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On a Conformal Field Theory Approach to Seiberg-Witten Models

The use of conformal field theory tools in string theory is, of course, not new. However, with the advent of logarithmic conformal field theory (LCFT, see [1] and referecnes therein) we may have a better machinery to describe physics whose underlying structure is determined by the geometry of Riemannian manifolds, in particular Riemannian surfaces. This is especially the case in modern string theory with its new family members such as branes, M-theory and the resulting exactly solvable low-energy effective field theories.

I. Seiberg-Witten models:

BPS states are given in terms of periods

$$a[\gamma] = \oint_\gamma \mathrm{d}S_\Sigma$$

over a meromorphic one-form $\mathrm{d}S_\Sigma$ associated to a Riemann surface Σ with induced metric $\mathrm{d}s^2 = |\mathrm{d}S_\Sigma|^2$. In all relevant cases,

$$\mathrm{d}S_\Sigma = \prod_i (z - x_i)^{-q_i}\mathrm{d}z$$

with $\sum_i q_i = 0$ and $2q_i \in \mathbb{Z}$ in case Σ is hyperelliptic. The precise form of Σ and $\mathrm{d}S_\Sigma$ can be derived in many ways from higher-dimensional physics, e.g. from wrapped 3-branes in type II string compactifications on fibred Calabi-Yau 3-folds (see Klemm et al., hep-th/9604034). The resulting effective string theory of self-dual strings has a 3-dimensional target space M with $\partial M = \Sigma$ its *euclidean* boundary. The BPS states of the low-energy effective SYM theory are created by such strings ending on Σ. Since they are localized with respect to the critical points of the metric $\mathrm{d}s^2$, they have non-trivial tension yielding the masses of the BPS states. The reduction of this effective 3-dimensional target space theory down to 2 dimensions should yield a *euclidean* CFT due to reparametrization invariance on ∂M. In fact,

$$a[\gamma] = \langle\!\langle Q_-[\gamma] \prod_i V_{q_i}(x_i) \rangle\!\rangle$$

yields a 1-to-1 correpondence of periods and conformal blocks with one screening charge of the rational LCFT with $c = -2$, which is the spin zero-one matter or ghost system. Here, $Q_-[\gamma] = \oint_\gamma \mathrm{d}z V_{-1}(z)$.

II. CFT Approach:

In the spin zero-one matter system the self-dual strings ending on Σ are represented by the spin one field $\phi^{(1)}$ (1-differential) localized on a curve γ. The action of this CFT is simply

$$S_{c=-2} = \int \phi^{(1)}\bar{\partial}\phi^{(0)}\mathrm{d}^2 z\,.$$

However, correlation functions evaluated on Riemannian surfaces vanish unless all zero modes are cancelled. Introduce, after bosonization with scalars φ, vertex operators $V_q = \exp(iq\varphi)$ and puncture operators $\Lambda_q = \varphi V_q$. For example, $\phi^{(1)} = V_{-1}$. The zeroes of $\mathrm{d}S_\Sigma$ take care of most of the zero modes, for the remaining ones one introduces the generating functional

$$Z[\{\rho_{\pm n}\}] = \\ \langle\Sigma_1, \bar{\mathrm{d}}S_1|\Lambda_{-1}(\infty)\exp(-\int\rho_0[D]V_{-1}(z)\mathrm{d}^2z - \sum_n\int\frac{\rho_n[D]}{n}V_n(\infty)V_{-1}(z)V_{-n}(0)\mathrm{d}^2z + \ldots)|\Sigma_2, \mathrm{d}S_2\rangle\,.$$

Here, $\rho[D]$ is localized at the divisor D, and includes the appropriate cocyle to ensure charge balance. Clearly, with $\Sigma_1 = \mathbb{P}^1$ with a sinmple pole at infinity, and $\Sigma_2 = \Sigma$, we have

$$a[\gamma] = \frac{\delta}{\delta\rho_0[\gamma]}\log(Z) = \langle\!\langle\mathbb{P}^1|\Lambda_2(\infty)Q_-[\gamma]|\Sigma, \mathrm{d}S_\Sigma\rangle\!\rangle\,.$$

These correlation functions can be determined explicitly. Assuming $x_1 = \infty, x_2 = 0, x_3 = 1$ to lighten the notation (otherwise the expressions are ornamented by appropriated transformation factors), we find for the period $\gamma(x_2, x_3)$ encircling x_2 and x_3 the result

$$a[\gamma(0,1)] = \frac{\Gamma(1-q_2)\Gamma(1-q_3)}{\Gamma(2-q_2-q_3)}F_D^{(n-3)}(1-q_2; q_4, \ldots, q_n; 2-q_2-q_3|1/x_4, \ldots, 1/x_n)\prod_{i=4}^n x_i^{-q_i}\,.$$

