, which all together results in $p_f n'_e + p'_f n_e$ fusion operations to get $\nu^* = \frac{n_e + n'_e}{2p_f + 2p'_f + 1}$. In fact, we not only tensorize the corresponding theories, but contract within the correlation functions in all possible ways fields from one theory with fields from the other via fusion F. If $\hat{\otimes}$ denotes tensoring with dividing out $\hat{\mathfrak{u}}(1)$, and if $p_f n'_e = p'_f n_e$, we have our second map

$$T:\; F^{p'_f n_e}\mathfrak{C}_{\nu=\frac{n_e}{2p_f+1}} \otimes F^{p_f n'_e}\mathfrak{C}_{\nu'=\frac{n'_e}{2p'_f+1}} \overset{\text{joining}}{\longmapsto} F^{p_f n'_e + p'_f n_e}\big(\mathfrak{C}_{\nu=\frac{n_e}{2p_f+1}} \hat{\otimes} \mathfrak{C}_{\nu'=\frac{n'_e}{2p'_f+1}}\big) \overset{\text{resizing}}{\longmapsto} \mathfrak{C}_{\nu^*=\frac{n_e+n'_e}{2p_f+2p'_f+1}}\,. \tag{3.5}$$

It is easy to see that the filling fractions of the Jain series are precisely the first order of continued fraction expansions

$$\nu = [n_1, 2p_1, \ldots, n_{k+1}] = n_{k+1} + \cfrac{1}{2p_k + \cfrac{1}{\ddots + \cfrac{1}{n_2 + \cfrac{1}{2p_1 + \cfrac{1}{n_1}}}}}\,. \tag{3.6}$$

As has been shown in [12, 15], the QHE can be described by a Chern-Simons theory. Then, due to large scale principles, only the one-loop diagrams contribute to the Hall conductivity. Viewing Chern-Simons theory as massive QED in three dimensions, where two flux quanta serve as a massive photon, this corresponds to a $1/N$ expansion [8], which naturally yields the filling factor as a continued fraction expansion and the Jain series as first order approximation to it.

4 Exceptional cases

The moduli space of the non-unitary RCFTs with $c_{\mathit{eff}} = 1$ has been explored in [7]. It is closely related to the modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ for $\Delta = 1$, and to $\Gamma(2)$ for $\Delta = 2$. So far, we described transitions, which change ν to first order in its continued fraction expansion, by our F (fusion) and T (tensoring) operation on RCFTs. Theoretically, every arbitrary filling factor can be obtained in this way. The F-map yields the $ST^{-2}S$ move of the modular group in our moduli space. Tensoring of theories yields the T move (which goes from the moduli space of fermionic theories to the space of bosonic theories and vice versa). Both together implement $\Gamma_T(2)$, one of the possible three subgroups $\Gamma(2) \subset \Gamma_X(2) \subset \Gamma$ of index two.

The appearance of a subgroup of the modular group in the QHE has already been noted earlier [22, 25, 26]. Our approach now yields a new description of the modular group as directly acting on a moduli space of RCFTs by the operations of F and T. More specifically we have for example

$$T: \ \mathfrak{C}\left[\begin{matrix} 1 & 1 \\ 2p+1 & 2p+3 \end{matrix}\right] \otimes \mathfrak{C}\left[\begin{matrix} 1 & 1 \\ 2p+1 & 2p+3 \end{matrix}\right] \otimes \mathfrak{C}[c_{6,1}]^{\otimes 2p}/\hat{\mathfrak{u}}(1) = \mathfrak{C}\left[\begin{matrix} 2 & 1 \\ 4p+1 & 2p+1 \end{matrix}\right] \otimes \mathfrak{C}[c_{3,2}]^{\otimes 2p}\,, \tag{4.1}$$

which describes the transitions $T(\nu = 1/(2p+1), \nu' = 1/(2p+1)) \mapsto \nu^* = 2/(4p+1)$, i.e. $1 \mapsto 2, 1/3 \mapsto 2/5, 1/5 \mapsto 2/9, \ldots$ As mentioned above, the partition function of the resulting theory is now partly non-diagonal, i.e. characters to different lowest-weight representations are combined, forcing that in the frustrated $Z_{1,0}$ model we have a non-trivial field creating the groundstate and thus yielding the wave function as its n-point function. Due to the second filled Landau level this correlator is no longer purely holomorphic but must be expanded in a specific way. This may be repeated in the following way: Starting with $\nu = n/(2pn+1)$ we have a transition to $\nu^* = (n+1)/(2p(n+1)+1)$ via the RCFT tensor product (3.4) with $m = 1$. Together with the fusion description of attaching flux quanta this yields all the Jain series – and more!

