Logarithmic torus amplitudes

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October 6, 2007

Abstract

For the example of the logarithmic triplet theory at $c = -2$ the chiral vacuum torus amplitudes are analysed. It is found that the space of these torus amplitudes is spanned by the characters of the irreducible representations, as well as a function that can be associated to the logarithmic extension of the vacuum representation. A few implications and generalisations of this result are discussed.

1. Introduction. During the last twenty years much has been understood about the structure of rational conformal field theories. Rational conformal field theories are characterised by the property that they have only finitely many irreducible highest weight representations of the chiral algebra (or vertex operator algebra), and that every highest weight representation is completely decomposable into irreducible representations. The structure of these theories is well understood: in particular, the characters of the irreducible representations transform into one another under modular transformations [1] (see also [2]), and the modular S-matrix determines the fusion rules via the Verlinde formula [3]. (A general proof for this has only recently been given in [4].)

On the other hand, it is clear that rational conformal field theories are rather special, and it is therefore important to understand the structure of more general classes of conformal field theories. One such class are the (rational) logarithmic theories that possess only finitely many indecomposable representations, but for which not all highest weight representations are completely decomposable. The name 'logarithmic' comes from the fact that their chiral correlation functions typically have logarithmic branch cuts. The first example of a (non-rational) logarithmic conformal field theory was found in [5] (see also [6]), and the first rational example (that shall also concern us in this paper) was constructed in [7]; for some recent reviews see [8, 9, 10]. From a physics point of view, logarithmic conformal field theories appear naturally in various models of statistical physics, for example in the theory

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of (multi)critical polymers [11, 12, 13], percolation [14, 15], two-dimensional turbulence [16, 17, 18], the quantum Hall effect [19] and various critical (disordered) models [20, 21, 22, 23, 24, 25, 26, 27]. There have also been applications in Seiberg-Witten models [28] and in string theory, in particular in the context of D-brane recoil [29, 30, 31, 32], and in pp-wave backgrounds [33]. Logarithmic vertex operator algebras have finally attracted some attention recently in mathematics [34, 35, 36, 37, 38]. Most examples that have been studied concern the $c = -2$ model (that shall also mainly concern us here), but logarithmic conformal field theories have also arisen in other contexts, see for example [39, 40, 41, 42].

As we have mentioned above, the characters of the irreducible representations of a rational conformal field theory close under the action of the modular group. This can be proven by showing that they span the space of (chiral) vacuum torus amplitudes which is modular invariant on general grounds [1]. On the other hand, for logarithmic conformal field theories it has been known for some time that the characters of the irreducible representations do not, by themselves, form a representation of the modular group [12, 43]. However, even for logarithmic theories the vacuum torus amplitudes should still be closed under the action of the modular group [36]. In order to see explicitly how this fits together, we study in this paper the space of vacuum torus amplitudes for the example of the triplet theory at $c = -2$ [44]. We explain how to derive the modular differential equation that characterises these amplitudes. (In the case of rational conformal field theories, such differential equations were first considered in [45].) As we shall see, the characters of the irreducible representation only account for a subspace of codimension one. Furthermore, we show that the remaining solution of the differential equation can be taken to agree with the 'logarithmic character' that can be formally associated to the indecomposable extension of the vacuum representation [43]; this clarifies its interpretation as a genuine vacuum torus amplitude (despite the fact that it is not actually a character). We also observe that this association of a vacuum torus amplitude to a logarithmic representation is not canonical. In particular, the indecomposable highest weight representations therefore do not give rise to a canonical basis for the space of these torus amplitudes. This explains why Verlinde's formula (that presupposes such a basis) cannot describe the fusion rules of the triplet theory correctly [7].

The modular properties of a logarithmic conformal field theory have played an important role in various applications of logarithmic conformal field theory, in particular in the analysis of the boundary theory (for some work in this direction see [46, 47, 48, 49, 50]) and the fusion rules [43, 51].

The paper is organised as follows. In section 2 we review briefly the main results of Zhu [1] that were generalised to the logarithmic case in [36]. In section 3 we recall the main properties of the $c = -2$ triplet theory. Putting these results together we derive, in section 4, the modular differential equation that characterises the vacuum torus amplitudes. The complete space of solutions is constructed in section 5. In section 6 we explain how the analysis of the modular differential equation can be generalised to arbitrary rational logarithmic conformal field theories. Finally, we sketch in section 7 how the analysis works for the other triplet theories, giving explicit details for the $c = -7$ example.