The functions $F_D^{(n)}$ are so-called Lauricella functions and generalize hypergeometric functions to multiple variables. One has

$$F_D^{(n)}(a; b_1, \ldots, b_n; c|z_1, \ldots, z_n) = \sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}}{(c)_{m_1+\ldots+m_n}}\prod_{i=1}^n\frac{(b_i)_{m_i}}{m_i!}z_i^{m_i}$$

whenever all $|z_i| < 1$. Analytic continuation of these functions can easily be infered from the well-known analytic continuation of ordinary hypergeometric functions making use of the symbolic differential operators

$$\nabla(h) = \frac{\Gamma(h)^{n-1}\Gamma(\sum_i\delta_i + h)}{\prod_i\Gamma(\delta_i + h)}\,,\quad \Delta(h) = \nabla(h)^{-1}\,,$$

where $\delta_i = z_i\partial_{z_i}$. As a result,

$$F_D^{(n)}(a; b_1, \ldots, b_n; c|z_1, \ldots, z_n) = \nabla(a)\Delta(c)\prod_i {}_2F_1(a, b_i; c; z_i)\,.$$

III. BPS States:

With the above sketched approach, the prepotential of the Seiberg-Witten models can be written suggestively as

$$\begin{aligned}\mathcal{F} &= \langle\!\langle\Sigma, \bar{\mathrm{d}}S^*_\Sigma|\Sigma, \mathrm{d}S_\Sigma\rangle\!\rangle \\ &= \sum_{\gamma,n}\frac{1}{n}\langle\!\langle\Sigma, \bar{\mathrm{d}}S^*_\Sigma|\Lambda_{2-n}(\infty)Q_-[\gamma]V_n(0)|\mathbb{P}^1\rangle\!\rangle\langle\!\langle\mathbb{P}^1|\Lambda_{2-n}(\infty)Q_-[\gamma]V_n(0)|\Sigma, \mathrm{d}S_\Sigma\rangle\!\rangle \\ &= \sum_{\gamma,n}\frac{1}{n}\langle\!\langle\mathbb{P}^1; n|Q_-[\gamma^*]|\Sigma, \mathrm{d}_\Sigma; 2+n\rangle\!\rangle\langle\!\langle\mathbb{P}^1; -n|Q_-[\gamma]|\Sigma, \mathrm{d}_\Sigma; 2-n\rangle\!\rangle\,,\end{aligned}$$

where we have absorbed the pole structure at $z=\infty$ and $z=0$ in the vacuum states, and where γ runs over the extended homology including cycles around the poles with γ^* the dual to γ with respect to the intersection form. Note that the CFT approach automatically yields $\mathcal{F}$ including contributions from the (truncated) Whitham hierarchy. The reason is that the CFT approach allows to introduce arbitrary boundary conditions for the fields on Σ, in particular for the vacuum states. Also, the CFT approach puts models with (massive) hypermultiplets on equal footing Stable BPS states are expected to correspond to geodesics running between zeroes of $\mathrm{d}S$. The zeroes correspond to branch points of the fibration of the Calabi-Yau compactification space. The geodesics are derived from the 3-cyles in the CY space by an essentially faithful map $f: H_3(CY) \to H_1(\Sigma)$ such that the integral of the holomorphic 3-form Ω over a 3-cylce C_3 is equivalent to $\int_{C_3}\Omega = \int_{f(C_3)}\mathrm{d}S$. As a result, non-trivial states (i.e. 3-cycles) correspond to strings on Σ localized at critical points of $\mathrm{d}S$. The mass is given by the tension which obvioulsy also yields the geodesic condition. The charges of the states are characterized by their homology class, since any geodesic must be equivalent to $\Gamma = \sum_i g_i\gamma_i$ with charges given as winding numbers g_i. Necessary conditions for the BPS state to be stable are

$$\begin{array}{ll} \text{(i)} & \gcd(\{g_i\}) = 1\,, \\ \text{(ii)} & \Gamma \cap \Gamma = 0\,, \end{array}$$