Consider for example the resulting theory in (4.1). Applying the fusion map we have a transition $F: \ \nu = 2/(4p+1) \mapsto \nu^* = 4/(2p+1)$. Form this we naturally obtain the filling fractions $4/5, 4/11$ and $4/13$ which are not members of the Jain hierarchy to first order[ii]. Also, $5/7$ is obtained by the fusion map from $3/11 = 3/(2\cdot 2\cdot 3-1)$. Note that the interpretation of the fusion map as purely attaching of flux quanta is no longer valid, if it is applied to QHE states of $n > 1$ Landau levels. Next, $7/11$ is obtained from $3/25 = 3/(2\cdot 4\cdot 3+1)$, $8/11$ from $3/29 = 3/(2\cdot 5\cdot 3-1)$, states with $p \geq 4$ flux quanta attached, which presumably is extremely difficult to observe, as is explained below. We propose that in such cases, where the state obtained from the fusion map has a significantly smaller number n_f of total flux quanta than the original state, the former is realized instead of the latter.

We even can get two whole new series out of our tensor operation (3.4) in the following way: The number of Landau levels must be additive, i.e. $(m+n)$ must be factored out from the rhs. This is possible for $p = 1$ even for a smaller power of $c_{6,1}$ theories. Namely, we could choose the powers $2(mn-m-n)$ and $2(mn-2m-2n)$ to tensorize with $\mathfrak{C}\left[\begin{smallmatrix} r & m \\ 2pm-1 & s \end{smallmatrix}\right] \otimes \mathfrak{C}\left[\begin{smallmatrix} r' & n \\ 2pm-1 & s' \end{smallmatrix}\right]$ which yields $\nu^* = n/(2n-3)$ and $\nu^* = n/(2n-5)$ respective. Note that lower members of the $n/(2n-5)$ series cannot be realized, because negative powers in the tensor product are meaningless, in particular if we use the physical case $m = 1$, i.e. addition of just one Landau level. Therefore, the first member of this series is the fraction $8/11$. Other cases such as $p > 1$ or a series $n/(2n-7)$ do not yield any new fraction in the "observable region" of the (n_e, n_f)-plane[iii]. Moreover, since it is physically unlikely that states realize,

[ii] The so called particle-hole duality $\nu \leftrightarrow 1-\nu$ (or $2-\nu$) is not considered in this work, since it is not supported by experiment in a sufficient way, i.e. for many QHE states the particle-hole duality conjugate state is not observed.

[iii] The observable region of the (n_e, n_f)-plane for $\nu < 1$ is more or less defined by $n_e \leq 10$, $n_f < 20$ and

where the power of $c_{6,1}$ models had to be higher then $2mn$ (which always is satisfactory to get additivity of Landau levels), one does indeed not observe states with $\nu = n/(2pn+k)$, $k = 3, 5, \ldots$. Note that we obtain all observed fractions within our scheme on the first level, i.e. by applying our fusion and tensor map each only once.

The huge amount of other possible transitions allowed by $\Gamma_T(2)$ all correspond to a change of ν to higher order in its continued fraction expansion. There are indeed experimentally observed filling fractions, which are not of first order, i.e. not members of the Jain series. But all these are of second order. They presumably belong to system configurations, where the number of particles is small enough such that second order effects become visible. All of them can be explained by the exceptional solutions described above.

The transitions between QHE plateaus presumably are disorder driven. In our treatment we found a minimal disorder coupling such that a transition can take place, which essentially is proportional as $\sigma \sim \frac{1}{n_e n_f}$. From this it follows that with increasing n, p in the Jain series $\nu = \frac{n}{2pn \pm 1}$ the plateau width decreases. In particular, it decreases fast for increasing p. Thus, higher members of the Jain series are more difficult to observe, since the parameters of the experiment have to be controlled with higher accuracy. It can even happen that second order states such as for $\nu = 4/(2p+1)$ can be seen without the corresponding first order states, here $\nu = 2/(4p+1)$ for $p > 3$, being observed, since the latter are much less stable against disorder. Also, if the total number of bound flux quanta n_f is very high, it seems likely that the system chooses a new configuration with lower n_f. In some cases this is possible via the fusion map which then in fact unbounds flux, e.g. $F : \frac{2}{4p+1} \mapsto \frac{4}{2p+1}$ as mentioned above. We explicitly checked that such possibilities are very rare and do not produce any unobserved filling fractions (in the sense that n_e, n_f are sufficiently small to yield states which should in principle be accessible by experiment).

To conclude, we think that our proposed way of describing transitions between QHE states by algebraic operations on the space of RCFTs strongly supports the composite fermion picture of Jain and may explain several experimentally observed facts. *In particular we have a frame in which precisely the observed fractions can be explained – and the completeness of the set of already observed series.* Our frame naturally produces the experimentally observed universal critical exponent 7/3. Moreover, it once more shows the deep rôle the modular group plays in nature. We hope that it may serve as a starting point for future investigations. A lot of questions remain open: Since our approach is heavily based on CFT numerology, it is urging to support it by a more close connection to first principle treatments of the FQHE. It also would be worthwhile to connect it with the classification of quantum Hall fluids obtained by Fröhlich, Studer, and Thiran [13, 14], and with topological explanations for the exclusivity of the Jain series [29]. This will be done in our future work [10].

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