2. Zhu's argument.   In the following we shall consider conformal field theories (or vertex operator algebras) that satisfy the C_2 condition, but we shall not assume that they define

rational conformal field theories. As is common in the mathematical literature, we call a conformal field theory rational if (i) it possesses only finitely many irreducible highest weight representations, each of which has finite-dimensional L_0 eigenspaces; and (ii) every highest weight representation can be decomposed into a direct sum of irreducible highest weight representations. The C_2 condition states that the quotient space $\mathcal{H}_0/C_2(\mathcal{H}_0)$ is finite dimensional, where $\mathcal{H}_0$ is the vacuum representation of the conformal field theory and $C_2(\mathcal{H}_0)$ is the space spanned by the states

$$V_{-h(\psi)-1}(\psi)\,\chi\,, \qquad \text{for } \psi, \chi \in \mathcal{H}_0. \tag{1}$$

The C_2 condition implies that Zhu's algebra $A(\mathcal{H}_0)$ is finite dimensional, and therefore that the conformal field theory has only finitely many irreducible highest weight representations (see also [52] for an introduction to these matters). However, it does not imply that the theory is rational in the above sense. Indeed, the example we shall mainly consider in this paper, the triplet algebra at $c = -2$ [44], satisfies the C_2 condition [53], yet is not rational since it possesses indecomposable representations [7]. It is natural to conjecture[1] that rational logarithmic conformal field theories are characterised by the condition that they are C_2-cofinite, but that Zhu's algebra is not semisimple. The results of this paper are certainly in agreement with this idea.

Let us briefly summarise the key results of Zhu [1] that were extended by Miyamoto [36] to theories that satisfy the C_2 condition but are not rational in the above sense. If the conformal field theory satisfies the C_2 condition, then every highest weight representation gives rise to a torus amplitude; in particular, the vacuum torus amplitude is just given by the usual character

$$\chi_{\mathcal{H}_j}(\tau) = \text{Tr}_{\mathcal{H}_j}\left(q^{L_0 - \frac{c}{24}}\right), \qquad q = e^{2\pi i\tau}, \tag{2}$$

which converges absolutely for $0 < |q| < 1$. Furthermore, the space of torus amplitudes is finite dimensional, and it carries a representation of $\text{SL}(2, \mathbb{Z})$ [1, 36]. As is explained in [1, 36], if the conformal field theory satisfies the C_2 condition then there exists a positive integer s so that every vacuum torus amplitude $T(q)$ satisfies

$$\left[\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^s + \sum_{r=0}^{s-1} h_r(q)\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^r\right] T(q) = 0\,. \tag{3}$$

Here the $h_r(q)$ are polynomials in the Eisenstein series $E_2(q)$, $E_4(q)$ and $E_6(q)$; we choose the convention that the Eisenstein series are defined by

$$E_k(q) = 1 - \frac{2\,k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\,q^n\,, \tag{4}$$

$$\sigma_k(n) = \sum_{d|n} d^k\,, \tag{5}$$

where B_k is the k-th Bernoulli number. Thus, the q-expansion of the Eisenstein series reads $E_2 = 1 - 24q - 72q^2 - 96q^3 - \cdots$, $E_4 = 1 + 240q + 2160q^2 + 6720q^3 + \cdots$, and $E_6 = 1 - 504q - 16632q^2 - 122976q^3 - \cdots$ in our normalisation.

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[1]A related conjecture was originally made by one of us (MRG) in collaboration with Peter Goddard — see [9].

For the following it is important (see Lemma 5.3.2 of [1]) that the functions h_r have the property that

$$\left(L - \frac{c}{24}\right)^s + \sum_{r=0}^{s-1} h_r(0) \left(L - \frac{c}{24}\right)^r = 0 \tag{6}$$

in Zhu's algebra $A(\mathcal{H}_0)$. This reflects the fact that for $q \to 0$, only the highest weight states contribute to the vacuum torus amplitudes, and that they must therefore satisfy the constraints of Zhu's algebra. As we shall argue below, the differential equation (3) can be identified with the modular differential equation that was first considered in [45].

If the conformal field theory is in addition rational in the above sense Zhu showed that the space of torus amplitudes is spanned by the characters of the irreducible representations. However, as already pointed out in [36], this is not longer the case if the theory is not rational. Indeed, we shall see this very explicitly for the case of the triplet algebra in the following.

3. The triplet theory. Let us briefly recall some of the properties of the triplet theory [44, 12, 13, 7]. The chiral algebra for this conformal field theory is generated by the Virasoro modes L_n, and the modes of a triplet of weight 3 fields W_n^a. The commutation relations are

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} - \frac{1}{6}m(m^2-1)\delta_{m+n}, \\
[L_m, W_n^a] &= (2m-n)W_{m+n}^a, \\
[W_m^a, W_n^b] &= g^{ab}\Big(2(m-n)\Lambda_{m+n} + \frac{1}{20}(m-n)(2m^2 + 2n^2 - mn - 8)L_{m+n} \\
&\quad - \frac{1}{120}m(m^2-1)(m^2-4)\delta_{m+n}\Big) \\
&\quad + f_c^{ab}\left(\frac{5}{14}(2m^2 + 2n^2 - 3mn - 4)W_{m+n}^c + \frac{12}{5}V_{m+n}^c\right),
\end{aligned}
$$

where $\Lambda = :L^2: - 3/10\,\partial^2 L$ and $V^a = :LW^a: - 3/14\,\partial^2 W^a$ are quasiprimary normal ordered fields. g^{ab} and f_c^{ab} are the metric and structure constants of $\mathfrak{su}(2)$. In an orthonormal basis we have $g^{ab} = \delta^{ab}$, $f_c^{ab} = i\epsilon^{abc}$.