i.e. the curve Γ is neither a multiple of another, nor does it self-intersect (and could thus be splitted). The CFT approach makes it easy to evaluate the geodesics, since the conformal blocks put poles and branch points in $\mathrm{d}S$ on equal footing with its zeroes. Moreover, a remarkably feature of the Lauricella F_D system is that its incomplete integral can be solved explicitly, which amounts in nothing else than inserting an "identity" $\Lambda_0(z)$ as an infinitesimal probe in the CFT correlator,

$$\begin{aligned} \int_0^z \mathrm{d}S &= \langle\!\langle \mathbb{P}^1 | Q_-[\gamma(0,z)]\Lambda_0(z) | \Sigma, \mathrm{d}S; 2\rangle\!\rangle \\ &= \frac{\Gamma(1-q_2)\Gamma(1-q_3)}{\Gamma(2-q_2-q_3)}\,\frac{z^{1-q_2}}{1-q_2} F_D^{(n-2)}(1-q_2; q_3, q_4, \ldots, q_n; 2-q_2 | z, 1/x_4, \ldots, 1/x_n)\,. \end{aligned}$$

Non-trivial geodesics result by the necessity to analytically continue the above formula for $|z| > 1$, and by encircling other critical points. Note that all physically relevant geodesics emmanate from a critical point of $\mathrm{d}S$ such that by an $SL(2,\mathbb{C})$ transformation the base point of the above incomplete integral can always be moved to $z=0$ without loss of generality.

IV. Further Remarks:

The necessary shift of the charge balance can also be achieved by using Q_+ which picks up the residue terms within its integration contour. For example, the insertion of $\Lambda_2(\infty)$ can be replaced by considering a conformal block with *two* screening charges:

$$a[\gamma] = \langle\!\langle \mathbb{P}^1 | Q_+[\gamma(\infty)] Q_-[\gamma] | \Sigma, \mathrm{d}S\rangle\!\rangle\,.$$

The stringy picture has a striking resemblance to some interesting condensed matter physics. What is essentially computed above is the interaction of a string on Σ with a gas of charges q_i on Σ. The string enters in the CFT approach as a spin one field which gets localized at some of the charges. This is precisely the same setting as in the quantum Hall effect. In particular the so-called Haldane-Rezayi state (see last reference in of [2]) is a quantum state of paired electrons forming singlets (spin zero fields). The wave functions for excitations of the Haldane-Rezayi state can be calculated by CFT methods, where again the CFT is considered to be the boundary theory for some conjectured 3-dimensional Chern-Simons theory. Depending on whether we are interested in the edge or bulk excitations, the boundary theory becomes a minkowskian $c = 1$ CFT in the former case, and a euclidean $c = -2$ CFT in the latter. The bulk excitations are entirely determined by the topological nature of the disorder which in fact is described by insertions of operators $V_{q_i}(x_i)$ creating a non-trivial topology of a Riemann surface from the plane and yielding such interesting features as non-abelian statistics. It is worth noting that the operators $V_{1/2}$, which create branch cuts for Σ, correspond to flux quanta piercing the plane.

One may now ask what the advantage is of using the LCFT description instead of more common and established mathematical tools to deal with computations on Riemannian surfaces. Firstly, many computations are much simpler in the LCFT picture. For example, all differential equations appearing in $SU(N_c)$ Seiberg-Witten N=2 super-symmetric models are only of second degree, even when $N_f < 2N_c$ massive hyper-multiplets are present. This is to be contrasted with the third order Picard-Fuchs equations one is otherwise confronted with. Also, the evaluation of the Seiberg-Witten periods in asymptotic regions of moduli space becomes just a simple and straight-forward task in applying operator product expansions.

Secondly, and more importantly, the LCFT picture might be much more physical. As already indicated above, the intersections of (e.g. in the type II framework) $D4$-branes on $NS5$-branes can be viewed in a very similar way as flux quanta piercing through a two-dimensional quantum Hall droplet. In the low-energy picture, one is not interested in the interaction among the intersecting branes, but only in their effect on an excitation (such as a wrapped string), as one is not so much interested in the interaction among the flux quanta than in their influence on a quasi-particle state.

In fact, the Seiberg-Witten differential can be rewritten as the exponential of a canonical potential a gas of charged particles would exert on a probe, where the charges are given by the precise nature of all singular fibres in the fibred Calabi-Yau compactification space. Hence, Laughlin's plasma-analogy supporting his quantum droplet hypothesis can be invoked in this setting, too.

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[3] M. Flohr, *Logarithmic Conformal Field Theory & Seiberg-Witten Models*, KCL-MTH-98-36 (1998) 13pp., to be published in Phys. Lett. **B** [hep-th/9808169]