The triplet algebra (at $c = -2$) is only associative, because certain states in the vacuum representation (which would generically violate associativity) are null. The relevant null vectors are

$$N^a = \left(2L_{-3}W_{-3}^a - \frac{4}{3}L_{-2}W_{-4}^a + W_{-6}^a\right)\Omega, \tag{7}$$

$$
\begin{aligned}
N^{ab} &= W_{-3}^a W_{-3}^b \Omega - g^{ab}\left(\frac{8}{9}L_{-2}^3 + \frac{19}{36}L_{-3}^2 + \frac{14}{9}L_{-4}L_{-2} - \frac{16}{9}L_{-6}\right)\Omega \\
&\quad - f_c^{ab}\left(-2L_{-2}W_{-4}^c + \frac{5}{4}W_{-6}^c\right)\Omega.
\end{aligned} \tag{8}
$$

We shall only be interested in representations which respect these relations, and for which the spectrum of L_0 is bounded from below. Evaluating the constraint coming from (8), we

find (see [7] for more details)

$$\left(W_0^a W_0^b - g^{ab}\frac{1}{9}L_0^2(8L_0 + 1) - f_c^{ab}\frac{1}{5}(6L_0 - 1)W_0^c\right)\psi = 0\,, \tag{9}$$

where ψ is any highest weight state, while the relation coming from the zero mode of (7) is satisfied identically. Furthermore, the constraint from $W_1^a N_{-1}^{bc}$, together with (9) implies that $W_0^a(8L_0 - 3)(L_0 - 1)\psi = 0$. Multiplying with W_0^a and using (9) again, this implies that

$$0 = L_0^2(8L_0 + 1)(8L_0 - 3)(L_0 - 1)\psi\,. \tag{10}$$

For irreducible representations, L_0 has to take a fixed value h on the highest weight states, and (10) then implies that h has to be either $h = 0, -1/8, 3/8$ or $h = 1$. However, it also follows from (10) that a logarithmic highest weight representation is allowed since we only have to have that $L_0^2 = 0$ but not necessarily that $L_0 = 0$. Thus, in particular, a two-dimensional space of highest weight states with relations

$$L_0\,\omega = \Omega \qquad\qquad L_0\,\Omega = 0\,. \tag{11}$$

satisfies (10). This highest weight space gives rise to the 'logarithmic' (indecomposable) representation $\mathcal{R}_0$ (see [7] for more details). [The other indecomposable representation $\mathcal{R}_1$ of [7] is not highest weight in the usual sense.]

It follows from the above analysis (and a similar analysis for the W^a modes — see for example [7]) that the triplet theory has only finitely many indecomposable highest weight representations. This suggests that it satisfies the C_2 condition, and in fact this has been proven recently [54] (see also [53]). Indeed, the space $\mathcal{H}_0/C_2(\mathcal{H}_0)$ has dimension 11, and it can be taken to be spanned by the vectors

$$\begin{aligned}
L_{-2}^s\Omega\,, &\qquad \text{where } s = 0, 1, 2, 3, 4 \\
L_{-2}^s W_{-3}^a\Omega\,, &\qquad \text{where } s = 0, 1 \text{ and } a \in \mathrm{adj}(\mathfrak{su}(2))\,.
\end{aligned} \tag{12}$$

As was already explained by Zhu [1], the dimension of this quotient space gives an upper bound on the dimension of Zhu's algebra, which is thus at most 11-dimensional. On the other hand, it also follows from the analysis of Zhu [1] that each irreducible representation whose space of ground states has dimension d, contributes d^2 states to Zhu's algebra. For the triplet algebra, the irreducible representations with highest weights $h = -1/8$ and $h = 0$ are singlet representations, while the irreducible representations with $h = 3/8$ and $h = 1$ are doublets. These irreducible representations therefore account for a $1^2 + 1^2 + 2^2 + 2^2 = 10$-dimensional (sub)space of Zhu's algebra. Since we have one additional highest weight representation — the logarithmic extension of the vacuum representation — we expect that Zhu's algebra is precisely 11-dimensional, and that the remaining state accounts for this logarithmic extension. We shall see below how this counting is in fact mirrored by our analysis of the vacuum torus amplitudes.

4. The modular differential equation. The above calculation leading to (10) implies that in Zhu's algebra we have the relation

$$L_0^2(8L_0 + 1)(8L_0 - 3)(L_0 - 1) = 0\,, \tag{13}$$

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where L_0 denotes the operator corresponding to the stress energy tensor, and the product is to be understood as the product in Zhu's algebra (see for example [52] for an explanation of this construction). In fact, such a relation had to hold in Zhu's algebra, given the structure of the homogeneous quotient space $\mathcal{H}_0/C_2(\mathcal{H}_0)$ in (12): it follows from (12) that $L_{-2}^5 = 0$ in $\mathcal{H}_0/C_2(\mathcal{H}_0)$. By the usual argument (see for example [1]), one can then show that

$$L_0^5 + \left(\text{terms of conformal weight} < 10 \right) = 0 \tag{14}$$

in Zhu's algebra. The terms of lower conformal weight can again be expressed in terms of the basis vectors of (12), as well as elements in $C_2(\mathcal{H}_0)$. Since all vectors that appear in (14) are $\mathfrak{su}(2)$ singlets, only the basis vectors in the first line of (12) contribute. Continuing this argument recursively, one then deduces that there is a fifth order polynomial relation involving only L_0 in Zhu's algebra, i.e. a relation of the form (13).

By the same token, it then also follows that the differential equation (3) that characterises the vacuum torus amplitudes for the triplet theory is (at most of) fifth order. Furthermore, (6) must actually reduce to (13), and thus the differential equation is precisely fifth order. Since the space of vacuum torus amplitudes is invariant under the action of the modular group $\mathrm{SL}(2,\mathbb{Z})$ (see section 2), the differential equation must be modular invariant as well. The most general modular invariant differential equation of degree five is

$$\left[D^5 + \sum_{r=0}^{4} f_r(q)\, D^r \right] T(q) = 0 \,, \tag{15}$$

where each $f_r(q)$ is a polynomial in $E_4(q)$ and $E_6(q)$ of modular weight $10 - 2r$, and

$$D^i = cod_{(2i)} \cdots cod_{(2)}\, cod_{(0)} \,, \tag{16}$$

with cod_s being the modular covariant derivative on weight s modular functions

$$cod_{(s)} = q\frac{\partial}{\partial q} - \frac{1}{12}(s-2)E_2(q) \,, \tag{17}$$

which increments the weight of a modular form by 2. Here E_2 is the second Eisenstein series, and $cod_{(0)}f = f$. For the case of rational conformal field theories, this differential equation was first considered in [45] (see also [55, 56] for further developments). It is often called the *modular differential equation*.

The first few of the D^i read to first order in q, i.e. where $E_2(q)$ is only taken as $1 - 24q + \mathcal{O}(q^2)$, and with the notation $D_q = q\frac{\partial}{\partial q}$, simply

$$
\begin{aligned}
D^0 &= 1 \,, \\
D^1 &= D_q \,, \\
D^2 &= D_q^2 - \frac{1}{6}D_q + q\, 4D_q \,, \\
D^3 &= D_q^3 - \frac{1}{2}D_q^2 + \frac{1}{18}D_q + q\left(12D_q^2 + \frac{4}{3}D_q \right) \,, \\
D^4 &= D_q^4 - D_q^3 + \frac{11}{36}D_q^2 - \frac{1}{36}D_q + q\left(24D_q^3 + \frac{4}{3}D_q^2 + \frac{4}{3}D_q \right) \,, \\
D^5 &= D_q^5 - \frac{5}{3}D_q^4 + \frac{35}{36}D_q^3 - \frac{25}{108}D_q^2 + \frac{1}{54}D_q + q\left(40D_q^4 - \frac{20}{3}D_q^3 + \frac{20}{3}D_q^2 \right) \,,
\end{aligned}
$$

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where all expressions are up to $\mathcal{O}(q^2)$. Of course, D^0 and D^1 are exact to all orders. The most general ansatz for the differential equation (15) is therefore

$$\sum_{k=0}^{5} \sum_{\substack{r,s \\ 4r+6s=10-2k}} a_{r,s}(E_4)^r(E_6)^s \left(\prod_{m=0}^{k} cod_{(2m)} \right) T(q) = 0 \, . \tag{18}$$

This differential equation must be satisfied by the characters of the irreducible highest weight representations of the triplet algebra. As we have explained before, there are four irreducible highest weight representations with conformal weights $h = 0, -1/8, 3/8$ and $h = 1$, and their corresponding characters are known [12, 13, 43]. In terms of the functions

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \, , \tag{19}$$

$$\theta_{\lambda,k}(q) = \sum_{n \in \mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}} \, , \tag{20}$$

as well as

$$(\partial\theta)_{\lambda,k}(q) = \sum_{n \in \mathbb{Z}} (2kn + \lambda) q^{\frac{(2kn+\lambda)^2}{4k}} \, , \tag{21}$$

they are given as

$$\chi_{-\frac{1}{8}}(q) = \theta_{0,2}(q)/\eta(q) \, , \tag{22}$$
$$\chi_0(q) = \tfrac{1}{2}(\theta_{1,2}(q) + (\partial\theta)_{1,2}(q))/\eta(q) \, , \tag{23}$$
$$\chi_{\frac{3}{8}}(q) = \theta_{2,2}(q)/\eta(q) \, , \tag{24}$$
$$\chi_1(q) = \tfrac{1}{2}(\theta_{1,2}(q) - (\partial\theta)_{1,2}(q))/\eta(q) \, . \tag{25}$$

Putting these pieces of information together we find that (up to an overall normalisation constant) (18) is uniquely determined to be the differential equation

$$\begin{aligned}
0 &= \left[\frac{143}{995328} E_4(q) E_6(q) + \frac{121}{82944} (E_4(q))^2 \, cod_{(2)} + \frac{65}{2304} E_6(q) \, cod_{(4)} \, cod_{(2)} \right. \\
&\quad \left. - \frac{163}{576} E_4(q) \, cod_{(6)} \, cod_{(4)} \, cod_{(2)} + cod_{(10)} \, cod_{(8)} \, cod_{(6)} \, cod_{(4)} \, cod_{(2)} \right] T(q) \, .
\end{aligned}$$

It is instructive to look at the leading order of the above equation. If we expand the Eisenstein series $E_n = 1 + g_{n,1}q + \mathcal{O}(q^2)$ with $g_{n,1}$ given by $g_{2,1} = -24, g_{4,1} = 240, g_{6,1} = -504$, we obtain

$$\begin{aligned}
0 &= \left(D_q^5 - \frac{5}{3} D_q^4 + \frac{397}{576} D_q^3 - \frac{427}{6912} D_q^2 - \frac{37}{82944} D_q + \frac{143}{995328} \right) T(q) \\
&\quad + q \left(40 D_q^4 - \frac{895}{12} D_q^3 + \frac{2209}{96} D_q^2 - \frac{209}{216} D_q - \frac{1573}{41472} \right) T(q) + \mathcal{O}(q^2) \, .
\end{aligned}$$

The zero-order term in q can be factorised as

$$\frac{1}{995328} (24 D_q - 11)(12 D_q - 13)(24 D_q + 1)(12 D_q - 1)^2 \, . \tag{26}$$

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Recalling that D_q has to be replaced by $L_0 - \frac{c}{24} = L_0 + \frac{1}{12}$ in order to relate (3) to (6), this therefore reduces, as required, to (13). If we make the ansatz

$$T(q) = q^{h+\frac{1}{12}} \left(1 + c_1 q + c_2 q^2 + c_3 q^3 + \mathcal{O}(q^4) \right), \qquad (27)$$

the above differential equation becomes, up to third order,

$$
\begin{aligned}
0 \;=\; & \frac{q^{h+1/12}}{64} \Big[q^0 \left(h^2(h-1)(8h+1)(8h-3) \right) \\
& + \; q^1 \left(c_1(h+1)^2 h(8h+9)(8h+5) + 2h(32h-45)(40h^2 - 5h - 1) \right) \\
& + \; q^2 \big(c_2(h+2)^2(h+1)(8h+17)(8h+13) + 2c_1(32h-13)(h+1)(40h^2 + 75h + 34) \\
& \qquad + 2(3840h^4 + 2840h^3 - 17331h^2 + 706h - 442) \big) \\
& + \; q^3 \big(c_3(h+3)^2(h+2)(8h+25)(8h+21) + 2c_2(h+2)(32h+19)(40h^2 + 155h + 149) \\
& \qquad + 2c_1(3840q^4 + 18200q^3 + 14229q^2 - 10076q - 10387) \\
& \qquad + 4(2560h^4 + 28880h^3 - 66574h^2 - 9772h - 12281) \big) + \mathcal{O}(q^4) \Big].
\end{aligned}
$$

5. Solving the modular differential equation.   As we have argued above, the modular differential equation is of fifth order for the triplet theory, and the space of vacuum torus amplitudes is therefore five-dimensional. On the other hand, we have only got four irreducible representations that give rise, via their characters, to four vacuum torus amplitudes (that solve the differential equation). Let us now analyse how to obtain a fifth, linearly independent, vacuum torus amplitude. First let us try to find a solution of the form (27). Because of the lowest order equation (26), this will only give rise to a solution provided that $h = -\frac{1}{8}, \frac{3}{8}, 0$ or $h = 1$. For each fixed h, one then finds that there is only one such solution, which therefore agrees with the corresponding character of the irreducible representation (*i.e.* with (22) – (25)). By the way, this conclusion was not automatic *a priori*, since there exist cases where the modular differential equation has two linearly independent solutions with the same conformal weight, both of which are of power series form. The simplest example is provided by the two $h = 0$ characters of the $c = 1 - 24k$ series of rational CFTs, $k \in \mathbb{N}$, with extended symmetry algebra $\mathcal{W}(2, 8k)$. One of these solutions belongs to the vacuum representation, the other to a second $h = 0$ representation (whose highest weight state has a non-zero W_0 eigenvalue).

The character of any highest weight representation always gives rise to a torus amplitude as in (27), and thus we have shown that the space of vacuum torus amplitudes for the triplet theory is not spanned by the characters of the (irreducible) highest weight representations. This was to be expected, given the analysis of [36].

It is not difficult to show that the missing, linearly independent solution can be taken to be

$$T_5(q) = \log(q)(\partial\theta)_{1,2}(q)/\eta(q). \qquad (28)$$

It is tempting to associate this vacuum torus amplitude with the logarithmic (indecomposable) highest weight representation $\mathcal{R}_0$ whose ground state conformal weight is $h = 0$, and this is indeed what was suggested in [43]. However, strictly speaking, this identification is only formal since $T_5(q)$ is not canonically determined by the above analysis. In particular, we could have equally replaced $T_5(q)$ by $T_5'(q) = T_5(q) + \alpha_0\chi_0(q) + \alpha_1\chi_1(q)$ for any (real) α_i,

$i = 1, 2$. It is therefore not clear which choice of the α_i should (formally) describe the character of the logarithmic representation $\mathcal{R}_0$. [It is also clear that the conventional character of $\mathcal{R}_0$ is in fact just

$$\chi_{\mathcal{R}_0}(q) = \chi_0(q) + \chi_1(q) = \theta_{1,2}(q)/\eta(q)\,, \tag{29}$$

and therefore does not account for the additional solution. The same is also true for the other indecomposable representation $\mathcal{R}_1$.] The additional logarithmic solution will appear in the description of the full torus amplitudes of the theory (see for example [58]). Indeed, for logarithmic theories the full vacuum torus amplitude is in general not just the partition function, but is built from the chiral torus amplitudes that were discussed above. The logarithmic solution also appears in a complete description of the boundary CFT [48].

One important consequence of this analysis is that the space of torus amplitudes does not have a canonical basis. This is unlike the case of a rational conformal field theory where the canonical basis for the space of vacuum torus amplitudes is given in terms of the characters of the irreducible representations. This canonical basis plays a crucial role in the Verlinde formula, where the matrix elements of the modular S-matrix with respect to this basis enters. It is therefore not surprising that the Verlinde formula does not work for this logarithmic conformal field theory: as was shown in [7], the fusion rules of the triplet theory cannot be diagonalised, and thus no Verlinde formula can exist.

Finally, we note that the solution $T_5(q)$ is in fact a torus amplitude in a slightly different sense. As we have seen above, $T_5(q)$ is proportional to $\tau\eta^2(q)$ and thus, up to the Liouville factor, proportional to one of the periods of the torus. Following an approach of Knizhnik [57], one can show that this torus amplitude is precisely one of the two conformal blocks one finds for the four-point function $\langle\mu\mu\mu\mu\rangle$ of the $h = -1/8$ field on the plane, provided we express it in terms of τ instead of the crossing ratio x with the help of the elliptic modulus $\kappa^2(\tau) = x$, see [58]. Actually, this four-point function gives the complex plane the geometry of a double covering with two branch cuts, *i.e.* of a torus.

6. A general analysis. For any (logarithmic) conformal field theory which satisfies the C_2 condition, the mere existence of a finite order differential equation allows us to derive some relations and bounds for the highest weights. As argued above, the torus amplitudes of such a theory have to satisfy an n-th order holomorphic modular invariant differential equation of the form (15),

$$\left[D^n + \sum_{r=0}^{n-1} f_r(q)D^r\right] T(q) = 0\,, \tag{30}$$

where the $f_r(q) \in \mathbb{C}[E_4, E_6]$ are modular functions of weight $2(n-r)$. These coefficient functions may be expressed in terms of a set of n linearly independent solutions $T_1(q), \dots, T_n(q)$ of the differential equation (30). However, in contrast to [45], these solutions cannot in general be identified with the characters of representations. In particular, we cannot assume that the $T_i(q)$ have a good power series expansion in q up to a common fractional power $h_i - c/24$ mod 1.[2] Instead we want to assume that they lie in $\mathbb{C}((q))[\tau]$, *i.e.* that they are

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[2]We will in the following always speak of power series expansions in q with the silent understanding that a common fractional power is allowed, *i.e.* that the functions can be expanded as $T(q) = q^\alpha \sum_{k=0}^{\infty} a_k q^k$, $\alpha \in \mathbb{Q}$.

power series in q times a polynomial in $\tau \equiv \frac{1}{2\pi i} \log(q)$. This is certainly the case for the triplet theory discussed before.

With this in mind we can adapt the analysis of [45] to this more general setting. The main difference will be that we shall *not* assume in the following that the highest weights are all different, $h_i \neq h_j$ for $i \neq j$, but only that $T_i(q) \neq T_j(q)$ for $i \neq j$. Note that the asymptotic behaviour of two functions $T_i(q)$ and $T_j(q)$ in the limit $q \to 0$ (or $\tau \to +i\infty$) is the same whenever $T_j(q) = p(\tau)T_i(q)$ for a polynomial p, provided $T_i(q) \sim q^\alpha$ with $\alpha \neq 0$. The case $\alpha = 0$ occurs precisely when $h_i - c/24 = 0$. We note that all known logarithmic conformal field theories, except for $c = 0$, do not have any logarithmic representations with $h = c/24$.

As in [45] we now express the coefficients of the modular differential equation in terms of the Wronskian of a set of n linearly independent solutions as

$$f_r(q) \;=\; (-1)^{n-r} W_r(q)/W_n(q)\,, \tag{31}$$

$$W_r(q) \;=\; \det \begin{pmatrix} T_1(q) & \cdots & T_n(q) \\ D^1 T_1(q) & \cdots & D^1 T_n(q) \\ \vdots & & \vdots \\ D^{r-1} T_1(q) & \cdots & D^{r-1} T_n(q) \\ D^{r+1} T_1(q) & \cdots & D^{r+1} T_n(q) \\ \vdots & & \vdots \\ D^n T_1(q) & \cdots & D^n T_n(q) \end{pmatrix}. \tag{32}$$

The torus amplitudes, considered as functions in τ, are non-singular in $\mathbb{H}$. As a consequence, the same applies for the W_r. Therefore, the coefficients f_r can have singularities only at the zeroes of W_n. We will now show that the total number of zeroes of W_n can be expressed in terms of the number n of linearly independent torus amplitudes, the central charge c and the conformal weights h_i associated to the torus amplitudes $T_i(q)$. In order to do so, we note that in the $\tau \to +i\infty$ limit, the torus amplitudes behave as $\exp(2\pi i(h_i - \frac{c}{24})\tau)$. With the above caveat concerning the case $h = c/24$, this applies to all torus amplitudes independently of whether they are pure power series in q, or whether they have a τ-polynomial as additional factor. This implies that $W_n \sim \exp(2\pi i(\sum_i h_i - n\frac{c}{24})\tau)$, which says that W_n has a pole of order $n\frac{c}{24} - \sum_i h_i$ at $\tau = i\infty$. Now, W_n involves precisely $\frac{1}{2}n(n-1)$ derivatives meaning that it transforms as a modular form of weight $n(n-1)$. Both facts together allow us to compute the total number of zeroes of W_n, which is

$$\frac{1}{6}\ell \equiv -\sum_{i=1}^{n} h_i + \frac{1}{24}nc + \frac{1}{12}n(n-1) \geq 0\,, \qquad \ell \in \mathbb{Z}_+ - \{1\}\,. \tag{33}$$

This number cannot be negative since W_n must not have a pole in the interior of moduli space. We note that (33) is always a multiple of $\frac{1}{6}$ since W_n, as a single valued function in Teichmüller space, may have zeroes at the ramification points $\exp(\frac{1}{3}\pi i)$ and $\exp(\frac{1}{2}\pi i)$ of order $\frac{1}{3}$ and $\frac{1}{2}$, respectively. Equation (33) provides a simple bound on the sum of the conformal weights.

For example, for the case of the $c = -2$ triplet theory, we have

$$-\left[(-\frac{1}{8}) + (0) + (0) + (\frac{3}{8}) + (1)\right] + \frac{1}{24}(5)(-2) + \frac{1}{12}(5)(4) = 0\,, \tag{34}$$

in agreement with the above analysis.

7. The other triplet theories. The analysis presented so far can in principle be generalised to all members of the $c_{p,1}$ series of triplet models. In practice, however, we have not found it possible to give uniform explicit expressions. The pattern which emerges in the treatment of the $c = -2$ case, *i.e.* the case $p = 2$, however, seems to be of a generic nature. Indeed, all the $c_{p,1}$ models are C_2 cofinite [53] and the characters of their irreducible representations are all known. They close under modular transformations provided that a certain number of 'logarithmic vacuum torus amplitudes' (the analogues of $T_5(q)$) are added to the set. In fact, the characters of the irreducible representation, together with additional torus amplitudes which we may again associate to the indecomposable representations, read [43]

$$\chi_{0,p}(q) = \frac{1}{\eta(q)} \Theta_{0,p}(q) \, , \tag{35}$$

$$\chi_{p,p}(q) = \frac{1}{\eta(q)} \Theta_{p,p}(q) \, , \tag{36}$$

$$\chi_{\lambda,p}^+(q) = \frac{1}{p\eta(q)} \left[(p - \lambda)\Theta_{\lambda,p}(q) + (\partial\Theta)_{\lambda,p}(q) \right] \, , \tag{37}$$

$$\chi_{\lambda,p}^-(q) = \frac{1}{p\eta(q)} \left[\lambda\Theta_{\lambda,p}(q) - (\partial\Theta)_{\lambda,p}(q) \right] \, , \tag{38}$$

$$\tilde{\chi}_{\lambda,p}(q) = \frac{1}{\eta(q)} \left[2\Theta_{\lambda,p}(q) - i\alpha\log(q)(\partial\Theta)_{\lambda,p}(q) \right] \, , \tag{39}$$

where $0 < \lambda < p$ and where we made use of the definitions (19) to (21). As before, the 'logarithmic' torus amplitudes $\tilde{\chi}_{\lambda,p}$ are not uniquely determined by these considerations since α is a free constant; the form given above is convenient for constructing modular invariant partition functions. One should note, however, that for logarithmic conformal field theories the complete space of states of the full non-chiral theory is not simply the direct sum of tensor products of chiral representations (see for example [59]).

The congruence subgroup for the $c_{p,1}$ model is $\Gamma(2p)$. There are $2p$ characters corresponding to irreducible representations, and $(p - 1)$ 'logarithmic' torus amplitudes, giving rise to a $(3p - 1)$ dimensional representation of the modular group. In particular, we therefore expect that the order of the modular differential equation is $(3p - 1)$. Furthermore, we expect that the dimension of Zhu's algebra is $6p - 1$: it follows from the structure of the above vacuum torus amplitudes that p of the irreducible representations have a one-dimensional ground state space, while the other p irreducible representations have ground state multiplicity two; as above one may furthermore expect that each of the $(p - 1)$ logarithmic representations probably leads to one additional state, thus giving altogether the dimension $p + 4p + (p - 1) = 6p - 1$.

While we have not managed to write down a general expression for the modular differential equation for all p, we can give support for these conjectures by analysing the $p = 3$ triplet model with $c = -7$. The vacuum character of this theory is $\chi_{2,3}^+(q)$. Under the assumption that the modular differential equation is in fact of order $3p - 1 = 8$, we can determine it uniquely by requiring it to be solved by this vacuum character. Explicitly we find

$$0 = \left[\left(\frac{833}{53747712} E_4(q)(E_6(q))^2 - \frac{990437}{36691771392} (E_4(q))^4 \right) \right.$$

$$- \frac{40091}{143327232}(E_4(q))^2 E_6(q) \, cod_{(2)}$$

$$+ \left(\frac{115}{746496}(E_6(q))^2 + \frac{53467}{47775744}(E_4(q))^3 \right) cod_{(4)} \, cod_{(2)}$$

$$- \frac{5897}{124416} E_4(q) E_6(q) \, cod_{(6)} \, cod_{(4)} \, cod_{(2)}$$

$$+ \frac{10889}{55296}(E_4(q))^2 \, cod_{(8)} \, cod_{(6)} \, cod_{(4)} \, cod_{(2)}$$

$$+ \frac{157}{432} E_6(q) \, cod_{(10)} \, cod_{(8)} \, cod_{(6)} \, cod_{(4)} \, cod_{(2)}$$

$$- \frac{21}{16} E_4(q) \, cod_{(12)} \, cod_{(10)} \, cod_{(8)} \, cod_{(6)} \, cod_{(4)} \, cod_{(2)}$$

$$+ \left. cod_{(16)} \, cod_{(14)} \, cod_{(12)} \, cod_{(10)} \, cod_{(8)} \, cod_{(6)} \, cod_{(4)} \, cod_{(2)} \right] T(q) \, . \qquad (40)$$

If we make the ansatz that $T(q)$ is of the form

$$T(q) = q^{h - \frac{c}{24}} \sum_{n=0}^{\infty} \sum_{k=0}^{1} c_{k,n} \tau^k \, q^m \, , \qquad (41)$$

we obtain, to lowest order the polynomial condition

$$
\begin{aligned}
0 \;=\;\; & \frac{1}{2304}(1 + 4h)^2 h^2 (h-1)(12h - 5)(3h + 1)(4h - 7)(c_{1,0}\tau + c_{0,0}) \\
+\; & \frac{1}{1152}(1 + 4h)h(2304h^5 - 5280h^4 + 2160h^3 + 870h^2 - 229h - 35)c_{1,0} \;+\; \mathcal{O}(q) \, . \quad (42)
\end{aligned}
$$

As expected, we can read off from this expression the allowed conformal weights: if the character does not involve any powers of τ ($c_{1,0} = 0$), then h needs to be from the set $h \in \{0, -1/4, 1, 5/12, -1/3, 7/4\}$. Furthermore, we have two 'logarithmic' torus amplitudes with $h = 0$ and $h = -1/4$. This then fits nicely together with the fact that there are in fact two indecomposable highest weight representations with these conformal weights [60].

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